Real perturbations of complex polynomials

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October 24, 2006

Abstract

In this article the dynamics of generic C^r $(r \geq 3)$ perturbations of complex polynomials are considered. The attention is focused on the determination of the existence of large or invariant components of the complement of the basin of ∞, where the interesting dynamics occur.[1]

1 Introduction

Given a manifold M, let $End^r(M)$ denote the set of differentiable endomorphisms of M, of class C^r, endowed with the Whitney or strong topology. For $f \in End^r(M)$, say that a point x is critical for f if the differential of f at x, Df_x, is not invertible. Denote by S_f the set of critical points of f. The study of the dynamics of endomorphisms has caught the attention of many authors. The question of stability, still not solved, was considered in [MP], [I], [AMS] and [DRRV]. A lot of work has been done also because these maps have abundant appearance in applications; the existence of critical points and multiple preimages is a source of creation of chaotic dynamics, as many authors shown by means of numerical experiments. The one dimensional theory was mostly considered, in the real context (see [MS]) as well as in the complex one, beginning with the works of Fatou and Julia ([Mi], [S]). This theory is now very rich and elegant and most conjectures have been proved. This is not the case in higher dimensions. In the attempt to understand the dynamics of maps with critical points in dimension two, we have considered here C^r perturbations of complex polynomials. The first problem consists in the description of the critical sets and critical values, and the regions where the number of preimages is constant.

It is well known that great part of the dynamical structure of a complex polynomial depends on the future orbits of the critical points. Properties such as

[1] *Keywords.* Critical points, real perturbations, complex polynomials.
2000 AMS subject classification: 37C05 37D05

connectivity of the Julia set or of the basin of ∞, existence of invariant domains and stability depend on simple assumptions about S_f. Two main difficulties arise when a perturbation of a polynomial is considered: first, as the conformality is lost, the usual techniques of complex analysis are no more available; on the other hand, as will be shown later, the set of critical points becomes generically a finite union of circles. For example, for the case of a polynomial P and a connected set K, the number of components of $P^{-1}(K)$ depends on a very simple way on the relative positions of K and $P(S_P)$. This problem becomes interesting and difficult when the set of critical points is a one dimensional manifold and then the possibilities for the intersections of K and $f(S_f)$ explode. One of the main questions developed in this article consists in determine under which conditions the preimage of a connected set is connected.
If P is a complex polynomial then the Julia set of P is connected if and only if all the critical points of P have bounded orbit; moreover, if all the critical points have unbounded orbit, then the Julia set of P is totally disconnected. Using some new techniques, sometimes inspired in [RRV] and obviously unusual in the complex domain, partial generalizations of these results will be given here.

Theorem 1. *Let P be a polynomial. There exists a C^1 neighborhood $\mathcal{U}$ of P such that for every $f \in \mathcal{U}$, either some critical point has unbounded orbit or the set of points with bounded orbit is connected and simply connected.*

Denote by B_∞ the set of points with unbounded orbit, that is, the basin of attraction of ∞. Strong perturbations of polynomial mappings always have ∞ as an attractor, so the dynamics occur in the complement B_∞^c of B_∞. There exists an open and dense set $\mathcal{G}$ in $End^r(M)$ $(r \geq 3)$ such that, for maps in $\mathcal{G}$, the critical points are nondegenerate (see next section). The next objective is to determine the invariant components of B_∞^c.
Theorem 2. *Let h be a quadratic polynomial. There exist a C^3 neighborhood $\mathcal{U}$ of h such that if $f \in \mathcal{U} \cap \mathcal{G}$ and B_∞^c has more than one invariant component, then B_∞^c has uncountably many components.*

The proof of this theorem is given at the end of section 4. Note that for a complex quadratic polynomial the fact that the critical point belongs to B_∞ implies that B_∞^c (the Julia set) is a Cantor expanding set. This is not the case for real perturbations; indeed, in the last section will be given an example of a map close to a quadratic polynomial such that B_∞^c has uncountably many components and saddle or attracting type periodic points. In that section it is also given a sufficient condition for a map f to satisfy that B_∞^c has uncountable many bounded components. The set of critical points S_f of a generic perturbation f of a complex quadratic polynomial h is a small circle (this will be proved in section 2). The set

$f^{-1}(f(S_f))$, denoted $\widetilde{S_f}$, is also contained in a small disc if f is a small perturbation of h. If the critical point of h is not fixed (otherwise h conjugated to $q(z) = z^2$) then S_f and $f(S_f)$ will be disjoint for any perturbation. A component of B^c_∞ is called large if it intersects S_f and $f(S_f)$. A set is small if it is contained in the complement of $ext(\widetilde{S_f})$, where $ext(A)$ denotes the unbounded component of the complement of A.

Theorem 3: *Let h be quadratic polynomial with connected Julia set and repelling fixed points.*

a) If f is a small C^1 perturbation of h and B^c_∞ contains a large component, then it is the unique large component and is invariant. In this case, for every other component K of B^c_∞ there exists $n > 0$ such that $f^n(K)$ is small.

b) If f is a generic C^3 perturbation of h, then B^c_∞ contains at most two invariant components; if it has exactly two, then one of them is a fixed point.

Part (a) says that the existence of a large component in B^c_∞ implies that it is invariant and that every other component has a small image. This shows that the dynamics of f is determined heavily by its behavior in this component. On the other hand, if there is no large component of B^c_∞, then by theorem 2 there exist uncountably many components of B^c_∞. It remains open the question if $S_f \subset B_\infty$ implies that B^c_∞ is a Cantor expanding set.

2 Preliminaries

In this work the strong or Whitney topology is considered. A neighborhood of $f \in End^r(\mathbb{R}^n)$ is determined by: a continuos function $\varepsilon : \mathbb{R}^n \to \mathbb{R}^+$ such that each derivative partial order r evaluate at x is $\varepsilon(x)-$closed to the respective derivative of f. For example, if f is a C^r proper map of $\mathbb{R}^n$ (that is, the preimage of a compact set under f is compact or equivalently, for every sequence x_n converging to ∞ it holds that $f(x_n)$ also converges to ∞), then there exists a C^0 strong neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is proper. In other words, if f has continuos extension to the one point compactification of $\mathbb{R}^n$, then g also has this property. It is also clear that if ∞ is, in addition an attractor for the extension of f, then the same will be true for every small C^0 strong perturbation of it.

Given $f \in End^r(M)$, denote by S_f the set of critical points of f. It is known, see for example [W], that if M is two dimensional manifold, then there exists an open and dense subset $\mathcal{G} = \mathcal{G}(M)$ of $End^r(M)$ ($r \geq 3$) such that for every $f \in \mathcal{G}$, the set S_f of critical points of f is empty or it is a one dimensional embedded submanifold of M. Moreover, for each critical point x of a mapping $f \in \mathcal{G}$, there exist local canonical forms:

Definition 2.1. *Let $f \in End^r(M)$, M two dimensional.*

A point $x \in S_f$ is a fold if there exist neighborhoods U of x and V of $f(x)$, and diffeomorphism $\varphi : I\!R^2 \to U$ and $\psi : V \to I\!R^2$ such that $\psi f \varphi$ is equal to the map $(x, y) \to (x^2, y)$.
A point $x \in S_f$ is a cusp if there exist neighborhoods and diffeomorphism as above, but now the composition $\psi f \varphi$ equals the map $(x, y) \to (x, -xy + y^3)$.

Theorem 2.1. *(Whitney, [W]) There exists an open and dense subset $\mathcal{G}(M)$ of $End^r(M)$ $(r \geq 3)$ such that for every $f \in \mathcal{G}(M)$:*

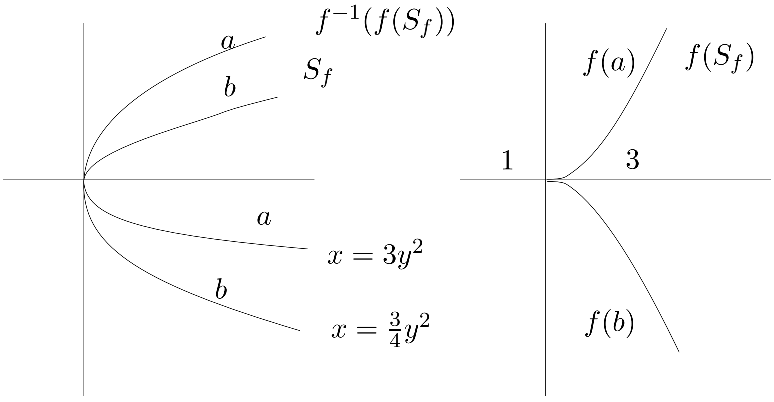
- *S_f is a one dimensional submanifold of M or is empty.*
- *Every critical point of f is a fold or a cusp.*
- *The set of cusp type points is isolated.*
- *If S_f' is a component of S_f, then $f(S_f')$ is a curve with transversal intersections, no one of which contains the image of a cusp.*

In figure 1 a sketch of the local behaviour of a map near a cusp type critical point is shown. Observe that if V is a neighborhood of $f(x)$, then $V \setminus f(S_f)$ is the union of two components; in one of them, the points have three preimages near x while in the other each point has only one preimage near x. At a cusp point x the kernel of Df_x coincides with the tangent space of $T_x S_f$. When x is a fold the kernel of Df_x is transverse to $T_x S_f$. Note that $f^{-1}(f(S_f))$ strictly contains S_f whenever S_f contains a cusp type point. Denote by $\widetilde{S_f} = f^{-1}(f(S_f))$.

Proposition 1. *Let M be a manifold of any dimension, compact or not, and $f \in End^r(M)$. Suppose that f is a C^1 proper map. The following facts can be easily verified:*

1. *The image of f, $Im(f)$ is a closed set.*
2. *$M \setminus \widetilde{S_f}$ is an open set.*
3. *If $x \in Im(f) \setminus f(S_f)$ then $f^{-1}(x)$ is finite.*
4. *If U is a connected component of $M \setminus f(S_f)$ and $A_k = \{x \in U : \sharp f^{-1}(x) = k\}$ then A_k is equal to U for some nonnegative k.*

Proposition 2. . *Suppose as above that f is a proper map of class C^1 on a manifold M. If V is a connected component of $M \setminus \widetilde{S_f}$ then $f(V)$ is a connected component of $M \setminus f(S_f)$ and $f\big|_V : V \to f(V)$ is a covering map.*



(a) (b)

Figure 1:

Proof. Clearly $f(V)$ is contained in a component U of $M \setminus f(S_f)$. So it suffices to prove that $f(V)$ is open and close in U. It is open because f is local diffeomorphism in V. If $x_n \in V$ for every $n > 0$ and $f(x_n) \to y \in U$, the sequence $\{x_n\}$ must be bounded because f is a proper map. If x is the limit of a subsequence of $\{x_n\}$. Then $f(x) = y$. Note the $x \in \overline{V}$, and $\partial V \subset \widetilde{S_f}$; so $f(x) \in f(S_f)$ which is absurd. If follows that $x \in V$. $\square$

Corollary 1. *If V is a connected component of $M \setminus f^{-1}(f(S_f))$ and $f(V)$ is simply connected, then $f\big|_V : V \to f(V)$ is a diffeomorphism and V is simply connected.*

The next step is to describe the set of critical points of a generic C^3-perturbation of a holomorphic map. Assume that z_0 is a non degenerate critical point of a holomorphic map p (non degenerate means $p''(z_0) \neq 0$).

Proposition 3. *There exists a C^3 neighborhood $\mathcal{U}$ of p and a neighborhood U of z_0 such that if $f \in \mathcal{U} \cap \mathcal{G}$, then $S_f \cap U$ is diffeomorphic to the circle S^1.*

Proof. It is well known that there exists a conformal map φ defined in a neighborhood of z_0 such that $q \circ \varphi = p$ in U, where $q(z) = z^2$. So it suffices to suppose that $p = q$ and the critical point is 0.
As $q(x, y) = (x^2 - y^2, 2xy)$ in real coordinates, observe that

$$Dq_{(x,y)} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} \quad \text{and} \quad detDq = 4x^2 + 4y^2 = H(x, y).$$

Let f be a C^3 perturbation of q. Then $detDf = 4x^2+4y^2+\alpha(x,y) = H(x,y)+\alpha(x,y) = H_1(x,y)$, where α is $\varepsilon - C^2$ close to 0. Now, as the gradient of H_1 is

$$\nabla H_1 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix}$$

then ∇H_1 is a diffeomorphism if the perturbation is small. It follows that H_1 has a unique critical point. Therefore there are just three possibilities for $H_1^{-1}(0)$:
i) $H_1^{-1}(0) = \emptyset$.
ii) $H_1^{-1}(0) = \{c\}$, and c is a critical point.
iii) 0 is a regular value of H_1 and $H_1^{-1}(0)$ is the union of a finite number of copies of S^1.
The first possibility is discarded by corollary 1. The second one is discarded because $f \in \mathcal{G}$ by hypothesis. Then iii) holds; moreover $H_1^{-1}(0)$ has to be only one copy of S^1 because H_1 has only one critical point. $\square$

2.1 Description of the set $\widetilde{S_f}$

The final part of this section gives a description of the image of S_f, thus determining the regions where the number of preimages is constant, and the coverings described in proposition 2.

Proposition 4. *Let f be a small generic C^3-perturbation of $q(z) = z^2$, and assume that $f\big|_{S_f}$ is injective. Then S_f contains three cusp type points and $R^2 \setminus \widetilde{S_f}$ has four bounded simply connected components. These components are simply connected, and each one of them is one to one mapped by f onto the bounded component of $I\!R^2 \setminus f(S_f)$. The restriction of f to the unbounded component of $I\!R^2 \setminus \widetilde{S_f}$ is a double covering of $I\!R^2 \setminus f(S_f)$.*

Proof. As f is small C^3-perturbation of q and it is a proper map, each point in the unbounded component of $I\!R^2 \setminus f(S_f)$ has two preimages. As $f\big|_{S_f}$ is injective, $f(S_f)$ is homeomorphic to S^1. As the images of the set of fold type points is dense in $f(S_f)$, it is clear that each point in the bounded component of $I\!R^2 \setminus f(S_f)$ has 0 or 4 preimages. But $I\!R^2 \setminus \widetilde{S_f}$ has at least one bounded components and these are mapped to bounded component, so it follows that the points in bounded component of $I\!R^2 \setminus f(S_f)$ have four preimages.

Let A be the unbounded component of $I\!R^2 \setminus \widetilde{S_f}$ and let C be the boundary of A. The following properties are satisfied:
i) S_f is contained in the closure of the bounded component of $I\!R^2 \setminus C$.
ii) C is compact and connected. Assume is not connected. If C is not connected

then fundamental group of A contains the free product of two elements. But f is a covering form component onto complement of S_f, whose fundamental group is Z: Absurd.
iii) $C \cup S_f = \widetilde{S_f}$. It is clear that $C \cup S_f \subset \widetilde{S_f}$. Let $x \in \widetilde{S_f}$; if $x \notin S_f$ then f is locally a diffeomorphism at x. It follows that $f(U)$ intersects the unbounded component of $I\!R^2 \setminus f(S_f)$ that for each neighborhood U of x so $x \in C$.
iv) $\widetilde{S_f} \setminus S_f$ is locally an arc, because f is locally invertible in $\widetilde{S_f} \setminus S_f$.
v)The generecity of f implies that every point in S_f is fold type or cusp type. This implies that for every $x \in C \cap S_f$ is holds that x is a cusp type point and S_f and C are tangent at x. On the other hand, f maps each component of $R^2 \setminus A$ onto the bounded component of $I\!R^2 \setminus f(S_f)$; as this one is simply connected, it follows that $C \setminus S_f$ has exactly four components. Then there must be three cusp type point and this proves the proposition. $\square$

Corollary 2. *Let f be a generic C^3 perturbation of a complex polynomial with non degenerate critical points . If $f\big|_{S_f}$ is injective then each component $\mathcal{L}$ of S_f contains exactly three cusp type points and there exists a neighborhood U of $\mathcal{L}$ such that $f^{-1}(f(\mathcal{L})) \cap U$ is mapped onto $f(U)$ like in the previous proposition. (See figure 2).*

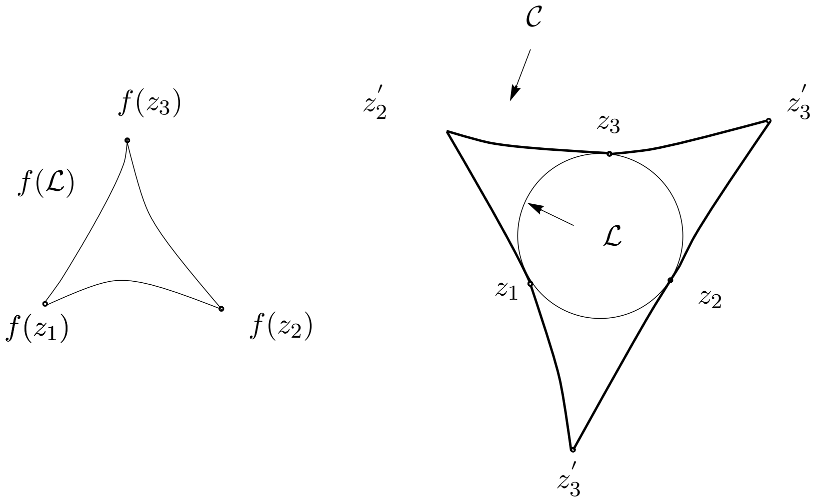
In the hypothesis of both the proposition and the corollary it was included the assumption that the restriction of f to S_f is injective. We do not know examples of perturbation of z^2 such that $f\big|_{S_f}$ not injective. In [DRRV] for example, it was proved that for generic real quadratic polynomials (each coordinates is a quadratic polynomial of x and y) of the plane the restriction of f to S_f is injective. See also [MST1] and [MST2] where some examples are shown of maps f drawing S_f homeomorphic to a circle but $f\big|_{S_f}$ not injective.

3 When all critical points have bounded orbit

In this section the proof of Theorem 1 of the introduction is given. The techniques are also used in subsequent sections and prove in fact a more general result that will be explained at the end of this section.

Observe first if $f \in End^1(R^2)$ is a strong C^1-perturbation of a complex polynomial, then ∞ is an attractor for f. There exists a compact disk K_0 centered at 0 such that the complement K_0^c of K_0 is contained in B_∞ and $f^{-1}(K_0) \subset K_0$. The proof of the first theorem is based on two simple ideas: (1) The nested sequence of successive preimages of K_0 converges to the complement of B_∞, B_∞^c. (2) If K is a connected and simply connected set that contains all the critical values of f, then

Figure 2:



z_1,z_2,z_3 are the cusp points.

$f^{-1}(K)$ is connected and simply connected. The statement in (2) is not necessarily true when there are critical values outside of K (see figure 3) . In subsequent sections this result will be refined.
Begin defining $K_n = f^{-n}(K_0)$. As f is a proper map, K_n is compact for every $n \geq 0$. Given a compact set K, denote by $ext(K)$ the unbounded component of $R^2 \setminus K$ and by $int(K)$ the interior of the complement of $ext(K)$.

Lemma 1. *If K is a compact set, then $ext(f^{-1}(K)) \subset f^{-1}(ext(K))$ and $f^{-1}(int(K)) \subset int(f^{-1}(K))$.*

Proof. Let $x \in ext(f^{-1}(K))$ and α a curve in $ext(f^{-1}(K))$ joining x with ∞; in particular, $\alpha \cap f^{-1}(K) = \emptyset$. As f is proper $f(\alpha)$ joins $f(x)$ with ∞, and since $f(\alpha) \cap K = \emptyset$ it follows that $f(x) \in ext(K)$. The other statement is dual. $\square$

Remark: Figure 3 shows that it is not true in general that $f^{-1}(ext(K)) \subset ext(f^{-1}(K))$.

Lemma 2. *The sequence of compact sets $\{K_n\}$ satisfies the following properties:*
i) $K_{n+1} \subset K_n$, for every $n \in I\!N$.
ii) $ext(K_n) \subset ext(K_{n+1})$, for every $n \in I\!N$.
iii) $ext(K_n) \subset B_\infty$, for every $n \in I\!N$.

iv) $B_\infty^c = \cap_{n\geq 0} K_n$.
v) If B_∞^o *is the immediate basin of* ∞ *(the unbounded component of* B_∞*) then*

$$\bigcup_{n=0}^{\infty} ext(K_n) = B_\infty^o$$

Proof. Parts *i)* and *ii)* are obvious. Part *iii)* is consequence of lemma 1. Part *iv)* follows immediately by invariance of B_∞^c: $f^{-n}(B_\infty^c) = B_\infty^c$ for every $n \in I\!\!N$. To prove *v)*, observe that, as $ext(K_n)$ is connected, *iii)* implies that $\cup_{n\geq 0}^{\infty} ext(K_n) \subset B_\infty^o$. Suppose that the other inclusion does not hold. Then there exists a point x in the boundary of $\cup_{n\geq 0}^{\infty} ext(K_n)$ such that $x \in B_\infty^0$. Let δ be such that $\overline{B(x,\delta)} \subset B_\infty^0$ (where $\overline{B(x,\delta)}$ denotes the closed disc of center x and radius δ). Observe that there exists $n_0 \in I\!\!N$ such that $B(x,\delta) \cap ext(K_n) \neq \emptyset$ for every $n > n_0$ and $B(x,\delta)$ is not contained in $ext(K_n)$. Therefore $B(x,\delta) \cap K_n \neq \emptyset$ for every $n > n_0$ which implies that $f^n(B(x,\delta)) \cap K_0 \neq \emptyset$. On the other hand, there exists $n_1 \in I\!\!N$ such that $f^n(B(x,\delta)) \subset ext(K_0)$ every $n > n_1$ because $\overline{B(x,\delta)} \subset B_\infty^0$ and $\overline{B(x,\delta)}$ is compact. But this implies that $f^n(B(x,\delta)) \cap K_0 = \emptyset$ for every $n > n_1$ which is a contradiction.

□

Lemma 3. *Suppose that* K *is a compact connected and simply connected set such that every critical value of* f *is contained in* K*. Then* $f^{-1}(K)$ *is connected and simply connected.*

Proof. As $f(S_f) \subset K$ then $\widetilde{S_f} \subset f^{-1}(K)$; it follows that $f : I\!\!R^2 \setminus f^{-1}(K) \to I\!\!R^2 \setminus K$ is a covering map. As K is simply connected, $I\!\!R^2 \setminus f^{-1}(K)$ as no bounded component (f maps bounded components to bounded components because the critical point of f are in K). Moreover, as the homomorphism $f_\sharp$ that f induces on fundamental groups ($f_\sharp : \Pi_1(I\!\!R^2 \setminus f^{-1}(K)) \to \Pi_1(I\!\!R^2 \setminus K)$) is injective, then $\Pi_1(I\!\!R^2 \setminus f^{-1}(K))$ is isomorphic to Z. It follows that $f^{-1}(K)$ is connected and simply connected. □

Remark: See figure 3 where it is shown that the hypothesis that the critical values are contained in K is necessary.

The argument used proves in fact a more general result:
Let f be a proper self mapping of R^2. Assume that ∞ is attracting for f and that B_∞ does not contain critical point. Then B_∞^c is simply connected.
Proof of theorem 1: If every critical point has bounded orbit, it suffices to show that K_n is connected and simply connected for all $n \in I\!\!N$, because $B_\infty^c(f) = \cap K_n$

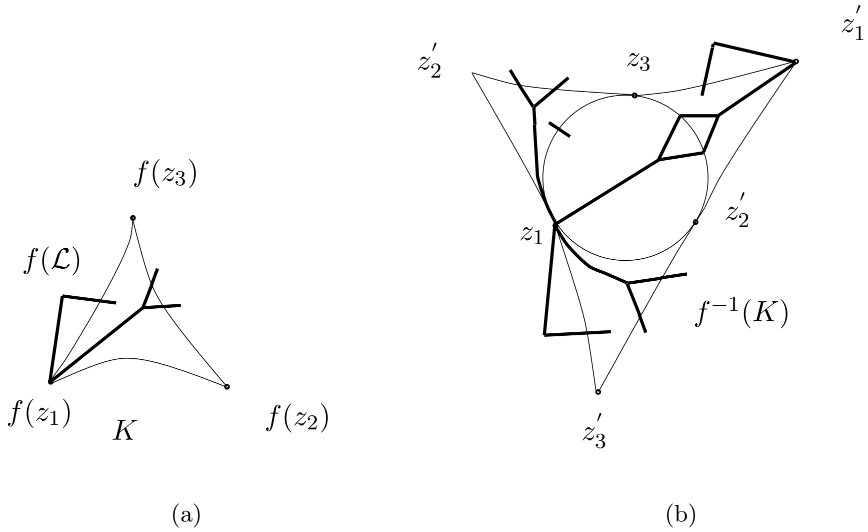


Figure 3:

(lemma 2, iv)). To prove this, proceed by induction: indeed, K_0 is a disc, and as $S_f \subset B^c_\infty(f)$, then S_f and also $f(S_f)$ are contained in K_n for every n. Then apply the previous lemma.

Corollary 3. *Let P be a complex polynomial such that every finite critical point is contained in the basin of a finite attractor. Then there exists a neighborhood $\mathcal{U} \subset End^1(R^2)$ of P such that $B^c_\infty(f)$ is connected and simply connected for every $f \in \mathcal{U}$.*

Proof. This follows by the theorem because in the above case, every critical point of f is contained in the basin of a finite attractor, and so $S_f \subset B^c_\infty(f)$. $\square$

Corollary 4. *Let f be a small C^3-perturbation of a holomorphic polynomial with non degenerate critical points. If $B_\infty \cap S_f \neq \emptyset$ then $B^0_\infty \cap f(S_f) \neq \emptyset$.*

Proof. Let $n_0 \in I\!\!N$ be the first number such that K_{n_0} does not contain $f(S_f)$. Applying the lemma 3 K_{n_0} is simply connected. Therefore $f(S_f) \setminus K_{n_0} \subset B^0_\infty$. $\square$

4 Preimages of connected sets

In this section it is assumed that B_∞ contains critical points. In this case there exists some $n > 0$ such that $K_n = f^{-n}(K_0)$ (defined in the previous section) does not contain the set of critical values. It becomes important to determine when the preimage of a connected set is connected. Note in figure 3 that if a connected set K has disconnected intersection with the interior of $f(S_f)$, then $f^{-1}(K)$ is not connected, but only one of the components of $f^{-1}(K)$, say K_1, is surjective, in the sense that $f(K_1) = K$.
Observe that for a polynomial P it is easy to determine the number of components of $P^{-1}(K)$ for any connected set K: it dependes on the relative location of the critical values. For example, if P is a quadratic polynomial and K is bounded and connected, then $P^{-1}(K)$ is connected if and only if the critical value of P does not belong to the unbounded component of K^c, and $P^{-1}(K)$ has two components otherwise. The remaining of this section is devoted to determine a similar result for perturbation of quadratic polynomials.

It will be assumed throughout this section that f is a generic C^3 perturbation of some quadratic polynomial and that the restriction of f to S_f is injective.

Definition 4.1. *A quadruple (Δ, z_1, z_2, z_3) will be called a triangle if*
-Δ is homeomorphic to the disc and its boundary homeomorphic to S^1.
- z_1, z_2, z_3 are different points in the boundary of Δ.

Whenever no confusion is possible, say Δ is a triangle without specifying the tree points. The points z_1, z_2, z_3 are called the vertices of Δ and the three closed curves $[z_i, z_j]$ in the boundary of Δ are called the sides of Δ.

Definition 4.2. *A subset K of the plane connects a triangle Δ if $K \cap \Delta$ contains a component which intersects the three sides of Δ.*

The following result, of intuitive meaning, is central in the development of the techniques.

Lemma 4. *Let K be a compact connected subset of R^2 and Δ a triangle. Then exactly one of following conditions hold:*

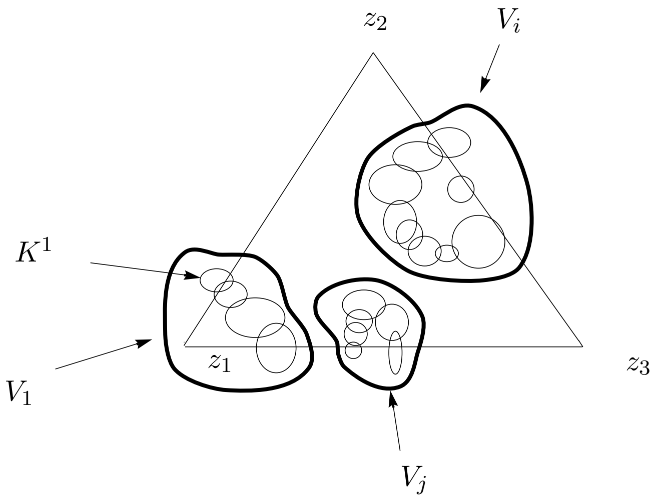
- *K connects Δ.*
- *K^c connects Δ.*

Proof. The proof is divided in several claims.
Claim 1 : The lemma is true if K is a finite union of discs. Suppose first that

K connects Δ. Then (as K is finite union of discs) there exists a simple curve $\alpha : [0,1] \to K \cap \Delta$ such that α intersects each side of Δ in exactly one point. It is clear that $\Delta \setminus \alpha$ has three components, no one of which intersects the three sides of Δ. Hence K^c does not connect Δ. Suppose now that K does not connect Δ. Then there exists a finite disjoint collection of regions, diffeomorphic to closed discs, $\mathcal{V} = \{V_i\}$ satisfying the following properties:

1. There exist at most three elements of $\mathcal{V}$ that intersect more than one side of Δ and no one of them intersect the three sides.
2. If $V_i \in \mathcal{V}$ intersects two sides, then V_i contains the common vertex of these two sides.
3. The intersection $V_i \cap \partial\Delta$ is connected for every $V_i \in \mathcal{V}$.
4. $K \cap \Delta \subset \cup V_i$.

Figure 4:



Observe that the quotient space $\Delta/\sim$ where $x \sim y$ iff there exists $V_i \in \mathcal{V}$ containing both x and y is again a triangle (the vertices are the three V_i that contain a vertex of Δ). As no point of this triangle belongs to the quotient projection of K, it follows that K^c connects $\Delta/\sim$, but this implies that it connects Δ. To prove de existence of such $\mathcal{V}$ suppose first that K^1 is a component of $K \cap \Delta$ containing points of $[z_1, z_2]$ and $[z_1, z_3]$; then there exists a simple closed curve $\gamma \subset K^1$ joining $[z_1, z_2]$ with $[z_1, z_3]$ (here it is used that K is finite union of discs). Note that

$\Delta \setminus \gamma$ has two connected components, one of them contains z_1. Define $\widetilde{V_1}$ as the set of points x such that there exists a curve γ_x as above leaving x and z_1 at the same side. Analogously define $\widetilde{V_2}$ and $\widetilde{V_3}$. These sets are open, connected, simply connected and disjoint (because the contrary assumption implies that K connects). Next define disjoints regions V_1, V_2, V_3 containing $\widetilde{V_1}$, $\widetilde{V_2}$ and $\widetilde{V_3}$ respectively.

Again using the fact that K is a finite union of discs one can easily see that for each side there exists a region V_j such that V_j contains all the components of $K \cap \Delta$ that intersect only this side; this can be done in such a way that $\mathcal{V}$ contains (at most) six disjoint elements.

Claim 2: There exists a nested sequence of compacts sets K_n such that $K = \bigcap K_n$ and each K_n is a finite union of discs. For the proof, just take K_n from a cover of K with discs of radio $1/n$.

Claim 3: If every K_n connects Δ then K connects Δ. As K_1 connects, there exists a component K_1^1 of K_1 that connects Δ and this is unique because the theorem is known for finite union of discs. For the same reason there exits a component K_2^1 of $K_2 \subset K_1$ such that K_2^1 connects Δ. Obviously $K_2^1 \subset K_1^1$. By induction, there exits a nested sequence K_n^1 of compact connected sets each one of which connects Δ. Then the intersection $\bigcap K_n^1 = K^1$ is connected, is contained in K, and connects Δ.

Now the proof the theorem finishes a follows:

If K does not connect Δ, be claim 3 there exists n such that K_n does not connect Δ; by claim 1, K_n^c connects Δ; so $K^c \supset K_n^c$ also connects Δ.

If K connects Δ then K_n connects Δ for every n. This implies, again by claim 1, that for every n $U_n = K_n^c$ does not connect Δ. But each U_n is open and the sequence is increasing; so a compact set contained in $U = \bigcup_{n \geq 0} U_n$ must be contained in some U_n. If U connects Δ, then there is a compact subset of U that connects Δ, but this is absurd. It follows that $U = K^c$ does not connect Δ. $\square$

Start assuming that $f \in \mathcal{G}$, with f a small C^3-perturbation of z^2+c. Recall that in this case $f(S_f)$ is closed curve with a finite number of transverse intersections, each one of which contains no cusp.

Definition 4.3. *The set A is a surjective component of $f^{-1}(B)$ if A is a connected component of $f^{-1}(B)$ and $f(A) = B$.*

Proposition 5. *Let f be a generic map such that S_f is diffeomorphic to the circle S^1. If K is a compact connected set and intersects $ext(f(S_f))$, then $f^{-1}(K)$ has at most two surjective components. The other components of $f^{-1}(K)$ are contained in $\overline{int(\widetilde{S_f})}$.*

Recall $\widetilde{S_f} = f^{-1}(f(S_f))$. This set is small if f is a small perturbation of complex polynomial. A set is called small if it is contained in $\overline{int(\widetilde{S_f})}$. Some previous results will be needed to prove this proposition

Lemma 5. *Let $\{K_n\}$ be a nested sequence of compact connected sets and $K = \cap K_n$. If for every n there exists a surjective component K^1_{n+1} of $f^{-1}(K_n)$ such that $K^1_{n+1} \subset K^1_n$, then there exists a surjective component K^1 of $f^{-1}(K)$.*

Proof. $K = \cap K_n$ is compact and connected. If $K^1 = \cap K^1_n$ then K^1 is a compact and connected subset of $f^{-1}(K)$. Let $x \in K$; for every n there exists $y_n \in K^1_n$ such that $f(y_n) = x$. If y is the limit of a convergent subsequence of $\{y_n\}$, then $y \in K^1_n$ for every n whence $y \in K^1$. By continuity $f(y) = x$. This implies the lemma. □

Lemma 6. *Let α be a simple open curve, transverse to $f(S_f)$, not containing images of cusps and intersecting $ext(f(S_f))$. Then $f^{-1}(\alpha)$ has a finite number of connected components. The surjective components of $f^{-1}(\alpha)$ are those that intersect $ext(\widetilde{S_f})$. Therefore $f^{-1}(\alpha)$ has at most two surjective components.*

Proof. First observe that the hypothesis imply that $f^{-1}(\alpha)$ is a finite union of simple curves. Let β be a component of $f^{-1}(\alpha)$ that intersects $ext(\widetilde{S_f})$. Let $x \in \beta \cap ext(\widetilde{S_f})$. As β is a simple curve, fix a parametrization of $\beta : [0, 1] \to R^2$ and let t_0 be such that $\beta(t_0) = x$. If β is contained in $ext(\widetilde{S_f})$ then is clear that $f(\beta) = \alpha$ and the result follows because f is a covering map form $ext(\widetilde{S_f})$ in $ext(f(S_f))$. Suppose that there exists $t_1 > t_0$ such that $\beta(t_1) \in \partial\widetilde{S_f}$. Without loss of generality it can be also assumed that $f(\beta(t_i)) = \alpha(s_i)$, $i = 0, 1$ with $s_1 > s_0$. Let $s(t)$ be a continuous function of t such that $f(\beta(t)) = \alpha(s(t))$. It is claimed now that for every $t > t_1$ $s(t) \geq s(t_1) = s_1$. Observe that $s(t)$ is increasing whenever $\beta(t)$ does not belong to $int(S_f)$. If the claim is not true, then $f(\beta(t))$ must cross $\alpha(s_1)$ at a point $t_2 > t_1$ and with s decreasing in a neighborhood of t_2, which implies that $\beta(t_2) \in S_f$; this is absurd because $f(int(S_f)) \subset int(f(S_f))$. This proves the claim. The claim implies that β is not closed and so the image of the extreme points of β under f must be the extreme points of α. The last assertion is now obvious since every point in $ext(f(S_f))$ has two preimages. This implies the lemma. □

Proof of proposition 5 : Using lemma 5, if suffices to prove the proposition for K equal a finite union of discs. In this case K is arcwise connected and the assertion follows from the lemma 6.

Next assume that $f\big|_{S_f}$ is injective. Then $f(S_f)$ is homeomorphic to S^1, and it contains three cusp type points as proved in proposition 4 and its corollary. Form now on Δ will be the closure of the bounded component of $R^2 \setminus f(S_f)$ with the images of the cusps as vertices.

Obs: If $C \subset \Delta$ is connected and connects Δ then $f^{-1}(C)$ is connected. The main result of this section is

Proposition 6. *Let K be a compact connected set such that $K \cap ext(\Delta) \neq \emptyset$.*
a) If $ext(K)$ does not connect Δ, then $f^{-1}(K)$ has only one surjective component.
b) If $ext(K)$ connects Δ, then $f^{-1}(K)$ has two surjective components.
c) Every non surjective component of $f^{-1}(K)$ is contained in $\overline{int(\widetilde{S_f})}$.

The proof of this proposition needs the following result:

Lemma 7. *If the connected set K is a finite union of discs that connects Δ, then there exists a unique component K^1 of $f^{-1}(K)$ such that $f(K^1) = K$.*

Proof. Let C be the component of $K \cap \Delta$ that connects Δ.

If K^1 is a surjective component of $f^{-1}(K)$ then K^1 must contain the connected set $f^{-1}(C)$. This proves the uniqueness. Now let $x \in K$ and α be a curve in K that contains x and intersects C, having also a finite intersection with $\partial\Delta$. By the previous lemma 6, there is a curve contained in $f^{-1}(\alpha)$ that contains x and intersects $f^{-1}(C)$. So the component of $f^{-1}(K)$ that contains $f^{-1}(C)$ is surjective. □

Proof of proposition 6:
(**a**) Suppose first that K connects Δ. Construct a nested sequence of compacts sets K_n as in the proof of lemma 4 (claim 2). As each K_n is a finite union of discs, the lemma 7 implies that there exists K_n^1, component of $f^{-1}(K_n)$, such that $f(K_n^1) = K_n$ and the sequence K_n^1 is decreasing. Now apply lemma 5 to obtain K^1. The set K^1 is determined by the condition $K^1 \supset f^{-1}(C)$ where C is the component of $K \cap \Delta$ that connects Δ. This shows that K^1 is unique.

Suppose now that K does not connect Δ (and neither $ext(K)$ connects Δ). Without loss of generality it can be also assumed that K is a finite union of discs. Then there exists a simple closed curve α contained in K, containing a point $x \in ext(\Delta)$ and whose interior contains a connected set C that connects Δ. It is claimed now that $f^{-1}(\alpha)$ has only one surjective component. To prove this, define $\widetilde{K} = \overline{int(\alpha)}$; as $\widetilde{K}$ connects, the first part implies that $f^{-1}(\widetilde{K})$ has only one surjective component, denoted $\widetilde{K_1}$. Let δ be the boundary of the bounded component of $R^2 \setminus \widetilde{K}$. Observe that this is a connected set. Moreover, $f(\delta) \subset \alpha$ and the two preimages of x are contained in δ. Then, using lemma 6, it follows that $f^{-1}(\alpha)$

has only one surjective component and the claim is proved. Now let $y \in K$. Then there exists a curve $\beta \subset K$ joining y with a point $z \in \alpha$. Now using lemma 6 and the claim above the proof of (a) is concluded.
(**b**) As $ext(K)$ connects Δ, there exists an injective curve $\alpha : [0, +\infty) \to R^2$ such that $\alpha \subset ext(K)$, $\alpha([0,1))$ connects Δ and $|\alpha(t)| \to \infty$ as $t \to +\infty$. It is easily seen that $f^{-1}(\alpha)$ disconnects the plane in the sense that $R^2 \setminus f^{-1}(\alpha)$ has two connected components H_1 and H_2. From proposition 5, it follows that $f^{-1}(K)$ contains exactly one surjective component K^i in each H^i such that $f(K^i) = K$.
(**c**) Is immediate consequence of lemma 6.
This sequence of results gave a topological insight into the structure of the preimages of a set. The next results is the first conclusion of the results previously obtained.

Proof of Theorem 2:

If the critical point of h is fixed and f is small generic C^3-perturbation of h then, by proposition 3, $B^c_\infty(f)$ is simply connected. Then the critical point of h is not fixed and then $S_f \cap f(S_f) = \emptyset$ and the fixed points of f belong to $ext(\Delta)$ (because f is small perturbation of h).
As f has fixed points B^c_∞ has at least one invariant connected component. Note that as f is a generic C^3 perturbation of a quadratic polynomial, its set of critical point is homeomorphic to S^1.
Recall from the previous section that there exists a sequence $K_n = f^{-n}(K_0)$ of compact sets such that K_0 is a disc. The properties of this sequence are collected in lemma 2. Next define a sequence $\{K^1_j\}$ as follows (this family can be either finite or infinite). Let $K^1_0 = K_0$; as K_0 is a big disc, so $f(S_f) \subset K_0$ and this implies that $f^{-1}(K^1_0)$ has exactly one component K^1_1 such that $f(K^1_1) = K^1_0$ (see lemma 3).

If $f^{-1}(K^1_1)$ has two surjective components the construction is stopped; otherwise, use proposition 5 to define K^1_2 as the unique component of $f^{-1}(K^1_1)$ such that $f(K^1_2) = K^1_1$. Again, using proposition 5 and that $K_2 \subset K_1$ it comes that $K^1_2 \subset K^1_1 \subset K^1_0$. If $f^{-1}(K^1_2)$ has two surjective components the construction is stopped. Otherwise, and analogous to the first step, there exists $K^1_3 \subset K^1_2$ with K^1_3 the unique surjective component of $f^{-1}(K_3)$. An obvious induction argument gives a nested sequence of connected set $\{K^1_n\}$.
Case a) The family $\{K^1_n\}$ is infinite.
If p and r are the fixed points of f, using the proposition 5, it follows that the sets $f^{-1}(p) = \{p, p'\}$ and $f^{-1}(r) = \{r, r'\}$ are contained in K^1_n, for all $n \in I\!N$ (because p and r are not in $\overline{int(f(S_f))}$). Then $M = \cap K^1_n$ is an invariant component of B^c_∞ and the fixed points belong to it. Suppose that there exists another invariant component

N of B^c_∞. As $M \neq N$ there exists $n_0 \in I\!N$ such that N is contained in a component $K^i_{n_0}$ of K_{n_0} different of $K^1_{n_0}$. By the construction of $\{K_n\}$ and the proposition 5, there exists $k \in I\!N$ such that $f^k(K^i_{n_0}) \subset \overline{int((\widetilde{S_f})}$. Then $N \subset \overline{int((\widetilde{S_f})}$ and $f(N) = N \subset \overline{int(f(S_f))}$, which is absurd because $\overline{int(\widetilde{S_f})} \cap \overline{int(f(S_f))} = \emptyset$ if f is a sufficiently small perturbation.
Case b) The family $\{K^1_n\}$ is finite.
In this case, there exists $n_0 \in I\!N$ such that $f^{-1}(K^1_{n_0})$ has two surjective components H_1 and H_2 contained in K_{n_0}. These two preimages are contained in $K^1_{n_0}$ by the same argument used in the case (a). As $H_i \subset K^1_{n_0}$ for $i = 1, 2$, it follows each $f^{-1}(H_i)$ has two connected components contained in $H_1 \cup H_2$ and whose images give the corresponding H_i. It follows that $f^{-n}(H_i)$ has at least 2^n components. The construction follows standard arguments giving uncountable many components of B^c_∞.

Corollary 5. *Let $f \in \mathcal{G}$ be a small C^3-perturbation of a quadratic polynomial such that the restriction of f to S_f is injective. If the immediate basin of ∞ connects $\Delta = \overline{intf(S_f)}$ then the complement of B_∞ has uncountably many components.*

Proof. It is sufficient to prove that the family $\{K^1_n\}$ is finite.

If the family is infinite then, by proposition 6 (b), $ext(K^1_n)$ does not connect for every $n \in I\!N$. As B^0_∞ connects so there exists $n_0 \in I\!N$ such that $ext(K_n)$ connects. Therefore $ext(K^1_n)$ connects and this is a contradiction. □

Corollary 6. *Let $f \in \mathcal{G}$ be a small C^3-perturbation of a quadratic polynomial. If M is an invariant component of B^c_∞ then M contains a fixed point.*

Proof. If the family $\{K^1_n\}$ is infinite then there exists only one invariant component and this component contain both fixed points. If the family is finite then there exists two invariant components, each one of which contains a fixed point. □

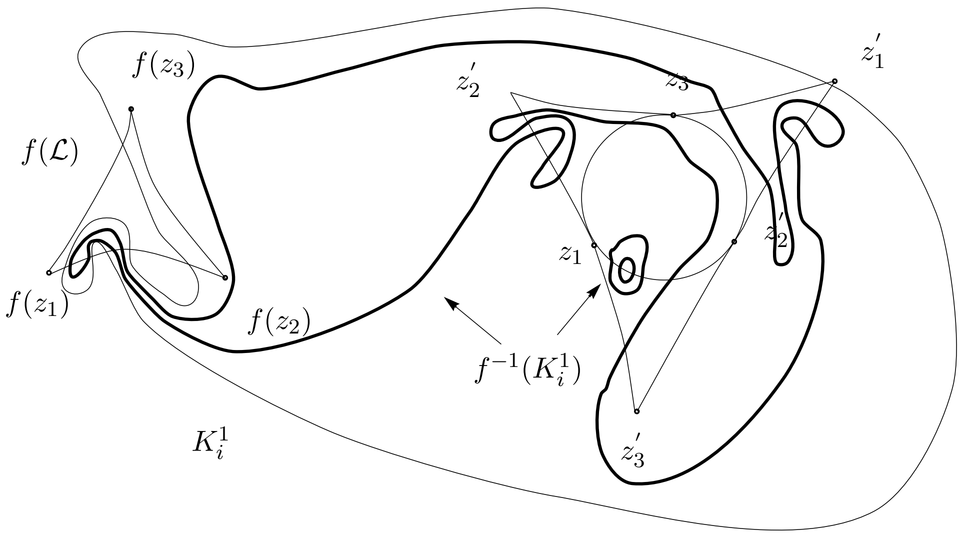
In the last section it will be shown that the condition in the hypothesis of corollary 5 do not imply that B^c_∞ is totally disconnected.

5 Invariant components of B^c_∞

The attention is focused on the determination of the existence of large components in B^c_∞, (that is a component that intersects both S_f and $f(S_f)$), and in the study of the invariant components of the complement of B_∞.

The construction of the previous section will be used; start with a quadratic polynomial $h_c(z) = z^2 + c$ such that the critical point 0 does not belong to $B_\infty(h_c)$.

Figure 5:



(Otherwise, the Julia set $\mathcal{J}_c$ of h_c is totally disconnected and the same holds for every small C^1 perturbation). Therefore the Julia set of h_c is connected. For each $c \in C$, let K_c denote the filled-in Julia set of h_c, i.e., the set of points having bounded forward orbit. In this case there exists a conformal map $\Phi_c : \{|z| > 1\} \to S^2 \setminus K_c$, a conjugacy between $q(z) = z^2$ and h_c such that $h_c\Phi_c = \Phi_c q$. It is also assumed that the fixed points of h_c are repellors. It follows that $h_c(0) \neq 0$ and then $S_f \cap f(S_f) = \emptyset$ for every small perturbation f of h_c.

The main ingredient in the proof of theorem 3 will be the following result of Douady and Hubbard (see [S]). Let $R(\theta) = \Phi_c(\{re^{2\pi i\theta} : r > 1\}$ for $0 \leq \theta < 1$. Each $R(\theta)$ is called the external ray of angle θ for h_c.

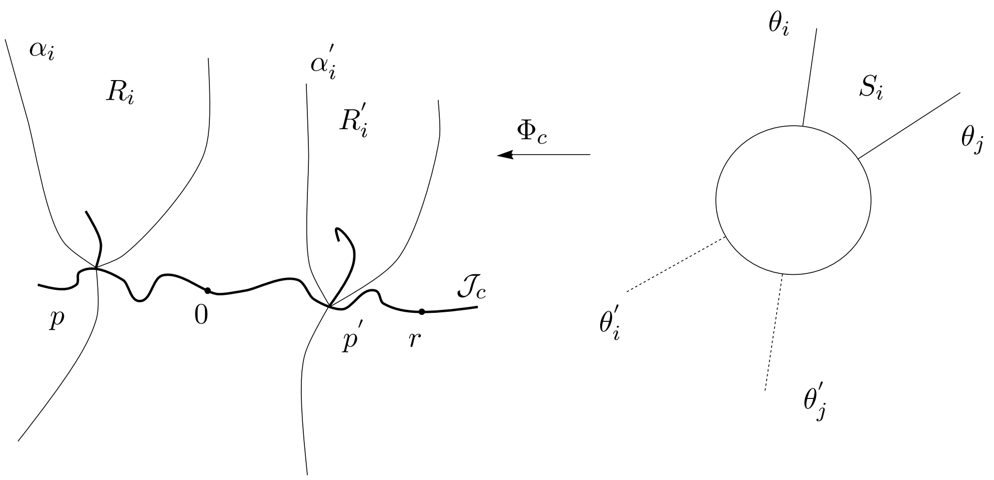
Theorem 5.1. *(Douady-Hubbard) If θ is rational then $R(\theta)$ lands at a point of the Julia set of h, this means that $\lim_{r\to 1^+} \Phi_c(re^{2\pi i\theta})$ exists and belongs to $\mathcal{J}_c$. This point is periodic or eventually periodic. Conversely, every repelling or parabolic periodic point of the Julia set of h_c is the landing point of a finite number of external rays, all with rational angles.*

Denote by p and r the fixed points of h_c. The fixed external ray $R(0)$ lands at a fixed point of h_c; let it be r. There is also an external ray that is not fixed, landing at p. It is a periodic orbit $\{\theta_1, \ldots, \theta_n\}$ of q such that the external rays $\alpha_i = R(\theta_i)$ land at p. Then $R^2 \setminus \cup_{i=1}^n \alpha_i$ is the union of n regions $\{R_i\}$; on the other side, there exist n regions S_i which are determined in the complement of the unit disc by the rays θ_i. It is clear that each of the regions R_i correspond to a unique S_i. It is not true that the image of R_i under Φ_c^{-1} is S_i, because Φ_c^{-1} is defined in $R_i \setminus K_c$. However it makes sense to say that a point $x \in K_c \setminus \{p\}$ correspond to a component S_i, because such a point is contained in a unique region R_i. Denoting by $\mathcal{J}_i = \mathcal{J} \cap R_i$, it comes that $\mathcal{J} \setminus \{p\} = \cup_{i=1}^{n} \mathcal{J}_i$.

Observe that $-p$ is the other preimage of p and that the external rays α_i have preimages $\alpha_i^{'}$ landing at $-p$. Denote by $R_i^{'}$ the components of the complement of the union of the $\alpha_i^{'}$. Denote also by $\theta_i^{'}$ the preimages under q of the angles θ_i and by $S_i^{'}$ the components of the complement of the rays $\theta_i^{'}$.

Lemma 8. *The following statements hold for h_c:*
(a) The critical point 0 of h_c and its image $h_c(0)$ cannot belong to the same $\mathcal{J}_i$.
(b) The points r and 0 belong to the same R_i, but to different $R_i^{'}$, and the points p and r belong to different $R_i^{'}$.
(c) Let $R_{i_r}^{'}$ the component of $R^2 \setminus \cup \alpha_i^{'}$ which contains r. If $CL = \overline{\{h_c^n(0) : n \in I\!N\}}$ is contained in $(R_{i_r}^{'})^c$, then there exists a finite set $\Lambda \subset B_\infty^c(h_c)$ such that every point $x \in B_\infty^c(h_c) \setminus \Lambda$, in $R_{i_r}^{'}$ leaves this component under iteration of h_c (i.e. there exists n such that $h_c^n(x) \notin R_{i_r}^{'}$).

Figure 6:



Proof. (a) Observe that $q^{-1}(S_i)$ has two surjective components, each one of which is contained in some S_j. Note also that $q^{-1}(S_i) \subset S_i$ is false, because this would imply

that q has two fixed rays in S_i. Note also that simple arguments of connectivity imply that the same applies to the regions R_i (even though the conjugacy does not extend from R_i to S_i).
Suppose by contradiction that 0 and $h_c(0)$ belong to the same component R_i. As 0 is the unique preimage of $h_c(0)$ it follows that $h_c^{-1}(R_i) \subset R_i$; but this is not possible because it implies that $q^{-1}(S_i) \subset S_i$, where $S_i = \Phi_c^{-1}(R_i \setminus K_c)$.
(b) There exists at most one S_i containing both preimages of another component (such an S_i must correspond to an angle greater than π). This actually occurs in the case under consideration, because the component that correspond to $h_c(0)$ must have both preimages in the component S_{i_0} that correspond to 0. Now observe that as the length of S_{i_0} is greater than π, it contains at least one component of each $q^{-1}(S_j)$ for $j = 1, ..., n$. In particular, the fixed ray of q belongs to this component because $q^{-1}(S_{i_0})$ has a surjective component contained in S_{i_0}. It follows that r and 0 belong to the same component R_{i_0} of $R^2 \setminus \cup \alpha_i$.
Observe also that one of the components of the preimage of S_{i_0} (the one containing the fixed ray) is contained in S_{i_0} but 0 does not correspond to this component by part (a). It follows that 0 and r belong to different components $R_i^{'}$. Now using that p and $-p$ are symmetric (so p and 0 belong to the same $R_i^{'}$), it follows that p and r belong to different $R_i^{'}$.
(c) Let $\Lambda = \{x \in R_{i_r}^{'} \cap B_\infty^c(h_c) : h_c^n(x) \in R_{i_r}^{'}, \forall n \in I\!N\}$. Observe that Λ is a compact set and is contained in $R_{i_r}^{'}$. Then $CL \cap \Lambda = \emptyset$. This implies that Λ is a hyperbolic set (see [MS]) and $h_c : \Lambda \to \Lambda$ is bijective, therefore Λ is a finite set.

$\square$

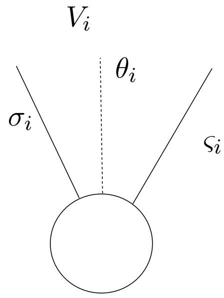
It will be proved next that there exists a sequence of open sets G_k such that

$$\bigcup_{i=1}^{n} \alpha_i = \bigcap_{k \geq 1} G_k \quad \text{and} \quad \overline{G_{k+1}} \subset G_k \quad \text{for every} \quad k \geq 0.$$

Let $B(p, \delta)$ be the disc centered at p, with radius δ small enough such that $\overline{h_c^{-1}(B(p,r)) \cap B(p,r)} \subset B(p,r)$ for every $r \leq \delta$. For $1 \leq i \leq n$ let V_i be a sector containing the ray θ_i. Denote by σ_i and ς_i the sides of V_i (see figure 7) and $V = \cup_{i=1}^n V_i$. It can V_i be taken such that:
i) $\overline{q^{-1}(V) \cap V} \subset V$.
ii) The end point of $\Phi_c(\sigma_i)$ and $\Phi_c(\varsigma_i)$ belongs in $B(p, \delta)$.
If $G_0 = \Phi_c(V) \cup B(p, \delta)$, then it is clear that $h_c^{-1}(G_0) \cap G_0 = G_1$ satisfies $\overline{G_1} \subset G_0$. The claim follows easily by induction, defining $G_k = h_c^{-1}(G_{k-1}) \cap G_{k-1}$.

Now the initial map h_c will be perturbed. Suppose that f is a map C^1 close to h_c such that the closure of $f^{-1}(G_0) \cap G_0 = \widetilde{G_1}$ is contained in G_0. It is then clear that there exists a sequence $\widetilde{G_k}$ such that $f(\widetilde{G_{k+1}}) = \widetilde{G_k}$ and $\overline{\widetilde{G_{k+1}}} \subset \widetilde{G_k}$. Let

Figure 7:



$C = \cap\widetilde{G_k}$; then C is invariant under f and connected and contains p_f the analytic continuation of p. The main property of C is that $C \setminus p_f$ is contained in $B_\infty(f)$: Indeed, this is trivial if $x \notin B(p,\delta)$; when $x \in B(p,\delta)$ there exists m such that $f^m(x) \notin B(p,\delta)$. By invariance of C and $B_\infty(f)$ the assertion follows. Thus the following result was completely proved:

Lemma 9. *For every small C^1 perturbation f of h_c there exists a connected set C such that:*
a) C separates the plane.
b) $p_f \in C$.
c) $C \setminus \{p_f\} \subset B_\infty(f)$.
d) C is invariant.

Now denote by $R_1, ..., R_n$ the components of the complement of C and by $R_1^{'}, ..., R_n^{'}$ the components of the complement of $C^{'}$, where $C^{'}$ is $f^{-1}(C) \setminus C$. The same conclusions of the lemma 8 hold for f.

Corollary 7. *If f is small perturbation of h_c:*
a) S_f and $f(S_f)$ belong to different component R.
b) p_f and r_f belong to different $R^{'}$.

As $h_c(z) = z^2 + c$ has no attracting fixed point, it is clear that $c = h_c(0) \neq 0$. If f is C^1 close to h_c, then $S_f \cap f(S_f) = \emptyset$. Thus a way of saying that a connected set M is large is to prove that it intersect both S_f and $f(S_f)$.

Definition 5.1. *A connected set M is large for f if $M \cap S_f \neq \emptyset$ and $M \cap f(S_f) \neq \emptyset$.*

Proof of Theorem 3:

a) By lemma 8, the points 0 and $h_c(0)$ belong to different components of $\mathcal{J}\backslash\{p\}$, where $\mathcal{J}$ is the Julia set of h_c and p its fixed point. The same holds for the map f if $\mathcal{U}^1$ is sufficiently small. Therefore, if M is a component of $B^c_\infty(f)$ that intersects both S_f and $f(S_f)$, then $p_f \in M$,(because by lema 9 M cannot intersect C outside p_f). This implies the uniqueness of M. Also $f(M)$ is connected, contains p_f and is contained in $B^c_\infty(f)$ so $f(M) \subset M$. The last assertion follows form proposition 6

b) From corollary 6 it follows that there exists at most two invariant components of $B_\infty(f)$, each one of which contains a fixed point. Asume now that there exists $M_{p_f} \neq M_{r_f}$ two invariant components of $B^c_\infty(f)$ such that M_{p_f} contains p_f and M_{r_f} contains r_f. We are going to prove that $M_{r_f} = \{r_f\}$.
That $p'_f \notin M_{r_f}$ is obvious since $f(p'_f) = p_f$ and $M_{p_f} \cap M_{r_f} = \emptyset$. It follows that the whole M_{r_f} is contained in the component R'_j of $R^2\backslash C'$ that contains r_f. Let V be a neighborhood of r such that for every perturbation of h_c, the analytic continuation of this fixed point r_f is contained in V and $f\big|_V$ is conjugate to its linear part. This is used only to assert that if $M_{r_f} \subset V$ then $M_{r_f} = \{r_f\}$.

If $CL = \overline{\{h^n_c(0) : n \in I\!N\}}$ is contained in $(R'_i)^c$, then using the lemma 8 (c), we have that Λ is a hyperbolic finite set. This implies that M_{r_f} is contained in V, so $M_{r_f} = \{r_f\}$. If CL is not contained in $(R'_i)^c$, so there exists $n_0 \in I\!N$ such that that $h_c(0)$ and $h^{n_0}_c(0)$ belong to different components R'. Thus if f is close to h_c then $f(S_f)$ and $f^{n_0}(S_f)$ are contained in different components of $R^2 \setminus (C')$. As $f(M) \subset M$ and intersects both $f(S_f)$ and $f^{n_0}(S_f)$, it follows that M contains p'_f, and then K^1_n has only one surjective preimage for every n. This implies that in fact $M_{p_f} = M_{r_f}$.

Corollary 8. *Let f be a C^3 perturbation of $h_c(z) = z^2 + c$ with $c \in (-2, c_0)$ (where $c_0^2 + c_0 + (1 - \sqrt{1-4c_0})/2) = 0$ and $c_0 \simeq -3/2$). If $B^c_\infty(f)$ has one large component M then M is the unique invariant component.*

Proof. The condition on c implies that $h_c(0)$ and $h^2_c(0)$ belong to different components R'. Thus if f is close to h_c then $f(S_f)$ and $f^2(S_f)$ are contained in different components of $R^2 \setminus (C')$. As $f(M) \subset M$ and intersects both $f(S_f)$ and $f^2(S_f)$, follows that M contains p'_f, and then K^1_n has only one surjective preimagen for every n. This implies, that in fact $M_{p_f} = M_{r_f}$. □

Example : It will be shown now that there exists a map f, perturbation of $z^2 - 2$ such that:

- The set of critical points of f is connected and $f\big|_{S_f}$ is one to one.
- The immediate basin of ∞ connects $f(S_f)$.

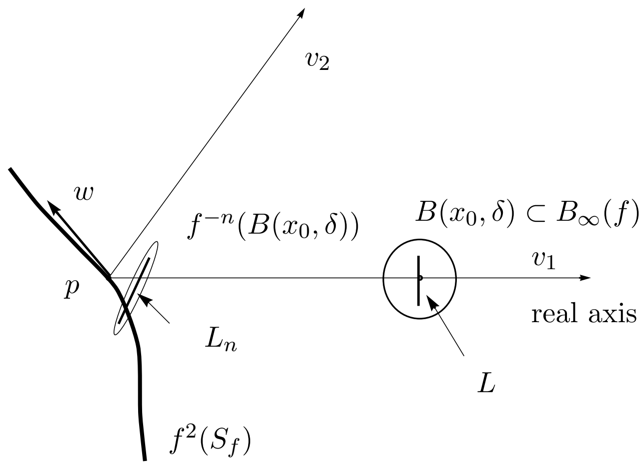
- The complement of B_∞ has uncountably many components but is not totally disconnected, in fact, it contains a hyperbolic periodic point which is not a repellor.

This map will be found near the family:

$$f_{(\lambda,\mu,\varepsilon)}(x,y) = (x^2 - y^2 - 2 + \lambda y + \mu, (2-\varepsilon)xy).$$

The set of critical point is $\mathcal{L} = \{(x,y) : \; x^2 + y^2 - (\lambda/2)y = 0\}$. The cusp are $(0,0)$ and $(\pm\sqrt{3}\lambda/8, 3\lambda/8)$. One of the fixed point of f is $(p,0)$, close to $(2,0)$; note that as $f(x,0) = (x^2 - 2 + \mu, 0)$ then $\{(x,0) : |x| > p\}$ is contained in $B_\infty(f)$. Moreover, as μ will be negative, then the intersection of $B^c_\infty(f)$ with the real axis will be a Cantor set. It is easy to see that for every λ, $(0,\lambda/2) \in S_f$ and that μ can be chosen negative in such way that $f^2(0,\lambda/2) = (p,0)$.
It is claimed now that exists $\varepsilon > 0$ such that $B^0_\infty(f)$ connects $f(S_f)$; in fact it will be proved that it connects $f^2(S_f)$, which is equivalent to the above. A simple calculation shows that

Figure 8:



v_1,v_2 are the eigenvectors .

$$w = \begin{pmatrix} \lambda \\ -(2-\varepsilon)p \end{pmatrix}$$

is tangent to $f^2(S_f)$ at the point $(p, 0)$. If follows that a eigenvectors of

$$Df_{(p,0)} = \begin{pmatrix} 2p & \lambda \\ 0 & (2-\varepsilon)p \end{pmatrix}$$

are $(1, 0)$ associated to the eigenvalue $2p$ and $(-\lambda, \varepsilon p))$ associated to the eigenvalue $(2-\varepsilon)p < 2p$. Choose ε so that w is not an eigenvector of $Df_{(r,0)}$.

Thus the situation is as in the figure 8. Now the claim can easily be proved. As was noted above, B_∞^0 contains $\{(x, 0) : |x| > p\}$. This implies that the image $f^2(0, 0)$ of the cusp $(0, 0)$ belongs to B_∞^0. So it remains to prove that the opposite side of the triangle $f^2(S_f)$ can be connected to $f^2(0, 0)$ within $B_\infty^0 \cap f(\Delta)$. To prove this take a small segment L contained in B_∞ and transverse to the real axis at a point $(x_0, 0)$ with $x_0 > p$. There is a sequence L_n of preimages of L converging to p, and as the eigenvector $(-\lambda, \varepsilon p)$ is associated to the weak eigenvalue of $Df_{(p,0)}$, it follows that the tangent to L_n is close to this direction when n is large. Therefore L_n intersects the side of $f^2(S_f)$ that contains $(p, 0)$. This proves the claim. So B_∞^0 connects the triangle $f(S_f)$ and it follows that B_∞^c has uncountably many components (by corollary 5). On the other hand, recall that a critical point $(0, \lambda/2)$ is preperiodic, namely its second image is $(p, 0)$. But the unstable set of $(p, 0)$ contains the point $(0, \lambda/2)$; thus there exists a critical homoclinic orbit associated to $(p, 0)$. Let f_0 be a C^1 map of a manifold M and z_0 a repelling fixed point of f_0. A point $x_0 \in M$ is homoclinic to z_0 if there exists $m > 0$ such that $x_m = f_0^m(x_0) = z_0$ and a sequence $\{x_n\}_{n<0}$ (preorbit of x_0) such that $f(x_{n-1}) = x_n$ and $x_n \to z_0$, to $n \to -\infty$. The orbit $\{x_n\}_{-\infty<n\leq m}$ is called homoclinic to z_0; if the at least one of point is a critical point of f, then the orbit $\{x_n\}$ is critical homoclinic to z_0.

Then the following, a generalization of a well knout one dimensional result, will be used here:

Theorem 5.2. *[A] Let x_0 be homoclinic to a fixed repellor z_0 for a map f_0. Then in any generic one parameter family $\{f_\mu\}$ through f_0 there exists close to 0 a parameter μ_0 such that f_{μ_0} has a critical periodic point.*

Then by a result previous, it follows that there exists a perturbation f' of f such that a critical point of f' is periodic. Of course, the property that $B_\infty^0(f)$ connects $f(S_f)$ is open, so the perturbation can be made small orden to obtain that $B_\infty^0(f')$ still connects the triangle $f'(S_{f'})$. A final perturbation can be made to make the critical periodic point hyperbolic. This finishes the construction.

Acknowledgment: We would like to thank Alvaro Rovella for helpful discussions and for many useful remarks on mathematical structure, style, references, etc.

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