Ergod. Th. & Dynam. Sys. (2006), **26**, 281–289
doi:10.1017/S0143385705000568

A sufficient condition for robustly minimal foliations

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(*Received* 30 *October* 2004 *and accepted in revised form* 22 *September* 2005)

Abstract. Let $f : M \to M$ be a partially hyperbolic diffeomorphism, $TM = E^{ss} \oplus E^c \oplus E^{uu}$ such that the stable foliation $\mathcal{F}^{ss}(f)$ is minimal. We give a sufficient condition so that this foliation remains minimal after perturbations, i.e. $\mathcal{F}^{ss}(g)$ is minimal for every g sufficiently close to f.

1. *Introduction*

In the theory of differentiable dynamical systems, it is an important problem to recognize when a dynamic feature of a system is also present in all nearby systems (with respect to some topology) and what are the conditions on the initial system that guarantee this fact. In other words, we look for a dynamic property of a particular system which is *robust* (or *stable*) under perturbations. As a guiding principle, a robust property should be reflected into some property of the tangent map, that is, we look for a property on the tangent map that guarantees that some phenomenon is robust.

There are many examples of the situation above. In particular, to start focusing on our goal, let us mention the robust transitivity: there are many examples of transitive systems that also remain transitive under C^r perturbations. The most well-known example is a transitive Anosov diffeomorphism. In the category of non-hyperbolic diffeomorphisms, the first example was given by Shub [**Sh**] on the torus $\mathbb{T}^4$. Another one by Mañé [**Ma**] followed on the torus $\mathbb{T}^3$. Bonatti and Díaz [**BD**] give a general geometric construction that leads to robustly transitive systems. All these (non-hyperbolic) examples are *partially hyperbolic systems* (although some new examples [**BV**] were shown to exhibit just a dominated splitting): the tangent bundle TM splits into three invariant sub-bundles $E^{ss} \oplus E^c \oplus E^{uu}$ where vectors in E^{ss} are forward contracted, vectors in E^{uu} are backward contracted, and vectors in E^c are less contracted than any vector in E^{ss} and less expanded than vectors in E^u (see definition below). The distributions E^{ss} and E^{uu} of a partially hyperbolic system are always uniquely integrable and lead to foliations on the manifold (i.e. a partition of the manifold into C^r leaves) called the (strong) stable and unstable foliations.

We will show that the mentioned examples share a stronger property: not only are they robustly transitive but also (at least) one of these foliations is robustly minimal, that is, every leaf is dense in the manifold. Let us mention here that in [**BDU**] it is shown that, C^1 generically, one of the (strong) foliations of a partially hyperbolic robustly transitive diffeomorphism on a three-dimensional manifold M^3 is minimal.

In this paper we are concerned with partially hyperbolic systems for which one of the strong foliations, say the stable one, is minimal and we will give sufficient conditions for this foliation to remain minimal under C^r perturbations.

Let us be more precise. A diffeomorphism $f : M \to M$ is said to be *partially hyperbolic* provided the tangent bundle splits into three non-trivial sub-bundles $TM = E^{ss} \oplus E^c \oplus E^{uu}$ which are invariant under the tangent map Df and there are numbers $0 < \lambda < \mu < 1$ such that for all $x \in M$,

$$\|Df_{|E^{ss}(x)}\| < \lambda, \quad \|Df^{-1}_{|E^{uu}(x)}\| < \lambda, \quad \mu < \|Df^{-1}_{|E^c(x)}\|, \quad \|Df_{|E^c(x)}\| < \mu^{-1}.$$

It is well known (see §2) that the sub-bundles E^{ss} and E^{uu} are uniquely integrable and hence we have two foliations in M called the (strong) stable one, denoted by $\mathcal{F}^{ss}(f)$, and the (strong) unstable one, denoted by $\mathcal{F}^{uu}(f)$, which are tangent to E^{ss} and E^{uu} respectively. We shall denote by $\mathcal{F}^{ss}(x, f)$ and by $\mathcal{F}^{uu}(x, f)$ the leaves of these foliations passing through the point x. One of these foliations is said to be *minimal* provided every leaf of this foliation is dense in M.

On the other hand, every diffeomorphism $g : M \to M$ sufficiently C^1 close to a partially hyperbolic diffeomorphism f is also partially hyperbolic and therefore it has two invariant foliations $\mathcal{F}^{ss}(g)$ and $\mathcal{F}^{uu}(g)$.

Definition 1.1. Let $f : M \to M$ be a C^r partially hyperbolic diffeomorphism. We say that $\mathcal{F}^{ss}(f)$ is C^r*-robustly minimal* if there exist a C^r neighborhood $\mathcal{U}(f)$ such that $\mathcal{F}^{ss}(g)$ is minimal for every diffeomorphism $g \in \mathcal{U}(f)$.

It is not difficult to see that if (for instance) $\mathcal{F}^{ss}(f)$ is robustly minimal then f is robustly transitive, i.e. every diffeomorphism C^1 close to f is transitive.

Next, we will define the key property that guarantees the robustness of a stable foliation of a partially hyperbolic diffeomorphism: some hyperbolicity (SH) on the central distribution E^c at some points. Before we do, let us introduce some notation: if $L : V \to W$ is a linear isomorphism between normed vector spaces we denote by $m\{L\}$ the minimum norm of L, i.e. $m\{L\} = \|L^{-1}\|^{-1}$.

Definition 1.2. (Property SH) Let $f \in \mathrm{Diff}^r(M)$ be a partial hyperbolic diffeomorphism. We say that f exhibits the property SH (or has the property SH) if there exist $\lambda_0 > 1$, $C > 0$ such that for any $x \in M$ there exists $y^u(x) \in \mathcal{F}^{uu}_1(x, f)$ (the ball of radius 1 in $\mathcal{F}^{uu}(x, f)$ centered at x) satisfying

$$m\{Df^n_{|E^c(f^\ell(y^u(x)))}\} > C\lambda_0^n \quad \text{for any } n > 0,\ \ell > 0.$$

In other words, we require that in any disk of radius 1 in any leaf of $\mathcal{F}^{uu}(f)$ there is a point y^u where the central bundle E^c has a uniform expanding behavior along the future orbit of y^u. A nice image of the above is the existence of a hyperbolic set (with E^c being part of the unstable bundle) such that the local stable manifold of this hyperbolic set is a global section to the foliation $\mathcal{F}^{uu}(f)$.

THEOREM A. *Let $r \geq 1$ and let $f \in \mathrm{Diff}^r(M)$ be a partial hyperbolic diffeomorphism satisfying Property SH and such that the (strong) stable foliation $\mathcal{F}^{ss}(f)$ is minimal. Then, $\mathcal{F}^{ss}(f)$ is C^1 (and hence C^r) robustly minimal.*

A similar result for the foliation $\mathcal{F}^{uu}(f)$ holds provided f^{-1} satisfies Property SH. As an immediate consequence we have that if f satisfies the conditions of Theorem A then it is robustly transitive. The proof of Theorem A will be given in §4.

We will use our theorem to reconstruct the examples by Shub and Mañé (see §§5 and 6) and showing that indeed one of the strong foliations is robustly minimal. It can be proven also that the examples by Bonatti and Díaz also satisfy Property SH, but to do so we have to go into the details of their geometric construction and this exceeds the purpose of this paper. Let us end this introduction by asking if a C^r robustly transitive partially hyperbolic diffeomorphism must exhibit Property SH (at least generically). In other words, is Property SH a necessary condition for generic robustly transitive partially hyperbolic systems?

2. *Preliminaries*

In this section we recall some well-known results regarding partially hyperbolic systems (that we have already mentioned in the introduction). We refer to [**HPS**] for a general background on the topics we will review.

As we said in the introduction, a diffeomorphism $f : M \to M$ is partially hyperbolic provided the tangent bundle splits into three non-trivial sub-bundles $TM = E^{ss} \oplus E^c \oplus E^{uu}$ which are invariant under the tangent map Df and there are $0 < \lambda < \mu < 1$ such that for all $x \in M$

$$\|Df_{|E^{ss}(x)}\| < \lambda, \quad \|Df^{-1}_{|E^{uu}(x)}\| < \lambda, \quad \mu < \|Df^{-1}_{|E^{c}(x)}\|, \quad \|Df_{|E^{c}(x)}\| < \mu^{-1}.$$

LEMMA 2.0.1. *Let $f \in \mathrm{Diff}^r(M)$ be a partially hyperbolic diffeomorphism. Then there exist a C^r neighborhood of f, say $\mathcal{U}$, $0 < \lambda < \lambda_1 < \mu_1 < \mu < 1$ and continuous functions $E^{ss} : \mathcal{U} \to C(M, TM)$, $E^c : \mathrm{Diff}(M) \to C(M, TM)$ and $E^{uu} : \mathcal{U} \to C(M, TM)$ such that, for any $g \in \mathcal{U}$ and $x \in M$, we have the following:*

1. *$TM = E^{ss}_g \oplus E^c_g \oplus E^{uu}_g$, this decomposition is invariant under Dg and no one of these sub-bundles is trivial;*
2. *$\|Dg_{|E^{ss}(x)}\| < \lambda_1$, $\|Dg^{-1}_{|E^{uu}(x)}\| < \lambda_1$;*
3. *$\mu_1 < \|Dg^{-1}_{|E^{c}(x)}\|$, $\|Dg_{|E^{c}(x)}\| < \mu_1^{-1}$.*

The sub-bundles E^{ss}_g and E^{uu}_g are uniquely integrable and form two foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$.

THEOREM 2.1. *Let $\mathcal{U}$ be as in Lemma 2.0.1. Then, for each $g \in \mathcal{U}$ there are two partitions $\mathcal{F}^{ss}(g)$ and $\mathcal{F}^{uu}(g)$ of M such that for each $x \in M$ the elements of the partitions that contain x, denoted by $\mathcal{F}^{ss}(x, g)$ and $\mathcal{F}^{uu}(x, g)$, are C^1 submanifolds (called leaves) such* *that* $T_x\mathcal{F}^{ss}(x, g) = E^{ss}_g(x)$ *and* $T_x\mathcal{F}^{uu}(x, g) = E^{uu}_g(x)$. *These submanifolds depend continuously (on compact subsets) on* $x \in M$ *and* $g \in \mathcal{U}$.

These submanifolds $\mathcal{F}^{ss}(x, g)$ and $\mathcal{F}^{uu}(x, g)$ inherit the Riemannian metric on M. We shall denote by $\mathcal{F}^{ss}_r(x, g)$ (respectively $\mathcal{F}^{uu}_r(x, g)$) the ball in $\mathcal{F}^{ss}(x, g)$ (respectively $\mathcal{F}^{uu}(x, g)$) of radius r centered at x.

The sub-bundle $E^{cu} = E^c \oplus E^{uu}$ (called center-unstable) is not (in general) integrable. However, we can choose a continuous family of locally invariant manifolds tangent to it. Let $\dim E^{cu} = l$ and denote by I_ϵ the ball of radius ϵ in $\mathbb{R}^l$.

LEMMA 2.0.2. *Let* $\mathcal{U}$ *be as in Lemma 2.0.1. There exists a continuous map* $\Phi : M \times \mathcal{U} \to \mathrm{Emb}_1(I_1, M)$ *such that, if we set* $W^{cu}_\epsilon(x, g) = \Phi(x, g)I_\epsilon$, *then the following hold:*

(1) $T_x W^{cu}_\epsilon(x, g) = E^{cu}(x, g)$;

(2) *given* $\epsilon > 0$ *there exists* $r = r(\epsilon)$ *such that* $g^{-1}(W^{cu}_r(x, g)) \subset W^{cu}_\epsilon(g^{-1}(x), g)$.

For the sake of simplicity we shall identify $W^{cu}_\epsilon(x, g)$ with the ball of radius ϵ in $W^{cu}_1(x, g)$.

LEMMA 2.0.3. *Let* $\mathcal{U}$ *be as in Lemmas 2.0.1 and 2.0.2. Given* $0 < \lambda < \lambda_1 < 1$ *there exists* r_0 *such that if* $g \in \mathcal{U}$ *and* $x \in M$ *satisfy*

$$\prod_{j=0}^{n} \|Dg^{-1}_{|E^{cu}(g^{-j}(x))}\| < \lambda^n, \quad 0 \le n \le m,$$

then $g^{-m}(W^{cu}_{r_0}(x, g)) \subset W^{cu}_{\lambda_1^m r_0}(g^{-m}(x), g)$.

Proof. Let $c > 0$ be such that $c\lambda < \lambda_1$ and let ϵ be such that if $\mathrm{dist}(z, y) < \epsilon$ then

$$c^{-1} < \frac{\|Dg^{-1}_{|E^{cu}(z)}\|}{\|Dg^{-1}_{|E^{cu}(y)}\|} < c. \tag{1}$$

Let ϵ be such that if $z \in W^{cu}_{\epsilon_0}(y, g)$ then $\mathrm{dist}(z, y) < \epsilon$. Let $r_0 = r(\epsilon_0)$ from Lemma 2.0.2. It follows that $g^{-1}(W^{cu}_{r_0})(x, g) \subset W^{cu}_{\epsilon_0}(g^{-1}(x), g)$. On the other hand, by (1), we conclude that

$$g^{-1}(W^{cu}_{r_0})(x, g) \subset W^{cu}_{\lambda_1 r_0}(g^{-1}(x), g) \subset W^{cu}_{r_0}(g^{-1}(x), g).$$

Arguing by induction up to m the result follows. □

3. *Robustness of Property SH*

In this section we prove that Property SH persists under slight perturbations.

THEOREM 3.1. *Let* $f \in \mathrm{Diff}^r(M)$ *be a partially hyperbolic diffeomorphisms exhibiting Property SH. Then, there are* $\mathcal{U}(f)$, $C' > 0$ *and* $\sigma > 1$ *such that for any* $g \in \mathcal{U}$ *it follows that for any* $x \in M$ *there exist* $y^u \in \mathcal{F}^{uu}_1(x, g)$ *satisfying*

$$m\{Dg^n_{|E^c(g^\ell(y^u))}\} > C'\sigma^n \quad \textit{for any } n > 0, \ell > 0.$$

Proof. Observe that to prove the theorem it is enough to find $\mathcal{U}$ and k_0 such that for any $x \in M$ there exist $y^u \in \mathcal{F}_2^{uu}(x, g)$ satisfying

$$m\{Dg^{k_0 n}_{|E^c(g^{k_0 \ell}(y^u))}\} > 2^n \quad \text{for any } n > 0, \ell > 0. \tag{2}$$

Changing f by a power of itself, we can assume that there is $\lambda_0 > 1$ such that for any $x \in M$ there exists $y^u \in \mathcal{F}_1^{uu}(x)$ such that

$$m\{Df^n_{|E^c(f^\ell(y^u))}\} > \lambda_0^n$$

for any $n > 0, \ell > 0$. We define the set $H^u_{\lambda_0}(f)$ as

$$H^u_{\lambda_0}(f) = \{y \in M : m\{Df^n_{|E^c(f^\ell(y))}\} \geq \lambda_0^n \text{ for } n > 0, \ell > 0\}. \tag{3}$$

Observe that this set is closed and $\mathcal{F}_1^{uu}(x) \cap H^u_{\lambda_0}(f)$ is not empty.

Notice that there exist λ_1, $1 < \lambda_1 < \lambda_0$, $\epsilon_0 > 0$ and $\mathcal{U}_0(f)$ such that

$$\begin{gathered}\text{if } x \in H^u_{\lambda_0}(f),\ y \in M \text{ and } g \in \mathcal{U}_0(f)\\ \text{satisfy } \operatorname{dist}(f^i(x),\ g^i(y)) < \epsilon_0 \text{ for } 0 \leq i \leq n,\\ \text{then } m\{Dg^k_{|E^c(g^j(y))}\} > \lambda_1^k \quad \text{for } 0 \leq j \leq n \text{ and } 0 \leq k \leq n - j.\end{gathered} \tag{4}$$

Next, we choose a positive integer m_0 such that for any $g \in \mathcal{U}_0$ and $x \in M$ it follows that

$$\mathcal{F}_2^{uu}(g^{m_0}(x), g) \subset g^{m_0}(\mathcal{F}^{uu}_{\epsilon_0/2}(x, g)). \tag{5}$$

Afterwards, we consider n_0 such that for any $g \in \mathcal{U}_0$ and $z \in M$ it follows that

$$\lambda_1^{n_0} \sup\{m\{Dg^{m_0}_{|E^c_z}\} : z \in M\} > 2. \tag{6}$$

Now, given ϵ_0 from (4) and n_0 from (6) we take ϵ_1 and $\mathcal{U}_1 \subset \mathcal{U}_0$ such that for any $g \in \mathcal{U}_1$ we have

$$\text{if } \operatorname{dist}(x, y) < \epsilon_1 \quad \text{then} \quad \operatorname{dist}(g^i(z), f^i(y)) < \epsilon_0/2 \quad \text{for } 1 \leq i \leq n_0. \tag{7}$$

Moreover, we can choose $\mathcal{U}_2 \subset \mathcal{U}_1$ such that

$$\text{if } g \in \mathcal{U}_2 \quad \text{then} \quad \operatorname{dist}(\mathcal{F}_1^{uu}(x, g), H_{\lambda_0}(f)) < \epsilon_1 \quad \text{for any } x \in M. \tag{8}$$

Set $k_0 = m_0 + n_0$ where m_0 and n_0 are as in (5) and (6) respectively. We will prove that for $g \in \mathcal{U} = \mathcal{U}_2$ and k_0 we have (2) for some $y^u \in \mathcal{F}_2^{uu}(x, g)$ and any $x \in M$. In order to find such a point y^u we will proceed by induction to construct sequences $\{z^u_n\}_{n\geq 0}$ and $\{x_n\}_{n\geq 0}$ such that:

(i) $z^u_{j+1} \in \mathcal{F}_1^{uu}(g^{k_0}(z^u_j), g)$ for $0 \leq j \leq n-1$;

(ii) $x_j \in H^u_{\lambda_0}(f)$;

(iii) $\operatorname{dist}(z^u_j, x_j) < \epsilon_1$.

Assume for the moment that we have already constructed such sequences and set $y^u_n = g^{-k_0 n}(z^u_n)$. Let us show that $y^u_n \in \mathcal{F}_2^{uu}(x, g)$ and for any $0 \leq j \leq n$, $0 < i \leq n - j$ we have

$$m\{Dg^{k_0 i}_{|E^c(g^{k_0 j}(y^u_n))}\} > 2^i. \tag{9}$$

First observe that, from (5), $g^{-k_0}(z_n^u) \in \mathcal{F}^{uu}_{\epsilon_0/2}(z^u_{n-1}) \subset \mathcal{F}^{uu}_2(g^{k_0}(z^u_{n-2}))$. By induction, it follows that $g^{-k_0(j-1)}(z_n^u) \in \mathcal{F}^{uu}_2(g^{k_0}(z^u_{n-j}))$ and hence $y_n^u \in \mathcal{F}^{uu}_2(x, g)$. Moreover, again by (5), we have that $g^{-m_0-k_0(j-1)}(z_n^u) \in \mathcal{F}^{uu}_{\epsilon_0/2}(g^{n_0}(z_j^u))$. Therefore, for $l = j, \dots, i-1$ and $d = 0, 1, \dots, n_0$ we have

$$\operatorname{dist}(g^{d+k_0 l}(y_n^u), g^d(z_l^u)) < \epsilon_0/2.$$

On the other hand, since $\operatorname{dist}(z_l^u, x_l) < \epsilon_1$, from (7) we conclude that $\operatorname{dist}(g^d(z_l^u), f^d(x_l)) < \epsilon_0/2$ and so

$$\operatorname{dist}(g^{d+k_0 l}(y_n^u), f^d(x_l)) < \epsilon_0.$$

Thus, using (4) and (6) we get $m\{Dg^{k_0}_{|E^c(g^{k_0 l}(y_n^u))}\} > 2$ and (9) follows. The desired point y^u is obtained as any accumulation point of the sequence $\{y_n^u\}$.

To finish the proof of our theorem we have just to show how to construct the sequences $\{z_n^u\}$ and $\{x_n\}$. However, the construction is straightforward: since $\operatorname{dist}(\mathcal{F}^{uu}_1(x, g), H^u_{\lambda_0}(f)) < \epsilon_1$ pick $z_0^u \in \mathcal{F}^{uu}_1(x, g)$ and $x_0 \in H^u_{\lambda_0}(f)$ such that $\operatorname{dist}(z_0^u, x_0) < \epsilon_1$. Once we have constructed z_j^u and x_j, pick $z^u_{j+1} \in \mathcal{F}^{uu}_1(g^{k_0}(z_j^u), g)$ and $x_{j+1} \in H^u_{\lambda_0}(f)$ such that $\operatorname{dist}(z^u_{j+1}, x_{j+1}) < \epsilon_1$. □

4. *Proof of Theorem A*

Let f be a partially hyperbolic diffeomorphism satisfying Property SH and such that the (strong) stable foliation $\mathcal{F}^{ss}$ is minimal, i.e. every leaf $\mathcal{F}^{ss}(x, f)$ is dense in M. We have to show that there is a neighborhood $\mathcal{U}(f)$ such that for every $g \in \mathcal{U}(f)$ the (strong) stable foliation of g is dense, i.e. every leaf $\mathcal{F}^{ss}(x, g)$ is dense.

From Theorem 3.1 we know that there exists a neighborhood $\mathcal{U}_1(f)$ and $C > 0$, $1 < \sigma$ such that for every $g \in \mathcal{U}_1$ and $x \in M$ there exists a point $y^u \in \mathcal{F}^{uu}_1(x, g)$ such that

$$m\{Dg^n_{|E^c(g^\ell(y^u))}\} > C\sigma^n \quad \text{for any } n > 0,\ \ell > 0. \tag{10}$$

We may assume that $C = 1$ (otherwise we take a fixed power of every $g \in \mathcal{U}_1(f)$). Let $\lambda = \sigma^{-1}$ and fix $0 < \lambda < \lambda_1 < 1$ and let r_0 be as in Lemma 2.0.3.

On the other hand, fix a number $\delta > 0$ arbitrarily small. Since the stable foliation $\mathcal{F}^{ss}(f)$ is minimal there exists $K_1 > 0$ such that $\mathcal{F}^{ss}_{K_1}(x, f)$ is $\delta/2$ dense, meaning that $\mathcal{F}^{ss}_{K_1}(x, f) \cap B(z, \delta/2) \neq \emptyset$ for every $x, z \in M$. On the other hand, if δ is small enough, it follows that if $y \in B(z, \delta/2)$ then $\mathcal{F}^{ss}_\delta(y, f) \cap W^{cu}_{r_0/2}(z, f) \neq \emptyset$. Thus, setting $K := K_1 + \delta$, we may assume that $\mathcal{F}^{ss}_K(x, f) \cap W^{cu}_{r_0/2}(z, f) \neq \emptyset$ for every $x, z \in M$. Hence there exists a neighborhood $\mathcal{U}(f) \subset \mathcal{U}_1(f)$ such that for every $g \in \mathcal{U}(f)$ and $x, z \in M$ the following holds:

$$\mathcal{F}^{ss}_{2K}(x, g) \cap W^{cu}_{r_0}(z, g) \neq \emptyset. \tag{11}$$

Let us prove that for $g \in \mathcal{U}(f)$ the stable foliation $\mathcal{F}^{ss}(g)$ is minimal. For this purpose, fix $x \in M$ and U an open subset of M. Let $z \in U$ and let $\beta > 0$ be such that $\mathcal{F}^{uu}_\beta(z, g) \subset U$. Take n_0 such that $g^{n_0}(\mathcal{F}^{uu}_\beta(z, g)) \supset \mathcal{F}^{uu}_1(g^{n_0}(z), g)$. Consider the point $y^u \in \mathcal{F}^{uu}_1(g^{n_0}(z), g)$ given by Theorem 3.1 and let $\eta > 0$ be such that

$$g^{-n_0}(W^{cu}_\eta(y^u, g)) \subset U. \tag{12}$$

Next, choose a positive integer m such that $\lambda_1^m r_0 < \eta$ and set $k = n_0 + m$. From (11) we have that

$$\mathcal{F}^{ss}_{2K}(g^k(x), g) \cap W^{cu}_{r_0}(g^m(y^u), g) \neq \emptyset. \tag{13}$$

Since $E^{cu} = E^c \oplus E^u$ and this decomposition is dominated, there is $L > 0$ such that $\|Dg^{-n}_{|E^{cu}}\| \leq L \sup\{\|Dg^{-n}_{|E^u}\|, \|Dg^{-n}_{|E^c}\|\}$. For the sake of simplicity, we will assume that $L = 1$. From (10) we know that

$$\prod_{j=0}^{n} \|Dg^{-1}_{|E^c(g^m(y^u))}\| < \lambda^n, \quad 0 \leq n \leq m$$

and therefore

$$\prod_{j=0}^{n} \|Dg^{-1}_{|E^{cu}(g^m(y^u))}\| < \lambda^n, \quad 0 \leq n \leq m.$$

From Lemma 2.0.3 we conclude that

$$g^{-m}(W^{cu}_{r_0}(g^m(y^u), g)) \subset W^{cu}_{\lambda_1^m r_0}(y^u, g) \subset W^{cu}_{\eta}(y^u, g)$$

and hence, using (12), we have $g^{-k}(W^{cu}_{r_0}(g^m(y^u), g)) \subset U$. Finally, this and (13) imply that $\mathcal{F}^{ss}(x, g) \cap U \neq \emptyset$ and the proof of Theorem A is completed.

5. *Shub's example on* $\mathbb{T}^4$

Here we will show that Shub's example of a non-hyperbolic robustly transitive diffeomorphism on $\mathbb{T}^4$ can also be derived from our methods.

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov diffeomorphism having two fixed points p and q. Since f is Anosov, $T\mathbb{T}^2 = E^{ss} \oplus E^{uu}$ with $\|Df_{|E^{ss}}\| < \lambda < 1$ and $\|Df^{-1}_{|E^{uu}}\| < \lambda$.

Now, consider a smooth family of torus diffeomorphisms $g_x : \mathbb{T}^2 \to \mathbb{T}^2$ indexed in $x \in \mathbb{T}^2$ such that the following hold:

- $T\mathbb{T}^2 = E^s(g_x) \oplus E^c(g_x)$ invariant under $D(g_x)$ and such that $\|D(g_x)_{|E^s(g_x)}\| < \mu < \mu_1 < 1$ and $\mu < \mu_1 < \|D(g_x)_{|E^c(g_x)}\| \leq \mu^{-1}$;
- for all $x \in \mathbb{T}^2$, g_x preserves a cone field C^s and C^{cu};
- g_p is Anosov and g_q is a DA (derived from Anosov) map;
- $g_x(p) = p$ for every x and p is an attractor for g_q.

We assume (taking a power of f if necessary) that $\lambda < \mu$. Next, we define the map on $\mathbb{T}^4$ which is the candidate to be robustly transitive:

$$F : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2, \quad F(x, y) = (f(x), g_x(y)).$$

It is not difficult to see that F is partially hyperbolic $T_{(x,y)}\mathbb{T}^4 = E^{ss}(x, y) \oplus E^s(x, y) \oplus E^c(x, y) \oplus E^u(x, y)$. We set $E^s = E^{ss} \oplus E^s$. Let us show that the stable foliation (tangent to $E^{ss} \oplus E^s$) is minimal. First observe that

$$W^s(\{p\} \times \mathbb{T}^2) = \bigcup_{z \in \mathbb{T}^2} W^{ss}(p, z) = W^{ss}(p, f) \times \mathbb{T}^2$$

and hence is dense in $\mathbb{T}^2 \times \mathbb{T}^2$. Moreover, since g_p is Anosov, we have that

$$W^s(p, z) = \bigcup_{y \in W^s(z, g_p)} W^{ss}(p, y)$$

is dense on $\mathbb{T}^2 \times \mathbb{T}^2$ for all $(p, z) \in \{p\} \times \mathbb{T}^2$.

On the other hand, for every $(z, w) \in \mathbb{T}^2 \times \mathbb{T}^2$ we have that

$$W^{uu}((z, w)) \cap W^s(\{p\} \times \mathbb{T}^2) \neq \emptyset.$$

From this it follows that the stable foliation $\mathcal{F}^{ss}$ (whose leaves are tangent to $E^{ss} \oplus E^s$) is minimal.

Finally, as we observed above for some $\epsilon > 0$ and $r > 0$ we have for every $(z, w) \in \mathbb{T}^2 \times \mathbb{T}^2$ that

$$W^{uu}_\epsilon((z, w)) \cap W^s_r(\{p\} \times \mathbb{T}^2) \neq \emptyset, \tag{14}$$

which implies (since g_p is Anosov and $DF_{|\{0\} \times E^c(p,z)}$ is uniformly expanding in the future) that Property SH is satisfied, i.e. if y^u is a point in the intersection given by (14) we may find constants $C > 0$ and $\sigma > 1$ such that

$$DF^n_{/E^c(F^m(y^u))} > C\sigma^n \quad \text{for all } n, m \geq 0$$

(recall that here E^c is one-dimensional). Thus, $F : \mathbb{T}^4 \to \mathbb{T}^4$ as above is in the hypothesis of Theorem A and so the stable foliation of F is robustly minimal and therefore F is robustly transitive. Since F has a hyperbolic periodic point of index 2 and another one of index 3 we conclude that F is non-hyperbolic.

6. *Mañé's example on* $\mathbb{T}^3$

In this section we will prove that certain DA (derived from Anosov) maps on $\mathbb{T}^n$ have the stable foliation robustly minimal and hence they are robustly transitive. This is an alternative approach to the one given by Mañé in [**Ma**].

Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an Anosov diffeomorphism such that $T\mathbb{T}^n = E^{ss} \oplus E^u \oplus E^{uu}$, none of the above trivial (and hence $n \geq 3$). Since f is Anosov (on a torus) it is transitive and so the stable foliation is minimal. Let p be a fixed point of f and let $U(p)$ be a small open ball containing p. Let $g_t : \mathbb{T}^n \to \mathbb{T}^n$, $-1 \leq t \leq 0$, be a smooth isotopy, supported in U and such that g_t is Anosov for $t < 0$, g_0 is not Anosov but conjugated to f, and $T\mathbb{T}^n = E^{ss}(g_t) \oplus E^c(g_t) \oplus E^{uu}(g_t)$. It follows that $\mathcal{F}^{ss}(g_0)$ is minimal. Let $0 < \lambda < 1$ be such that $\|D(g_t)^{-1}_{|E^{uu}}\| < \lambda$. By replacing f by a power of itself we may assume that $\lambda < 1/4$. Now, if the ball $U(p)$ is small enough, we have that for any $x \in \mathbb{T}^n$ there is a point $z_x \in \mathcal{F}^{uu}_1(x, g_0)$ such that $\mathcal{F}^{uu}_{1/4}(z_x) \cap U = \emptyset$. Therefore, we have that in any $\mathcal{F}^{uu}_1(x, g_0)$ there is a point y^u whose forward orbit never meets U. Since the maximal invariant set outside U,

$$\bigcap_{n \in \mathbb{Z}} g_0(\mathbb{T}^n \backslash U)$$

is hyperbolic (recall that the support of the isotopy is inside U), we conclude that g_0 satisfies Property SH. Therefore $\mathcal{F}^{ss}(g_0)$ is robustly minimal and g_0 is robustly transitive. Notice that there are DA maps as close as we wish to g_0 and therefore we have DA maps on $\mathbb{T}^n$ such that their (strong) stable foliation is robustly minimal (and so they are robustly transitive).

Acknowledgement. MS is grateful to CIMAT–Guanajuato for support during a visit in 2003 during which part of this work took place.

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