Density of hyperbolicity and tangencies in sectional dissipative regions.

Enrique R. Pujals   Martin Sambarino

Preliminary Draft

Abstract

In the present paper we show that in sectionally dissipative region of the Limit set (noted with $L(f,1)$) either hold that f can be C^1-approximated by a diffeomorphism exhibiting a sectional dissipative homoclinic tangency or by another one such that $L(f,1)$ is a hyperbolic compact set. The prof goes extending some results on dominated splitting obtained in compact surfaces.

1 Introduction and statements.

For a long time (mainly after Poincaré) it has been a goal of the theory of dynamical systems to describe the dynamics from the generic viewpoint, that is, to describe the dynamics of "big sets" (residual, dense, etc.) within the space of all dynamical systems.

It was briefly thought in the sixties that this could be realized by the so-called hyperbolic ones: systems with the assumption that the tangent bundle over the limit set $L(f)$ (the closure of the accumulations points of any orbit) splits into two complementary subbundles $T_{L(f)}M = E^s \oplus E^u$ so that vectors in E^s (respectively E^u) are uniformly forward (respectively backward) contracted by the tangent map Df. Under this assumption, it was proved that the limit set decomposes into a finite number of disjoint transitive sets such that the asymptotic behavior of any orbit is described by the dynamics in the trajectories in those finite transitive sets (see [S]).

Uniform hyperbolicity was soon realized to be a less universal property than was initially thought: In fact, it was through the seminal works of Newhouse (see [N1], [N2], [N3]) that hyperbolicity was shown not to be dense in the space of C^r diffeomorphisms ($r \geq 2$) of compact surfaces. The underlying mechanism here was the presence of a *homoclinic tangency*: non-transversal intersection of the stable and unstable manifold of a periodic point. The unfolding of a homoclinic tangencies leads to the nowadays so-called "Newhouse phenomena", i.e., residual subsets of diffeomorphisms displaying infinitely many periodic attractors. In particular, this shows that the unfolding of tangencies "destroys in a robust way" transitive sets.

In few words, for surface diffeomorphisms, *homolicinic tangencies* can be understood as an obstruction to hyperbolicity and in particular as an obstruction to decompose the Limit set into a finite number of transitive isolated sets. In this direction, around the early 80's, Palis conjectured (see [P] and [PT]) that homoclinic tangencies are very common in the complement of the hyperbolic systems:

Any C^r diffeomorphism on a surface can be C^r approximated by one which is hyperbolic or by one exhibiting a homoclinic tangency.

The above conjecture was proved to be true for the case of surfaces and the C^1 topology (see [PS1]).

Theorem ([PS1]): *Let M^2 be a surface. Every $f \in Diff^1(M^2)$ can be C^1-approximated either by a diffeomorphism exhibiting a homoclinic tangency or by an Axiom A diffeomorphism.*

In higher dimension, robust transitive sets (sets that remains transitive after perturbation of the dynamic), can coexist with the presence of a homoclinic tangency (see for instances the examples showed in [BV] of robust transitive systems). In particular, it follows that tangencies are not the universal obstruction to decompose the Limit set in a finite number of transitive isolated sets.

However, it was shown in [PV] that for smooth diffeomorphisms on manifold with dimension larger than two, the unfold of tangencies associated to sectional dissipative periodic points (tangencies associated to a periodic point such that the modulus of the product of any pair of eigenvalues is smaller than one) leads to the same Newhouse phenomena that holds in dimension two.

Regarding the previous comments, it is naturally to ask if it holds that *any diffeomorphisms on a finite dimensional manifold can be either C^r-approximated by another one such its dynamic is hyperbolic restricted to a sectionally dissipative regions of the limit set, or it is C^r-approximated by a system exhibiting a sectional dissipative homoclinic tangency.*

To be precise, let us introduce some definitions.

A hyperbolic diffeomorphism means a diffeomorphism such that its limit set is hyperbolic. The limit set of f is the closure of the forward and backward accumulation points of all orbits and we note it with $L(f)$. A set Λ is called hyperbolic for f if it is compact, f-invariant and the tangent bundle $T_\Lambda M$ can be decomposed as $T_\Lambda M = E^s \oplus E^u$ invariant under Df and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$|Df^n_{/E^s(x)}| \leq C\lambda^n$$

and

$$|Df^{-n}_{/E^u(x)}| \leq C\lambda^n$$

for all $x \in \Lambda$ and for every positive integer n.

Moreover, a diffeomorphism is called Axiom A, if the non-wandering set is hyperbolic and it is the closure of the periodic points.

We recall that the stable and unstable sets

$$W^s(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to \infty\},$$

$$W^u(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to -\infty\}$$

are C^r-injectively immersed submanifolds when p is a hyperbolic periodic point of f. A point of intersection of these manifolds is called a homoclinic point.

Definition 1 Homoclinic tangency. *We say that f exhibits a homoclinic tangency if there is a periodic point p such that there is a point $x \in W^s(p) \cap W^u(p)$ with $T_xW^s(p) + T_xW^u(p) \neq T_xM$. Given an open set V, we say that the tangency holds in V if p and x belong to V.*

Definition 2 *Given $f : M \to M$ be a C^1-diffeomorphism of a finite dimensional compact riemannian manifold M, we say that f is sectionally dissipative at a point x if for any two dimensional subspace L hold that*

$$|det(Df_{x|L})| < 1.$$

Given $\lambda > 0$, we note with $SD_f(\lambda)$ the set

$$SD_f(\lambda) := \{x \in M : |det(Df_{x|L})| < \lambda \text{ for any two dimensional subspace } L \subset T_xM\}.$$

We take

$$\mathcal{SD}_f(\lambda) := \overline{\{x : \mathcal{O}(x) \subset SD_f(\lambda)\}},$$

where $\mathcal{O}(x)$ is the orbit of x by f. We define the sectionally dissipative limit set as

$$L(f, 1) := L(f) \cap \mathcal{SD}_f(1).$$

Definition 3 *We say that a tangency is sectionally dissipative if the tangency is associated to a sectionally dissipative periodic point.*

Related to these notions we can prove the following theorem.

Theorem A: *Let $f : M \to M$ be a C^2-diffeomorphism of a finite dimensional compact riemannian manifold M. Let us assume that $L(f, 1)$ is an isolated set in $L(f)$. Then, f can be C^1-approximated by a diffeomorphism exhibiting a sectional dissipative homoclinic tangency or by another one such that $L(f, 1)$ is a hyperbolic compact set.*

Roughly speaking, in this paper we deal with the "sectionally dissipative region of the Limit set".

In the direction to prove the previous theorem, we shall extend some results on dominated splitting we have obtained in compact surfaces. Let $f : M \to M$ be a C^1 diffeomorphism of a compact riemmanian manifold M. An f-invariant set Λ is said to have dominated splitting if we can decompose its tangent bundle in two invariant subbundles $T_\Lambda M = E \oplus F$, such that:

$$\|Df^n_{/E(x)}\| \|Df^{-n}_{/F(f^n(x))}\| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0.$$

with $C > 0$ and $0 < \lambda < 1$.

We say that the dominated splitting is a *codimension one dominated splitting* if dimension$(F) = 1$. We say that a codimension one dominated splitting is a *contractive codimension one dominated splitting* if the direction E is a contractive direction, i.e.: there exists $C > 0$ and $\lambda < 1$ such that for any x and any n holds that $|Df^n_{|E_x}| < C\lambda^n$. We denote the direction E as E^s.

The strategy of the proofs of theorem A consists first in showing that if $L(f, 1)$ cannot be approximated by another system exhibiting a sectionally dissipative homoclinic tangency, then $L(f, 1)$ exhibits a contractive codimension one dominated splitting $E^s \oplus F$ (see theorem B).

Definition 4 *Given a compact inavariant set Λ in $L(f, 1)$ we say that f_Λ is C^1-far from sectionally dissipative homoclinic tangencies, if there is a neighborhood $\mathcal{U} \subset Diff^1(M)$ of f and a neighborhood V of Λ such that any $g \in \mathcal{U}$ does not exhibit a sectionally dissipative tangency in V.*

In particular, we say that $f_{|L(f,1)}$ is C^1-far from sectionally dissipative homoclinic tangencies, if there is a neighborhood $\mathcal{U} \subset Diff^1(M)$ of f such that any $g \in \mathcal{U}$ does not exhibit a sectionally dissipative tangency in $L(g, 1)$. Moreover, Before to obtain theorem A, we get a weak version of it in terms of the following dichotomy: either f is approximated by a system having a sectionaly dissipative homoclinic tangeny, or $L(f,1)$ exhibits a contractive codimension one dominated splitting.

Theorem B:

Let Λ be a compact invariant set in $L(f,1)$. Let us assume that f_Λ is C^1-far from sectionally dissipative tangencies. Then, Λ has a contractive codimension one dominated splitting.

Let us assume that $f_{|L(f,1)}$ is C^1-far from sectionally dissipative tangencies. Then, $|L(f,1)$ has a contractive codimension one dominated splitting.

Later, we prove that under certain conditions, contractive codimension one dominated splitting are actually hyperbolic. Before to state it, we note with $P_0(f)$ the sets of sinks of f and we get the set $L_0(f)$ defined as

$$L_0(f) = Closure(L(f) \setminus P_0(f)).$$

Theorem C: *Let $f : M \to M$ be a C^2-diffeomorphism. Let Λ be a compact invariant set contained in $L(f)$ and exhibiting contractive codimension one dominated splitting. Let also assume that either Λ is isolated in $L(f)$ or is isolated in $L_0(f)$ and all the periodic points in Λ are hyperbolic. Then, we get that,*

$$\Lambda = \Lambda_1 \cup \Lambda_2$$

where Λ_1 is a hyperbolic set and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, ...\mathcal{C}_n$, normally hyperbolic and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).

Remark 1.1 *Observe that in theorem C we are not assuming that the set Λ is contained in $L(f,1)$.*

The next corollary follows if it assume that the whole manifold has has contractive codimension one dominated splitting.

Corollary 1: *Let $f : M \to M$ be a C^2-diffeomorphism. Assume that M has contractive codimension one dominated splitting and all the hyperbolic periodic points are of saddle type. Then f is an Anosov diffeomorphism and $M = T^n$.*

Now we are in condition to show how the proof of theorem A follows from theorem B and C.

Proof of theorem A:

To conclude theorem A, we assume that $f_{|L(f,1)}$ is C^1-far from sectionally dissipative tangencies. Therefore, by theorem B holds that $f_{|L(f,1)}$ exhibits a contractive codimension one dominated splitting. Later, we approximated f by another C^2-diffeomorphisms g such that $L(g,1)$ remains isolated in $L(g)$, all the periodic points in $L(g,1)$ are hyperbolic and $L(g,1)$ is close (in the Hausdorff topology) to $L(f,1)$. Therefore $L(g,1)$ has a contractive codimension one dominated splitting such that all the periodic points in $L(g,1)$ are hyperbolic. Then, we can apply the theorem C, concluding that $L(g,1)$ is decomposed in a finite number of hyperbolic sets and a finite number of normally hyperbolic invariant curve with dynamics conjugated to an irrational rotation. After a second perturbation, we can assume that the dynamics in the normally hyperbolic curves are Morse-Smale, concluding so that $L(g,1)$ is hyperbolic.

∎

As we said before, there is a srong relation between the presences of infinitely many sinks with unbounded period and the unfolding of a sectionally dissipative homoclinic tangency. In fact, in [PV] it is prove that any smooth diffeomorphisms with a sectionally dissipative homoclinic tangency can be C^r−approximated ($r \leq 2$) by another one having infinitely many sinks with unbounded period (and also contained in a sectionally dissipative region).

The next theorem shows a weak converse to the theorem states in [PV].

Theorem D: *Let $f \in Diff^2(M)$ having infinitely many sinks with unbounded period and contained in $L(f,1)$. Let Λ be the accumulation set of the sinks of f. Let us also assume that Λ is isolated in $L_0(f)$ and such that all the periodic points in Λ are hyperbolic. Then, for any neighborhood V of Λ, it follows that f can be C^1 approximated by another diffeomorphism exhibiting a sectional dissipative tangency in V.*

Proof of theorem D:

To conclude theorem D, we assume that f_Λ is C^1−far from sectionally dissipative tangencies. Therefore, by theorem B holds that Λ exhibits a contractive codimension one dominated splitting. Then, we can apply the theorem C, concluding that Λ is decomposed in a finite number of hyperbolic sets and a finite number of normally hyperbolic invariant curve with dynamics conjugated to an irrational rotation. Since any normally hyperbolic curve is isolated and since Λ is the accumulation set of periodic points, it follows that Λ is hyperbolic such that the unstable subbundle has dimension one. Therefore, the same holds for the closure of the maximal invariant set of some neighborhood of Λ. Since the sinks of largest period remains in a neighboprhood of Λ, we get a contradiction.

■

The paper is organized as follows: In section 2 we show that if it cannot be created sectionally dissipative homoclinic tangencies by C^1−perturbations, it follows that $L(f,1)$ exhibits a contractive codimension one dominated splitting. In section 3 we show the existence of Markov partition for a general class of sets that include the homoclinic classes under the hypothes of contractive codimension one dominated splitting. This results are a fundamental tool in the proof of the rest of the theorems. In section 4 is done the proof of theorem C.

2 Dominated splitting for systems far from sectionally dissipative homoclinic tangencies. Proof of theorem B.

We give the proof for the case that $\Lambda = L(f,1)$. The general case, is similar. The theorem follows from techniques introduced in [PS1], and it goes in two steps:
Step I. If $f_{|L(f,1)}$ is C^1-far from sectionally dissipative tangencies. Then, $f_{|L(f,1)}$ exhibits a codimension one dominated splitting.
Step II. Any codimension one dominated splitting over $L(f,1)$ is a contractive codimension one dominated splitting.

First we recall some definitions:

Definition 5 *It is said that a hyperbolic periodic point has stable index d if the number of stable eigenvalues (or eigenvalues with modulus smaller than one) counted with multiplicity is d.*

Definition 6 *We note with*

$$Per(f,\lambda)$$

the set of periodic point of f such that they have stable index $n-1$ and they belong to $\mathcal{SD}(\lambda)$.

Lemma 2.0.1 *Let $x \in L(f,1)$. Then, there exist a sequence of diffeormorphisms $\{g_m\}_{\{m>0\}}$ converging to f in the C^1-topology, a sequences of periodic points $\{q_m\}_{\{m>0\}}$ converging to x such that $q_m \in Per(g_m,1)$ and q_m has stable index $n-1$.*

First we introduce the notion of angle between a vector and a hyperplane:

Definition 7 *Let v a vector in $\mathbb{R}^n$ and S a hyperplane in $\mathbb{R}^n$. It is defined the angle $\alpha(v,S)$ as the unique positive number in $[0,\frac{\pi}{2}]$ such that*

$$\cos(\alpha(v,S)) = \frac{<v,w>}{|v||w|}$$

where $<.,.>$ is the internal product induced by the riemanniam metric and w is the orthogonal projection of v over S.

The following lemma is a atraightforward adaptation of lemma 2.2.2 of [PS1].

Lemma 2.0.2 *Let us assume that $f_{|L(f,1)}$ is C^1-far from sectionally dissipative tangencies. Then there exists a neighborhood $\mathcal{U}$ of f and a positive constant α such that for any $g \in \mathcal{U}$ and any $q \in Per(g,1)$ follows that*

$$\alpha(E^s_q, E^u_q) > \alpha,$$

where E^s_q is the stable eigenspace of $D^{n_q}_q$ and E^u_q is the unstable eigenspace of $D^{n_q}_q$ (n_q is the period of q).

Proof:

The proof is similar to the proof of lemma 2.2.2 in [PS1]. The basic idea is that if there is not possible to get an uniform angle bounded away by zero, then it is possible to perform a C^1-perturbation in a way to create a homoclinic tangency associated to a sectionally dissipative periodic point.

$\blacksquare$

End of Proof of theorem B:

To conclude the proof, we follow [M2]. In fact, the goal is to show that there exist a positive integer n_0 and a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ and any $q \in Per(g,1)$ follows that

$$\|Dg^{n_0}(E^s(g^j(q)))\|\|Dg^{-n_0}(E^u(g^{j+n_0}(q)))\| < \frac{1}{2}.$$

If this does not hold, then by a C^1-perturbation it is contradicted the lemma 2.0.2. To conclude that $L(f,1)$ exhibits a codimension one dominated splitting, $E \oplus F$, we use lemma 2.0.1 to extend the domination property over $Per(g,1)$ to $L(f,1)$.

Therefore we have concluded that there is a splitting $E \oplus F$ over $T_{|L(f,1)}M$, a positive integer k and a positive constant $\lambda < 1$ such that

$$\|Df_{|E(x)}\|\|Df^{-k}_{|F(f^k(x))}\| < \lambda.$$

To show that the the subbundle E is contractive, we use that the splitting holds in a sectionally dissipative region. In fact, and without lose of generality assuming that $k = 1$, given $x \in L(f,1)$, let $v \in E_x$ be a vector of norm one such that $\|Df_{|E(x)}\| = \|Df(v)\|$. Let $w \in F_{f(x)}$ be a vector of norm one such that $\|Df^{-1}_{|F(x)}\| = \|Df^{-1}(w)\|$. Therefore

$$\|Df_{|E(x)}\|\|Df^{-1}_{|F(f(x))}\| = \|Df(v)\|\|Df^{-1}(w)\| < \lambda.$$

On the other hand, if we take L the subspace generated by v and $Df^{-1}(w)$ it follows that

$$\det(Df_{|L})| = \frac{\|Df(v)\|}{\|Df^{-1}(w)\|} \leq 1$$

and so

$$\|Df_{|E(x)}\|^2 = \|Df(v)\|\|Df^{-1}(w)\|\frac{\|Df(v)\|}{\|Df^{-1}(w)\|} < \lambda.$$

$\blacksquare$

Proof of lemma 2.0.1:

Before to give the proof we recall a technical version of the closing lemma that appears in [Pg].

Lemma 2.0.3 *[Pg] Let $x \in L(f)$, then for any $\epsilon > 0$ there exist a diffeomorphism g $C^1 - \epsilon-$close to f, a periodic point q and a positive integer n such that*

$$dist(f^{n+j}(x), g^j(q)) < \epsilon \quad 0 \leq j \leq n_q,$$

where n_q is the period of q.

The next lemma, is a simple yet powerful perturbation technique (in the C^1 topology). This results says, for instance, that any small perturbation of the linear maps along a periodic orbit can be realized through a diffeomorphism C^1-nearby:

Lemma 2.0.4 *[Fr, Lemma 1.1] Let M be a closed n-manifold and $f : M \to M$ be a C^1 diffeomorphism, and let $\mathcal{U}(f)$ a neighborhood of f. Then, there exist $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ and $\delta > 0$ such that if $g \in \mathcal{U}_0(f)$, $S \subset M$ is a finite set, $S = \{p_1, p_2, \dots p_m\}$ and L_i, $i = 1, \dots, m$ are linear maps $L_i : T_{p_i}M \to T_{f(p_i)}M$ satisfying $\|L_i - D_{p_i}g\| \leq \delta$, $i = 1, \dots, m$ then there exists $\tilde{g} \in \mathcal{U}(f)$ satisfying $\tilde{g}(p_i) = g(p_i)$ and $D_{p_i}\tilde{g} = L_i$, $i = 1, ..., m$. Moreover, if U is any neighborhood of S then we may chose $\tilde{g}$ so that $\tilde{g}(x) = g(x)$ for all $x \in \{p_1, p_2 \dots p_m\} \cup (M \backslash U)$.*

Now we state a slight modification of a theorem due to Pliss proved in [Pl].

Theorem 2.1 *Let $x \in L(f)$ and let us assume that there is a sequences of diffeormorphisms $\{g_m\}$ converging to f in the C^1-topology, and sequences of attracting periodic points $\{q_m\}$ converging to x with unbounded period. Then, for any $\epsilon > 0$ there exists m_0 such that for any g_m with $m > m_0$ follows that there is $\hat{g}_m$ C^1-close to g_m such that q_m is a periodic point of $\hat{g}_m$ with an eigenvalue with modulus greater than one.*

Now we can proceed to prove the lemma 2.0.1 using the three previous result listed. By lemma 2.0.3, follows that there is a sequences of periodic of diffeormorphisms $\{g_m\}_{\{m>0\}}$ converging to f in the C^1-topology, and a sequences of attracting periodic points $\{q_m\}_{\{m>0\}}$ converging to x such that for each m there exists a sequences $\{\epsilon_m\}_{\{m>0\}}$ converging to zero and for each m there are two positive integer $k_m < n_m$ such that

$$dist(f^{k_n+j}(x), g_m^{k_n+n_m}(q_n)) < \epsilon_m \quad 0 \leq j \leq n_m,$$

where n_m is the period of q_m. Therefore, it follows that there is a sequences $\{\delta_m\}_{\{m>0\}}$ converging to zero such that

$$q_m \in \mathcal{SD}(g_m, (1+\delta_m)).$$

In particular, this implies that the product of any eigenvalues of q_n is smaller than $(1+\delta_m)^{n_m}$. By theorem 2.1, we can assume that q_m has at least one eigenvalues has modulus largest than one. Using lemma 2.0.4, for a perturbation of g_m we can assume that there is only one eigenvalue with modulus largest than one and that $q_m \in Per(g_m, 1)$. In fact, to perform that, we consider $\{E^i_{q_m}\}_{\{1 \leq i \leq n\}}$ the eigenspace of $D_{q_m}g_m^{n_m}$ associated to each eigenvalue, noting with $E^n_{q_m}$ the one associated to the eigenvalue with largest modulus. Then we consider the set of linear maps $\{A_j\}_{\{1 \leq j \leq n_m-1\}}$ defined as

$$A_{j|D^{j-1}(E^i_{q_m})} = \frac{1}{1+2\delta_m} D_{g^j_m(q_m)} g_{m|D^{j-1}(E^i_{q_m})} \quad 1 \leq i \leq n-1, \;\; 1 \leq j \leq n_m - 1;$$

$$A_{j|D^{j-1}(E^n_{q_m})} = D_{g^j_m(q_m)} g_{m|D^{j-1}(E^n_{q_m})} \quad 1 \leq j \leq n_m - 1.$$

Then, by lemma 2.0.4 there is $\hat{g}_m$ close to g_m such that the orbit of q_m by g_m remains the same for $\hat{g}_m$ and such that

$$D_{q_m}\hat{g}_m^{n_m} = \Pi_{j=0}^{n_m-1} A_j.$$

Therefore, it follows that

$$q_m \in \mathcal{SD}(\hat{g}_m, 1)$$

and q_m is a saddle point (meaning that $D^{n_m}_{q_m}\hat{g}_m$ has at least one eigenvalue largest than one) and therefore $D^{n_m}_{q_m}\hat{g}_m$ has only one eigenvalue with modulus larger and

$$q_m \in Per(\hat{g}_m, 1).$$

■

3 Markov partitions for contractive codimension one dominated splitting.

in this section, we show the existences of Markov partition for "basic sets" exhibiting a contractive codimension one dominated dominated splitting (see subsection 3.2 for the correct definitions). First, in the next subsection we show some dynamical properties that the center unstable manifold exhibits.

3.1 Some dynamical properties.

In the sequel, we will prove that the tangent manifold has dynamical meaning and we will use this to prove in subsection 3.2 that for some special sets can exhibit a Markov Partition

Let $I_1 = (-1, 1)$ and $I_\epsilon = (-\epsilon, \epsilon)$, and denote by $Emb^2(I_1, M)$ the set of C^2-embedings of I_1 on M, and denote by $Emb^2(I_1^{n-1}, M)$ the set of C^2-embedings of I_1^{m-1} on M, where n is the dimension of M.

Recall by [HPS], that a contractive codimension one dominated splitting imply the next

Lemma 3.1.1 *There exist two continuous functions $\phi^s : \Lambda \to Emb^2(I_1^{n-1}, M)$ and $\phi^{cu} : \Lambda \to Emb^2(I_1, M)$ such that if define $W^s_\epsilon(x) = \phi^s(x)I_\epsilon^{n-1}$ and $W^{cu}_\epsilon(x) = \phi^{cu}(x)I_\epsilon$ the following properties holds:*

a) $T_xW^s_\epsilon(x) = E(x)$ and $T_xW^{cu}_\epsilon(x) = F(x)$,

b) There is $\lambda < 1$ such that

$$f(W^s_\epsilon(x)) \subset W^s_{\lambda\epsilon}(f(x)),$$

c) for all $0 < \epsilon_1 < 1$ there exist ϵ_2 such that and

$$f^{-1}(W^{cu}_{\epsilon_2}(x)) \subset W^{cu}_{\epsilon_1}(f^{-1}(x)).$$

We will conclude some dynamical properties for the center unstable manifold tangent to the F direction.

First, we appeal to some results and definitions proved in [PS3] for "codimension one dominated splitting". It what follows with $\ell(I)$ it is denoted the usual length of an arc I.

Definition 8 *Let $f : M \to M$ be a C^2 diffeomorphism and let Λ be a compact invariant set having dominated splitting $E \oplus F$ with $dim(F) = 1$. Let U be an open set containing Λ where is possible to extend the previous dominated splitting. We say that a C^2-arc I in M (i.e, a C^2-embedding of the interval (-1,1)) is a δ-E-arc provided the next two conditions holds:*

1. *$f^n(I) \subset U$, and $\ell(f^n(I)) \leq \delta$ for all $n \geq 0$.*
2. *$f^n(I)$ is always transverse to the E-subbundle.*

Related to this kind of arcs it is proved in [PS3] the following result.

Theorem 3.1 Denjoy theorem. *There exists δ_0 such that if I is a δ-E-interval with $\delta \leq \delta_0$, then one of the following properties holds:*

1. $\omega(I) = \cup_{\{x \in I\}} \omega(x)$ *is a periodic simple closed curve and* $f^m_{/\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ *(where* m *is the period of* $\mathcal{C}$*) is conjugated to an irrational rotation,*

2. $\omega(I) \subset J$ *where* J *is a periodic arc.*

As a consequence of the Denjoy Theorem, we can conclude the following lemma related to the center unstable manifolds.

Lemma 3.1.2 *For all* $\gamma < \delta_0$ *there exists* $r = r(\gamma)$ *such that:*

1. *for any positive integer* n *follows that* $f^{-n}(W^{cu}_r(x)) \subset W^{cu}_\gamma(f^{-n}(x))$.

2. *For every* $r \leq r(\delta_0)$*, either*

   (a) $\ell(f^{-n}(W^{cu}_r(x))) \to 0$ *as* $n \to +\infty$*,*

   (b) *or* $x \in W^u_r(p)$ *for some* $p \in Per(f_{/\Lambda})$ *such that* $p \in W^{cu}_r(x)$ *and there exists a another periodic points* $q \in \overline{W^u_r(p)}$ *which is a sink or a non-hyperbolic periodic point,*

   (c) $x \in \mathcal{C}$ *such that* $\mathcal{C}$ *is a periodic simple closed curve and* $f^m_{/\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ *(where* m *is the period of* $\mathcal{C}$*) is conjugated to an irrational rotation.*

3.2 Markov partitions.

Definition 9 *We say that* Λ *has local product structure if exists* $\alpha > 0$ *such that if for any* $x, y \in \Lambda$ *with* $d(x, y) < \alpha$ *holds that* $W^s_\epsilon(x) \cap W^{cu}_\epsilon(y) \in \Lambda$. *We denote with* $[x, y] = W^s_\epsilon(x) \cap W^{cu}_\epsilon(y)$.

Definition 10 *A subset* B *is called a box if*

1. $[x, y] \in B$ *whenever* $x, y \in B \cap \Lambda$*,*

2. $B = \overline{int(B)}$.

We also define the diameter of B *as the maximum distance between points in* B.

Definition 11 *Let* Λ *be a compact and invariant set having contractive codimension one dominated splitting. A Markov partition of* Λ *is a collection of boxes* $\mathcal{P} = \{B_1, ..., B_n\}$ *such that:*

1. $\Lambda \subset \cup_{1 \leq i \leq n} B_i$*,*

2. $int(B_i) \cap int(B_j) = \emptyset$ *if* $i \neq j$*, where* $int(B_i)$ *denotes the interior of* B_i *in* Λ*,*

3. *for any* $x \in \Lambda$*, if* $x \in B_i$ *for some* $B_i \in \mathcal{P}$ *follow that:*

   (a) *there exist* $B_j \in \mathcal{P}$ *such that* $f^{-1}(W^{cu}_\epsilon(x) \cap B_i) \subset B_j$*,*

   (b) *there exist* $B_k \in \mathcal{P}$ *such that* $f(W^s_\epsilon(x) \cap B_i) \subset B_k$*,* *Moreover, we define the size of the Markov partition as the maximum of the diameters of B_i.*

Definition 12 *We say that a point x in the limit set $L(f)$ is isolated if there exists a neighborhood U_x of x such that $U_x \cap L(f) \subset Per(f)$. Let $\tilde{L}(f) \subset L(f)$ be the sets of the non-isolated points.*

Definition 13 *We say that a compact and invariant set Λ with contractive codimension one dominated splitting is a Basic piece if it is transitive and has local product structure.*

Theorem 3.2 *Let Λ be a basic piece of $\tilde{L}(f)$. Then, there exists a Markov partition of Λ of arbitrarily small size.*

To prove this, first we will prove that there exists Markov partition for the unstable direction. Using this, we will prove a shadowing lemma. And from that we will construct a semiconjugacy with a subshift of finite type, and that will allow us to construct the Markov partition. We want to point out, that classically, the shadowing lemma is used to prove the existence of a Markov partition. Here, we do the opposite.

Definition 14 *We say that a collection of sets $\mathcal{P}^u = \{B_1, ..., B_n\}$ is an unstable Markov partition for Λ if:*

1. $\Lambda \subset \cup_{1 \leq i \leq n} B_i$,
2. *$int(B_i) \cap int(B_j) = \emptyset$ if $i \neq j$, where $int(B_i)$ denotes the interior of B_i,*
3. *if $x \in B_i$ for some $B_i \in \mathcal{P}$ follows that there exist B_j such that $f^{-1}(W^{cu}_\epsilon(x) \cap B_i) \subset B_j$,*

Moreover, we define the size of the Markov partition as the maximum of the diameters of B_i

Remark 3.1 *Given an unstable Markov partition $\mathcal{P}^u = \{B_1, ..., B_n\}$ and a positive integer k, then $\mathcal{P}_k = \{f^{-k}(B_i) \cap B_j : 1 \geq i, j \geq n\}$ is also an unstable Markov partition. We say in this case, that $\mathcal{P}_k$ refine $\mathcal{P}$ and we call this unstable Markov partition, a refining of $\mathcal{P}$.*

Proposition 3.1 *Let Λ be a basic piece of $\tilde{L}(f)$, and $\beta > 0$. Then, there exists an unstable Markov partition of Λ with size smaller than β.*

Before to give the proof we need a series of definitions and lemmas.

Now, we fix β and we will prove that there exists an unstable Markov partition of Λ of size smaller than β. Before to prove the Proposition we introduce some news definitions.

Definition 15 Boundary points. *Let Λ be a basic piece of $\tilde{L}(f)$. We say that x is boundary point, if there exists $\epsilon_1 < \epsilon$ such that one of the connected components of $W^{cu}_{\epsilon_1}(x) \setminus \{x\}$ does not contain points in Λ.*

We say that x is a $\gamma-$boundary point if one of the connected components of $W^{cu}_{\gamma}(x) \setminus \{x\}$ does not contain points in Λ but the closure of this connected component contain a point in Λ, i.e.: the two extremal point, of the connected component of $W^{cu}_{\gamma}(x) \setminus \{x\}$ are in Λ.

Lemma 3.2.1 *Let Λ be a basic piece of $\tilde{L}(f)$. The followings holds:*

1. *if x is a boundary points then it belongs to the stable manifold of a periodic point p in Λ;*
2. *$\gamma-$boundary periodic points are finite;*
3. *if there are not boundaries points, then $\ell(f^{-n}(W^{cu}_{\gamma}(x))) \to 0$ for any $x \in \Lambda$.*

Proof: Let x be a boundary point. Then, there is $\epsilon_1 < \epsilon$ such that one of the connected components of $W^{cu}_{\epsilon_1}(x) \setminus \{x\}$ do not contain points in Λ. By the lemma 3.1.2, we have that there is $r = r(\epsilon_1)$ such that for any $y \in \Lambda$ $f^{-n}(W^{cu}_{r}(y)) \subset W^{cu}_{\epsilon_1}(f^{-n}(y))$.

To see the first item, it is enough to show that there are positive integers $m < n$ such that $f^n(x) \in W^s_{\epsilon}(f^m(x))$. If this does not hold, we would have positive integers n_1, n_2, n_3 such that $W^{cu}_r(f^{n_2}(x)) \setminus \{f^{n_2}(x)\} \cap W^s_{\epsilon}(f^{n_1}(x)) \neq \emptyset$ and $W^{cu}_r(f^{n_2}(x)) \setminus \{f^{n_2}(x)\} \cap W^s_{\epsilon}(f^{n_3}(x)) \neq \emptyset$, such that these intersections hold at both side of $f^{n_2}(x)$. But this implies that $W^{cu}_{\epsilon_1}(x)$ has points in Λ in both sides of x which is a contradiction with the assumption that x is a boundary point.

The second item is inmediat.

To prove the last one, we use the sublemma that follows from lemma 3.1.2.

Sublemma 1 *If for some γ and some x $\ell(f^{-n}(W^{cu}_{\gamma}(x)))$ does not converge to zero, them there is a periodic point p in Λ such that one of the component of the local central unstable manifold of p contains a sink or a non-hyperbolic periodic point.*

Proof:

If $\ell(f^{-n}(W^{cu}_{\gamma}(x)))$ does not converge to zero, we take a strictly increasing sequences of positive integers k_n such that

$$\ell(f^{-k_n}(W^{cu}_{\gamma}(x))) = \gamma$$

and

$$\ell(f^{-j}(W^{cu}_{\gamma}(x))) < \gamma \quad 0 \leq j \leq k_n.$$

Taking

$$I = \lim_{n \to +\infty} f^{-k_n}(W^{cu}_{\gamma}(x))$$

follows that I does not growth for positive iteration and it is transversal to $E^s \oplus F$; i.e.: I is a $\gamma - E^s \oplus F$ arc.

Then, we can apply theorem 3.1 and follows that either $\omega(I)$ is a periodic curve with dynamic conjugated to an irrational rotation or it is contained in a periodic arc. Since we are assuming that there are not closed curves, then it holds the second option.

■

Applying the previous sublemma it follows also that p is a boundary point.

■

Lemma 3.2.2 *For any periodic point p in Λ, follows that $W^s(p)$ is dense.*

Proof:
Let z such that $\omega(z) = \alpha(z) = \Lambda$. It follows that this point the central unstable manifold is dynamically defined. In fact, if it not the case, by sublemma 3.2.1 follows that $z \in W^s(q)$ for some periodic point q and therefore, $\omega(z) = \mathcal{O}(q)$; a contradiction.

Then, given any periodic point p, there exists $n > 0$ such that $dist(f^n(z), p) < \frac{\epsilon}{2}$ and therefore, $W^s_\epsilon(p) \cap W^{cu}_\epsilon(f^n(z)) \neq \emptyset$. Noting with z' the point of intersection, it follows that

$$dist(f^{-m}(f^n(z)), f^{-m}(z')) \to 0 \;\; n \to +\infty$$

and since $\alpha(z) = \Lambda$ it follows that $\alpha(z') = \Lambda$.

■

Lemma 3.2.3 *Let Λ be a basic piece of $\tilde{L}(f)$.*
Given β, there are a finite number of periodic points $p_1, \ldots.., p_r$ and $D_1, \ldots, D_r$ compact disc contained in $\cup_{1\leq i\leq r} W^s(p_i)$ such if $x \notin D = \cup_{1\leq i\leq r} D_i$ then:

1. *$W^{cu}_\epsilon(x)$ has intersection with D at both sides of x;*
2. *the connected component of $W^{cu}_\epsilon(x) \setminus D$ containing x has length smaller than β.*

Proof: Take $\epsilon_1 < \beta/2$. Take $\epsilon_2 < \beta/2$ and such that $\ell(f^{-n}(W^{cu}_{\epsilon_2}(x))) < \epsilon_1$. Take $\gamma < \epsilon_1, \epsilon_2$ and take all the $\gamma-$boundary periodic points $p_1, \ldots., p_r$. Let us assume that the lemma is not true. Then, there exists a sequence x_n of points in Λ and compacts disks $D_n = \cup_i D_{i,n}$ such that the conclusion 1) of the lemma does not holds for any x_n and D_n. Take x and accumulation point of $\{x_n\}$. If x is in the stable manifold of some p_i, from the fact that p_i is a boundary point, then all points x_n are converging either from one side of the stable compact disk D_x of $W^s(p_i)$ or are contained in D_x. Using the lemma 3.2.2 we get that there are compact disks $\hat{D}_n$ contained in the stable manifold of p_i converging to D_x, and so the points x_n are enclosed by compact disks of the stables manifolds of the points p_i getting a contradiction. If x do not belong to any of the stables manifolds of the points p_i, we get two alternatives; either x is a boundary point, or it is not a boundary point. In the first case, x belong to the stable manifold of some $\delta-$boundary periodic point q with $\delta < \gamma$. This implies that on one of the connected components of $W^{cu}_\epsilon(x) \setminus \{x\}$ we get points of Λ converging to q and on the other components there are points of Λ also contained in $W^{cu}_\gamma(q)$. Taking n large enough such that $f^n(x)$ is close to q we get that there are points of Λ contained in both side of $W^{cu}_\gamma(f^n(x))$, and this implies that there are points of Λ on both sides of $W^{cu}_\gamma(x)$. Again, using that the stables manifolds of the periodic points are dense, we conclude the points x_n are closed by compact disks of the stables manifolds of the points p_i getting a contradiction. In the case that x is not a boundary point, there are points of Λ on both sides of $W^{cu}_{\gamma}(x)$, and again we get a contradiction.

■

Definition 16 Boxes around x.

Given a family of disks $\{D_i\}$ as the one obtained in previous lemma, for each $x \in \Lambda$ we take the connected component of $W^{cu}_{\epsilon}(x) \setminus D$ containing x. Observe that this arc intersect D in two points x^+ and x^-. For $\gamma > 0$, take $D^+(x,\gamma) = B_\gamma(x^+) \cap D$ and $D^-(x,\gamma) = B_\gamma(x^-) \cap D$, where $B_\gamma(z)$ note the ball of radius γ and center x. Take the set

$$C = \{y \in D^-(x,\gamma) \cap \Lambda : W^{cu}_{\epsilon}(y) \cap D^+(x,\gamma) \neq \emptyset\}$$

and define $W^{cu}(y,x)$ as the connected component of $W^{cu}_{\epsilon}(y)$ containing y and intersecting $D^+(x,\gamma)$ and $D^-(x,\gamma)$. We define the box $B(x)$ around x in the following way:

$$B(x) = \cup_{y \in C} W^{cu}(y,x).$$

Lemma 3.2.4 *Given two boxes $B(x)$ and $B(z)$ such that $B(x) \cap B(z) \neq \emptyset$ then we can subdivide $B(x) \cap B(z)$ in a finite number of boxes (at most seven boxes) $B_1(x,z), ..., B_k(x,z)$ which are pair disjoint and such that:*

1. $\partial^{cu} B_i(x,z) \subset D$,
2. *if* $y \in \partial^{cu,+} B_i(x,z)$ *then* $\partial^{cu,-} B_i(x,z) \cap W^{cu}_{\epsilon}(y) \neq \emptyset$.

Proof:

Given $B(x)$ and $B(y)$ with non-empty intersection, we subdivide $B(x)$ in the following four boxes (observe that one of them could be empty):

1. $B_1(x,z) = \{y \in B(x) \cap \overline{B(z)}\}$;
2. $B_2(x,z) = \{y \in B(x) \setminus \overline{B(z)} : W^{cu}(y,x) \cap B(z) = \emptyset\}$;
3. $B_3(x,z) = \{y \in B(x) \setminus \overline{B(z)} : W^{cu,+}(y,x) \cap B(z) \neq \emptyset\}$;
4. $B_4(x,z) = \{y \in B(x) \setminus \overline{B(z)} : W^{cu,-}(y,x) \cap B(z) \neq \emptyset\}$.

The same is done with $B(z)$. By definition, follows the thesis of the lemma.

■

Proof of proposition 3.1:

We take the collection of sets $\{B(x)\}_{x \in \Lambda}$ defined in definition 16. Then we take a finite covering of Λ and then we "refine" this collection as is done in previous lemma. After that, we obtained the following:

A finite covering $\{B_i\}$ of Λ defined as in definition 16 and refined by disjointness as in lemma 3.2.4, is an unstable Markov partition of size smaller than β.

So, the proof of Proposition 3.1 is finished.

∎

Now, we will use this unstable Markov partition to obtain a shadowing lemma. Before to do that we need some definitions:

Definition 17 *We say tha $\{x_i\}$ is a $f^{n_0}-\alpha$-pseudo orbit in Λ if for any integer i follows that*

1. $x_i \in \Lambda$ *and*
2. $dist(f^{n_0}(x_i), x_{i+1}) < \alpha$.

Definition 18 *Given a $f^{n_0}-\alpha$-pseudo orbit $\{x_i\}$ in Λ we say that the f^{n_0}-orbit of x $\beta-$shadows $\{x_i\}$ if for any integer i follows that*

$$dist(f^{n_0 i}(x), x_i) < \beta.$$

Theorem 3.3 *Let Λ be a basic piece of $\tilde{L}(f)$. Then, it has the shadowing property, that is, given $\beta > 0$ there exists $\alpha > 0$ such that any α-pseudo orbit in Λ is β- shadow by an orbit in Λ*

Proof:

First we will prove that given $\beta_1 < \beta/2$, there exist $n_0 = n_0(\beta_1)$ and α such that any f^{n_0} α-pseudo orbit in Λ is β_1- shadowed by a true f^{n_0} orbit in Λ. From there, we will conclude the shadowing lemma.

Let $\beta_1 > 0$ be given. Choose $\epsilon > 0$ such that $\ell(f^{-n}(W^{cu}_\epsilon(x))) < \beta_1/3$. Then choose an unstable Markov partition $\{B_i\}$ of size less than $\beta_1/4$. We say that two boxes B_i and B_j of the Markov partition are adjacent if $B_i \cap B_j \neq \emptyset$. For a box B_i denote by $\mathcal{B}_i$ the collection formed by the boxes B_i and all its adjacent; we note with $\hat{B}_i$ the union of all the boxes in $\mathcal{B}_i$. Moreover, we take the diameter of the Markov partition small enough such that given any $x \in B_i$ for some i then the connected component of $W^{cu}_{2\epsilon}(x) \cap \hat{B}_i$ that contains x is contained in $W^{cu}_\epsilon(x)$. On the other hand, let $B_{i_1} ... B_{i_k}$ be the elements of the Markov partition such that there is a periodic point in the stable boundary. Recall that the boundary of any of this boxes are given by the stable manifolds of those periodic points.

Sublemma 2 *Refining the Markov partition by negative iteration, we can assume that the elements of the Markov partition having periodic points in the boundary are not adjacent.*

This follows combining the followings facts:

1. the boundary of the boxes obtained after refining are stable manifolds of the previous periodic points and so no new periodic points can appear;
2. the diameter of the boxes is arbitrarily small.

After that, we get the next sublema:

Sublemma 3 *There exists k_0 such that if for some x there exists an arc $l^{cu} \subset W^{cu}_{2\epsilon}(x)$ that is contained in at most three boxes it follows that for any $n \geq k_0$ it is verified that $f^{-n}(l^{cu})$ is contained in at most two boxes.*

Proof of the sublemma:

The proof of the sublemma goes as follows: first, lets show first that there exists k_0 such that for any l^{cu} contained in at most two (adjacent) boxes then, either there exists a box B_i such that $f^{-k_0}(l^{cu}) \in B_i$ or there exists a box $B'_{i_j}s$ containing a periodic point in its boundary such that $l^{cu} \cap \partial^s B_{i_j} \neq \emptyset$.

Take $k_0 = 2r + 1$ where r is the number of boxes involved in the Markov partition, and assume that $f^{-k_0}(l^{cu})$ belongs to two adjacent boxes. Observe that the same holds for $f^{-i}(l^{cu})$, $0 \leq i \leq k_0$. From the election of k_0 follows that there exists $0 \leq k_1 < k_2 \leq k_0$ such that $f^{-k_1}(l^{cu})$ and $f^{-k_2}(l^{cu})$ intersects a boundary component of some box B_{i_0}. Denote this boundary component by $\partial^s_1(B_{i_0})$. Set $w = f^{-k_2}(l^{cu}) \cap \partial^s_1(B_{i_0})$ and notice that w is an interior point of $f^{-k_2}(l^{cu})$. From the fact that $f^{k_2-k_1}(f^{-k_2}(l^{cu})) = f^{-k_1}(l^{cu})$, it follows that $f^{k_2-k_1}(w) \in f^{-k_1}(l^{cu})$ and it is an interior point of it. Hence $f^{k_2-k_1}(w) \in \partial^{cs}_1(B_{i_0})$. Since the direction E^s is a contractive direction it follows that $f^{k_2-k_1}(\partial^s_1(B_{i_0})) \subset \partial^s_1(B_{i_0})$. Therefore there is a periodic point in the boundary on this box and so it is one of the boxes B_{i_j}. From this follows immediately that l^{cu} intersects one of the boxes B_{i_j}. Secondly, to conclude the sublemma, recall that the boxes having periodic points in their boundaries are not adjacent and let l^{cu} be is as in the hypothesis of the sublemma. Then we may write $l^{cu} = l^{cu}_1 \cup l^{cu}_2$ where $l^{cu}_i, i = 1, 2$ is contained in at most two adjacent boxes and so only one of them can intersects the boundary of some B_{i_j} and the result follows.

Finally, notice that if the property in the conclusion holds for k_0 then it holds for any $n \geq k_0$. This completes the proof of the sublemma.

■

Continuing with the proof of our shadowing lemma, choose $\gamma > 0$ such that $\gamma < \text{diam}_s(B_i)$ for any i where $\text{diam}_s(B_i) = \inf\{\ell(W^s_\epsilon(x) \cap B_i) : x \in B_i\}$.

Let m_0 be such that $f^m(W^s_\gamma)(x) \subset W^s_{\gamma/2}(f^m(x))$ for any x and $m \geq m_0$. Let k_0 be the positive integer chosen in sublemma 3.

Take $g = f^{n_0}$ where $n_0 = \max\{m_0, k_0\}$. We will prove that there is α_1 such that any α_1- pseudo orbit (for g) is β_1-shadowed by a true orbit (of g). Later, we will prove that it is enough to prove the shadowing property for f.

Choose $\alpha_1, 0 < \alpha_1 < \beta_1/2$ such that if $x, y \in \Lambda,\ x \in B_i$ and $d(x, y) < \alpha_1$ then $y \in \hat{B}_i$ and such that if $x \in B_i$ and $d(x, y) < \alpha_1$ then $W^s_\epsilon(x) \cap W^{cu}_\epsilon(y) \in \hat{B}_i$. Moreover, choose α_1 so small that if $d(x, y) < \alpha_1$ then $W^s_\epsilon(x) \cap W^{cu}_\epsilon(y) \in W^{cu}_\gamma(y)$ and $W^s_\epsilon(x) \cap W^{cu}_\epsilon(z) \in W^s_\gamma(x)$ for any $z \in W^s_{\gamma/2}(y)$.

Now let us proceed by induction. Let $\{x_i\}_{i=0}^\infty$ is a "forward" α_1-pseudo orbit. Assume that for $k \leq n$ we have a point z_k with the following properties:

1. $z_k \in W^s_\gamma(x_k)$ and both points belong to the same box or to adjacent boxes.

2. $g^{-j}(z_k)$ and z_{k-j} belong to a same box ort o adjacent ones for each $j = 1, .., k$. Moreover $g^{-j}(z_k) \in W^{cu}_\epsilon(z_{k-j})$.

This two conditions imply that $y_n = g^{-n}(z_n)$ shadows $\{x_i\}_{i=0}^n$. Indeed,

$$\begin{aligned} d(g^i(y_n), x_i) &= d(g^{i-n}(z_n), x_i) \\ &\leq d(g^{i-n}(z_n), z_{n-(n-i)}) + d(z_{n-(n-i)}, x_i) \\ &\leq \frac{\beta_1}{2} + \frac{\beta_1}{2} = \beta_1 \end{aligned}$$

(remember that the size of the partition is less than $\beta_1/4$).

Let us show how to construct z_{n+1}. Since $z_n \in W^s_\gamma(x_n)$ then $g(z_n) \in W^s_{\gamma/2}(g(x_n))$. Set

$$z_{n+1} = W^s_\epsilon(x_{n+1}) \cap W^{cu}_\epsilon(g(z_n)).$$

Since $d(g(x_n), x_{n+1}) < \alpha_1$ it follows that $z_{n+1} \in W^s_\gamma(x_{n+1})$ (and in particular this implies that z_{n+1} and x_{n+1} are in the same box or in adjacent ones). This shows that property 1) of the induction hypothesis is satisfied for z_{n+1}. Moreover, by the election of α_1 it follows that z_{n+1} is adjacent to the box that contains $g(z_n)$. Then, $g^{-1}(z_{n+1})$ and $g^{-1}(g(z_n)) = z_n$ belong to the same box or to adjacent ones.

To continue with the proof of the property 2), assume that given $j \geq 1$ we get that $g^{-j}(z_{n+1})$ and z_{n+1-j} belong to the same box or to adjacent ones. We will prove that the same holds for $j+1$. It follows from the assumption that the unstable arc containing $g^{-j}(z_{n+1}), z_{n+1-j}$ and $g(z_{n-j})$ is contained in at most three boxes of the Markov partition (and contained in $W^{cu}_{2\epsilon}(g(z_{n-j}))$. By the sublemma 3 it follows that the pre image is contained in at most two, that is, $g^{-(j+1)}(z_{n+1})$ and z_{n-j} belongs at most to two adjacent boxes and by the size of the Markov partition it follows that $g^{-(j+1)}(z_{n+1}) \in W^{cu}_\epsilon(z_{n-j})$. This completes the proof of the item 2).

Notice that we have constructed a sequence $\{z_n\}_{\{n\geq 0\}}$ such that

1. $z_n \in W^s_\epsilon(x_n)$.
2. $z_n \in W^{cu}_\epsilon(g(z_{n-1}))$
3. $g^{-j}(z_n) \in W^{cu}_\epsilon(z_{n-j})$

Therefore it follows that for any $l, 0 \leq l \leq n_0 - 1$ it is satisfied

$$d(f^l(x_k), f^l(g^k(y_n))) < \beta_1.$$

Indeed,

$$\begin{aligned} d(f^l(x_k), f^l(g^k(y_n)) &\leq d(f^l(x_k), f^l(z_{n-k})) + d(f^{l-n_0}(f^{n_0}(z_{n-k})), f^{l-n_0}(z_{n-k+1})) \\ &\quad + d(f^{l-n_0}(z_{n-k+1})), f^{l-n_0}(g^{-k+1}z_n)) \\ &\leq \beta_1/3 + \beta_1/3 + \beta_1/3 \leq \beta_1 \end{aligned}$$

This means that if $\{x_i\}_{i=0}^n$ is a α-f^{n_0}-pseudo orbit then there is a point y_n such that the f^{n_0}-orbit of this point β_1-shadows the pseudo orbit $\{x_i\}_{i=0}^n$ and this point also shadows the f-pseudo orbit $\{\{f^i(x_{jn_0})\}_{i=0}^{n_0-1}\}_{j=0}^n$.

To prove the Lemma for f, given $\beta_1 > 0$ consider $\beta_1/2$ and n_0 and α such that any α-f^{n_0}-pseudo orbit is $\beta_1/2$ shadowed by f^{n_0}.

Take α such that if $\{w_i\}_{i\geq 0}$ is a α-f-pseudo orbit then $\{w_{jn_0}\}_{j\geq 0}$ is an α_1-f^{n_0}-pseudo orbit. Moreover, choose α so small such that $d(f^i(w_{jn_0}), w_{jn_0+i}) < \beta/2$ for any j and $i = 0, ..., n_0 - 1$. Then $\{\{f^i(w_{jn_0})\}_{i=0}^{n_0-1}\}_{j=0}$ is $\beta_1/2$-f-shadowed by a point y. So, we get that this point also β_1-f-shadows the pseudo orbit $\{w_i\}$.

■

To conclude the proof of theorem 3.2, observe first that it may happen that the shadowing point is not unique, since we do not have, a priori, expansivity. For this reason we argue as follows to construct a Markov partition of size β. Let $\beta_1 < \beta/2$ and define a relation in Λ :

$$x \sim y \text{ iff } d(f^n(x), f^n(y) \leq \beta_1, \ \forall n \in \mathbb{Z}.$$

Lemma 3.2.5 *We have that $x \sim y$ if and only if:*

1. *$x \in W^{cu}_\gamma(y)$ and $y \in W^{cu}_\gamma(x)$.*

2. *Denoting by (x, y) the (open) arc in $W^{cu}_\gamma(y)$ whose endpoints are x and y we have that $(x, y) \cap \Lambda = \emptyset$.*

3. *$x \in W^s(p_1)$ and $y \in W^s(p_2)$ and p_1, p_2 are η boundary points with $\eta \leq \beta_1$.*

Proof:

The converse follows immediately from the definition of η boundary point.

To prove the two first item listed in the lemma, assuming that $x \sim y$, we argue by contradiction.

If $x \in W^{cu}_\gamma(y)$, we take $z = W^{cu}_\gamma(y)\mathcal{W}^f_\epsilon(\S)$ and observe that for some positive integer n follows that $dist(f^{-n}(z), f^{-n}(x)) > \epsilon$ and for any positive integer m, $dist(f^{-m}(z), f^{-m}(y)) < \eta$. Therefore, if β_1 is small enough we get that $\beta_1 > dist(f^{-n}(y), f^{-n}(x)) > dist(f^{-n}(z), f^{-n}(x)) - dist(f^{-n}(z), f^{-n}(y)) > \epsilon - \eta$. Which is a contradiction.

If $(x, y) \cap \Lambda \neq \emptyset$ it follows that $W^s(q) \cap (x, y) \neq \emptyset$ for some q with unbounded unstable manifold. Therefore, for any open connected compact arc W of the unstable manifold of q that contains q, it follows that there is a positive iterate of (x, y) C^r-close to W Hence the arc length of (x, y) growths by positive iteration. Again a contradiction.

To prove the last item, observe that by item two it follows that x and y are boundary points. Therefore they belongs to the stable manifold of some periodic points p_1 and p_2 respectively. Since $f^n(x) \to p_1$ and $f^n(y) \to p_2$ and $dist(f^n(x), f^n(y)) < \beta_1$ the last item follows.

■

Corolary 3.1 *$\sim$ is an equivalence relation.*

Let

$$\tilde{\Lambda} = \Lambda / \sim$$

and

$$p : \Lambda \to \tilde{\Lambda}$$

the canonical projection and endow $\tilde{\Lambda}$ with the quotient topology. Denote by $[x] = p(x)$. Also denote by

$$W^{cu}_{\gamma}([x]) = p(W^{cu}_{\gamma}(x) \cap \Lambda)$$

and by

$$W^{s}_{\gamma}([x]) = p(W^{s}_{\gamma}(x) \cap \Lambda).$$

Moreover, denote by

$$\tilde{f} : \tilde{\Lambda} \to \tilde{\Lambda}$$

the induced homeomorphism.

Lemma 3.2.6 *With the notations above the following holds:*

1. *p is closed and $\tilde{\Lambda}$ is a compact Hausdorff metrizabel space.*
2. *$\tilde{f}$ is expansive and has the shadowing property.*
3. *$W^{cu}_{\gamma}([x])$ is an unstable set and $W^{s}_{\gamma}([x])$ is a stable set and $\tilde{\Lambda}$ has local product structure.*

Proof:
It follows immediately from the definition of p and lemma 3.2.5.

■

End of proof of theorem 3.2:
Using the above lemma and arguing exactly the same way as [B] we can construct a Markov partition $\tilde{\mathcal{P}} = \{\tilde{B}_1, \ldots.., \tilde{B}_n\}$ on $\tilde{\Lambda}$ of size less than β_1. Define $B_i = p^{-1}(\tilde{B}_i)$. It is straightforward to verify that $\mathcal{P} = \{B_1, \ldots.., B_n\}$ is a Markov partition of Λ. It remains the question if it has size less than β. It may occur that the this Markov partition does not have size smaller than β. Therefore, we construct a new one with this property. To do that, we will refine the Markov partition $\mathcal{P}^u$ using that there is a an unstable Markov partition $\mathcal{P}^u$ of arbitrarily small size (see proposition 3.1). Recall that the boundaries of an unstable Markov partition is given by stable discs contained in the stable manifold of some periodic points. Then, given a box B_i of $\mathcal{P}$ with size larger than β, we consider the points $x_i^1, \ldots., x_i^{k_i}$ contained in B_i which are boundary points of $\mathcal{P}^u$ and we take the local stable manifold $W^s_{\epsilon}(x_i^j)$. Then we take the connected components $B_i^1, \ldots, B_i^{m_i}$ of $B_i \setminus \{W^s_{\epsilon}(x_i^j)\}_{\{1 \leq j \leq k_i\}}$. Now we take a new partition

$$\mathcal{P}_* = \{B_i^j\}_{\{1 \leq j \leq m_i \ 1 \leq i \leq n\}}.$$

It follows immediately that the partition is a Markov partition of size smaller than β.

■

In the sequel, we will consider the especial case of homoclinic class, and we will show that they exhibit Markov partition.

Definition 19 *We define the homoclinic class of a saddle hyperbolic periodic point as the closure of intersection of the stable and unstable manifold of p and will be denoted with $H(p) = \overline{W^s(p) \cap W^u(p)}$.*

Proposition 3.2 *If a homoclinic class has codimension one contractive dominated splitting, then it is a Basic piece. In particular, it has Markov partition.*

Proof: It is well known that a homoclinic class is transitive. Let us prove the local product structure. If one of the components $W^u(p) \setminus \{p\}$ has finite length, take γ small and less than the this length. It follows that for any x in the intersection of the stable and unstable manifold of p we have that $W^{cu}_\gamma(x) \subset W^u(p)$. And also $W^s_\gamma(x) \subset W^s(p)$. Thus, if $x, y \in W^s(p) \cap W^u(p)$ and $dist(x, y)$ is small then $W^{cu}_\gamma(x) \cap W^s_\gamma(y) \in W^s(p) \cap W^u(p)$, i.e., $[x, y] \in \Lambda = H(p)$. Since $H(p)$ is the closure of the intersection of the stable and unstable manifold of p we conclude, by continuity, the local product structure on $H(p)$.

■

4 Proof of Theorem C:

The Theorem A follows from the next Theorem.

Theorem 4.1 *Let $f : M \to M$ be a C^2-diffeomorphism. Let $\Lambda \subset L(f)$ be a compact invariant set such that it is isolated in the limit set, all the periodic points are hyperbolic, and has contractive codimension one dominated splitting. Then, one of the following statements holds:*

1. *Λ is a hyperbolic set;*
2. *there exist a simple closed curve $\mathcal{C} \subset \Lambda$ which is invariant under f^m for some m and it is normally hyperbolic. Moreover $f^m : \mathcal{C} \to \mathcal{C}$ is conjugated to an irrational rotation.*

Assuming that this last theorem is true we show that in this case, the number of periodic simple closed curves normally hyperbolic and conjugated to an irrational rotation contained in Λ is finite. This will imply Theorem A. For more details see [PS1].

The first step in the proof of the Theorem 4.1 is the following elementary lemma.

Lemma 4.0.7 *Let Λ_0 be a compact invariant set having a contractive codimension one dominated splitting $T_{/\Lambda}M = E^s \oplus F$. If $\|Df^{-n}_{/F(x)}\| \to 0$ as $n \to \infty$ then Λ_0 is a hyperbolic set.*

Now, using the previous lemma, we will prove Theorem 4.1 based on the next lemma.

To prove how the Maim Lemma implies theorem 4.1 we argue as follows: First we take a compact invariant subset $\Lambda_0 \subset \Lambda$ which is the minimal set, in the Zorn's lemma sense, such that Λ_0 is not hyperbolic. To prove the existence of this set, it is enough to show that given a sequences of nonhyperbolic compacts invariant sets $\{\Lambda_\alpha\}_{\alpha \in \mathcal{A}}$ ordered by inclusion follows that $\cap_{\alpha \in \mathcal{A}} \Lambda_\alpha$ is a nonhyperbolic compact invariant set.

can be found in [PS1].

Main Lemma

Let $f : M \to M$ be a C^2-diffeomorphism of a finite dimensional compact riemannian manifold M. Let Λ_0 be non-trivial transitive and compact invariant set contained in a set Λ such that it is isolated in the limit set and not containing a periodic simple closed curve normally hyperbolic $\mathcal{C}$ conjugated to an irrational rotation. Assume that every properly compact invariant subset of Λ_0 is hyperbolic. Then, Λ_0 is a hyperbolic set.

To prove how the Maim Lemma implies theorem 4.1 we argue as follows: First we take a compact invariant subset $\Lambda_0 \subset \Lambda$ which is the minimal set, in the Zorn's lemma sense, such that Λ_0 is not hyperbolic. To prove the existence of this set, it is enough to show that given a sequences of nonhyperbolic compacts invariant sets $\{\Lambda_\alpha\}_{\alpha \in \mathcal{A}}$ ordered by inclusion follows that $\cap_{\alpha \in \mathcal{A}} \Lambda_\alpha$ is a nonhyperbolic compact invariant set. By election of Λ_0 it follows that every properly compact invariant subset of Λ_0 is hyperbolic. More details can be found in [PS1].

The proof of the maim lemma will be given in the next subsection. Nevertheless we give here the basics steps of it proof:

1. The central unstable manifolds (which are of class C^2) have dynamics properties. In fact for every $x \in \Lambda_0$ there exist $\epsilon(x)$ such that $W^{cu}_{\epsilon(x)}(x)$ is an unstable manifold of x, meaning that $\ell(f^{-n}(W^{cu}_{\epsilon(x)}(x))) \to 0$ as $n \to \infty$.
2. For point x in an open set B in Λ_0 we have
$$\sum_{n\geq 0} \ell(f^{-n}(W^{cu}_{\epsilon(x)}(x))) < \infty.$$
3. For every point $x \in \Lambda_0$ we have
$$\|Df^{-n}_{/F(x)}\| \to 0$$
when $n \to \infty$.

4.1 Proof of the Maim Lemma.

In this section we shall assume that $\Lambda_0 = \Lambda$ is in the hypothesis of the main lemma, i.e., Λ is a nontrivial compact invariant transitive set isolated in the limit set, such that every proper compact invariant subset is hyperbolic and it is not a periodic simple curve normally hyperbolic conjugated to an irrational rotation. Under this conditions, we will prove that for every $x \in \Lambda$. $\|Df^{-n}_{/F(x)}\| \to 0$.

To show that, it is enough to find an open set B_0 such that for every $y \in B_0 \cap \Lambda$ we have $\|Df^{-n}_{/F(y)}\| \to_{n\to\infty} 0$. Let us show, that this is enough: let z be any point in Λ. There are two possibilities:

- The α- limit set $\alpha(z)$ is properly contained in Λ. Then, $\alpha(z)$ is an hyperbolic set, thus
$$\|Df^{-n}_{/F(z)}\| \to_{n\to\infty} 0.$$
- $\alpha(z) = \Lambda$. Then, there exist m_0 such that $f^{-m_0}(z) \in B_0$, implying that
$$\|Df^{-n}_{/F(f^{-m_0}(z))}\| \to_{n\to\infty} 0$$
and so
$$\|Df^{-n}_{/F(z)}\| \to_{n\to\infty} 0.$$

The first lemma of this subsection is classical in one dimensional dynamics(see for example [dMS]) and the proof is left to the reader. We only have to remark, since the diffeomorphism f is of class C^2, the center unstable manifolds are also, and moreover this center unstable manifolds varies continuously in the C^2 topology, we have a uniform Lipchitz constant K_0 of $\log(Df)$ along this manifolds.

Lemma 4.1.1 *there exist K_0 such that for all $x \in \Lambda$ and $J \subset W^{cu}_{\gamma}(x)$ we have for all $z, y \in J$ and $n \geq 0$:*

1. $\frac{\|Df^{-n}_{/\tilde{F}(y)}\|}{\|Df^{-n}_{/\tilde{F}(z)}\|} \leq \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J)))$

2. $\|Df^{-n}_{/\tilde{F}(x)}\| \leq \frac{\ell(f^{-n}(J))}{\ell(J)} \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J)))$

where $\tilde{F}(z) = T_z W^{cu}_{\epsilon}(x)$

First, we will show that we can assume that Λ_0 is contained in an homoclinic class.

Lemma 4.1.2 *Let $f : M \to M$ be a C^2-diffeomorphism of a finite dimensional compact riemannian manifold M, and let Λ be a transitive compact and invariant set with contractive codimension one dominated splitting. Then either Λ is is a periodic simple closed curve such the dynamic is conjugated to an irrational rotation or Λ is contained in the homoclinic class of a periodic point p.*

Proof: Let us assume that Λ is not a periodic simple closed curve with dynamic conjugated to an irrational rotation. Let $x \in \Lambda$ such that $\Lambda = \alpha(x)$. So, there is a subsequence m_i of positive integers such that $f^{-m_i}(x) \to x$. We can assume that x does not belong to the unstable manifold of a periodic point (in other case, Λ would be a periodic point) and so by lemma 3.1.2 we get that there is γ such that $\ell(f^{-n}(W^{cu}_{\gamma}(x))) \to 0$. Then, for m_{i_0} large enough, we get that for any $y \in f^{-m_{i_0}}(W^{cu}_{\gamma}(x))$ follows that $W^s_{\epsilon}(y) \cap W^{cu}_{\gamma/3}(x) \neq \emptyset$. Then, from standard arguments, we get a periodic point p with orbit in a neighborhood of Λ and such that $W^{cu}_{\gamma}(p) \subset W^u(p)$. Moreover, we get that for any $y \in \Lambda$ in a box of radius $\gamma/3$ around x we get that $W^s_{\epsilon}(p) \cap W^{cu}_{\gamma}(y) \neq \emptyset$ and $W^{cu}_{\gamma}(p) \cap W^s_{\epsilon}(y) \neq \emptyset$. In particular we get that $W^s_{\epsilon}(p) \cap W^{cu}_{\gamma}(f^{-m_i}(x)) \neq \emptyset$ and $W^{cu}_{\gamma}(p) \cap W^s_{\epsilon}(f^{-m_i}(x)) \neq \emptyset$ for any m_i large enough.

From the fact that $W^{cu}_{\gamma}(p) \cap W^s_{\epsilon}(f^{-m_i}(x)) \neq \emptyset$ for any m_i and that $W^{cu}_{\gamma}(p) \subset W^u(p)$, we conclude that there are compact disks of $W^u(p)$ converging to the central unstable manifold of x. On th other hand, since $W^s_{\epsilon}(p) \cap W^{cu}_{\gamma}(f^{-m_i}(x)) \neq \emptyset$, $f^{-m_i}(x) \to x$ and the dynamical properties of the central unstable manifold, we get that there are compact disks of $W^s(p)$ converging to the local stable manifold of x. The two fact together, imply that there are homoclinic points of p converging to x.

∎

Observe that in the previous lemma, we not assume that the whole limit set has contractive codimension one dominated splitting.

Lemma 4.1.3 *Let $\Lambda \subset L(f)$ be a compact invariant set either isolated in the limit set or in $L_0(f)$. Then, there is a neighborhood U of Λ such that for any $z \in \Lambda \cap \omega(x)$ for some $x \in M$ it follows that there exists a positive integer n_0 such that $f^n(x) \in U$ for any $n > n_0$.*

Proof: Let us assume that it is false. Then for any closed neighborhood U of Λ such that $\Lambda \subset$ *interior*(Λ) and $L(f) \cap U = \Lambda$, and there exists x and $y \in \Lambda \cap \omega(x)$ such that $\mathcal{O}^+_n(x) = \{f^k(x) : k > n\}$ is not contained in U for any positive integer n. Let $n_i \to +\infty$ such that $f^{n_i}(x) \to y$. Let for each n_i the first positive k_i such that $f^{n_i+k_i}(x) \notin U$ and let z be an accumulation point of $\{f^{n_i+k_i-1}(x)\}_{i>0}$. We can assume that $f^{n_i+k_i-1}(x) \to z$. It follows that $z \in U$ and $z \in \omega(x)$. Therefore, $z \in \Lambda$ and so $f(z) \in \Lambda$. However, $f^{n_i+k_i}(x) \to f(z)$ and so $f(z) \in$ *interior*$(U)^c$. A contradiction.

∎

Corolary 4.1 *Let $f : M \to M$ be a C^2-diffeomorphism of a finite dimensional compact riemannian manifold M, and let Λ be a transitive compact and invariant set either isolated in the limit set or in $L_0(f)$ and having a contractive codimension one dominated splitting. Then Λ is contained in the homoclinic class $H(p)$ of a periodic point p with $H(p) \subset \Lambda$.*

Now, we can also assume that $\Lambda_0 \subset H(p)$ for some hyperbolic periodic point p. Moreover, since $H(p) \subset \Lambda$ has codimension one contractive dominated splitting, we can obtain a Markov $\mathcal{P} = \{B_1, ..., B_n\}$ partition associated for $H(p)$ and we will use it to conclude the Main Lemma.

Definition 20 *Given a Markov Partition $\mathcal{P} = \{B_1, ..., B_n\}$ we say that a set B is a Markov box if there exist $k \geq 0$ and two rectangles B_i and B_j of $\mathcal{P}$ such that B is a connected component of $f^{-k}(B_i) \cap B_j$.*

Now, given a Markov box B, for any $y \in B$ we define:

$$J_B(y) = W^{cu}_{\gamma}(y) \cap B.$$

Moreover, since $\mathcal{P}$ is a Markov partition we get that for any $y \in B$ and any $k \geq 0$ either,

1. $f^{-k}(J_B(y)) \cap B = \emptyset$ or
2. $f^{-k}(J_B(y)) \subset B$.

In many occasions, we will need to estimate the length between different central unstable arcs in a Markov box. In this direction, we introduce the following definitions.

Definition 21 *We say that a Markov box B has distortion (or cu-distortion) C if for any two intervals J_1, J_2 in B transversal to the E^s-direction and whose endpoints are in the same local stable manifold, the following holds:*

$$\frac{1}{C} \leq \frac{\ell(J_1)}{\ell(J_2)} \leq C.$$

Lemma 4.1.4 *Let Λ be a compact invariant set having a contractive codimension one dominated splitting. It follows that the local stable foliation defined on Λ is a C^1-foliation.*

Notice that, in order to guarantee distortion C on a box B, it is sufficient to find a C^1 foliation close to the E^s-direction in the box, such that, for any two intervals J_1, J_2 (taken as in the definition of distortion),

$$\frac{1}{C} \leq \|\Pi'\| \leq C$$

holds, where $\Pi = \Pi(J_1, J_2)$ is the projection along the foliation between these intervals.

Now, given a Markov partition $\mathcal{P} = \{B_1, ..., B_n\}$, we introduce a stable foliation in the following way:

For each box B, and since the stable foliation is a C^1 foliation, we can extended it to a C^1-foliation in B, C^1-close to the E^s-direction (recall that the stable foliation is only defined for points in Λ) and such that it coincides with the stable one on the boundary of the box. Then, we take the foliation given the now obtained in each box.

Denote this foliation by

$$\mathcal{F}^s.$$

For any $x \in B$ let $\mathcal{F}^s(x)$ the leave passing through x. Notice that there exists C such that

$$\frac{1}{C} \leq \|\Pi'\| \leq C$$

where $\Pi = \Pi(J_1, J_2)$ the projection along this foliation between two intervals transversal to the E^s-direction.

The following lemma will be useful in the sequel.

Lemma 4.1.5 *Let $\mathcal{P} = \{B_1, ..., B_n\}$ be a Markov partition. Then, there exist $C = C(\mathcal{P})$ such that any Markov box of any refinement of this Markov partition has distortion C.*

Proof:

For each box $B_i \in \mathcal{P}$, we consider a C^1- stable foliation with distortion C_i. From standard arguments about foliations and contractive direction, follows that any negative iterate of this foliation has distortion smaller than DC_i. Then we take, $C = \max\{DC_1, ..., DC_n\}$ and so, any Markov box in the refinement has distortion smaller than C.

■

This Lemma, will help us to prove the following.

Lemma 4.1.6 *Let $\mathcal{P} = \{B_1, ..., B_n\}$ be a Markov Partition. Then, exists $K = K(\mathcal{P})$ such that for any Markov box B and any $z \in B \cap \Lambda$ holds that*

$$\sum_{i=0}^{n} \ell(f^{-i}(J_B(z))) \leq K$$

provided $f^{-i}(z) \notin B, 1 \leq i \leq n$.

Proof:

Let $\mathcal{P} = \{B_1, ..., B_n\}$ be a Markov partition of H_p. For each box B_j we choose a point $x_j \in B_j \cap \Lambda$ and we take $J_j = J_{B_j}(x_j)$.

Let B be a Markov box and let $z \in B \cap \Lambda$ such that $f^{-i}(z) \notin B$ for $i = 1, ..., n$. Let $B(i)$ the connected component of $f^{-i}(B) \cap B_{k_i}$ where k_i is such that $f^{-i}(J_B(z)) \subset B_{k_i}$, and we consider $J_{i,k_i}(z)$ the projection of the arc $f^{-i}(J_B(z))$ over J_{k_i} along the stable foliation. Observe that:

1. Since each $B \in \mathcal{P}$ is Markovian in the stable direction follows that for $i \neq j$ $B(i) \cap B(j) = \emptyset$, and in particular $J_{i,k_i}(z) \cap J_{j,k_j}(z) = \emptyset$;
2. for any i,
$$\frac{1}{C} \leq \frac{\ell(J_{i,k_i}(z))}{\ell(f^{-i}(J_B(z)))} \leq C.$$

Then,

$$\sum_{i=0}^{n} \ell(f^{-i}(J_B(z))) < \frac{1}{C}\sum_{i=0}^{n} \ell(J_{i,k_i}(z)) \leq \frac{1}{C}\sum_{j} \ell(J_j) < K.$$

■

Now, we will prove that in the case that the central unstable manifolds are not dynamically defined, then we can prove the Main Lemma.

Lemma 4.1.7 *Assume that $\Lambda \subset L(f)$ is a transitive and invariant set such that every proper compact subset is hyperbolic. Then, either there is $\gamma > 0$ such that*

$$\ell(f^{-n}(W^{cu}_{\gamma}(x))) \to 0 \text{ as } n \to \infty \ \forall x \in \Lambda$$

or F is expanding (i.e, $\|Df^{-n}_{/F(x)}\| \to 0 \ \forall x \in \Lambda$).

Proof: First, we will take a Markov Partition $\mathcal{P} = \{B_1, ..., B_n\}$ for the homoclinic class where Λ is contained. Recall that if $\ell(f^{-n}(W^{cu}_{\gamma}(x)))$ does not converge to 0, then there is a periodic point $p_x \in Per(f_{/\Lambda})$ such that $x \in W^u(p_x)$ and $p_x \in W^{cu}_{\gamma}(x)$ having one of the separatrix of $W^u(p_x)$ with length less than δ_0. Moreover, the endpoint of this separatrix, q_x, different from p_x is a sink or a nonhyperbolic periodic point. If we have only finite of this arcs, then reducing γ we get that the central unstable manifold are dynamically defined. On the other hand, if there are infinite of these separatrixes, then the size of them go to zero.

Moreover, since Λ is transitive (and is not a periodic orbit), there exists $y_0 \in \Lambda$ such that $y_0 \in W^s_{\epsilon}(p_x) - p_x$. Since the size of the separatrixes go to zero and the boundaries of the Markov boxes are given by a compact arcs of the stable manifold of a finite number of periodic points, we can chose one of the separatrixes, such that y_0 and $z_{y_0} = W^{cu}_{\epsilon}(y_0) \cap W^s_{\epsilon}(q_x)$ belong to same box B of the Markov partition. We can take also an small neighborhood U of y_0 such that $U \cap \Lambda \subset B$ and for any $y \in U \cap \Lambda$ follows that y and $z_y = W^{cu}_{\epsilon}(y_0) \cap W^s_{\epsilon}(q_x)$ belong to same box B. Now, for each $y \in U \cap \Lambda$ we consider the arc $I(y)$ contained in $J_B(y)$ with endpoints z_y and $W^{cu}_{\epsilon}(y_0) \cap W^s_{\epsilon}(p_x)$. Observe that this arcs are contained in the local stable manifold of the arc that link p_x and q_x.

Arguing as in the previous lemma, we consider the arcs $I_j(y)$ obtained as the projection of $f^{-j}(I(y))$ along the stable foliation over the arc J_{k_j} where k_j is such that $f^{-j}(I(y)) \subset B_{k_j}$. These arcs are contained in the stable manifold of the arc that link p_x and q_x, and so they are all pair disjoint. Hence, the sum of their lengths is bounded. Again, using the distortion property, we get that

$$\sum_{i=0}^{\infty} \ell(f^{-i}(I(y))) < \frac{1}{C}\sum_{i=0}^{\infty} \ell(I_j(y)) \leq \sum_{j} \ell(J_j) < K.$$

Now, as in the Schwarz's proof of the Denjoy Theorem ([Sch]), we conclude that $\forall y \in U \cap \Lambda$ there exist $I_1(y) \subset J_B(y)$ such that the length of $I_1(y) - I(y)$ is bounded away from zero (independently of y) and such that, for some $\tilde{K}$,

$$\sum_{n=0}^{\infty} \ell(f^{-n}(I_1(y))) < \tilde{K}.$$

Now, taking the set $\hat{U} = \cup_{y\in\Lambda\cap U} I_1(y)$ follows using lemma 4.1.1, that for any $z \in \hat{U}$, holds

$$\|Df^{-n}_{/\tilde{F}(z)}\| \to 0 \text{ as } n \to \infty$$

completing the proof of the lemma.

∎

The previous lemma, shows that we can assume that the central unstable manifold are dynamically defined (otherwise, from the previous lemma, there is nothing to prove). In particular, from the fact that the central unstable manifold are dynamically defined, then we have Markov boxes which are arbitrarily small in the vertical direction. Furthermore, as a consequence of lemma 3.1.2, given $\delta > 0$, then, if the box is small enough,

$$\ell(f^{-n}(J(y))) \le \delta$$

for any $y \in B \cap \Lambda$ and $n \ge 0$.

Now, we proceed to conclude the proof of the Main Lemmma. We will make a distinction if Λ is either a minimal or is not a minimal set. In the next subsubsection is done the minimal case and in subsection 4.1.2 the other one.

4.1.1 Λ is not a minimal set.

Observe that if Λ is not a minimal set, there are points $x \in \Lambda$ such that $x \notin \omega(x)$. We will explore this properties to prove the Main Lemma when Λ is not minimal.

Definition 22 *Given a Markov box B, we say that it has infinite returns if there are points $x_n \in U \cap \Lambda$ such that $f^{-k_n}(x_n) \in B$, $f^{-j} \notin B$ for $j = 1, ..., k_n - 1$ and $k_n \to \infty$. For the point x_n we call the integer k_n, the time of return of x_n.*

Since Λ is not a minimal set, then there exist a point $x \in \Lambda$ such that $x \notin \omega(x)$. Take now a small box B associated to x such that $B \cap \{f^n(x) : n \ge 1\} = \emptyset$. Then, since Λ is transitive, we conclude that for this box we have infinite returns (notice that if the point x is a boundary point of $\tilde{\Lambda}$ the same conclusion holds also).

Lemma 4.1.8 *Let B be a Markov box and assume that there is $\xi < 1$ such that for every $y \in B \cap \Lambda$ we have $\|Df^{-k}_{/\tilde{F}(y)}\| < \xi$ for all $y \in J_B(y)$ where k is such that $f^{-k}(y) \in B$ and $f^{-i}(x) \notin B$ for $i \le i < k$. Then for all $y \in B \cap \Lambda$ the following holds:*

$$\sum_{n\ge 0} \ell(f^{-n}(J_B(y))) < \infty.$$

In particular this implies that

$$\|Df^{-n}_{/F(y)}\| \to_{n\to\infty} 0.$$

The proof of the previous lemma is the same as the proof of Lemma 3.7.2. in [PS1]

Now, we show that given a Markov box, it is possible to find another one verifying the condition of lemma 4.1.8.

Lemma 4.1.9 *Let B be Markov box with infinite returns. Then there exist another Markov box B_0 contained in B such that satisfies the conditions of lemma 4.1.8.*

Proved this lemma, it is finished the proof in the case that Λ is not minimal, since in this case we have a box with infinite returns. And so, by the previous lemma, there exist an Markov box B_0 such that satisfies the conditions of lemma 4.1.8.

Now we give the proof of lemma 4.1.9.

Proof:

Let B be a Markov box as in the hypothesis of the lemma, and let K, K_0, C be as in lemmas 4.1.6, 4.1.1 and corollary 4.1.5 respectively. Consider also $L = \min\{\ell(J(z)) : z \in B \cap \Lambda\}$.

Let $r > 0$ such that

$$r\frac{C_1}{L}\exp(2K_0K) < \frac{1}{2}.$$

Since B has infinite returns, there exist $y \in B \cap \Lambda$ such that if we take B_0 the connected component of $f^{-k_0}(B) \cap B$ that contains $f^{-k_0}(y)$, where k_0 is the return time of y, then follows that

$$\ell(f^{-j}(J_{B_0}(z))) < r, \, \forall j \geq 0, \, \forall z \in B_0 \cap \Lambda.$$

Let us prove that the box B_0 satisfied the thesis of the lemma. Observe that if $z \in f^{k_0}(B_0) \cap \Lambda$, then for $y \in J(z)$

$$\|Df^{-k_0}_{/\tilde{F}(y)}\| \leq \frac{\ell(f^{-k_0}(J(z)))}{J(z)}\exp(K_0K)$$

Let now $y \in B_0 \cap \Lambda$ with return time to B_0 equal to k Setting $n_0 = k - k_0$, $(k \geq k_0)$ we have $f^{-n_0}(y) \in f^{k_0}(B_0)$.

Then, for $y \in J_{B_0}(z)$

$$\begin{aligned}
\|Df^{-k}_{/\tilde{F}(y)}\| &\leq \|Df^{-k_0}_{/\tilde{F}(f^{-n_0}(y))}\|\|Df^{-n_0}_{/\tilde{F}(y)}\| \\
&\leq \frac{\ell(f^{-k_0}(J(f^{-n_0}(z))))}{\ell(J(f^{-n_0}(z)))}\exp(K_0K)\frac{\ell(f^{-n_0}(J_{B_0}(z)))}{\ell(J_{B_0}(z))}\exp(K_0K) \\
&= \ell(f^{-n_0}(J_{B_0}(z)))\frac{\ell(J_{B_0}(f^{-k}(z)))}{\ell(J_{B_0}(z))}\frac{1}{\ell(J(f^{-n_0}(z)))}\exp(2K_0K) \\
&\leq rC_1\frac{1}{L}\exp(2K_0K) < \frac{1}{2}.
\end{aligned}$$

So, the proof is finished.

∎

This completes the proof of the main lemma in case Λ is not a minimal set.

4.1.2 Λ is a minimal set.

We begin remarking that we can not expect to do the same argument here as in the preceding case, due to the fact that, if Λ is a minimal set, then for every Markov box, the set of returns of this box is always finite. Nevertheless we shall exploit the fact that in the case Λ is a minimal set, the central unstable manifold is in fact an unstable together with the existence of "boundary points". First, we introduce some notations. Given a central unstable arc J, we order J in some way and we denote $J^+ = \{y \in J : y > x\}$, $J^- = \{y \in J : y < x\}$. Also, giving $x \in B$ we shall denote by B^+ (say the upper part of the box) the connected component of $B - W^s_{\epsilon(x)}$ which contains J^+, and by B^- (the bottom one) the one containing J^-.

Lemma 4.1.10 *Assume Λ is minimal set. Then, there exist a Markov box arbitrary small B such that $B^+ \cap \Lambda = \emptyset$ or $B^- \cap \Lambda = \emptyset$.*

Proof: Given a box $B_i \in \mathcal{P}$, follows that there is $x_1, x_2 \in B_i \cap \Lambda$ such that the strip in B, bounded by $W^s_\epsilon(x_1)$ and $W^s_\epsilon(x_2)$ has empty intersection with Λ. If it is not the case, we would get that there is a periodic point in Λ which is a contradiction since Λ is minimal. Now, take for instance the point x_1 in B_i and take an small Markov box B containing this point, i.e.: take k large enough $B_j \in \mathcal{P}$ and the connected component of $f^{-k}(B_j) \cap B_i$ that contains x_1. Observe that the point x_1 does not belong to the boundary of B, since the boundary of the Markov box are given by the stable manifold of periodic points and Λ is minimal. On the other hand, if k is large enough, the vertical size of B is small enough such that one of the side of $B \setminus W^s_\epsilon(x)$ is contained in the strip bounded by $W^s_\epsilon(x_1)$ and $W^s_\epsilon(x_2)$. These two facts imply that the box B satisfies the properties required.

■

Related to this box we will get the following lemma that will imply the Main Lemma when Λ is minimal:

Lemma 4.1.11 *Let B be a Markov box such that $B^+ \cap \Lambda = \emptyset$. Then there exist K such that for every $y \in B \cap \Lambda$,*

$$\sum_{j \geq 0} \ell(f^{-j}(J^+(y))) < K.$$

In particular there exist $J_1(y), J^+(y) \subset J_1(y) \subset J(y)$ such that the length of $J_1(y) - J^+(y)$ is bounded away from zero (independently of y) and such that

$$\sum_{n=0}^{\infty} \ell(f^{-n}(J_1(y))) < \infty.$$

Assuming this lemma, we can prove the Main Lemma when Λ is a minimal set. We shall proceed as in lemma 4.1.7. Using the notation of the preceding lemma, take

$$B = \bigcup_{y \in B \cap \Lambda} J_1(y).$$

Notice that B is an open set of Λ, and for every $y \in B \cap \Lambda$ (i.e. $y \in J_1(y)$), we have

$$\sum_{n=0}^{\infty} \ell(f^{-n}(J_1(y))) < \infty$$

and so

$$\|Df^{-n}_{/F(y)}\| \to_{n\to\infty} 0.$$

Let z be any point in Λ. Since Λ is a minimal set there exist $m_0 = m_0(z)$ such that $f^{-m_0}(z) \in B$ and so

$$\|Df^{-n}_{/F(f^{-m_0}(z))}\| \to_{n\to\infty} 0$$

implying that

$$\|Df^{-n}_{/F(z)}\| \to_{n\to\infty} 0.$$

This completes the proof of the Main Lemma.

■

To finish the section, we have to conclude the proof of lemma 4.1.11. First we show that the box introduced in 4.1.10, has some similar property as the Markov box.

Lemma 4.1.12 *Assume that Λ is a minimal set and let B be a Markov box such that $B^+ \cap \Lambda = \emptyset$. Then B^+ verifies that for all $y \in B_\epsilon(J) \cap \Lambda$,*

$$f^{-n}(J^+(y)) \cap B^+ = \emptyset \text{ or } f^{-n}(J^+(y)) \subset B^+$$

where $J^+(y) = J(y) \cap B^+$. Moreover, there exist K_1 such that if $y \in B_\epsilon(J) \cap \Lambda$ and $f^{-j}(J^+(y)) \cap B^+ = \emptyset, 1 \leq j < n$ then

$$\sum_{j=0}^{n} \ell(f^{-j}(J^+(y))) < K_1.$$

Proof:

Assume that for some $y \in B \cap \Lambda$ and $n > 0$ $f^{-n}(J^+(y)) \cap B^+ \neq \emptyset$ holds. As B is a Markov box we conclude that $f^{-n}(J(y)) \subset B$. If $f^{-n}(J^+(y))$ is not contained in B^+, then $f^{-n}(J^+(y)) \cap W^s_\epsilon(x) \neq \emptyset$. However, this implies that $f^n(x) \in B^+$. Since $x \in \Lambda$ this is a contradiction, and completes the proof of the first part.

The existence of K_1 it can be proved with the same arguments as in lemma 4.1.6.

■

Now, we conclude the proof of lemma 4.1.11.

Proof of lemma 4.1.11:

First, if we have that there are not points in B^+ that for some negative iterates are also in B^+, by the preceding lemma, we conclude the thesis.

Assume now that there is k_0 such that if $y \in B$ such that $f^{-k}(J^+(y)) \subset B^+$ (where k is the first return time of y to B^+), then $k < k_0$. In this case, we assert that there exist $r > 0$ such that

$$dist(W^s_\epsilon(x), f^{-k(y)}(J^+(y))) > r, \forall y \in B^+ \ f^{-k(y)}(y) \in B^+.$$

If it is not the case, since that the return time is bounded, we get a point $y \in B \cap \Lambda$ such that $f^{-k(y)}(J^+(y))) \cap W^s_\epsilon(x) \neq \emptyset$; then, using that B is a Markov box, and that the extremal points of $J^+(y)$ and $f^{-k(y)}(J^+(y))$ are $W^s_\epsilon(x)$, we conclude that $w(x)$ is a periodic point, which is an absurd because $z \in \Lambda$ and Λ is minimal.

On the other hand, there exist N such that for every $y \in B \cap \Lambda$ and $n \geq N$ we have

$$\ell(f^{-n}(J(y))) < r.$$

This implies that for $n \geq N$ we get that

$$f^{-n}(J(y)) \cap B^+ = \emptyset.$$

In fact, since $\ell(f^{-n}(J(y))) < r$ and $dist(W^s_\epsilon(x), f^{-k(y)}(J^+(y))) > r$, we conclude that $f^{-n}(J(y)) \subset B^+$ implying that $f^{-n}(y) \in B^+ \cap \Lambda = \emptyset$, which is impossible.

Let $n_0 = \max\{n \geq 0 : \ f^{-n}(J(y)) \cap B^+ \neq \emptyset\}$. Thus, $n_0 \leq N$ and, since $f^{-n_0}(J^+(y)) \subset J^+(f^{-n_0}(y))$, we conclude that

$$f^{-j}(J^+(f^{-n_0}(y))) \cap B^+ = \emptyset, \ \forall j \geq 1$$

and so, by the previous lemma,

$$\sum_{j\geq 0} \ell(f^{-j}(J^+(f^{-n_0}(y)))) < K_1.$$

Therefore

$$\sum_{j\geq 0} \ell(f^{-n}(J^+(y))) < Ndiam(M) + K_1 = K.$$

Finally, let us assume that there is a sequences $y_n \in B \cap \Lambda$ such that $f^{-k_n(y)}(J^+(y_n)) \subset B^+$ and $k_n(y) \to \infty$.

Take $r > 0$ such that

$$\frac{r}{L} \exp(K_0 K_1) < \frac{1}{2}$$

where $L = \min\{\ell(J^+(y)) : y \in B \cap \Lambda\}$. Let $N > 0$ be such that if $n > N$ then $\ell(f^{-n}(J(y))) < r$. We assert that there is j_0 such that given $y \in \Lambda$, if $k_i(y), k_2(y), \ldots.., k_n(y), \ldots$ are the return times of $J^+(f^{k_1(y)+\ldots+k_i(y)}(y))$, then for $i > j_0$ holds that $k_{i+1}(y) - k_i(y) > N$. It this is not the case, arguing as in the previous case, we get a point $y \in B \cap \Lambda$ such that $f^{-k(y)}(J^+(y))) \cap W^s_\epsilon(x) \neq \emptyset$, which leads to a contradiction. Now, follows from the assertion, that

$$\|Df^{-k_j}_{/F(f^{k_1(y)+\ldots+k_i(y)}(y))}\| < \frac{1}{2}$$

provided that $j > j_0$.

Applying the same argument as before, we can conclude that for every $y \in B \cap \Lambda$ we have

$$\sum_{j\geq 0} \ell(f^{-j}) \leq Ndiam(M) + \sum_{j\geq 0} diam(M) \exp(2K_0K_1)(\frac{1}{2})^j = K < \infty.$$

In particular, as in the Schwarz's proof of the Denjoy Theorem, we conclude that $\forall y \in B \cap \Lambda$ there exist $J_1(y), J^+(y) \subset J_1(y) \subset J(y)$ such that the length of $J_1(y) - J^+(y)$ is bounded away from zero (independently of y) and such that

$$\sum_{n=0}^{\infty} \ell(f^{-n}(J_1(y))) < \infty$$

and the proof of the lemma is finished.

■

References

[ABV] J. Alves, C. Bonatti, M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.* **140** (2000), 351–398.

[B] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms *Springer-Verlag* (1975).

[BV] C. Bonatti, M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly contracting, *Israel J. Math.* **115** (2000), 157–193.

[BP] M. I.Brin, Ja. B. Pesin, . Partially hyperbolic dynamical systems. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 170–212.

[C] Costa, M. J., Saddle-node horseshoes giving rise to global Hénon-like attractors. *An. Acad. Brasil. Cienc.***70** (1998), no. 3, 393–400.

[CY] Cowieson, B.; Young, L-S.; SRB measures as zero noise limits, Talk at Conference on Partially Hyperbolicity, Northwestern University, May 2001.

[dMS] W. de Melo, S van Strien, One-dimensional dynamics, *Springer Verlag*, 1993.

[DRV] Díaz, L. J.; Rocha, J.; Viana, M. Strange attractors in saddle-node cycles: prevalence and globality. *Invent. Math.* **125** (1996), no. 1, 37–74.

[Fr] J. Franks, Necessary conditions for stability of diffeomorphisms, *Trans. Amer. Math. Soc.* **158** (1971) 301–308.

[HPS] M. Hirsch, C. Pugh, M. Shub, Invariant manifolds, *Springer Lecture Notes in Math.*, **583** (1977).

[K] I. Kupka, Contribution à la théorie des chaps génériques, *Cont. Diff. Equ.* **2** (1963), 457-484.

[LW] L. Wen. Homoclinic tangencies and dominated splittings. *Nonlinearity* **15** (2002), no. 5, 1445–1469

[M] Mañé, R., Ergodic Theory and Differential Dynamics, *Springer-Verlag* New York, 1987.

[M2] Mañé, R., An ergodic closing lemma, *Ann. of Math,* **116**, 1982, 503-540.

[N1] S. Newhouse, Non-density of Axiom A(a) on S^2, *Proc. A.M.S. Symp. Pure Math.* **14** (1970), 191-202.

[N2] S. Newhouse, Diffeomorphism with infinitely many sinks, *Topology* **13** (1974), 9-18.

[N3] S. Newhouse, The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms, *Publ. Math. I.H.E.S.* **50** (1979), 101-151.

[N4] S. Newhouse, Hyperbolic limit sets, *Trans. A.M.S.* **167** (1972), 125-150.

[NP] Newhouse, S.; Palis, J. Hyperbolic nonwandering sets on two-dimensional manifolds. Dynamical systems *Proc. Sympos., Univ. Bahia, Salvador*, 1971, pp. 293–301. Academic Press, New York, 1973.

[P] J. Palis, On Morse-Smale dynamical systems, *Topology* **8** (1969), 385-405.

[Pg] C. Pugh, The closing lemma, *Amer. J. Math* (1967).

[Pl] V. A. Pliss, On a conjecture due to Smale, *Diff. Uravnenija*, **8** (1972), 268-282.

[PPV] M. J. Pacifico, E. R. Pujals, J. Vieitez, Robustly expansive homoclinic classes *Ergodic Theory and Dynamical Systems,* **25** n. 1, (2004), p. 271-300.

[PuSh] C. Pugh, M. Shub, Stable ergodicity and julienne quasi-conformality. *J. Eur. Math. Soc. (JEMS)* **2** (2000), no. 1, 1–52.

[PS1] E. Pujals, M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, *Ann. of Math.* **151** (2000), no. 3, 961–1023.

[PS2] E. Pujals, M. Sambarino, On homoclinic tangencies, hyperbolicity, creation of homoclinic orbits and varation of entropy. *Nonlinearity* **13** (2000), 921–926.

[PS3] E. Pujals, M. Sambarino, Integrability on codimension one dominated splitting. *preprint.*

[PT] J.Palis and F.Takens Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations, *Cambridge University Press*, (1993).

[PV] J.Palis, M. Viana High dimension diffeomorphisms displaying infinitely many periodic attractors. *Ann. of Math. (2)* **140**, (1994), no. 1, 207–250.

[S] S. Smale, Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

[S3] S. Smale, Stable manifolds for differential equation and diffeomorphisms, *Ann. Scuola Norm. Super. Pisa* **17** (1963) 97-116.

[Sch] A. J. Schwartz, A generalization of a Poincaré-Bendixon Theorem to closed two-dimensional manifolds, *Amer. J. Math.* **85** (1963), 453-458.

[Sh] M. Shub, Global Stability of Dynamical Systems, *Springer-Verlag*, 1987.

Enrique R. Pujals
(enrique@impa.br)
Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina 110, CEP: 22460-320
Rio de Janeiro, R.J., Brazil

Martín Sambarino
(samba@cmat.edu.uy)
Centro de Matemática
Universidad de la República
Iguá 4225 esq. Mataojo, CP:11400
Montevideo-Uruguay