TRANSITIVE ANOSOV FLOWS AND AXIOM A DIFFEOMORPHISMS

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Abstract. Let M be a smooth compact Riemannian manifold without boundary, and $\phi : M \times \mathbb{R} \to M$ a transitive Anosov flow.

We prove that if the time one map of ϕ is C^1-approximated by Axiom A diffeomorphisms with more than one attractor, then ϕ is topologically equivalent to the suspension of an Anosov diffeomorphism.

Introduction

A flow ϕ on a closed manifold M is called an *Anosov flow* if it is *hyperbolic*: the transverse bundle of the flow splits into two invariant bundles E^s and E^u, where the vectors in E^s and E^u are exponentially contracted and expanded, respectively, by the action of the flow. In the same way, a diffeomorphism on compact manifold is an *Anosov diffeomorphism* if the whole manifold is a hyperbolic set. Given an Anosov diffeomorphism $f\colon N \to N$ there is an associated Anosov flow ϕ called *suspension* of f: the suspension manifold N_f is obtained from the direct product $N \times [0,1]$ by identifying pairs of points of the form $(x,0)$ and $(f(x),1)$ for $x \in N$. The suspension flow $\varphi(x,t)$ is determined by the vector field $\frac{\partial}{\partial t}$.

The global hyperbolic structure of Anosov flows or diffeomorphisms is a very strong geometric property, so that manifolds carrying such dynamics satisfy strong topological conditions and the list of known examples is not so long. For example, all the known Anosov diffeomorphisms are conjugated to algebraic automorphisms of infranil manifolds, and Franks [7] and Newhouse [14] proved that any codimension 1 Anosov diffeomorphism (i.e. the stable bundle E^s of f has codimension 1) is conjugated to a linear automorphism of the torus T^n. In the same way Verjovsky conjectured in [22], that every codimension 1 Anosov flow, on a compact manifold M with $dim M \geq 4$, is topologically equivalent to the suspension of an Anosov diffeomorphism. There are many partial results in the direction of this conjecture, and the proof of the conjecture for conservative flows has been recently announced by S. Simić [19].

However, in dimension 3 there are many examples of Anosov flow with unexpected behavior: for example there are non-transitive Anosov flow, see [8]; see also [4]. We refer to [1] and [6] for works in the direction of a classification of transitive Anosov flows in dimension 3.

So, a natural problem is

Problem 1. *Give a dynamical characterization of the Anosov flows which are obtained by the suspension of Anosov diffeomorphisms.*

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The time one map $f_1 = \phi(.,1)$ of an Anosov flow ϕ is not hyperbolic, since the tangent direction of the orbits of the flow is neither contracted nor expanded by the differential of the iterates of f_1. Hence f_1 is not structurally stable: small perturbations of f_1 may be not conjugated to f. However [10] shows that perturbations of f_1 only change the dynamic along the orbit of the flow ϕ: every diffeomorphism g, sufficiently C^1-close to f_1, is conjugated to a homeomorphism of the form $x \mapsto \phi(x, \tau_g(x))$, where $\tau_g(x) > 0$ is a continuous function. In this paper we investigate the dynamics of diffeomorphisms C^1-close to f_1.

Question 1. *What kind of dynamical system can appear under perturbations of the time one map of an Anosov flows?*

We are specially interested in the case of a *transitive* Anosov flow (i.e. the case when the non-wandering set of ϕ is the whole manifold). More specifically we would like to understand when the time one map of a transitive Anosov flow can be approximated by hyperbolic (i.e. Axiom A) diffeomorphisms.

At the end of the sixties, Abraham and Smale constructed a diffeomorphism $f : T^2 \times S^2 \to T^2 \times S^2$ and a C^1- neighbourhood $N(f)$ of f, such that if $g \in N(f)$ then the non-wandering set of g is non-hyperbolic: there is no Axiom A diffeomorphisms in $N(f)$. This example can be extended in a simple way to higher dimensions, and has been extended to dimension 3 in [19]. Later, C^1-open sets of non Axiom A and robustly transitive diffeomorphisms were described on T^4 by Shub (see [17]), on T^3 by Mañé (see [13]). All these examples show that Axiom A diffeomorphisms are not dense, if the dimension of the manifold is greater or equal to 3 (it is not known if Axiom A diffeomorphisms are C^1- dense in surfaces). In other words, many diffeomorphisms cannot be approximated by Axiom A, and an ambitious general question is to characterize diffeomorphisms in the C^1-closure of the set of Axiom A diffeomorphisms.

Another example, built in [2], gives a partial answer of Question 1: for any transitive Anosov flow ϕ, any C^1-neighborhood of the time one map f_1 of ϕ contains a (non-empty) open set of nonhyperbolic and transitive diffeomorphisms.

It is a well known fact that the time one map f_1 belongs to the closure of Axiom A diffeomorphisms in the case of an Anosov flow ϕ is obtained as the suspension of an Anosov diffeomorphism $g : N \to N$. The suspension manifold, N_g, is fibered over S^1 and the projection of the time one map onto S^1 is the identity map. There are diffeomorphisms f preserving fibers, C^1- close to f_1, and such that the projection of f over S^1 is a Morse-Smale map. Then such a diffeomorphism f is an Axiom A diffeomorphism.

Palis and Pugh asked in [15, Problem 20]:

Question 2. *Can the time one map of an Anosov flow be approximated by an Axiom A diffeomorphism? If the flow is a suspension of an Anosov diffeomorphism the answer is yes.*

Our main result is:

Theorem 1. *Let M be a smooth compact Riemannian manifold without boundary. If the time one map of a transitive Anosov flow ϕ is C^1-approximated by Axiom A diffeomorphisms having more than one attractor, then ϕ is topologically equivalent*[1] *to the suspension of a hyperbolic diffeomorphism.*

Indeed we prove a slightly stronger statement: if ϕ is not topologically equivalent to a suspension, and if f is an Axiom A diffeomorphism C^1-close to the time one map of ϕ, then the unique transitive attractor and the unique transitive repeller of f are connected (hence topologically mixing).

A partial result was given previously in ([9]) for codimension one Anosov flows under some technical and restrictive assumptions related to periodic points: we asked that the number of periodic points of the Axiom A diffeomorphism in a "fundamental domain " of any closed central leaf is constant (see [9] for more details). This technical hypothesis allowed us to prove that a repeller basic set was a global section, so that the initial Anosov flow was topologically equivalent to a suspension.

An important improvement is that we remove not only this "technical" assumption but the codimension one hypothesis as well. Although in the general case the proof is quite different, we include some results that had appeared in ([9]). This is done because they contain basic ideas of the proof of the main theorem.

0.1. **A stronger version of Theorem 1.**

Let us restate our main result in a slightly stronger version:

Denote by $\mathcal{F}^c$ the 1-dimensional foliation of M whose leaves are the orbits of ϕ. Consider the set $\mathcal{E}_\phi$ of diffeomorphisms $f\colon M \to M$ satisfying the following hypotheses:

- f is a partially hyperbolic diffeomorphisms with one dimensional central direction: there exists a Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$, such that $Df|E^s$ is uniformly contracting, $Df|E^u$ is uniformly expanding, and E^c is a nonhyperbolic central direction with $dim(E^c) = 1$
- there is an f-invariant foliation $\mathcal{F}^c_f$ tangent to E^c.
- the central foliation $\mathcal{F}^c_f$ is topologically conjugated to $\mathcal{F}^c$: there is a homeomorphism $h\colon M \to M$ such that $h(\mathcal{F}^c) = \mathcal{F}^c_f$.
- each leaf $F^c_f(x)$ is invariant by f. Furthermore, the distance $d^c(x, f(x))$ in the leaf $F^c_f(x)$ is uniformly bounded on M: there is $K_f > 0$ such that, for any $x \in M$ there is a path $\gamma \subset F^c_f(x)$ with length $\ell(\gamma) < K_f$ joining x to $f(x)$ in the central leaf $F^c_f(x)$.

We will show that, for $f \in \mathcal{E}_\phi$ there is a continuous function $\tau(x) \neq 0$ such that f is conjugated to the homeomorphism $x \mapsto \phi(x, \tau(x))$.

Hirsch Pugh and Shub (see Section 1) proved that $\mathcal{E}_\phi$ contains a C^1-neighborhood of f_1. Hence Theorem 1 is a direct consequence of:

Theorem 2. *Let ϕ be a transitive Anosov flow on a smooth compact Riemannian manifold M without boundary. If the set $\mathcal{E}_\phi$ contains Axiom A diffeomorphisms which have more than one attractor, then ϕ is topologically equivalent to the suspension of a hyperbolic diffeomorphism.*

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[1]Some authors use *orbit equivalent* instead of topologically equivalent.

In a future work, we will show that for a transitive Anosov flow on a 3-manifold M, the set $\mathcal{E}_\phi$ contains Axiom A diffeomorphism whose non-wandering set is the union of only two basic sets. One of them is an attractor and the other is a repeller set.

0.2. **Sketch of the proof of Theorem 2 and organization of the paper.**
Assume that $f \in \mathcal{E}_\phi$ is an Axiom A diffeomorphism.
Using arguments in [9] we prove in Section 3 that:

- every attracting or repelling basic sets of f meets every central segment (i.e. segment of $\mathcal{F}^c_f$) of length K_f; in particular they meet every compact central leaf.
- the local central direction is contracting for every point in the attracting basic sets and it is expanding for the points in the repelling basic sets.

Then, in Section 4 we prove:

- Let A be an attracting basic set and $W^s(A)$ its basin. Then there is an open and dense subset U of $W^s(A)$ such that for $x \in U$ the connected component of $F^c_f(x) \cap W^s(A)$ containing x meets A. Furthermore, there is a residual subset of U (hence of $W^s(A)$) for which this connected component meets A in precisely one point.

Using these properties, in Section 5, we prove that for any attractor A there are two repellers Λ_- and Λ_+, called the predecessor and the successor of A, respectively, such that, for generic point $x \in W^s(A)$ the closure of the connected component of $F^c_f(x) \cap W^s(A)$ containing x is a segment $[a(x), b(x)]$ of central leaf with $a(x) \in \Lambda_-$ and $b(x) \in \Lambda_+$. An analogous fact holds for generic points in the basin of repeller.

As f is an Axiom A diffeomorphism, the union of the basin of attracting basic sets is a dense open subset of M. As a consequence one proves that for a generic point x of M, the intersection of the (dynamically oriented) central leaf with the union of attracting and repelling basic sets of f form a sequence x_i which belongs alternately to attracting and repelling basic sets; furthermore if x_i belongs to an attractor A (resp. a repeller Λ), then x_{i+1} belongs to the repeller Λ_+ (resp. the attractor A_+) which is the successor of A (resp. of Λ) (see Lemma 5.3).

This relation of successor induces a family of cycles in the set of attracting and repelling basic sets of f. We prove that there is a unique cycle (Lemma 5.4); in other words there is an indexation A_i, Λ_i of the set of attractors and repellers such that Λ_i is the successor of A_i and the predecessor of A_{i+1}. As a consequence, generic central leaves intersect the attractors and repellers following this cycle order. One deduces (Proposition 5.1) that the same holds for all the central leaves, up to allowing repetitions: a leaf may cut an attractor in more than one point before crossing the successor.

Assume now that f has more than one attracting basic set, (or equivalently more than one hyperbolic repelling basic set). We show (Corollary 7.1) that the boundary of the basin of a repelling basic set, Λ_0 is the union of two disjoint compact sets K_0 and K_1, verifying that $A_0 \subset K_0$ and $A_1 \subset K_1$.

Then we build a continuous and suryective function $\rho\colon M \to S^1 = \mathbb{R}/\mathbb{Z}$ mapping K_0 on $0 = 1$ and K_1 on $\frac{1}{2}$. Since every segment of central leaf of length greater than K_f meets every attractor A_i, then there exists L such that every segment of central leaf γ of length greater than L verifies that $\rho(\gamma) = S^1$.

Let $\Pi\colon \tilde{M} \to M$ be the infinite cyclic cover of M, obtained by pullback by ρ of the universal cover $\mathbb{R} \to S^1$. Consider the lift $\tilde{\mathcal{F}}^c_f$ of the foliation $\mathcal{F}^c_f$ on this cyclic cover. The foliation $\tilde{\mathcal{F}}^c_f$ has all its leaves going uniformly from $-\infty$ to $+\infty$(Lemma 7.6). The same holds for the lift of the initial foliation $\mathcal{F}^c$ generated by the Anosov flow ϕ. An argument of Schwartzman (see [23]) allows to conclude: the flow ϕ admits a global section, hence it is topologically equivalent to the suspension of an Anosov diffeomorphisms.

1. Definitions and classical results

We begin recalling some basic definitions about flows and diffeomorphisms.

1.1. **Anosov flows.** A good reference for basic properties of Anosov flows is [23]

Definition 1.1. *Let $\phi\colon M \times \mathbb{R} \to \mathbb{R}$ be a C^1-flow on a compact manifold M. A compact ϕ_t-invariant set, $\Lambda \subset M$, is called a* hyperbolic set for the flow ϕ *if there exist a Riemannian metric on an open neighborhood $\mathcal{U}$ of Λ, and $0 < \lambda < 1 < \mu$ such that for all $x \in \Lambda$ there is a decomposition*

$$T_x(M) = E^{ss}_x \oplus E^{uu}_x \oplus E^c_x$$

such that $\partial_t\phi(x,t)|_{t=0} \in E^c_x - \{0\}$, $dim(E^c(x)) = 1$, $D_x\phi_t(x)(E^i_x) \subset E^i_{\phi(x,t)}$, with $i = ss, uu$, and

$$\|D_x\phi(x,t)|_{E^{ss}(x)}\| \le \lambda^t \text{ for } t \ge 0$$

$$\|D_x\phi(x,t)|_{E^{uu}(x)}\| \le \mu^t \text{ for } t \le 0.$$

A C^1 flow $\phi : M \times \mathbb{R} \to M$, is called an Anosov flow *if M is a hyperbolic set for ϕ.*

Let ϕ be an Anosov flow on a compact manifold M. The bundles E^{ss} and E^{uu} are called the strong stable and strong unstable bundles of ϕ. We fix k such that $dim E^{ss}_x = n - k - 1$ and $dim E^{uu}_x = k$ for all $x \in M$.

We call $E^{cs} = E^{ss} \oplus E^c$ and $E^{cu} = E^{uu} \oplus E^c$ the *central stable* and *central unstable bundles*, respectively.

There are ϕ-invariant foliations $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$, $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$ and $\mathcal{F}^c$ tangent to the bundles E^{cs}, E^{cu}, E^{ss}, E^{uu} and E^c, respectively, and called the center stable, center unstable, strong stable, strong unstable and central foliations, respectively. The leaves of these foliations are C^1-manifolds varying continuously in the C^1 topology, but in general they fail to be C^1-foliations: the holonomy maps are in general not C^1.

The leaves of the central foliation (called the central leaves) are the orbits of ϕ, so that the central foliation $\mathcal{F}^c$ is a C^1 foliation. For any point $x \in M$ the strong stable leaf $F^{ss}(x)$ is the stable manifold $W^{ss}(x)$ of x, that is the set of points y such that the distance $d(\phi(x,t), \phi(y,t))$ tends to 0 when t tends to $+\infty$. The central stable leaf $F^{cs}(x)$ is the stable manifold of the orbit of x, that is union of the strong stable leaves through the orbit of x:

$$F^{cs}(x) = \bigcup_{y \in F^c(x)} F^{ss}(y).$$

In the same way, the central unstable leaf of x is the union of the strong unstable leaves through the orbit of x:

$$F^{cu}(x) = \bigcup_{y \in F^c(x)} F^{uu}(y).$$

In particular, if $\mathcal{O}$ is a closed orbit of ϕ then, for $x \in \mathcal{O}$ the central stable and central unstable leaves $F^{cs}(x)$ and $F^{cu}(x)$ are the stable and unstable manifolds $W^s(\mathcal{O})$ and $W^u(\mathcal{O})$, respectively.

Assume now that the Anosov flow ϕ is transitive. Then :

- the periodic orbits of ϕ are dense in M. In other words:

$$\{x|F^c(x) \text{ is a closed set }\} \text{ is dense in } M.$$

- generic points of M have a dense forward and backward orbits. That is:

$$\{x|F^c(x) \text{ is dense in } M\} \text{ is a residual set.}$$

- for every point $x \in M$ the central stable and central unstable leaves $F^{cs}(x)$ and $F^{cu}(x)$ are both dense in M.

1.2. **Axiom A diffeomorphisms.** The proof of Theorems 1 and 2 uses many properties of Axiom A diffeomorphisms. Here we just recall briefly some basic definition and properties of Smale's hyperbolic theory. The reader will find more complete information on hyperbolic dynamics in [21], [18], [11, Part 4], [16, Chapter 0].

Definition 1.2. *Let $f : M \to M$ be a C^r diffeomorphism . An f-invariant set Λ is called* hyperbolic *if there exists a Df-invariant decomposition of $T_\Lambda M$ such that*

$$T_\Lambda M = E^s \oplus E^u$$

and $Df|E^s$ is uniformly contracting and $Df|E^u$ is uniformly expanding. More precisely, there are $c > 0$ and λ, with $0 < \lambda < 1$ such that for all $x \in \Lambda$

$$\|D_x f^n|E^s(x)\| < c\lambda^n$$

and

$$\|D_x f^{-n}|E^u(x)\| < c\lambda^n.$$

A diffeomorphism $f : M \to M$ is called an Anosov diffeomorphism *if M is a hyperbolic set for f.*
A diffeomorphism $f : M \to M$ satisfies the Axiom A *if the non-wandering set $\Omega(f)$ is hyperbolic and the set of periodic points is dense in $\Omega(f)$.*

A compact hyperbolic set K of f is called a *basic set* if it is *transitive* (i.e. there is a point $x \in K$ whose positive orbit is dense in K) and it is the maximal invariant set of f in a open neighborhood U of K, i.e. $K = \bigcap_{n\in\mathbb{Z}} f^n(U)$. The stable manifold $W^s(K)$ of a basic set K is the set of points whose ω-limit (limit of the forward orbit) is contained in K; according to the shadowing lemma, $W^s(K)$ is the union of the stable manifolds $W^s(x)$ of the points $x \in K$, where the stable manifold of x is the set of points y such that the distance $d(f^n(x), f^n(y))$ tends to 0 when n tends to $+\infty$. The stable manifolds $W^s(x)$ is C^1-immersion of $E^s(x)$, and is tangent at x to $E^s(x)$. Furthermore it depends continuously on x for the C^1 topology.

Smale proved in [21] that:

- the non-wandering set of an Axiom A diffeomorphism is a the union of finitely many disjoint basic sets K_i;
- for any point $x \in M$ exist i, j such that x belongs to the stable manifold of K_i and to the unstable manifold of K_j;
- for any i the intersection between the stable and the unstable manifold of K_i is equal to K_i:
$$W^s(K_i) \cap W^u(K_i) = K_i.$$
- some of the K_i are *attractors*: this means that K_i admits an open neighborhood U such that $f(\bar{U}) \subset U$ and such that $K_i = \bigcap_{n>0} f^n(U)$. The basin (i.e. stable manifold) of the attractors are open sets whose union is dense in M.
- a *repeller* of f is an attractor of the Axiom A diffeomorphism f^{-1}.

Furthermore, a basic set K is an attractor if and only if it contains its unstable manifold, that is $K = W^u(K)$.

Remark 1.1. (1) *A transitive hyperbolic attractor (i.e. an attracting basic set) has finitely many connected components, which are exchanged by f.*
(2) *For any $n > 0$, f^n is an Axiom A diffeomorphism.*
(3) *Any transitive hyperbolic attractor of f^n is the orbit by f^n of a connected component of an attractor of f.*
(4) *Hence, there is $n > 0$ such that each transitive attractor and repeller of f^n is connected.*

1.3. **Partially hyperbolic diffeomorphisms.** We refer to [10] and to [3, Appendix B] for the basic properties of partially hyperbolic dynamics.

Definition 1.3. *A C^1 diffeomorphism $f : M \to M$ is called* partially hyperbolic *if there exists a Df-invariant decomposition of*

$$T_xM = E^s_x \oplus E^c_x \oplus E^u_x$$

such that the dimensions of the spaces E^s_x, E^c_x, and E^u_x do not depend on $x \in M$, furthermore $Df|E^s$ is uniformly contracting , $Df|E^u$ is uniformly expanding, and the expansion in E^c is stronger than the expansion in E^s and less than the expansion in E^u. More precisely, there are $c > 0$ and λ, with $0 < \lambda < 1$ such that for all $x \in \Lambda$

$$\begin{aligned} \|D_xf^n|E^s(x)\| &< c\lambda^n, \\ \|D_xf^{-n}|E^u(x)\| &< c\lambda^n, \\ \|D_xf^n|E^s(x)\|\|D_xf^{-n}|E^c(x)\| &< c\lambda^n, \text{ and} \\ \|D_xf^n|E^c(x)\|\|D_xf^{-n}|E^u(x)\| &< c\lambda^n. \end{aligned}$$

The bundles E^s, E^c and E^u are always continuous. The partial hyperbolicity is an C^1-open structure: if f is a partially hyperbolic then there is a C^1-neighborhood $\mathcal{U}$ of f such that any $g \in \mathcal{U}$ admits a (unique) partially hyperbolic structure $T_xM = E^s_{x,g} \oplus E^c_{x,g} \oplus E^u_{x,g}$ such that the dimension of the spaces are the same for f and for g. Furthermore the bundles E^s_g E^c_g and E^u_g depend continuously on g.

As in the hyperbolic case, if f is partially hyperbolic there are invariant foliations $\mathcal{F}^{ss}_f$ and $\mathcal{F}^{uu}_f$ tangent to E^s_f and E^u_f, respectively, whose leaves are C^1 immersed manifold. In general there is no invariant foliation tangent to E^c (there are counter-examples when $dim E^c > 1$, and the existence of a central foliation is unknown for $dim E^c = 1$). However, the existence of a *central foliation* $\mathcal{F}^c$ tangent to E^c and f invariant leads to a strong rigidity property of f. Hirsch Pugh and Shub showed:

Theorem 3. *Let f be a partially hyperbolic diffeomorphism of a compact manifold. Assume that f admits a C^1-foliation $\mathcal{F}^c_f$ tangent to E^c. Then, there is a C^1-neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is a partially hyperbolic diffeomorphism having an invariant central foliation $\mathcal{F}^c_g$; furthermore, there is a homeomorphism $h_g\colon M \to M$ and a constant $c_g > 0$ such that*

$$h_g(\mathcal{F}^c_g) = \mathcal{F}^c_f,$$

and the homeomorphism $h_g \circ g \circ h_g^{-1}$ satisfies the following property: for every $x \in M$ the point $h_g \circ g \circ h_g^{-1}(x)$ belongs to the same leaf of $\mathcal{F}^c_f$ as $f(x)$ and the distance between $h_g \circ g \circ h_g^{-1}(x)$ and $f(x)$ in the central leaf is bounded by c_g:

$$d^c(h_g \circ g \circ h_g^{-1}(x), f(x)) < c_g.$$

Finally, if the neighborhood $\mathcal{U}$ is small enough, then h_g is close to the identity and c_g is very small.

This Theorem has been stated in [10, Theorem 7.1] with the hypothesis " $\mathcal{F}^c$ is *plaque expansive*" instead of "$\mathcal{F}^c$ is C^1". However, [10, Theorem 7.2] shows that the central foliation is always plaque expansive if it is a C^1 foliation.

Finally, a partially hyperbolic diffeomorphism f is called *dynamically coherent* if there are f-invariant foliations $\mathcal{F}^{cs}_f$, $\mathcal{F}^{cu}_f$ an $\mathcal{F}^c_f$ tangent to $E^s_f \oplus E^c_f$, $E^u_f \oplus E^c_f$ and E^c_f, respectively.

1.4. **C^1-small perturbations of the time one map of an Anosov flow.** Let ϕ be a C^1-Anosov flow on a compact manifold M.

We will denote by $f_1 : M \to M$, the time one diffeomorphism of ϕ defined as

$$f_1(x) = \phi(x, 1),\ \forall x \in M.$$

The diffeomorphism f_1 has no hyperbolic set because the direction tangent to the flow is invariant but neither expanded nor contracted. However, it is partially hyperbolic, the invariant bundles are those E^{ss}, E^c and E^{uu} of the flow ϕ. Furthermore, the central foliation $\mathcal{F}^c$ of the Anosov flow ϕ is a C^1 foliation[2]. Hence one may apply Theorem 3 to $\mathcal{F}^c$.

Hence there is a C^1-neighborhood $\mathcal{U}$ of f_1 such that any $f \in \mathcal{U}$ satisfies the following properties:

(1) it is a partially hyperbolic diffeomorphism with a splitting $TM = E^{uu}_f \oplus E^c_f \oplus E^{uu}_f$ of the same dimensions as those of f_1 (i.e. of ϕ);
(2) there is a 1-dimensional f-invariant central foliation $\mathcal{F}^c_f$ tangent to E^c_f
(3) there is a homeomorphism h_f such that $h_f(\mathcal{F}^c_f) = \mathcal{F}^c$
(4) furthermore, for every $x \in M$ the point $h_f \circ f \circ h_f^{-1}(x)$ belongs to the same leaf of $\mathcal{F}^c$ as $f_1(x)$, that is the leaf $F^c(x)$ of x; as a direct consequence, each leaf of $\mathcal{F}^c_f$ is f-invariant

[2] One also verify that the foliation $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ are plaque expansive. Theorem 3 may be applied (independently) to each of the foliations $\mathcal{F}^c$, $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$.

(5) finally there is a constant $c_f > 0$ such that the distance between $h_f \circ f \circ h_f^{-1}(x)$ and $f_1(x) = \phi(x,1)$ in the central leaf is bounded by c_f. As a consequence the distance in the leaf $F_f^c(x)$ between x and $f(x)$ is uniformly bounded.

This shows that any diffeomorphism $f \in \mathcal{U}$ belongs to the set $\mathcal{E}_\phi$ defined in the introduction.

1.5. **Perturbations of the time one map of an Anosov flow along the central foliation.** Let ϕ be an Anosov flow and $\mathcal{E}_\phi$ be the set, defined in the introduction, of partially hyperbolic diffeomorphisms f, $TM = E_f^s \oplus E^c \oplus E_f^u$ satisfying the following properties:

- f has a one dimensional central foliation $\mathcal{F}_f^c$ conjugated to the central foliation $\mathcal{F}^c$ of ϕ by a homeomorphism h_f,
- each leaf $F_f^c(x)$ is invariant by f. Furthermore, there is $K_f > 0$ such that, for any $x \in M$ there is a path $\gamma \subset F_f^c(x)$ with length $\ell(\gamma) < K_f$ joining x to $f(x)$ in the central leaf $F_f^c(x)$.

Lemma 1.1. *$h_f \circ f \circ h_f^{-1}$ is an homeomorphism of M of the form $x \mapsto \phi(x,\tau(x))$ where $\tau\colon M \to \mathbb{R} \setminus \{0\}$ is a continuous function.*

Proof: First notice that $\tau(x)$ is uniquely defined by $h_f \circ f \circ h_f^{-1}(x) = \phi(x,\tau(x))$, on non compact central leaves. Furthermore, τ is continuous on the complement of the compact leaves of $\mathcal{F}^c$.

On a compact leaf $F^c(x)$ the equation on $h_f \circ f \circ h_f^{-1} = \phi(x,\tau)$ admits infinitely many solutions τ_i, $i \in \mathbb{Z}$, and the difference $\tau_i - \tau_j$ is precisely $(i-j)$ times the period of the ϕ-orbit $F^c(x)$. Let us show now that τ admits a unique extension on $F^c(x)$: the flow ϕ has a countable family of periodic orbits then the complement of the compact leaves of $\mathcal{F}_f^c$ satisfies that the intersection with any connected open set of M is connected. As a consequence, the accumulation values of $\tau(y)$ when y tends to x is an interval of $\mathbb{R}$ contained in $\{\tau_i\}_{i\in\mathbb{Z}}$, that is, there is a unique τ_i. We just proved that the function τ admits a unique continuous extension on M.

For ending the proof of Lemma 1.1, it remains to prove that τ does not vanish. We will use the following lemma:

Lemma 1.2. *Let $t\colon M \to \mathbb{R}$ be a continuous function and let $\varphi\colon M \to M$ be the homeomorphism defined by $\varphi(x) = \phi(x,t(x))$. Assume that $t(x_0) = 0$ for some point x_0. Then there is $\delta > 0$ such that the $\delta-$local stable manifold of x_0 for φ (that is, the set of points y such that the distance $d(\varphi^n(y),\varphi^n(x))$ remains smaller than δ and tends to 0 when $n > 0$ tends to $+\infty$) is included in the ϕ orbit of x_0.*

Proof: Recall that ϕ has no fixed point, so that there is a flow box at x_0. As t is very small in the neighborhood of x_0, the orbit of each point close to x_0 remains in the same local central leaf until it goes out the flow box. Choosing $\delta > 0$ such that the ball of radius δ centered at x_0 is contained in the flow box, we get the statement of the lemma.
■

The partial hyperbolicity of f ensures the existence of local strong stable and strong unstable manifolds at each point x of M and these manifolds are not contained in the central leaf $F^c_f(x)$. This implies that every point $x \in M$ has a local stable manifold for $h_f \circ f \circ h_f^{-1}$ which is not contained in the ϕ orbit of x. One concludes that τ cannot vanish ending the proof of Lemma 1.1. ■

Notation

For every $f \in \mathcal{E}_\phi$, we denote $\tau_f = \tau \circ h_f$. Then $h_f \circ f(x) = \phi(h_f(x), \tau_f(x))$

Remark 1.2. (1) *If f is a diffeomorphism which belongs to $\mathcal{E}_\phi$ then its inverse f^{-1} belongs to $\mathcal{E}_\phi$ too; furthermore one can choose $h_{f^{-1}} = h_f$ and $\tau_{f^{-1}} = -\tau_f \circ f^{-1}$. Hence we will now assume (up to replace f by f^{-1}), that $\tau_f > 0$.*

(2) *If f is a diffeomorphism in $\mathcal{E}_\phi$ then for any $n > 0$ the diffeomorphism f^n belongs to $\mathcal{E}_\phi$, with $h_{f^n} = h_f$ and $\tau_{f^n} = \tau_f + (\tau_f \circ f) + \cdots (\tau_f \circ f^{n-1})$.*

Notice that $\mathcal{F}^c$ is naturally oriented by the flow ϕ and that the foliation $\mathcal{F}^c_f = h_f^{-1}(\mathcal{F}^c)$ inherits the image orientation. This orientation coincides, on the non-compact leaves, with the orientation given by the dynamics of f, that is the leaf $F^c_f(x)$ is oriented from x to $f(x)$.

From now on, we choose the dynamical orientation for $\mathcal{F}^c_f$.

A *parametrized central arc* $\gamma\colon [0,1] \to M$ is an immersion of $[0,1]$ in a central leaf. Two parametrized central arcs γ_1 and γ_2 define the same *oriented central arc* if there is a orientation preserving homeomorphism σ of $[0,1]$ such that $\gamma_2 = \gamma_1 \circ \sigma$. The C^0-topology on the set of parametrized central arcs induces a topology, already called the *C^0-topology*, on the set of oriented central arcs. We denote by $\ell(\gamma)$ the length of the arc γ.

Let $b \in \mathcal{F}^c_f(a)$ in the positive direction starting from a. We will denote C^a_b the arc included in $\mathcal{F}^c_f(a)$ between a and b.

Let us denote by $D(x)$ the arc of central curve positively oriented and joining x to $f(x)$, and whose image by h_f is the arc $\phi(h_f(x), [0, \tau_f(x)])$. Notice that $D(x)$ and $C^x_{f(x)}$ are equal with a possible exception when $\mathcal{F}^c_f(x)$ is closed. In fact, just in the cases where $W^c(x)$ is closed and $D(x)$ winds around itself more than once we have that $D(x) \neq C^x_{f(x)}$.

Remark 1.3. *As there are finitely many compact central leaves of length less K_f, we get that $D(x) = C^x_{f(x)}$ excepted for x in finitely many closed central leaves.*

The family $C^x_{f(x)}$ is not *a priori* continuous. However, as the function $x \mapsto \tau_f(x)$ is continuous, the arcs $D(x)$, $x \in M$ form a continuous family of compact central arcs.

By the continuity of the family $D(x)$, there is $K_f > 0$ such that the length $\ell(D(x))$ is upper bounded by K_f for every $x \in M$. Furthermore, as $\tau_f > 0$ there is a lower bound $c_f > 0$ of $\ell(D(x))$. Finally, $D(f(x)) = f(D(x))$. As a consequence one gets the following properties:

Lemma 1.3. *Let γ be a central arc with $\ell(\gamma) \leq c_f$. Then for any $n \in \mathbb{Z}$, the length $\ell(f^n(\gamma))$ is upper bounded by K_f.*

Proof: Let denote $x = \gamma(0)$. Then $\gamma \subset D(x)$. Hence $f^n(\gamma) \subset D(f^n(x))$. So $\ell(f^n(\gamma)) \leq \ell(D(f^n(x)) \leq K_f$. ■

Lemma 1.4. *For any $x \in M$ and any $y \in F^c_f(x)$, there is $n \in \mathbb{Z}$ with $f^n(y) \in D(x)$.*

Proof: Just notice that one gets a central curve with infinite length in the both positive an negative direction by putting together the segments $D(f^n(x)) = f^n(D(x))$, which are all of length greater than $c_f > 0$. Hence this curve is the whole central leaf $F^c_f(x)$. So y belongs to some $f^n(D(x))$ that is $f^{-n}(y) \in D(x)$ ∎

1.6. **Dynamical coherence of diffeomorphisms $f \in \mathcal{E}_\phi$.**

Two leaves $F^c_f(x)$ and $F^c_f(y)$ will be called *asymptotic at* $+\infty$ if there are parametrizations $\gamma_x \colon \mathbb{R} \to F^c_f(x)$ and $\gamma_y \colon \mathbb{R} \to F^c_f(x)$ preserving the orientation and such that the distance $d(\gamma_x(t), \gamma_y(t))$ tends to 0 when $t \to +\infty$. If $y \in F^{cs}(x)$ we say that $F^c_f(x)$ and $F^c_f(y)$ are *asymptotic at* $+\infty$ *in* $F^{cs}(x)$ if the central stable distance $d^{cs}(\gamma_x(t), \gamma_y(t))$ tends to 0 when $t \to +\infty$.

The map f is normally hyperbolic (see [10]) then there exist strong stable and strong unstable foliations $\mathcal{F}^{ss}_f$, $\mathcal{F}^{uu}_f$

Lemma 1.5. *Consider $f \in \mathcal{E}_\phi$ and h_f the homeomorphism associated to f, conjugating $\mathcal{F}^c_f$ to $\mathcal{F}^c$. Then:*

- *there is an f-invariant foliation $\mathcal{F}^{cs}_f$ tangent to the bundle $E^{ss}_f \oplus E^c_f$;*
- *there is an f-invariant foliation $\mathcal{F}^{cu}_f$ tangent to the bundle $E^{uu} \oplus E^c$;*
- *for every point x the leaf $F^{cs}_f(x)$ contains the leaves $F^{ss}_f(x)$ and $F^c_f(x)$;*
- *for every point x the leaf $F^{cu}_f(x)$ contains the leaves $F^{uu}_f(x)$ and $F^c_f(x)$;*
- *$\mathcal{F}^{cs}_f = h_f^{-1}(\mathcal{F}^{cs})$ and $\mathcal{F}^{cu}_f = h_f^{-1}(\mathcal{F}^{cu})$.*

Proof: We first prove that, for every $x \in M$ the image $h_f^{-1}(F^{cs}(h_f(x)))$ of the leaf of $\mathcal{F}^{cs}$ through $h_f(x)$ contains the leaves $F^{ss}_f(x)$ and $F^c_f(x)$. Notice that $F^{cs}(h_f(x))$ contains the leaf $F^c(h_f(x))$, and $h_f^{-1}(F^c(h_f(x))) = F^c_f(x)$, by definition of h_f. Hence $h_f^{-1}(F^{cs}(h_f(x)))$ contains $F^c_f(x)$.

Consider $y \in F^{ss}_f(x)$. Then the distance $d(f^n(x), f^n(y))$ tends to 0 when $n \to +\infty$. As the central distance $d^c(f^n(x), f^{n+1}(x))$ and $d^c(f^n(y), f^{n+1}(y))$ are uniformly bounded, it follows that the oriented leaves $F^c_f(y)$ and $F^c_f(x)$ are asymptotic (when one follows the foliation in the positive direction). This property persists by conjugacy so that the oriented leaves $F^c(h_f(y))$ and $F^c(h_f(x))$ are asymptotic. This means that $h_f(y) \in F^{cs}(h_f(x))$. Hence $y \in h_f^{-1}(F^{cs}(h_f(x)))$, proving the claim.

This implies that the dimension of $E^s_f \oplus E^c_f = 1 + dimE^s_f$ is less than (or equal to) the dimension of $E^s \oplus E^c = 1 + dimE^s$. In the same way one proves that the image $h_f^{-1}(F^{cu}(h_f(x)))$ contains the leaves $F^{uu}_f(x)$ and $F^c_f(x)$, implying $dim(E^u_f) \leq \dim E^u$. One conclude that $dimE^s_f = dimE^s$ and $dimE^u_f = dimE^u$.

One deduces that $h_f^{-1}(F^{cs}(h_f(x)))$ is a C^1-immersed submanifold tangent to $E^s_f \oplus E^c_f(x)$. Let us denote $\mathcal{F}^{cs}_f = h_f^{-1}(\mathcal{F}^{cs})$. It is a foliation tangent to $E^s_f \oplus E^c_f$ and subfoliated by $\mathcal{F}^c_f$ and $\mathcal{F}^{ss}_f$. As the leaves of $\mathcal{F}^c_f$ are each f-invariant , each leaf of $\mathcal{F}^{cs}_f$ is invariant.

Analogously it can be proven that $\mathcal{F}^{cu}_f = h_f^{-1}(\mathcal{F}^c)$ is a foliation tangent to $E^u_f \oplus E^c_f$ and subfoliated by $\mathcal{F}^c_f$ and $\mathcal{F}^{uu}_f$. ∎

Lemma 1.6. *For every point $x \in M$, the central stable leaf $F^{cs}_f(x)$ is the union of the strong stable leaves crossing the central leaf $F^c_f(x)$.*

$$F^{cs}_f(x) = \bigcup_{y \in F^c_f(x)} F^{ss}_f(y)$$

In the same way,
$$F_f^{cu}(x) = \bigcup_{y \in F_f^c(x)} F_f^{uu}(y).$$

Proof: We proved the inclusion $\bigcup_{y\in F_f^c(x)} F_f^{ss}(y) \subset F_f^{cs}(x)$. It remains to prove the converse inclusion.

Using Lemma 1.5 one can prove that there is $\delta > 0$ such that, for every $x \in M$ the ball $B_f^{cs}(x,\delta)$ of radius δ centered at x in the leaf $F_f^{cs}(x)$ is contained in $\bigcup_{y\in F_f^c(x)} F_f^{ss}(y)$.

Now consider $y \in F_f^{cs}(x)$. By Lemma 1.5 the points $h_f(x)$ and $h_f(y)$ belongs to the same central stable leaf of the flow ϕ. This means that the central leaf of ϕ through $h_f(x)$ and $h_f(y)$ are asymptotic at $+\infty$ in $F^{cs}(h_f(x))$.

Let t_n such that $h_f(f^n(y)) = \phi(h_f(y), t_n)$. Since $t_n \to \infty$ it follows that there exist t'_n such that the distance $d^{cs}(\phi(h_f(y), t_n), \phi(h_f(x), t'_n))$ (in the central stable leaf $F^{cs}(h_f(x))$) tends to 0. Let $x_n = h_f^{-1}(\phi(h_f(x), t'_n))$. As a consequence $x_n \in F_f^c(x)$ and one can prove that for $n \to +\infty$, the distance $d_f^{cs}(f^n(y), x_n)$ (in the leaf $F_f^{cs}(x) = h_h^{-1}(F^{cs}(h_f(x)))$ tends to 0. In particular this distance is less that δ for large n. As a consequence, $f^n(y)$ belongs to the strong stable leaf through the point $x'_n \in F_f^c(x)$. One conclude: $y \in F_f^{ss}(f^{-n}(x'_n)) \subset \bigcup_{z\in F_f^c(x)} F_f^{ss}(z)$, proving the converse inclusion.

$\blacksquare$

Lemma 1.7. *Let x be a point of M and $\gamma\colon [0,1] \to F_f^c(x)$ be a path in the central leaf trough x, such that $\gamma(0) = x$. Let y be a point of $F_f^{ss}(x)$. Then there is a unique path $\sigma\colon [0,1] \to F_f^c(y)$ such that $\sigma(0) = y$ and for every $t \in [0,1]$ one has $\sigma(t) \in F_f^{ss}(\gamma(t))$.*

Proof: First notice that it is enough to prove Lemma 1.7 for central paths whose length is upper bounded by some constant c: one deduces the general case by cutting γ is pieces of length smaller that c.

Now, given some fixed constant c, there is $\varepsilon(c) > 0$ such that the conclusion of Lemma 1.7 holds with the following hypotheses:

- γ is a central path with length $\ell(\gamma) < c$, $\gamma(0) = x$
- y is a point in $F_f^{ss}(x)$ such that the distance $d^{ss}(x,y)$ (in $F_f^{ss}(x)$) is less than $\varepsilon(c)$.

Let γ be a parametrized arc of length less that c_f, where c_f is defined as before Lemma 1.3. Let $x = \gamma(0)$ and let y be a point of $F_f^{ss}(x)$. By Lemma 1.3 we have that for any $n \in \mathbb{N}$, $\ell(f^n(\gamma)) \leq K_f$. There is $n > 0$ such that the distance $d^{ss}(f^n(x), f^n(y)$ (in the leaf $F_f^{ss}(f^n(x))$ is less that $\varepsilon(K_f)$. So there is a central arc $\tilde\sigma \subset F_f^c(f^n(y))$ such that $\tilde\sigma(0) = f^n(y)$ and $\tilde\sigma(t) \in F_f^{ss}(f^n(\gamma(t)))$. One denotes $\sigma = f^{-n}(\tilde\sigma)$. It is a central arc in $F_f^s(y)$ starting at $\sigma(0) = y$ and $\sigma(t) \in F_f^{ss}(\gamma(t))$.

$\blacksquare$

In Lemma 1.7, we say that σ is the image of γ by the holonomy of the foliation $\mathcal{F}_f^{ss}$ from x to y and we denote $\sigma = \mathcal{H}_f^{ss}(\gamma, y)$. Next lemma asserts that action of the holonomy of $\mathcal{F}_f^{ss}$ on central arcs is continuous:

Lemma 1.8. *Given $K_1, K_2 > 0$, the map $(\gamma, y) \mapsto \mathcal{H}_f^{ss}(\gamma, y)$ is a continuous map, from the space of pairs (γ, y) where γ is a central arc with $\ell(\gamma) \leq K_1$ and $y \in F_f^{ss}(\gamma(0))$ satisfies $d^{ss}(\gamma(0), y) \leq K_2$, to the space of central arcs.*

Proof: As for Lemma 1.7, cutting γ in pieces of length less that c_f and iterating positively by f, it is enough to show Lemma 1.8 for segment γ of length bounded by K_f, and with $d^{ss}(\gamma(0), y)$ less that an arbitrarily small $\varepsilon > 0$. Then the statement follows by working in foliated charts of the foliation $\mathcal{F}^c_f$. ∎

Assume now that the Anosov flow ϕ is transitive. As we have seen, the periodic orbits are dense, the orbits of generic points are dense, all the central stable or unstable leaves are dense. All these properties holds for any $f \in \mathcal{E}_\phi$:

$$\{x|F^c_f(x) \text{ is a closed set}\} \text{ is dense in } M$$

and

$$\{x|F^c_f(x) \text{ is dense in } M\} \text{ is a residual set.}$$

For any $x \in M$, $F^{cs}_f(x)$ and $F^{cu}_f(x)$ are dense.

2. Periodic orbits and compact central leaves

From now on, ϕ is a transitive Anosov flow on a compact manifold, f_1 denotes the time one map of ϕ, and $f \in \mathcal{E}_\phi$ is an Axiom A diffeomorphism, with $\tau_f > 0$. Let us denote by $F^c_f(x)$ or by $W^c(x)$ the leaf of the central foliation through the point x. From now on, we choose the dynamical orientation for $\mathcal{F}^c_f$.

We denote by k the dimension of E^{uu}_f and by $per(f)$ the set of periodic points of f.

The metric induced by the Riemannian metric on the leaves of $\mathcal{F}^c_f$ will be denoted d^c. Analogously we define d^s and d^u.

Consider $x \in \Omega(f)$. As $\Omega(f)$ is hyperbolic, the central direction $E^c_f(x)$ is contained either in the unstable or in the stable space at x. In the first case, there is a neighborhood $F^{cu}_{f,loc}(x)$ of x in $F^{cu}_f(x)$ which coincides with the local unstable manifold $W^u_{loc}(x)$. In the second case, $F^{cs}_{f,loc}(x)$ coincides with the local stable manifold $W^s_{loc}(x)$.

Remark 2.1. *If x is a periodic point of f then the central leaf $E^c_f(x)$ is compact, because each central leaf is f-invariant and τ_f is strictly positive. Indeed $h_f(x)$ is a periodic point of ϕ; a period of $h_f(x)$ is the sum of the $\tau_f(y)$ for y in the f-orbit of x.*

Next proposition asserts that, conversely, every compact central leaf contains periodic orbits of f:

Proposition 2.1. *If $\mathcal{O} = F^c_f(x)$ is a closed curve then*

- *the rotation number of $f|\mathcal{O}$ is rational*
- *there exists at least two periodic points in $\mathcal{O}$ with different indices*
- *the points in $\Omega(f) \cap \mathcal{O}$ are the periodic ones.*

Proof:

Let $\mathcal{O} = F^c_f(x)$ be a closed curve. Let us prove that the rotation number of $f|\mathcal{O}$ is rational. Assume, by contradiction that it is irrational. Then there exists a unique minimal set $I \subset \mathcal{O}$ which is not periodic. As $I \subset \Omega(f)$ and f is Axiom A, I is hyperbolic and included in a basic set Λ.

Besides, $\forall y \in \mathcal{O}$, $\alpha(y) = \omega(y) = I$, where $\alpha(y)(\omega(y))$ is the set of limit points of $\{f^n(y)\}$ when $n \to -\infty$ $(n \to +\infty)$ (See, for example [5] page 34). Hence

$$y \in W^s(I) \cap W^u(I) \subset W^s(\Lambda) \cap W^u(\Lambda) \subset \Lambda,$$

therefore $y \in \Omega(f)$. Then $\mathcal{O} \subset \Lambda \subset \Omega(f)$ and it follows that $f|_{\mathcal{O}}$ is expansive which leads to a contradiction with the nonexistence of one dimensional expansive diffeomorphism (See [12]).

Hence, $f|\mathcal{O}$ has periodic points since the rotation number is rational. Since f is Axiom A, all the periodic points are hyperbolic and (restricted to $\mathcal{O}$) are alternately attractors or repellers. An attractor (resp.repeller) point correspond in M to periodic saddle whose stable manifold has dimension equal to $dimE^s + 1$ (resp. $dimE^s$): hence $\mathcal{O}$ contains periodic orbits of different indices.

All the points in $\Omega(f) \cap \mathcal{O}$ must be periodic, otherwise, if there were a nonperiodic point, $x \in \Omega(f) \cap \mathcal{O}$ then the invariance of $\Omega(f) \cap \mathcal{O}$ implies that $\alpha(x)$ and $\omega(x)$ would be periodic points of different indices so they would be in different basic sets. This is a contradiction with the fact that a non-wandering point of an Axiom A diffeomorphism must have the α and the ω limit set in the same basic set. ∎

Let $\mathcal{O}$ be a compact leaf of $\mathcal{F}^c_f$. Then the *stable manifold* $W^s(\mathcal{O})$ *of* $\mathcal{O}$ is the set of points whose ω-limit set is included in $\mathcal{O}$. Let us state some properties of $W^s(\mathcal{O})$

Remark 2.2.

- *The leaf $\mathcal{O}$ is a normally hyperbolic invariant compact manifold. As a consequence $W^s(\mathcal{O})$ is the union of the strong stable leaves through $\mathcal{O}$.*
- *Hence, according to Lemma 1.6, $W^s(\mathcal{O})$ is the leaf of $\mathcal{F}^{cs}_f$ containing $\mathcal{O}$.*
- *We have seen that every leaf of $\mathcal{F}^{cs}_f$ is dense in M; so the stable manifold $W^s(\mathcal{O})$ is dense in M.*
- *The stable manifold $W^s(\mathcal{O})$ is the union of the stable manifolds of the periodic points in $\mathcal{O}$. More precisely, let $per(\mathcal{O})$ denote $per(f) \cap \mathcal{O}$, the set of periodic points contained in $\mathcal{O}$. Then*

$$W^s(\mathcal{O}) = \bigcup_{x \in per(\mathcal{O})} W^s(x).$$

One defines analogously the invariant manifold $W^u(\mathcal{O})$ of $\mathcal{O}$ and it satisfies the corresponding properties.

3. Properties of attracting and repelling basic sets.

We include some statement in the present section that have been published in ([9]), assuming some extra hypotheses. Their proofs include simple ideas that are essential in the proof of the main theorem.

Let us recall that, as f is an Axiom A diffeomorphism, there is a finite number of attractors and repellers basic sets. We will show here that each attractor and each repeller meets every central leaf (so it looks like a complete cross section of the Anosov flow) .

Let A denote an attractor basic set of the spectral decomposition of f. Notice that $A \neq M$ because f is not an Anosov diffeomorphism. According to the remarks 1.1 and 1.2, every positive iterate f^n is an Axiom A diffeomorphism in $\mathcal{E}_\phi$ and there is $n > 0$ such that every transitive attractor and repeller is connected. Hence, up to replacing f by f^n we may assume that A is connected.

Recall that we denote $dimE^{uu}_f = k$.

Lemma 3.1. $dim(W^s(x)) = n - k, \forall x \in A$

Proof: By hypotheses $dim(E^{ss}_\phi) = n - k - 1$. For every $x \in A$, either its local central leaf $F^c_{f,loc}(x)$ is expanding (i.e. $dim(W^s(x)) = n - k - 1$), or its local central leaf is contracting (i.e. $dim(W^s(x)) = n - k$).

Consider a periodic point $x \in A \cap per(f)$. We have seen that it central leaf $F^c_f(x)$ is a closed curve.

We argue by contradiction, assuming that the central direction $E^c_f(x)$ is unstable. Since A is an attractor it contains its unstable manifold; in particular one has $W^u(x) \subset A$; hence the local central leaf $F^c_{f,loc}(x)$ is contained in A (See Figure 1).

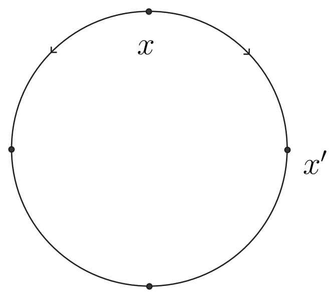

FIGURE 1.

Consider now the dynamic of f restricted to the circle $F^c_f(x)$. The f-orbit of x is a repeller for these dynamic. A point $y \in F^c_{f,loc}(x)$ has positive iterates which converge to an attracting point x' of $f|_{F^c_f(x)}$. The set A is closed and f-invariant then x' belongs to A. But $dim(W^s(x')) = n - k$ (because the central direction $E^c_f(x')$ is contracting on the orbit of x'). It follows that there exist two periodic points of different indices in A, which contradicts the hyperbolicity of A.

Then the local central leaf is stable for every point in A and the claim follows. ■

Recall that $K_f > 0$ is an upper bound of the length of $D(x)$, for $x \in M$.

Lemma 3.2. *It holds that*

- *For every closed curve $\mathcal{O}$ in $\mathcal{F}^c_f$ there exists a periodic point $p \in A \cap \mathcal{O}$.*
- *In every central arc γ with $length(\gamma) \geq K_f$, there exists a point $p \in \gamma \cap A$.*
  In particular, every leaf of $\mathcal{F}^c_f$ intersects A.

Proof: According to Remark 2.2 the stable manifold $W^s(\mathcal{O})$ is dense in M. Since $W^s(A)$ is an open set, there exists y in the intersection $W^s(\mathcal{O}) \cap W^s(A)$. By Remark 2.2 there is a periodic point $x \in \mathcal{O}$ such that $y \in W^s(x) \cap W^s(A)$. Hence $x \in A$, proving $A \cap \mathcal{O} \neq \emptyset$.

As A is f-invariant, the whole orbit of x is contained in A. By Lemma 1.4, for every $y \in \mathcal{O}$, the arc $D(y)$ contains at least one point of the orbit of x: hence, for every point y in a compact central leaf, the segment $D(y)$ meets A.

Then A meets $D(y)$ for every y in the dense subset of M equal to the union of the closed central leaves. Since the family of arcs $D(z)$, $z \in M$ is a continuous family of compact arcs and A is compact, it follows that $A \cap D(z) \neq \emptyset$ for every $z \in M$. ■

Remark 3.1. *Analogously we can show that there exists $q \in O \cap \Lambda$, where Λ is a repeller set. Moreover, in every segment γ of a central curve with length$(\gamma) \geq K_f$, there exists a periodic point $q \in \gamma \cap \Lambda$.*

As a direct consequence of Remark 3.1 one gets that every leaf of the central foliation "goes away " from the basin of attraction of any attractor.

Corollary 3.1. *In every leaf of $\mathcal{F}_f^c$ there exists at least one point outside of $W^s(A)$.*

Lemma 3.3. *For every attractor basic set A, every repeller basic set Λ, and every compact central segment γ, the intersections $\gamma \cap A$ and $\gamma \cap \Lambda$ are finite sets.*

Proof: Each central leaf is contracting at each point of the attracting basic sets. One deduces that there is $c > 0$ such that, for any two distinct points $x, y \in A$ in the same central leaf, the central distance $d^c(f^{-n}(x), f^{-n}(y))$ is larger than c for every $n > 0$ large enough.

We will show that , for any $x \in M$ the intersection $A \cap D(x)$ is finite.

Consider $x \in M$ and $\{x_i\}$ in $A \cap D(x)$, such that $x_0 < x_1 < x_2 < \ldots < x_l$ in the given orientation of $D(x)$.

Then there exists $n_i \in I\!N$, $i = 1, \ldots l$ verifying that $\ell(f^{-n_i}(C_{x_i}^{x_{i-1}})) > c$, for all $n \geq n_i$.

So for $n \geq \sup_i n_i$ one gets $\ell(f^{-n}(D(x)) = \ell(D(f^{-n}(x)))$ is larger than $l \cdot c$. However $\ell(D(f^{-n}(x))) \leq K_f$ by definition of K_f. So $l \leq \frac{K_f}{c}$, ending the proof of the claim.

Consider now a compact central segment γ, and let x be the origin of γ. Then there is $i > 0$ such that γ is contained in $\bigcup_1^i D(f^j(x))$. Hence $\gamma \cap A$ is finite. One proves in the same way (using positive iterations instead of negative ones) that $\gamma \cap \Lambda$ is finite. ∎

4. Properties of the central foliation in the basin of A.

Let A be an attractor of f. The aim of this section is to show that:

- For x in an open and dense subset of the basin $W^s(A)$, the connected component of x in $W^c(x) \cap W^s(A)$ contains at least a point $y_x \in A$ (Lemma 4.5).
- For generic points in $W^s(A)$ the connected component of x in $W^c(x) \cap W^s(A))$ contains exactly one point in A: the set

$$\{x \in W^s(A) | \sharp\{\text{conn. comp. of } x \text{ in } (W^c(x) \cap W^s(A)) \cap A\} = 1\}$$

  is a residual set of $W^s(A)$ (Lemma 4.6).

In order to get these properties we introduce, for every $x \in W^s(A)$, the *entering point* $\tilde{S}_A(x)$ and the *exit point* $S_A(x)$ of its center leaf $F_f^c(x)$ in $W^s(A)$, as follows:

Definition 4.1. *We denote by $S_A : W^s(A) \to \partial W^s(A)$ the map defined by: $S_A(x)$ is the nearest point of the central leaf of $x \in W^s(A)$ in the positive direction which is not in $W^s(A)$, i.e.,*

$$S_A(x) = \sup\{y \in F_f^c(x) | C_y^x \subset W^s(A)\}$$

In the same way $\tilde{S}_A : W^s(A) \to \partial W^s(A)$ is the map defined by: $\tilde{S}_A(x)$ is the nearest point of the central leaf of $x \in W^s(A)$ in the negative direction which is not in $W^s(A)$.

By Corollary 3.1, every segment of central leaf with length greater than K_f contains points out of $W^s(A)$ so that that maps S_A and $\tilde{S}_A$ are well defined.

We denote by $\widehat{W^c(x)} = C_{S_A(x)}^{\tilde{S}_A(x)}$ the arc of central curve which is the closure of the connected component of $W^c(x) \cap W^s(\mathcal{A})$ which contains x. Notice that the interior of this segment is contained, by definition, in $W^s(A)$, so that its length is bounded by K_f.

Lemma 4.1. *If $x \in W^s(A)$ belongs to a compact central leaf, then $\widehat{W^c(x)}$ cuts A in exactly one point. Furthermore $S_A(x)$ and $\tilde{S}_A(x)$ are periodic points for which the central direction is expanding.*

Proof: If $x \in W^s(A)$ is periodic, then $x \in A$; furthermore, by the proof of Lemma 3.1 the intersection $W^s(x) \cap F_f^c(x)$ is an interval in $W^s(A)$ ending at two periodic points of index different from the index of x, hence out of $W^s(A)$. Finally, by Proposition 2.1 x is the unique non-wandering point in that open interval, so that $\widehat{W^c(x)}$ is the closure of $W^s(x) \cap F_f^c(x)$.

If $F_f^c(x)$ is compact but x is not periodic, it belongs to the stable manifold of a periodic point $y \in F_f^c(x) = F_f^c(y)$, and $\widehat{W^c(x)}$ is the closure of $W^s(y) \cap F_f^c(y)$. ■

Lemma 4.2. *Let $\ell_+(x)$ and $\ell_-(x)$, $x \in W^s(A)$, denote the length of the central arcs $C_x^{S_A(x)}$ and $C_{\tilde{S}_A(x)}^x$ respectively. The maps $x \mapsto \ell_+(x)$ and $x \mapsto \ell_-(x)$ are lower semicontinuous.*

Proof: This is a classical consequence of the compactness of the complement of $W^s(A)$: let $x_n \in W^s(A)$ be a sequence converging to x. As the length ℓ_+ is uniformly bounded by K_f, by considering a subsequence, one may assume that the central arcs $C_{S_A(x_n)}^{x_n}$ converge to a central arc C_y^x, with $\ell(C_y^x) = \lim \ell_+(x_n)$, where $\ell(C)$ is the length of the arc C. Moreover, $y \notin W^s(A)$. Hence $C_{S_A(x)}^x$ is a sub arc of C_y^x, proving $\ell_+(x) \leq \lim \ell_+(x_n)$. This proves that ℓ_+ is lower semi-continuous, and the proof of the semi continuity of ℓ_- is identical. ■

We denote by $Q \subset W^s(A)$ the set of points such that the map $x \mapsto \widehat{W^c(x)}$ is continuous at x; this is equivalent to the fact that both ℓ_- and ℓ_+ are continuous at x. Let us state some properties of the sets Q :

Remark 4.1.
1. *Q is invariant by f.*
2. *As semicontinuous maps are continuous on generic points, the set Q is residual in $W^s(A)$.*
3. *if $x \in Q$ then the open arc $\widehat{W^c(x)} \setminus \{\tilde{S}_A(x), S_A(x)\}$ is contained in Q.*
4. *if $x, y \in W^s(A)$ belong to the same strong stable leaf, then $\widehat{W^c(y)}$ is the image of $\widehat{W^c(x)}$ by the holonomy map of the foliation $\mathcal{F}_f^{ss}$ from $F_f^c(x)$ to $F_f^c(y)$. As a consequence of the continuity of the holonomy map (see Lemma 1.8)*
5. *if $x \in Q$ then the strong stable leaf $F_f^{ss}(x)$ is contained in Q.*

Remark 4.2. *T. For any $\alpha > 0$ let us denote by $\widetilde{U}_\alpha \subset W^s(A)$ the set of points x such that there is a neighborhood $V_{x,\alpha}$ of x verifying $\ell_+(y) < \ell_+(x) + \alpha$ and $\ell_-(y) < \ell_-(x) + \alpha$ for every $y \in V_{x,\alpha}$. Since the functions ℓ_- and ℓ_+ are lower semi-continuous, positive and upper bounded by K_f it follows that $\widetilde{U}_\alpha$ is dense in $W^s(A)$ for any $\alpha > 0$.*

We denote by U_α the dense subset of $W^s(A)$ defined by $U_\alpha = \bigcup_{\alpha'<\alpha} \widetilde{U}_{\alpha'}$.

Lemma 4.3. *For any $\alpha > 0$ the set U_α is an open and dense subset of $W^s(A)$.*

Proof: Let x be a point of U_α. By definition of U_α there is $0 < \alpha' < \alpha$ such that x belongs to $\widetilde{U}_{\alpha'}$. Fixe $0 < \epsilon < \alpha - \alpha'$. Since the functions ℓ_- and ℓ_+ are lower semi-continuous, it follows that there exists an open neighborhood of x, $\widetilde{U}^\epsilon$, such that if $y \in \widetilde{U}^\epsilon$ then $\ell_+(y) > \ell_+(x) - \epsilon$ and $\ell_-(y) > \ell_-(x) - \epsilon$. Since x belongs to $\widetilde{U}_{\alpha'}$, there is a neighborhood $V_{x,\alpha'}$ of x verifying $\ell_+(y) < \ell_+(x) + \alpha'$ and $\ell_-(y) < \ell_-(x) + \alpha'$ for every $y \in V_{x,\alpha}$.

Let $y \in V_{x,\alpha} \cap U^\epsilon$. We will show that $y \in U_\alpha$. Let $z \in V_{x,\alpha'}$, then $\ell_+(z) < \ell_+(x) + \alpha'$ and $\ell_-(z) < \ell_-(x) + \alpha'$. Since $y \in U^\epsilon$ we have that $\ell_+(z) < \ell_+(y) + \epsilon + \alpha' < \ell_+(y) + \alpha$ and $\ell_-(z) < \ell_-(y) + \epsilon + \alpha < \ell_-(y) + \alpha$, hence $y \in U_\alpha$ and U_α is an open subset of $W^s(A)$. ∎

Remark 4.3. *Clearly, Q is contained in U_α for all $\alpha > 0$. More precisely, $Q = \bigcap_\alpha U_\alpha$.*

Lemma 4.4. *There is a dense open subset U of $W^s(A)$ such that for every closed central leaf $\mathcal{O}$ the intersection $\mathcal{O} \cap U$ is contained in Q.*

Proof: We just sketch the proof which is done in details in [9, lemma 2.4]. Let $\delta_f > 0$ such that $2\delta_f$ is less than the infimum distance between two different basic sets of f, and we denote $U = U_{\delta_f}$. Consider a closed leaf $\mathcal{O}$ and $x \in \mathcal{O} \cap U$. We argue by contradiction assuming that x is not a continuity point of ℓ_+.

Hence there is a sequence of points x_n converging to x such that $\ell_+(x_n)$ converge but $\lim \ell_+(x_n) \neq \ell_+(x)$; as ℓ_+ is lower-semicontinuous this implies that $\lim \ell_+(x_n) > \ell_+(x)$. The closed central curves are dense in M and ℓ_+ is lower semi continuous, hence there are y_n close to x_n such that $F^c_f(y_n)$ is closed and $\ell_+(y_n) \geq \ell_+(x_n) - \frac{1}{n}$. From Lemma 4.1 above, the point $z_n = S_A(y_n)$ is a periodic point for which the central direction is expanding; in particular it belongs to $\Omega(f)$. Up to choosing a subsequence, one may assume that the arcs $C^{y_n}_{z_n}$ converge to some arc γ strictly larger than $C^x_{S_A(x)}$, and joining x to a point $z = \lim z_n$. As $\Omega(f)$ is compact, the point z is non-wandering. As z belongs to the closed leaf $\mathcal{O}$ this implies that z is a periodic point; furthermore the central direction is expanding along the orbit of z. So z has the same index as the periodic point $S_A(x)$. Consider σ be the arc joining $S_A(x)$ to z and obtained by removing $C^x_{S_A(x)}$ to γ. The central arc σ is joining two periodic points with the same index in $\mathcal{O}$ (and its length is not 0). As the periodic points in $\mathcal{O}$ are alternately attracting and repelling, σ contains a periodic point with different index, which implies that $\ell(\sigma) > \delta_f$. So we proved $\lim \ell_+(y_n) > \ell_+(x) + \delta_f$ and $\lim y_n = x$, which contradicts $x \in U$. ∎

Lemma 4.5. *For every point $x \in U = U_{\delta_f}$ the arc $\widehat{W^c(x)}$ meets A:*

$$\widehat{W^c(z)} \cap A \neq \emptyset$$

Proof: Consider $x \in U$ and $x_n \to x$ a sequence of points converging to x and whose central leaf is compact. According to Lemma 4.1, $\widehat{W^c(x_n)}$ contains a unique periodic point y_n in A and its extremities are periodic points z_n^- and z_n^+ for which the central direction is expanding. In particular, by definition of δ_f (see the proof of the previous lemma), the distances $d(y_n, z_n^+)$ and $d(y_n, z_n^-)$ are larger that $2\delta_f$.

Since the length of $C_{z_n+}^{y_n}$ and $C_{y_n}^{z_n-}$ are bounded, up to considering a subsequence we may assume that the points z_n^-, y_n and z_n^+ converge to points z^-, y and z^+, such that the distance $d(y, z^+)$ and $d(y, z^-)$ are larger that $2\delta_f$. The points z^- and z^+ are on the central leaf of x, and by definition of the set U the central distance between them and the extremities $\tilde{S}_A(x)$ and $S_A(x)$ is less that δ_f, respectively. Furthermore, the point y belongs to the arc $C_{z^-}^{z^+}$ and it is at distance larger than $2\delta_f$ from the extremities, so that $y \in \widehat{W^c(x)}$. Finally y belongs to A by compactness of A, ending the proof. ■

Corollary 4.1. *For $x \in Q$, $\widehat{W^c(x)} \cap A \neq \emptyset$. Furthermore, the extremities $\tilde{S}_A$ and S_A are non-wandering points for which the central direction is expanding.*

Proof: The first part follows from $Q \subset U$. The second part comes from the proof of Lemma 4.5: as x is a continuity point of the map $z \mapsto \widehat{W^c(z)}$, the points $z^- = \lim z_n^-$ and $z^+ = \lim z_n^+$ (in the notation of the proof of Lemma 4.5) coincides with $\tilde{S}_A(x)$ and $S_A(x)$, respectively. Furthermore, z^- and z^+ are limit of periodic points hence are non-wandering. As f is axiom A, the points z^- and z^+ belong to the same basic sets as z_n^- and z_n^+, respectively, for large n. Hence the central direction is expanding along the orbit of z^- and z^+, ending the proof. ■

Lemma 4.6. *Let D_A be the subset of points of Q verifying that the intersection $\widehat{W^c(x)} \cap A$ is exactly one point. Then $D_A \subset Q$ is residual in Q (hence in U and in $W^s(A)$).*

Proof: According to Lemma 4.5, for every $x \in U$ the intersection $\widehat{W^c(x)} \cap A$ is not empty.

Let α_x denote the smallest arc in $\widehat{W^c(x)}$ containing $\widehat{W^c(x)} \cap A$, and let $a(x)$ denote the length $a(x) = \ell(\alpha_x)$. We will show that the restriction of the function $x \mapsto a(x)$ to Q is upper semi continuous:

Fix $x \in Q$ and consider a sequence $x_n \in Q$ converging to x. Then the points $z_n^- = \tilde{S}_A(x_n)$ and $z_n^+ = S_A(x_n)$ are non-wandering and the central direction is expanding along their orbits. As a consequence, the distance between the extremities of $\alpha_n = \alpha(x_n)$ and the points z_n^+ and z_n^- is larger that $2\delta_f$. Up to considering a subsequence one may assume that the central arcs α_n converge to a central arc α. Notice that the extremities of α belong to A, by compactness of A.

By definition of Q the segments $\widehat{W^c(x_n)}$ converge to $\widehat{W^c(x)}$, so that $\alpha \subset \widehat{W^c(x)}$. As the extremities of α belong to A one deduces $\alpha \subset \alpha(x)$. This proves $\lim a(x_n) \leq a(x)$ and so the upper semi continuity of the function a restricted to Q.

Now Lemma 4.4 asserts that the points in U whose central leaf is closed belong to Q. The union of the closed central leaf is dense in M, hence in the open set U. Furthermore Lemma 4.1 asserts that, for $x \in U$ on a closed central leaf, $\widehat{W^c(x)} \cap A$ is exactly one point, so that $a(x) = 0$. The restriction of the function a to Q is an upper continuous function which vanishes on a dense subset, so there is a residual subset D_A of Q (hence of $W^s(A)$ because Q is residual in $W^s(A)$), on which a vanishes. This means that for every $x \in D_A$ the intersection $\widehat{W^c(x)} \cap A$ is exactly one point. ■

Remark 4.4. *Recall that, if $x \in Q$ then the interior of the arc $\widehat{W^c(x)}$ is contained in Q. Hence, by definition of D_A, $x \in D_A$ then the interior of the arc $\widehat{W^c(x)}$ is contained in D_A.*

Lemma 4.7. *The set of periodic points $x \in D_A$ is dense in the attractor A.*

Proof: Recall that, every periodic point in Q belongs to A. Consider a periodic point x in the open set U. Then $A \cap U$ contains a neighborhood of x in A, hence $A \cap U$ is a non-empty open subset of A. The periodic points are dense in A, so they are dense in the open set $A \cap U$ of A. Since the periodic points in $A \cap U$ are included in Q, the periodic points of Q are dense in $A \cap U$.

Let V_A be the set of periodic point in D_A. Since $V_A = D_A \cap per(f) = Q \cap per(f)$, we get that V_A is dense in $A \cap U$. Notice that V_A in invariant by f. So V_A in dense in the union of the iterates $f^i(A \cap U)$, $i \in \mathbb{Z}$.

As A is transitive, $\bigcup_{i \in \mathbb{Z}} f^i(A \cap U)$is a dense open subset of A, proving that V_A is dense in A. ■

5. Predecessor and successor of an attractor

Lemma 5.1. *Let A be a transitive attractor of f. There are transitive repellers Λ_- and Λ_+ such that $\tilde{S}_A(x) \in \Lambda_-$ and $S_A(x) \in \Lambda_+$ for every $x \in Q$ (where $Q \subset W^s(A)$ is the residual subset of continuity points of the function $x \mapsto \widehat{W^c(x)}$).*

Proof: Fix $x \in Q$. Corollary 4.1 asserts that $S_A(x)$ and $\tilde{S}_A(x)$ are non-wandering points. Then, the closure $\overline{Im(S_A|Q)}$ of the image of S_A restricted to Q, is a compact invariant set included in the non wandering set.

By Lemma 4.5, we know that the image $Im(S_A|Q)$ of S_A restricted to Q is equal to the image of S_A restricted to $Q \cap A$.

Since the set of dense orbits is a residual set of A, it follows that there exists $x \in A \cap Q$ such that $\{f^n(x)\}_{n \in \mathbb{N}}$ is dense in A. Then $\{f^n(x)\}_{n \in \mathbb{N}}$ is dense in $A \cap Q$. As the map S_A is continuous restricted to Q, the sequence $\{S_A(f^n(x))\}_{n \in \mathbb{N}}$ is dense in $S_A(A \cap Q)$ and in $\overline{S_A(A \cap Q)}$. As $S_A(f^n(x)) = f^n(S_A(x))$, we have that there exists a dense orbit in $\overline{Im(S_A|A \cap Q)} = \overline{Im(S_A|Q)}$. Hence, there exists a basic set Λ_+ such that $\overline{Im(S_A|Q)} \subset \Lambda_+$.

In the same way there is a basic set Λ_- containing $\overline{Im(\tilde{S}_A|Q)}$. It remains to prove that Λ_+ and Λ_- are repellers.

Fix a point $x \in Q$. Remark 4.1 claims that Q is invariant by the foliation $\mathcal{F}_f^{ss}$ and that, for $y \in F_f^{ss}(x)$ the arc $\widehat{W^c(y)}$ is the image by holonomy of $\mathcal{F}_f^{ss}$ of the arc $\widehat{W^c(x)}$. Conversely, a point $z \in F_f^{ss}(S_A(x))$ is the end point of an arc which is the image by holonomy of $\mathcal{F}_f^{ss}$ of $\widehat{W^c(x)}$. This arc is of the form $\widehat{W^c(y)}$ for some $y \in F_f^{ss}(x)$. So z is the image $S_A(y)$ with $y \in Q$. Hence $z \in \Lambda_+$.

As a consequence, $F_f^{ss}(S_A(x))$ is contained in Λ_+. This implies that Λ_+ is a repeller. One proves in the same way that Λ_- is a repeller.

■

Putting together Lemma 5.1 and Lemma 4.6 one gets:

Corollary 5.1. *For every transitive attractor A there are two transitive repellers Λ_- and Λ_+ with the following property:*

For $x \in D_A$ the arc $\widehat{W^c(x)}$ meets $\Omega(f)$ in exactly $\tilde{S}_A(x) \in \Lambda_-$, $S_A(x) \in \Lambda_+$ and exactly one point in A in the interior of $\widehat{W^c(x)}$.

Definition 5.1. *Following the notation of the previous lemma, Λ_- and Λ_+ are called the* predecessor *and the* successor *of A respectively.*

Remark 5.1. *Analogously we can prove the following:*

Let Λ be a transitive repeller. There are two transitive attractors A_- and A_+ and a residual set D_Λ of $W^u(\Lambda)$, such that for every $x \in D_\Lambda$ one has:

- *the connected component of $(W^u(\Lambda) \cap W^c(x))$ that contains x intersects $\Omega(f)$ in just a point which belongs to Λ.*
- *Let $C^{a(x)}_{b(x)}$ be the closure of connected component of $W^u(\Lambda) \cap W^c(x)$ that contains x, then $a(x) \in A_-$, $b(x) \in A_+$.*

We call A_- and A_+ the predecessor *and* the successor of Λ, *respectively.*

Next Lemma proves that our definitions of successor and predecessor of attractors and repellers are coherent:

Lemma 5.2. *A repeller Λ is the successor (resp. the predecessor) of an attractor A if and only if A is the predecessor (resp. the successor) of Λ.*

Proof: Consider a repeller Λ and its predecessor A. Let x_0 be a point of D_Λ. By definition of the predecessor of Λ, the point $y_0 = \tilde{S}_\Lambda(x)$ belongs to A. Let $\delta > 0$ such that $\delta < \delta_f$ (recall that $2\delta_f$ is the infimum distance between two basic sets) and such that the ball $B(y_0, \delta)$ is contained in $W^s(A)$. We fix a point x_1 in the interior of $C^{y_0}_{x_0}$ at distance less than $\delta/2$ of y_0: more precisely we require $\ell(C^{y_0}_{x_1}) < \delta/2$. As x_1 belongs to the interior of the connected component of $(W^u(\Lambda) \cap W^c(x_0))$ that contains x_0 one gets:

- x_1 belongs to $W^u(\Lambda)$
- x_1 is a continuity point of the function $x \mapsto \ell(C^{\tilde{S}_\Lambda(x)}_x)$,
- $\tilde{S}_\Lambda(x_1) = y_0$

As a consequence, there is an open neighborhood V of x_1 such that, for every $x \in V$ one has:

- $x \in W^u(\Lambda) \cap W^s(A)$
- the arc $C^{\tilde{S}_\Lambda(x)}_x$ is contained in $B(y_0, \delta) \subset W^s(A)$
- $\ell(C^{\tilde{S}_\Lambda(x)}_x) < \frac{3}{4}\delta$

As the sets D_A and D_Λ are residual in $W^s(A)$ and $W^u(\Lambda)$, respectively, they are both residual em V. Hence $D_A \cap D_\Lambda \cap V \neq \emptyset$. Choose $x \in D_A \cap D_\Lambda \cap V$, and let us denote $y_- = \tilde{S}_\Lambda(x) \in A$, $y_+ = S_\Lambda(x)$. By definition of D_Λ the interior of the arc $C^{y_-}_{y_+}$ meets $\Omega(f)$ in an unique point $z \in \Lambda$.

By definition of V the arc $C^{y_-}_x$ is contained in $W^s(A)$ hence is disjoint from Λ. Hence the point x belongs to the interior of the central arc $C^{y_-}_z$; furthermore the interior of this arc is disjoint from $\Omega(f)$. As x belongs to D_A, this implies that $z = S_A(x)$ and that Λ is the successor of A. ∎

Denote by $X_{\mathcal{A}}$ and $X_{\mathcal{R}}$ the sets of transitive attractors and repellers of f , respectively. These sets are finite, and the function which maps an attractor to its successor induces a bijection between this two sets. Furthermore, the function on $X_{\mathcal{A}} \cup X_{\mathcal{R}}$ which maps any element to its successor is a permutation of $X_{\mathcal{A}} \cup X_{\mathcal{R}}$. An orbit of this permutation will be called a *cycle of attractors repellers*. The cycles of attractors and repellers form a partition of $X_{\mathcal{A}} \cup X_{\mathcal{R}}$ (we will see at Lemma 5.4 that there is a unique cycle).

Remark 5.2. *Let A and Λ be a transitive attractor and a transitive repeller of f. Then*

$$W^s(A) \cap W^u(\Lambda) \neq \emptyset \Longleftrightarrow (A \textit{ is the predecessor or the successor of } \Lambda)$$

Lemma 5.3. *There is a residual set D of M such that, $\forall x \in D$:*

(1) *the intersection $F_f^c(x) \cap \Omega(f)$ is contained in the union of the transitive attractors and repellers of f;*
(2) *Furthermore, considering an orientation preserving parametrization of $F_f^c(x)$ by $\mathbb{R}$, then $F_f^c(x) \cap \Omega(f)$ is an increasing sequence $\{x_i, i \in \mathbb{Z}\}$ such that $\lim_{i\to-\infty} x_i = -\infty$ and $\lim_{i\to+\infty} x_i = +\infty$;*
(3) *the points x_i belongs alternately to an attractor or to a repeller; for fixing the idea, we can chose the indexation such that $a_i = x_{2i}$ belongs to an attractor A_i and $r_i = x_{2i+1}$ belongs to a repeller Λ_i, for all $i \in \mathbb{Z}$.*
(4) *with the notation above, the attractor A_i is the predecessor of the repeller Λ_i and the successor of the repeller Λ_{i-1}.*
(5) *Moreover, the interior of the central arc $C_{r_i}^{a_i}$ is included in $W^u(\Lambda_i) \cap W^s(A_i)$, and the interior of $C_{a_{i+1}}^{r_i}$ is included in $W^u(\Lambda_i) \cap W^s(A_{i+1})$.*

Proof: Let A be an attractor and D_A be the set defined in Lemma 5.1. Let $M_A = \{x \in W^s(A) | x \notin D_A\}$. Since D_A is a residual set in $W^s(A)$ by Lemma 5.1 then we have that M_A is a meagre set (First Baire category). For every $x \in D_A$ we have that the interior of the arc $\widehat{W^c(x)}$ is contained in D_A, therefore for every $x \in M_A$ we have that the interior of $\widehat{W^c(x)}$ is contained in M_A.

Let $\mathbb{M}_A = \cup_{x\in M_A} F_f^c(x)$ be the union of the whole central leaves through M_A. We will show that $\mathbb{M}_A$ is meagre. For that, consider the homeomorphism h_f conjugating $\mathcal{F}_f^c$ to the central foliation $\mathcal{F}^c$ of the flow ϕ. Let denote by φ_t, for $t \in \mathbb{R}$ the homeomorphism defined by $\varphi_t(x) = h_f^{-1}\phi(h_f(x), t)$, that is the conjugated by h_f of the time t of the flow ϕ. Notice that φ_t is a topological flow whose orbits are the central leaves of $\mathcal{F}_f^c$.

As φ_t is a homeomorphism, one gets that $\varphi_t(M_A)$ is meagre for all t. As the union of countably many meagre sets is a meagre set, one gets that $\bigcup_{t\in\mathbb{Q}} \varphi_t(M_A)$ is a meagre set. One conclude the claim by noticing that $\mathbb{M}_A = \bigcup_{t\in\mathbb{Q}} \varphi_t(M_A)$: in fact for every $x \in M$ the set $\{\varphi_t(x), t \in \mathbb{Q}\}$ is dense in the leaf $F_f^c(x)$. However, if x belongs to $\mathbb{M}_A$ then, by definition, $F_f^c(x)$ contains a point $y \in M_A$ hence it contains the open arc interior of $\widehat{W^c(y)}$. Notice that there is $t \in \mathbb{Q}$ such that $\varphi_t(x)$ belongs to this open arc. This means that x belongs to $\varphi_{-t}(M_A)$ hence to $\bigcup_{t\in\mathbb{Q}} \varphi_t(M_A)$. This proves $\mathbb{M}_A \subset \bigcup_{t\in\mathbb{Q}} \varphi_t(M_A)$, implying that $\mathbb{M}_A$ is meagre (the other inclusion is straightforward , and we will not use it).

Analogously we construct the sets $\mathbb{M}_{A_i}$ for every attractor A_i and $\mathbb{M}_{\Lambda_i}$ for every repeller Λ_i. Since there exist finitely many attractors and repellers we have that

$$\Upsilon = \cup_{i=1,\ldots,n} \mathbb{M}_{\Lambda_i} \cup \mathbb{M}_{A_i}$$

is meagre.

Notice that the union $\bigcup_{A \text{ attractor of } f} W^s(A)$ of the basin of the transitive attractor of f is a dense open subset of M and in the same way the union $\bigcup_{\Lambda \text{ repeller of } f} W^u(\Lambda)$ of the basins of the repellers is a dense open set

Let denote

$$D = \left(\bigcup_{A \text{ attractor of } f} W^s(A) \cup \bigcup_{\Lambda \text{ repeller of } f} W^u(\Lambda) \right) \setminus \Upsilon.$$

It follows that D is a residual set of M.

Furthermore, for every $x \in D$ we have that each connected component of $W^c(x) \cap W^s(A_i)$ is included in D_{A_i} for every attractor A_i; in the same way, each connected component of $W^c(x) \cap W^u(\Lambda_j)$ is included in D_{Λ_j} for every attractor Λ_j. Moreover there is one attractor or one repeller such that x belongs to the basin of it. For instance $x \in W^s(A_0)$. As $x \in D_{A_0}$ the connected component of x in $W^c(x) \cap W^s(A_0)$ is an open central arc which meets A_0 in a (unique) point $x_0 = a_0$; furthermore the origin $x_{-1} = r_{-1}$ of the oriented arc belongs the predecessor Λ_{-1} of A_0 and its end point is a point $x_1 = r_0$ in the successor Λ_0 of A_0. Now, r_{-1} belongs to $D_{\Lambda_{-1}}$ and r_0 belongs to D_{Λ_0}; this allows to build inductively the sequence (a_i, r_i) , a_i in the successor of r_{i-1} and r_i in the successor of a_i, and by construction the open central arc joining a_i to r_i is contained in $W^u(\Lambda_i) \cap W^s(A_i)$ and the open central arc joining a_{i-1} to r_i is contained in $W^u(\Lambda_i) \cap W^s(A_{i-1})$.

For ending the proof it remains to remark that the central distance between x_i and x_{i+1} is larger than δ_f so that the union of the arc $C^{x_i}_{x_i+1}$ cover the whole central leaf $F^c_f(x)$.

Lemma 5.4. *There exists an unique cycle of attractors and repellers.*

Proof: Notice that, in Lemma 5.3, for any $x \in D$ the central leaf $F^c_f(x)$ meets $\Omega(f)$ along the sequence $\{a_i, r_i\}$ and the corresponding sequence of attractors repellers A_i, Λ_i is exactly one cycle of attractors and repellers.

However, according to Lemma 3.2, the central leaf $F^c_f(x)$ meets every attractor in $X_{\mathcal{A}}$ and every repeller in X_Λ: as a consequence, there is an unique cycle of attractors and repellers. In other words, the notion of successor induces a cyclic order on the set of attractors and repellers of f.

Let k denote the number of attractors of f. There is an indexation A_i, Λ_i, $i \in \mathbb{Z}/k\mathbb{Z}$, of the attractors and repellers of f such that Λ_i is the successor of A_i and the predecessor of A_{i+1}. ∎

So, the central leaves through the residual set D visit all the attractors and repellers, following the cyclic order on $X_{\mathcal{A}} \cup X_{\mathcal{R}}$ given the notion of successor. Next lemma show that this property holds for all central leaf, if we allow repetition (i.e. a central leaf may cross an attractor or a repeller in more than one point before crossing its successor):

Proposition 5.1. *Let $A_0, \Lambda_0, \ldots A_{k-1}, \Lambda_{k-1}$ be the sequence of attractors and repellers with the indexation compatible with the cycle.*

Then for every $x \in M$, $\mathcal{F}^c(x) \cap (\cup_{i\in\mathbb{Z}/k\mathbb{Z}} A_i \cup \Lambda_i)$ is a sequence $...a_1^i, ..., a_{n_i}^i, r_1^i, ..., r_{m_i}^i, a_1^{i+1}...$ such that $\{a_1^i, ..., a_{n_i}^i\} \subset A_i$, $\{r_1^i, ..., r_{m_i}^i\} \subset \Lambda_i$.

Proof:

Consider two points x, y in the same central leaf $F_f^c(x)$ such that the segment C_y^x is positively oriented. Assume that x belongs to an attractor A_i, and y belongs to an attractor or a repeller K which is neither A_i nor the successor Λ_i of A_i. We will prove that $C_y^x \cap \Lambda_i \neq \emptyset$.

According to Lemma 4.7 there is a sequence of periodic points $x_n \in D_{A_i}$ converging to x and a sequence of points $y_n \in F_f^c(x_n)$ such that the arcs $C_{y_n}^{x_n}$ converge to C_y^x. For n large enough, the points y_n belongs to the basin of K. However, according to Lemma 4.7, the point $z_n = S_{A_i}(x_n)$ belongs to the successor Λ_i of A_i and the interior of the arc $C_{z_n}^{x_n}$ is contained in $W^s(A_i) \cap W^u(\Lambda_i)$, hence disjoint from the basin of K. As a consequence, y_n does not belong to $C_{z_n}^{x_n}$, so that z_n belongs to the arc $C_{y_n}^{x_n}$. Now, any accumulation point z of the sequence z_n is a point of Λ_i in C_y^x.

Let γ be the connected component of the intersection of any central leaf with any basin of an attractor(or repeller). Since $l(\gamma)$ is bounded by K_f, Lemma 3.3 implies that there is finitely many points in the intersection of γ with the attractor (or the repeller). One concludes that every compact central arc meets $(\cup_{i\in\mathbb{Z}/k\mathbb{Z}} A_i \cup \Lambda_i)$ on a finite set, so that the intersection $F_f^c(x)$ is a sequence of points going from infinity to infinity. ∎

6. THE BASINS OF THE ATTRACTORS AND THE REPELLERS

In what follows, we will look at the relative positions of the basins of the attractors an repellers, the closures of these basins, and the interiors of these closures.

All the results in this section related to attractors admit analogous version for repellers. For instance:

Remark 6.1. *The basins of two different attractors A_i, A_j are disjoint open sets. As a direct consequence, the closure $\overline{W^s(A_i)}$ is disjoint from the interior of the closure $Int\overline{W^s(A_j)}$. On the other hand, the union of the closures of the basins cover M, (in formula: $M \subset \bigcup_{i=0}^k \overline{W^s(A_i)}$). As a consequence one gets*

$$Int\left(\overline{W^s(A_i)}\right) = M \setminus \bigcup_{j\neq i} \overline{W^s(A_j)}, \text{ and}$$

$$\overline{W^s(A_i)} = M \setminus Int\left(\bigcup_{j\neq i} \overline{W^s(A_j)}\right)$$

Let $\alpha, \beta : M \to M$ be the maps defined as follows, for every x in M such that x is not included in any attractor or repeller set, $\alpha(x)$ is the first point in its central leaf in the negative direction verifying that it is in any attractor or in any repeller and $\beta(x)$ is the first point in its central leaf in the positive direction verifying that it is in any attractor or in any repeller. In the case that x belongs to a attractor or a repeller we define $\alpha(x) = \beta(x) = x$.

According to Proposition 5.1, either $\alpha(x)$ and $\beta(x)$ belong to the same (attracting or repelling) basic set, or $\beta(x)$ belongs to the successor of the basic set containing $\alpha(x)$.

Lemma 6.1. *Assume that $\alpha(x)$ or $\beta(x)$ belong to an attractor A. Then $x \in Int(\overline{W^s(A)})$. In the same way, if $\alpha(x)$ or $\beta(x)$ belong to a repeller Λ, then $x \in Int(\overline{W^u(\Lambda)})$.*

Proof: Let prove the Lemma for $\alpha(x) \in A$. The other cases are analogous. If $x = \alpha(x)$ that is $x \in A$, the point x admits a neighborhood contained in $W^s(A)$ ending the proof. Let us now assume that $x \neq \alpha(x)$, and then $x \neq \beta(x)$. Let K be the attractor or repeller containing $\beta(x)$. We fix two points y_0, z_0 in the interior of the arc $C_{\beta(x)}^{\alpha(x)}$ in such a way that the arc $C_{z_0}^{y_0}$ is positively oriented, the point x belongs to $C_{z_0}^{y_0}$, the point y_0 belongs to $W^s(A)$ and z_0 belongs to the basin of K.

Consider two disk Δ_{y_0} and Δ_{z_0}, transverse to the foliation $\mathcal{F}_f^c$ and centered at y_0 and z_0, respectively. Up to shrinking the disks Δ_{y_0} and Δ_{z_0}, one may assume that the holonomy map of the foliation $\mathcal{F}_f^c$ along the path $C_{z_0}^{y_0}$ is well defined and it is a homeomorphism $h\colon \Delta_{y_0} \to \Delta_{z_0}$. For $y \in \Delta_{y_0}$ we denote by γ_y the central arc $C_{h(y)}^{y}$. The family γ_y is a continuous family of central arcs and $\gamma_{y_0} = C_{z_0}^{y_0}$. Notice that the union $V_x = \bigcup_{y \in \Delta_{y_0}} \gamma_y$ is a neighborhood of x.

By construction, the segment γ_{y_0} is disjoint from the compact set obtained as the union of all the attracting and repelling basic sets of f, hence, up to shrinking once more Δ_{y_0} one may assume that every arc γ_y, $y \in \Delta_{y_0}$, is disjoint from the union of the transitive attractors and repellers of f. Recall that the set D is residual and saturated for the central foliation $\mathcal{F}_f^c$; as a direct consequence, D meets any transverse section Δ of $\mathcal{F}_f^c$ in a residual subset of Δ. In particular generic points $y \in \Delta_{y_0}$ belong to D. Lemma 5.3 implies that for $y \in D \cap \Delta_{y_0}$, the arc γ_y is contained in $W^s(A)$. As $W^s(A)$ is open, and the union of the γ_y, $y \in D \cap \Delta_{y_0}$ is dense in V_x, one gets that $W^s(A) \cap V_x$ is a dense open subset of V_x. This implies that $V_x \subset \overline{W^s(A)}$ and then $x \in Int\overline{W^s(A)}$, concluding the proof of the lemma. ■

Lemma 6.2. *Assume that both $\alpha(x)$ and $\beta(x)$ belong to an attractor A, and let Λ_- and Λ_+ be the predecessor and the successor of A. Then*

(1) *For any transitive repeller Λ of f one has*

$$x \in \overline{W^u(\Lambda)} \Longleftrightarrow \Lambda \in \{\Lambda_-, \Lambda_+\}$$

(2) *Hence:*

$$x \in \overline{W^u(\Lambda_-)} \cap \overline{W^u(\Lambda_+)} \cap Int\left(\overline{W^u(\Lambda_-)} \cup \overline{W^u(\Lambda_+)}\right).$$

Proof: As $\alpha(x) \in A$ there is a sequence of periodic point points $y_n \in D_A$ converging to $\alpha(x)$. Then the points $z_n = S_A(y_n)$ belongs to Λ_+ and the interior of the arc $\gamma_n = C_{z_n}^{y_n}$ is contained in $W^u(\Lambda_+)$. Up to considering a subsequence, one may assume that the arc γ_n converges to a central arc $\gamma = C_z^y$ beginning at $y = \alpha(x)$. Notice that $z \in \Lambda_+$ and $\gamma \subset \overline{W^u(\Lambda_+)}$. As there is no points of Λ_+ in the arc $C_{\beta(x)}^{\alpha(x)}$ one gets that $C_{\beta(x)}^{\alpha(x)} \subset \gamma \subset \overline{W^u(\Lambda_+)}$. So $x \in \overline{W^u(\Lambda_+)}$. Using $\beta(x)$ instead of $\alpha(x)$ one prove in the same way that $x \in \overline{W^u(\Lambda_-)}$. For concluding the proof of the lemma, it remains to show $x \in Int\left(\overline{W^u(\Lambda_-)} \cup \overline{W^u(\Lambda_+)}\right)$. Notice that M is the union of the closure of the basins of the transitive repellers of f.

Hence

$$M \setminus \bigcup_{\Lambda \in X_R \setminus \{\Lambda_-, \Lambda_+\}} \overline{W^u(\Lambda)} \subset Int\left(\overline{W^u(\Lambda_-)} \cup \overline{W^u(\Lambda_+)}\right).$$

For concluding the proof of the lemma, it is enough to prove that $x \notin \overline{W^u(\Lambda)}$, for every repeller Λ different from Λ_+ and Λ_-.

Let Λ be a repeller such that $x \in \overline{W^u(\Lambda)}$. According to Lemma 6.1 the point x belongs to the interior of the closure of $W^s(A)$. One deduce that $W^u(\Lambda)$ meets the interior of the closure of $W^s(A)$. As $W^u(\Lambda)$ is an open set, this implies that $W^u(\Lambda) \cap W^s(A) \neq \emptyset$. Remark 5.2 implies that $\Lambda \in \{\Lambda_+, \Lambda_-\}$ ending the proof. ■

Corollary 6.1. *Let A be a transitive attractor. Then a point x belongs to $\overline{W^s(A)}$ if and only if $\{\alpha(x), \beta(x)\} \subset \Lambda_- \cup A \cup \Lambda_+$.*

Proof: First assume that $\{\alpha(x), \beta(x)\} \subset \Lambda_- \cup A \cup \Lambda_+$. If $\alpha(x)$ or $\beta(x)$ belong to A, Lemma 6.1 asserts that x belongs to the interior of $\overline{W^s(A)}$. Otherwise, $\alpha(x), \beta(x) \in \Lambda_- \cup \Lambda_+$; assume for instance $\alpha(x), \beta(x) \in \Lambda_-$. The version of Lemma 6.2 for repellers implies that x belongs to the intersections of the closures of the basins of the predecessor and of the successor of Λ_-. In particular, $x \in \overline{W^s(A)}$.

Conversely, consider a point x such that $\{\alpha(x), \beta(x)\} \not\subset \Lambda_- \cup A \cup \Lambda_+$. If $\alpha(x)$ or $\beta(x)$ belong to an attractor $A_i \neq A$ then Lemma 6.1 implies that $x \in Int(\overline{W^s(A_i)})$ which is disjoint from $\overline{W^s(A)}$. In the other case there is $\Lambda_i \notin \{\Lambda_-, \Lambda_+\}$ such that $\alpha(x), \beta(x) \in \Lambda_i$. Then the version of Lemma 6.2 for repellers implies that x belongs to $Int\left(\overline{W^s(A_i) \cup W^s(A_{i+1})}\right)$ which is disjoint from $\overline{W^s(A)}$ by remark 6.1, because $A \notin \{A_i, A_{i+1}\}$. ■

Corollary 6.2. *Let A be a transitive attractor, Λ_- its predecessor and Λ_+ its successor. Then:*

$$\overline{W^s(A)} \subset Int(\overline{W^u(\Lambda_-) \cup W^u(\Lambda_+)}).$$

Proof: Consider $x \in \overline{W^s(A)}$. We know from Corollary 6.1 that $\{\alpha(x), \beta(x)\} \subset \Lambda_- \cup A \cup \Lambda_+$.

If $\alpha(x)$ or $\beta(x)$ belongs to $\Lambda_- \cup \Lambda_+$ then Lemma 6.1 implies that $x \in Int\left(\overline{W^u(\Lambda_-)}\right) \cup Int\left(\overline{W^u(\Lambda_+)}\right) \subset Int(\overline{W^u(\Lambda_-) \cup W^u(\Lambda_+)})$. Otherwise, $\alpha(x)$ and $\beta(x)$ belong to A; then Lemma 6.2 claims that $x \in Int(\overline{W^u(\Lambda_-) \cup W^u(\Lambda_+)})$, which concludes the proof.

■

Lemma 6.3. *It holds that*

$\overline{W^s(A_i)} \cap \overline{W^s(A_j)} \neq \emptyset$ if and only if either $A_i = A_j$ or there is a repeller set Λ such that $\{A_i, A_j\} = \{A_-, A_+\}$ where A_- is the predecessor of Λ and A_+ is its successor. In other words,

$$\overline{W^s(A_i)} \cap \overline{W^s(A_j)} \neq \emptyset \Longleftrightarrow |i - j| \leq 1$$

Proof: Assume that $A_i \neq A_j$ and consider $x \in \overline{W^s(A_i)} \cap \overline{W^s(A_j)}$. Then Lemma 6.1 implies that neither $\alpha(x)$ nor $\beta(x)$ can belong to $A_i \cup A_j$. So Corollary 6.1 implies that $\alpha(x), \beta(x)$ belong to a repeller Λ which needs to be not only the successor or the predecessor of A_i but the successor or the predecessor of A_j, as well. It follows that $\{A_i, A_j\} = \{A_-, A_+\}$ where A_- is the predecessor of Λ and A_+ is its successor.

Conversely, if A_i and A_j are the predecessor and the successor of Λ, then $\overline{W^s(A_i)} \cap \overline{W^s(A_j)}$ contains Λ. ∎

Lemma 6.4. *Given any attractor A and any repeller Λ,*

$$\overline{W^s(A)} \cap \overline{W^u(\Lambda)} \neq \emptyset \Longleftrightarrow \Lambda \text{ is the successor or the predecessor of } A.$$

Proof: Let Λ_- and Λ_+ be the predecessor and the successor of A, respectively. According to Corollary 6.2 one has $\overline{W^s(A)} \subset Int(\overline{W^u(\Lambda_-)} \cup \overline{W^u(\Lambda_+)})$. If $\overline{W^s(A)} \cap \overline{W^u(\Lambda)} \neq \emptyset$ then $Int(\overline{W^u(\Lambda_-)} \cup \overline{W^u(\Lambda_+)}) \cap \overline{W^u(\Lambda)} \neq \emptyset$, implying that $\Lambda \in \{\Lambda_-, \Lambda_+\}$.

The converse implication is a direct consequence of the definition of successor or predecessor.

∎

7. Axiom A diffeomorphisms in $\mathcal{E}_\phi$ with more than one attractors and repellers

We now assume that $f \in \mathcal{E}_\phi$ is an Axiom A diffeomorphism having at least two transitive attractors. We consider the attractors A_i and the repellers Λ_i, $i \in \mathbb{Z}/k\mathbb{Z}$, $k > 1$, Λ_i being the successor or A_i and the predecessor of A_{i+1}.

Lemma 7.1. *For every i, the boundary $\partial\overline{W^u(\Lambda_i)}$ is*

$$\partial\overline{W^u(\Lambda_i)} = \left(\overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)}\right) \cup \left(\overline{W^u(\Lambda_i)} \cap \overline{W^u(\Lambda_{i+1})}\right)$$

Proof: Just notice that the $\overline{W^u(\Lambda_j)}$ are compact sets, equal to the closure of they interior, of interior pairwise disjoints, and whose union is M. Hence $Int\left(\overline{W^u(\Lambda_i)}\right)$ is the complement of $\bigcup_{j\neq i} \overline{W^u(\Lambda_j)}$, and $\partial\overline{W^u(\Lambda_i)} = \overline{W^u(\Lambda_i)} \cap \bigcup_{j\neq i} \overline{W^u(\Lambda_j)}$. The version of Lemma 6.3 for repeller sets implies that $\overline{W^u(\Lambda_i)} \cap \bigcup_{j\neq i} \overline{W^u(\Lambda_j)} = \overline{W^u(\Lambda_i)} \cap \left(\overline{W^u(\Lambda_{i-1})} \cup \overline{W^u(\Lambda_{i+1})}\right)$ (because $\overline{W^u(\Lambda_i)} \cap \overline{W^u(\Lambda_j)} = \emptyset$ if $j \notin \{i-1, i, i+1\}$), concluding the proof. ∎

Next lemma states some properties of the boundaries of the closures of the basins of the attractors and repellers:

Lemma 7.2.

(1) *The boundary $\partial\overline{W^u(\Lambda_i)}$ is contained in $\overline{W^s(A_i)} \cup \overline{W^s(A_{i+1})}$.*
(2) *Moreover, $\partial\overline{W^u(\Lambda_i)} \cap \overline{W^s(A_i)} = \left(\overline{W^u(\Lambda_{i-1})} \cap \overline{W^s(\Lambda_i)}\right) \cap \overline{W^s(A_i)}$; let K_i denote this compact set;*
(3) *$\partial\overline{W^u(\Lambda_i)} \cap \overline{W^s(A_{i+1})}$ is the compact set $K_{i+1} = \left(\overline{W^u(\Lambda_i)} \cap \overline{W^u(\Lambda_{i+1})}\right) \cap \overline{W^s(A_{i+1})}$*

(4) *for every i, the compact set K_i is contained in* $Int\left(\overline{W^s(A_i)}\right)$;
(5) *the compact set K_i is characterized by*

$$x \in K_i \Longleftrightarrow \alpha(x), \beta(x) \in A_i$$

Proof: The version of Corollary 6.1 for repeller sets asserts that $x \in \overline{W^u(\Lambda_i)}$ if and only if $\alpha(x)$ and $\beta(x)$ belong to $A_i \cup \Lambda_i \cup A_{i+1}$. As a consequence, $x \in \overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)}$ if and only if $\{\alpha(x), \beta(x)\} \subset (A_{i-1} \cup \Lambda_{i-1} \cup A_i) \cap (A_i \cup \Lambda_i \cap A_{i+1})$.

If $k \neq 2$, $(A_{i-1} \cup \Lambda_{i-1} \cup A_i) \cap (A_i \cup \Lambda_i \cap A_{i+1}) = A_i$. So

$$x \in \overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)} \Longleftrightarrow \{\alpha(x), \beta(x)\} \subset A_i.$$

Then, Lemma 6.1 implies that, for all $i \in \mathbb{Z}/k\mathbb{Z}$, $\overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)}$ is contained in $Int\overline{W^s(A_i)}$. Analogously $\overline{W^u(\Lambda_i)} \cap \overline{W^u(\Lambda_{i+1})}$ is contained in $Int\overline{W^s(A_{i+1})}$. Then Lemma 7.1 implies (1).

From Lemma 6.4, we have that $\overline{W^u(\Lambda_{i-1})} \cap \overline{W^s(A_{i+1})} = \emptyset$ and $\overline{W^u(\Lambda_{i+1})} \cap \overline{W^s(A_i)} = \emptyset$, then (2), (3) and (4) hold. To prove (5) is enough to show that if $\alpha(x), \beta(x) \in A_i$ then $x \in K_i$, but this is a consequence of Lemma 6.1 and Lemma 6.2.

If $k = 2$, $(A_{i-1} \cup \Lambda_{i-1} \cup A_i) \cap (A_i \cup \Lambda_i \cap A_{i+1}) = A_i \cup A_{i+1} = A_0 \cup A_1$. So

$$x \in \overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)} \Longleftrightarrow \{\alpha(x), \beta(x)\} \subset A_i \cup A_{i+1}$$

Notice that as a consequence of Lemma 6.1 $\{\alpha(x), \beta(x)\} \subset A_i \cup A_{i+1}$ if and only if $\{\alpha(x), \beta(x)\} \subset A_i$ or $\{\alpha(x), \beta(x)\} \subset A_{i+1}$. Then, Lemma 6.1 implies that, for all $i \in \mathbb{Z}/2\mathbb{Z}$, $\overline{W^u(\Lambda_{i-1})} \cap \overline{W^u(\Lambda_i)}$ is contained in $Int\overline{W^s(A_i)} \cup Int\overline{W^s(A_{i+1})}$. All the items of the lemma follow immediately.

■

By the last claim of the previous Lemma and by the definition of K_i, it follows

Remark 7.1. *Any connected component of the intersection of a central leaf with K_i is either a point in A_i or it is a central arc whose both extremities belong to A_i.*

As a direct consequence of Lemma 7.2 one has

Corollary 7.1. *For all different i, j in $\mathbb{Z}/k\mathbb{Z}$, K_i and K_j are disjoint compact sets and $\partial\overline{W^u(\Lambda_i)} = K_i \cup K_{i+1}$. Furthermore*

$$K_0 \cup K_1 = \partial\overline{W^u(\Lambda_0)} = \partial\left(\bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}\right)$$

Let $\theta\colon M \to [0,1]$ denote a continuous function such that $\theta^{-1}(0) = K_0$ and $\theta^{-1}(1) = K_1$: for instance one can choose θ defined by $\theta(x) = \frac{d(x,K_0)}{\sup\{d(x,K_0),d(x,K_1)\}}$. We denote by $\rho\colon M \to S^1 = \mathbb{R}/\mathbb{Z}$ the map defined as follows:

- for $x \in \overline{W^u(\Lambda_0)}$, $\rho(x)$ is the class modulo $\mathbb{Z}$ of $\frac{1}{2}\theta(x)$.
- for $x \in \bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}$, $\rho(x)$ is the class modulo $\mathbb{Z}$ of $1 - \frac{1}{2}\theta(x)$.

The map $\rho(x)$ is well defined: if $x \in \overline{W^u(\Lambda_0)} \cap \bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}$ then either $x \in K_0$ and $\rho(x) = 0 = 1 \in S^1$ or $x \in K_1$ and $\rho(x) = \frac{1}{2}$. The map ρ is continuous restricted to both compact sets $\overline{W^u(\Lambda_0)}$ and $\bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}$ whose union is M, hence is continuous on M.

Let us consider the universal cover $\pi\colon \mathbb{R} \to S^1$. Let $\Pi\colon \tilde{M} \to M$ be the pull back of the covering π by ρ. Recall that there is a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \stackrel{\tilde{\rho}}{\to} & \mathbb{R} \\ \Pi \downarrow & & \downarrow \pi \\ M & \stackrel{\rho}{\to} & S^1 \end{array}$$

Lemma 7.3. *Let $\gamma\colon [0,1] \to M$ be a central arc such that $x = \gamma(0)$ belongs to A_0, $y = \gamma(1)$ belongs to A_1, and γ is disjoint from Λ_i for $i \neq 0$. Let $\tilde{\gamma}$ be a lift of γ on $\tilde{M}$ and let denote $\tilde{x} = \tilde{\gamma}(0) \in \Pi^{-1}(x)$ and $\tilde{y} = \tilde{\gamma}(1) \in \Pi^{-1}(y)$. Then $\tilde{\rho}(\tilde{y}) - \tilde{\rho}(\tilde{x}) = \frac{1}{2}$.*

Proof: By hypotheses, $\rho(x) = 0$ and $\rho(y) = \frac{1}{2}$. We just need to prove that $\rho(\gamma)$ is contained in the arc $[0, \frac{1}{2}]$ of $S^1 = \mathbb{R}/\mathbb{Z}$ (hence is equal to that arc). For that, it is enough to see that γ is contained in $\overline{W^u(\Lambda_0)}$. Proposition 5.1 and the fact that γ joins the point $x \in A_0$ to $y \in A_1$ without crossing Λ_i for $i \neq 0$ implies that, for any $z \in \gamma$ one has $\{\alpha(z), \beta(z)\} \subset A_0 \cup \Lambda_0 \cup A_1$. We conclude using Corollary 6.1 that $z \in \overline{W^u(\Lambda_0)}$, ending the proof. ■

Lemma 7.4. *Let $\gamma\colon [0,1] \to M$ be a central arc such that $x = \gamma(0)$ belongs to A_1, $y = \gamma(1)$ belongs to A_0, and γ is disjoint from Λ_0. Let $\tilde{\gamma}$ be a lift of γ on $\tilde{M}$ and let denote $\tilde{x} = \tilde{\gamma}(0) \in \Pi^{-1}(x)$ and $\tilde{y} = \tilde{\gamma}(1) \in \Pi^{-1}(y)$. Then $\tilde{\rho}(\tilde{y}) - \tilde{\rho}(\tilde{x}) = \frac{1}{2}$.*

Proof: This time, $\rho(x) = \frac{1}{2}$ and $\rho(y) = 1 = 0 \in S^1$. We just need to prove that $\rho(\gamma)$ is contained in the arc $[\frac{1}{2}, 1]$ of $S^1 = \mathbb{R}/\mathbb{Z}$. In other words, we have to prove that γ is disjoint from $Int\left(\overline{W^u(\Lambda_0)}\right)$, that is it is included in $\bigcup_{i\neq 0} \overline{W^u(\Lambda_i)}$. Proposition 5.1 and the fact that γ joins the point $x \in A_1$ to $y \in A_0$ without crossing Λ_0 implies that, for any $z \in \gamma$, $\alpha(z) \notin \Lambda_0$ and $\beta(z) \notin \Lambda_0$. Using the fact that $\beta(z)$ belongs either to the basic set containing $\alpha(z)$ or to its successor, one shows that there is $i \neq 0$ in $\mathbb{Z}/k\mathbb{Z}$ such that $\{\alpha(z), \beta(z)\} \subset A_i \cup \Lambda_i \cup A_{i+1}$. We conclude using the version of Corollary 6.1 for repeller sets that $z \in \overline{W^u(\Lambda_i)}$ (with $i \neq 0$), ending the proof. ■

Lemma 7.5. *Let $\gamma\colon [0,1] \to M$ be a central arc meeting A_0 and A_1 at most at its extremities (in formula: $\gamma \cap (A_0 \cup A_1) \subset \{\gamma(0), \gamma(1)\}$). Let $\tilde{\gamma}$ be a lift of γ on $\tilde{M}$ and let denote $\tilde{x} = \tilde{\gamma}(0) \in \Pi^{-1}(x)$ and $\tilde{y} = \tilde{\gamma}(1) \in \Pi^{-1}(y)$. Then $|\tilde{\rho}(\tilde{y}) - \tilde{\rho}(\tilde{x})| \leq \frac{1}{2}$.*

Proof: We will prove that, either $\gamma \subset \overline{W^u(\Lambda_0)}$ or $\gamma \cap Int\overline{W^u(\Lambda_0)} = \emptyset$. Recall that the boundary $\partial\overline{W^u(\Lambda_0)}$ is the union $K_0 \cup K_1$. From Remark 7.1, we have that any connected component of the intersection of a central leaf with K_0 is either a point in A_0 or a central arc whose both extremities belong to A_0. As the interior of γ is disjoint from A_0 we get that γ is either contained in K_0 or its interior is disjoint from K_0. The same holds for K_1. So either γ is contained in $\partial\overline{W^u(\Lambda_0)}$ (hence in $\overline{W^u(\Lambda_0)}$) or the interior of γ is disjoint from $\partial\overline{W^u(\Lambda_0)}$; in that case the interior of γ is either contained in $Int\overline{W^u(\Lambda_0)}$ or is disjoint from $\overline{W^u(\Lambda_0)}$, ending the proof. ■

We denote by $\tilde{\mathcal{F}}^c_c$ the lift on $\tilde{M}$ of the foliation $\mathcal{F}^c_f$. Given a point $x \in M$ we denote by $\gamma_x \colon \mathbb{R} \to M$ the infinite positively oriented central arc, parametrized by the arc length, such that $\gamma_x(0) = x$. Consider $\tilde{x} \in \Pi^{-1}(x)$. We denote by $\gamma_{\tilde{x}}$ the lift of γ_x on $\tilde{M}$ with $\gamma_{\tilde{x}}(0) = \tilde{x}$. Recall that K_f is an upper bound of the length of the central arc $D(x)$, for every $x \in M$.

Lemma 7.6. *For every point $\tilde{x} \in \tilde{M}$ and for any $\ell > 4K_f$, the map $\varphi_x = \tilde{\rho} \circ \gamma_{\tilde{x}} \colon \mathbb{R} \to \mathbb{R}$ satisfies*

$$\varphi_x(t + \ell) \geq \varphi_x(t) + 1.$$

In particular, $\lim_{t \to -\infty} \varphi_x(t) = -\infty$ and $\lim_{t \to +\infty} \varphi_x(t) = +\infty$.

Proof: Let x be the projection of $\tilde{x}$ on M. Let γ be the restriction of γ_x to $[0, 4K_f]$, and let $\tilde{\gamma}$ be the lift of γ on $\tilde{M}$ starting at $\tilde{x}$. We want to prove $\tilde{\rho}(\tilde{\gamma}(4K_f)) - \tilde{\rho}(\tilde{\gamma}(0)) \geq 1$. For that, using Proposition 5.1 we write γ as being the concatenation $\gamma_0 \cdot \gamma_1 \cdots \gamma_m \cdot \gamma_{m+1}$ where :

- γ_0 is a central arc starting at x and joining x to the first point of γ in $A_0 \cup A_1$.
- by the proof of Lemma 7.5 for $i \in \{1, \dots, m\}$, γ_i is either
  - a central arc joining a point of A_0 to a point of A_1 included in $\overline{W^u(\Lambda_0)}$ and then it is disjoint of Λ_i, for $i \neq 0$, or
  - a central arc joining a point of A_1 to a point of A_0 included in the complement of $Int\overline{W^u(\Lambda_0)}$ and then it is disjoint of Λ_0,
- γ_{m+1} is the arc in γ joining the last point of γ in $A_0 \cup A_1$ to $\gamma(4K_f)$

Lemmas 7.3 7.4 7.5 imply that

$$\tilde{\rho}(\tilde{\gamma}(4K_f)) - \tilde{\rho}(\tilde{\gamma}(0)) \geq \frac{m}{2} - 1.$$

Recall that each arc $D(z)$, $z \in M$ meets any attractor and any repeller and its length is less than K_f. So every arc $D(z)$ contains at least one segment either joining A_0 to A_1 or joining A_1 to A_0. As the length of γ is larger that $4K_f$ it contains at least 4 disjoint arcs of the form $D(z)$. Hence $m \geq 4$, ending the proof. ∎

Let $\psi \colon M \times \mathbb{R} \to M$ be a smooth flow, C^1-close to ϕ so that every non-zero time map of the flow ψ belongs to $\mathcal{E}_\phi$. Let $h \colon M \to M$ be an homeomorphism such that $h(\mathcal{F}^c_\psi) = \mathcal{F}^c_f$. Let denote $\rho_\psi = h \circ \rho \colon M \to S^1$. Let $\Pi_\psi \colon \tilde{M}_\psi \to M$ be the pull-back by ρ_ψ of the universal cover $\pi \colon \mathbb{R} \to S^1$, and $\tilde{\rho}_\psi \colon \tilde{M}_\psi \to \mathbb{R}$ be the lift of ρ_ψ. Notice that $\tilde{\rho}_\psi$ splits in a product $\tilde{\rho}_\psi = \tilde{\rho} \circ \tilde{h}$, where $\tilde{h} \colon \tilde{M}_\psi \to \tilde{M}$ is a lift of h. One has the following abelian diagram:

$$\begin{tikzcd}
\tilde{M}_\psi \arrow[rr, "\tilde{\rho}_\psi"] \arrow[dr, "\tilde{h}"] \arrow[dd, "\Pi_\psi"'] & & \mathbb{R} \arrow[dd, "\pi"] \\
& \tilde{M} \arrow[ur, "\tilde{\rho}"] \arrow[d, "\Pi"] & \\
M \arrow[r, "h"] & M \arrow[r, "\rho"] & S^1
\end{tikzcd}$$

Let $\tilde{\psi}$ be the lift of the flow ψ on $\tilde{M}_\psi$. Then one gets

Corollary 7.2. *There is $L > 0$ such that for every $x \in \tilde{M}_\psi$ one has:*

$$\tilde{\rho}_\psi(\tilde{\psi}(x, L)) - \tilde{\rho}_\psi(x) > 1.$$

Then a nice argument of Schwartzman (see [23, Teorema 5.1]) allows us to conclude the proof of Theorem 2. We reproduce here this argument for completeness:

Proof of Theorem 2: Let $\mu\colon M \to S^1$ be a smooth map C^0-close to the map ρ_ψ, and $\tilde{\mu}$ be the lift of μ on $\tilde{M}_\psi$. If μ is close enough to ρ_ψ, for every $x \in \tilde{M}_\psi$ one has

$$\tilde{\mu}(\tilde{\psi}(x, L)) - \tilde{\mu}(x) > 1.$$

For $x \in \tilde{M}_\psi$ let us denote

$$\tilde{\lambda}(x) = \frac{1}{L}\int_0^L \mu(\tilde{\psi}(x,t))dt.$$

Notice that $\tilde{\lambda}\colon \tilde{M}_\psi \to \mathbb{R}$ is a smooth function which projects on M in a map $\lambda\colon M \to S^1$. Furthermore, for any $x \in \tilde{M}_\psi$ the derivative of $\tilde{\lambda}$ along the $\tilde{\psi}$-orbit is

$$\frac{\partial}{\partial t}\tilde{\lambda}(\psi(x,t))|_{t=0} = \tilde{\mu}(\tilde{\psi}(x,L)) - \tilde{\mu}(x) > \frac{1}{L} > 0.$$

This proves that the map $\lambda\colon M \to S^1$ is a submersion and that the orbits of ψ are transverse to the fibers; let denote by N the fiber of this fibration. So, ψ (hence also ϕ) is topologically equivalent to a suspension of a diffeomorphism g of N, which inherit the hyperbolic structure of the flow (the stable and unstable bundles of g are the intersections of the corresponding bundles for ψ with TN); so g is an Anosov diffeomorphism and the transitivity of ψ implies the transitivity of g.

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