On C^1-Persistently Expansive Homoclinic Classes

Martín Sambarino, José Vieitez *

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Abstract

Let $f : M \to M$ be a diffeomorphism defined in a d-dimensional compact boundary-less manifold M. We prove that C^1-persistently expansive homoclinic classes $H(p)$, p an f-hyperbolic periodic point, have a dominated splitting $E \oplus F$, $\dim(E) = \text{index}(p)$. Moreover, we prove that if the $H(p)$-germ of f is expansive (in particular if $H(p)$ is an attractor, repeller or maximal invariant) then it is hyperbolic.

1 Introduction

It has been a problem in differentiable dynamical systems during the last decades to understand the influence of a robust dynamic property (i.e. a property that holds for a system and all nearby ones) on the behavior of the tangent map of the system. For instance, to begin with the context of this paper, Mañé ([Ma1]) proved that if M is a compact manifold and a diffeomorphism $f : M \to M$ and all C^1 nearby ones are expansive then f is a quasi-Anosov system (any non vanishing vector growth exponentially in norm by forward or backward iterations of the tangent map), and in particular is a hyperbolic system (Axiom A). In this paper we are concerned when some natural invariant subset is robustly expansive. A first generalization would be to assume that the non-wandering set is robustly expansive. However, this immediately yields that f is in the C^1 interior of those systems having all the periodic points hyperbolic, and it follows by ([Ao],[Ha]) that the system is Axiom A. In this paper we study the case when a homoclinic class is robustly or persistently expansive (see definitions below) generalizing previous results in [PPV]. The difference of this case with that of [Ma1] is that no regards is done on

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the global aspects of the dynamics. For instance, it is not known, a priori whether a robustly expansive homoclinic class is a (locally) maximal invariant set or not. Let us be more precise.

Let M be a compact connected boundary-less Riemmanian d-dimensional manifold and $f : M \to M$ a homeomorphism. Let K be a compact invariant subset of M and dist : $M \times M \to I\!R$ a metric on M and $\alpha > 0$. We say that f restricted to K is α-expansive when $\text{dist}(f^n(x), f^n(y)) \leq \alpha$ for all $x, y \in K$ and all $n \in Z\!\!\!Z$ implies $x = y$. The number $\alpha > 0$ is called a constant of expansiveness for f and K, and sometimes we say that f is expansive in K if α is fixed. Expansiveness is a property shared by a large class of dynamical systems exhibiting chaotic behavior.

By *homoclinic class* we mean the closure of the (transverse) homoclinic points associated to a hyperbolic periodic orbit p. By *persistently expansive* homoclinic class we mean that for all nearby diffeomorphism g, the homoclinic class of the continuation of p is expansive. Before we state our results in a precise way, let us introduce some notations and definitions.

Throughout $\text{Diff}^r(M)$, $r \geq 1$, is the space of C^r diffeomorphisms on M with the C^r topology. It is well known that when p is a hyperbolic periodic point of f of period k (i.e. no eigenvalues of Df_p^k has modulus one) then the sets $W^s(p) = \{y : f^{kn}(y) \to_{n\to\infty} p\}$ and $W^u(p) = \{y : f^{-kn}(y) \to_{n\to\infty} p\}$ are C^r injectively immersed sub-manifolds. A point $x \in W^s(p) \cap W^u(p)$ is called a homoclinic point (and it is said to be transversal if the above intersection is transversal at x). The closure of the homoclinic points is called the homoclinic class associated to p, i.e.:

$$H(p, f) = \overline{W^s(p) \cap W^u(p)}.$$

The closure of the transverse homoclinic points is called the transverse homoclinic class associated to p, i.e.:

$$H_T(p, f) = \overline{W^s(p) \pitchfork W^u(p)}.$$

It turns out that both $H(p, f)$ and $H_T(p, f)$ are compact invariant sets and clearly $H(p) \supset H_T(p)$. We shall write just $H(p)$ and $H_T(p)$ when there is no confusion about which set we are referring to.

If $x \in M$ then $\mathcal{O}_f(x)$ denotes the orbit of x by f, $W^s(x, f)$ and $W^u(x, f)$ denote the stable and unstable manifolds of x. When no ambiguity is possible we denote these sets by $\mathcal{O}(x)$, $W^s(x)$ and $W^u(x)$ respectively. Similarly we denote by $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ the ϵ-local stable and unstable manifolds of points $x \in M$ respectively. When $\epsilon > 0$ is fixed we omit to mention it and just say local stable manifold, etc.

In the case that $H(p)$ has topological dimension zero like occurs for instance in Smale's horseshoe, where $H(p) = H_T(p)$, the fact that $H(p)$ is expansive has very poor implications from the hyperbolic point of view. For instance, we may perturb

Smale's horseshoe to obtain one in which we have a saddle node fixed point p where we still have $H(p) = H_T(p)$. It is easy to verify that this example is expansive but expansiveness disappears (on the continuation of $H(p)$) under an arbitrarily small C^1-perturbation.

For a different example let us consider again Smale's horseshoe H and $\Lambda \subset H$ a non-trivial minimal subset. Assume that this example is in the yOz plane included in $I\!R^3$. For any point $(0, y, z)$ in $H \setminus \Lambda$ we add in the direction Ox a contraction $c(y, z) < 1$ depending on the distance from $(0, y, z)$ to Λ that tends to 1 when $(0, y, z) \to \Lambda$. In this example all periodic points are hyperbolic and H is expansive but clearly it cannot be hyperbolic. On the other hand, under an arbitrarily small C^1-perturbation we can destroy expansiveness. Again in this example we have $H(p) = H_T(p)$.

To show an example in which $H(p) \neq H_T(p)$ consider a hyperbolic fixed point p of saddle type in $I\!R^2$ such that its stable manifold $W^s(p)$ is tangent to $W^u(p)$ in a point q, and $W^s(p) \cap W^u(p) \setminus \{p\}$ only exhibits tangencies. Then $H_T(p) = \{p\}$ which is hyperbolic but $H(p) \supset \mathcal{O}(q)$.

We will see that $H(p, f) = H_T(p, f)$ whenever f is C^1-persistently expansive (see definition below).

We will assume that not only $H(p, f)$ but the continuation $H(p, g)$ is expansive. Recall that when p is a hyperbolic periodic point of f, there is a neighbourhood U of p and a neighbourhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ there is a unique periodic point of g (with the same period of f) in U. This periodic point of g, denoted by p_g is called the continuation of $p = p_f$.

Definition 1.1. *Let p be a hyperbolic periodic point of f. We say that the homoclinic class $H(p, f)$ is C^r-robustly expansive ($r \geq 1$) iff there exist $\alpha > 0$ and a C^r-neighborhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$, there exists a continuation p_g of p such that g is α-expansive in $H(p_g)$.*

Definition 1.2. *We say that $H(p, f)$ is C^r-persistently expansive ($r \geq 1$) iff there exist a C^r-neighbourhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$, there exist p_g the continuation of p and $\alpha(g) > 0$ such that g is expansive on $H(p_g, g)$.*

The difference between the definitions above is that in the latter we do not require a uniform expansivity constant on $H(p_g, g)$ for $g \in \mathcal{U}(f)$. Hence, persistent expansiveness is a weaker notion than robust expansiveness.

Remark 1.1. *The same definitions can be applied to $H_T(p, f)$. For instance: $H_T(p, f)$ is C^r-robustly expansive ($r \geq 1$) when there is an $\alpha > 0$ and a C^r-neighborhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$, there exists a continuation p_g of p such that g is α-expansive in $H_T(p_g)$. Analogously with the definition of C^r-persistently expansive.*

In [PPV] it is studied the case when $H(p)$ is C^1-robustly expansive and p has index $d-1$, where $d = \dim(M)$, proving that in this case $H(p)$ has a dominated splitting and for an open dense subset of $\mathcal{U}(f)$ in the C^1-topology, $f|_{H(p)}$ is hyperbolic.

Our first aim is to prove that C^1-persistently expansive homoclinic classes have a dominated splitting $E \oplus F$ where $\dim(E) = \text{index}(p)$. This is a two-fold generalization of some results in [PPV]. First we just assume that $f|_{H(p)}$ is persistently expansive which as we said above is (a priori) weaker than the assumption that $f|_{H(p)}$ is robustly expansive. Second we drop the co-dimension one hypothesis assumed in [PPV]. Let us recall the definition of Dominated Splitting.

Definition 1.3. *We say that a compact f-invariant set Λ admits a dominated splitting if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that*

$$\|Df^n_{|E(x)}\| \cdot \|Df^{-n}_{|F(f^n(x))}\| \leq C\lambda^n \ \forall x \in \Lambda,\ n \geq 0.$$

Our first theorem is:

Theorem A. *Let $f \in \mathit{Diff}^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p)$ is C^1-persistently expansive. Then $H(p)$ has a dominated splitting $E \oplus F$.*

We can also give some information regarding the hyperbolicity of periodic points in a persistently expansive homoclinic class.

Definition 1.4. *Let p, q be two hyperbolic periodic points of f We say that p and q are homoclinically related, and write $p \sim q$ if index(p) = index(q) (i.e.: $\dim(W^s(p)) = \dim(W^s(q))$) and*

$$W^s(p) \pitchfork W^u(p) \neq \emptyset \neq W^s(q) \pitchfork W^u(p).$$

Theorem B. *Let $f \in \mathit{Diff}^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p)$ is C^1-persistently expansive. Then there exist $C > 0$, $0 < \lambda < 1$ and $m > 0$ such that if q is a hyperbolic periodic point of period $\pi(q)$ and $q \sim p$ then*

$$\prod_{i=0}^{k-1} \|Df^m_{/E^s(f^{im}(q))}\| < C\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \|Df^{-m}_{/E^u(f^{-im}(q))}\| < C\lambda^k \tag{1}$$

where $k = [\pi(q)/m]$ ([.] represents the integer part.) Moreover, the constants C, λ and m holds uniformly in a neighbourhood of f.

Next, we shall prove hyperbolicity of a C^1-persistently expansive homoclinic class under supplementary assumptions on $H(p,f)$. The key concept is germ-expansiveness.

Definition 1.5. *Given a compact f-invariant set $\Lambda \subset M$ we say that the Λ-germ of f is expansive if there exists $\delta > 0$ such that if $x \in \Lambda$, $y \in M$ and $\operatorname{dist}(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ then $x = y$.*

It is not difficult to see that if $H(p,f)$ is both expansive and an attractor, repeller or maximal invariant on some neighbourhood, then it is germ-expansive (see section 6). However, germ-expansiveness does not imply any of the above concepts. Indeed, the definition says that no orbit on a neighbourhood of $H(p,f)$ is shadowed by an orbit in $H(p,f)$. Before stating our last main theorem, let us recall the definition of hyperbolicity.

Definition 1.6. *We say that a compact f-invariant set Λ is hyperbolic if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that*

$$\|Df^n_{|E(x)}\| \leq C\lambda^n \ \forall x \in \Lambda,\ n \geq 0,$$

and

$$\|Df^{-n}_{|F(x)}\| \leq C\lambda^n \ \forall x \in \Lambda,\ n \geq 0.$$

Theorem C. *Let $f \in Diff^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p,f)$ is C^1-persistently expansive and $H(p,f)$ is germ-expansive. Then $H(p,f)$ is hyperbolic.*

Remark 1.2. *We do not assume in the above theorem that $H(p_g, g)$ is germ-expansive for g C^1-close to f.*

As an immediate consequence we have:

Corollary 1.1. *Let $f \in Diff^r(M)$, $r \geq 1$, with a hyperbolic periodic point p such that its homoclinic class $H(p,f)$ is C^1-persistently expansive and $H(p,f)$ is either an attractor, repeller or maximal invariant. Then $H(p,f)$ is hyperbolic.*

1.1 Sketch of the proof

We now give an idea of the proof of Theorem A.
The basic idea is the standard one after the work of Mañé (see [Ma3]): to control the angles between natural candidates to the splitting on an appropriate dense subset. This is usually done on the periodic points. However we shall do it on the

transversal homoclinic points and if x is an homoclinic point the candidates to the splitting at x will be the tangent spaces $T_xW^s(p)$ and $T_xW^u(p)$.
We first prove that we cannot have tangencies between stable and unstable manifolds of homoclinic points in $H(p)$. If not we may create a flat intersection and after it infinitely many transversal intersections between a local stable manifold and a local unstable manifold of a homoclinic point. This will contradict the persistence of the expansiveness. Thus we prove that the angle between the tangent spaces $T_xW^s(x)$ and $T_xW^u(x)$ is bounded away from zero for all $x \in W^s(p) \cap W^u(p)$. Moreover, this property persists for any g C^1-near f. After this we may repeat the arguments given in [Ma3] and [PS1] to prove the existence of domination.

The basic idea to prove Theorem B is that it is enough to show that the modulus of the (normalized) eigenvalues of the $Df_q^{\pi(q)}$ for every hyperbolic periodic point q are bounded away from one. If this is not the case we can bifurcate a periodic point in order to obtain two periodic points in the (continuation) of the homoclinic class whose orbit remains close in the future and in the past contradicting the expansive properties of $f/H(p)$. We need to make an induction argument in order to prove that expansiveness fails with any positive constant of expansivity.

In order to prove Theorem C, we follow some ideas in [Ma3, Theorem C]. We use Theorem B which gives a control of the norm of $\|Df^m_{/E^s}\|$ and $\|Df^m_{/E^u}\|$, m fixed, as we mentioned above. This, together with the germ-expansiveness will imply that for q homoclinically related with p their center-stable and center-unstable manifolds will be indeed stable and unstable manifolds of uniform size. This means that any $y \in W^{cs}_\epsilon(x)$ satisfies that for all $n \geq 0$ $\mathrm{dist}(f^n(x), f^n(y)) < \epsilon$ and $\lim_{n\to+\infty} \mathrm{dist}(f^n(x), f^n(y)) = 0$. Similarly for $y \in W^{cu}_\epsilon(x)$ taking $n \leq 0$ and limits with $n \to -\infty$. Moreover, $H(p)$ satisfies the shadowing lemma. Next we show that all periodic points in $H(p)$ have the same index and that (1) holds for all them. This, together with the shadowing property will guarantee the hyperbolicity of $H(p)$.

2 Perturbation Lemmas in the C^1-Topology

We assume in the sequel that p is a hyperbolic fixed point of index $= k$ (i.e.: Df_p has k eigenvalues μ_j of modulus less than 1 and $(d-k)$ eigenvalues λ_j of modulus greater than 1).

The next lemma, known as Franks' Lemma, is a simple yet powerful result allowing us to perturb the tangent map along a finite set with an arbitrarily small support.

Lemma 2.1. *Let M be a closed n-manifold and $f : M \to M$ be a C^1 diffeomorphism, and let a neighbourhood of f, $\mathcal{U}(f)$ be given. Then, there exist $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ and $\delta > 0$ such that if $g \in \mathcal{U}_0(f)$, $S \subset M$ is a finite set, $S = \{p_1, p_2, \ldots p_m\}$ and L_i, $i = 1, \ldots, m$ are linear maps $L_i : TM_{p_i} \to TM_{f(p_i)}$ satisfying $\|L_i - D_{p_i}g\| \le \delta$, $i = 1, \ldots, m$ then there exists $\tilde{g} \in \mathcal{U}(f)$ satisfying $\tilde{g}(p_i) = g(p_i)$ and $D_{p_i}\tilde{g} = L_i$. Moreover, if U is any neighborhood of S then we may chose $\tilde{g}$ so that $\tilde{g}(x) = g(x)$ for all $x \in \{p_1, p_2 \ldots p_m\} \cup (M \backslash U)$.*

Proof. See [Fr, Lemma 1.1]. The statement given there is slightly different from that above, but the proof of our statement is contained in [Fr]. □

The following lemma says that with an arbitrarily small perturbation of the identity we can linearize a manifold W around a point $x \in M$. Although the content is somehow folkloric, we include the proof for the sake of completeness.

Lemma 2.2. *Let $W \subset \mathbb{R}^d$ be a sub-manifold, $0 \in W$. Then, given $\epsilon > 0$ there exists a diffeomorphism $h = h_W : \mathbb{R}^d \to \mathbb{R}^d$ and ϵ' arbitrarily small such that h is ϵ-C^1 close to the identity, $h = id$ outside the ball $B(0, \epsilon')$ and such that $h(W \cap B(0, \delta)) \subset T_0W$ for some $\delta > 0$. (T_pW stands for the tangent space of W at the point $p \in W$.)*

Proof. Let $E = T_0W$ and denote a ball of radius r in the subspace E centered at the origin by $B_E(r)$. Let $\eta > 0$ and $\Gamma : B_E(\eta) \to E^\perp$ be such that $graph(\Gamma)$ is a neighbourhood of O in W (in other words, W is locally the graph of the map Γ). Notice that $D\Gamma(0) = 0$.

Let $\epsilon' < \eta$ be small enough so that for $y \in E$, $\|y\| < \epsilon'$ we have $\|\Gamma(y)\| \le \frac{\epsilon}{4}\|y\|$ and $\|D\Gamma(y)\| < \frac{\epsilon}{2}$.

Let B be a C^∞-bump function , $0 \le B \le 1$, that vanishes outside of $B(0, \epsilon')$, is equal to 1 in $B(0, \epsilon'/4)$, and such that the norm of its gradient ∇B is less than $\frac{2}{\epsilon'}$. Let $h_W : \mathbb{R}^d = E \oplus E^\perp \to \mathbb{R}^d$ defined by $h(y_1, y_2) = (y_1, y_2 - B(y_1, y_2)\Gamma(y_1))$ if $\|y_1\| < \eta$ and $h(y_1, y_2) = (y_1, y_2)$ otherwise. Thus, if $(y_1, y_2) \notin B(0, \epsilon')$ then $h = id$. On the other hand, if $\|y_1\| < \epsilon'$ then $\|h(y_1, y_2) - (y_1, y_2)\| \le \|\Gamma(y_1)\| \le \frac{\epsilon}{2}$. Moreover

$$Dh = \begin{pmatrix} Id & \vdots & 0 \\ \ldots & \vdots & \ldots \\ -BD\Gamma(y_1) - \Gamma^t \frac{\partial B}{\partial y_1} & \vdots & Id - \Gamma^t \frac{\partial B}{\partial y_2} \end{pmatrix}$$

and notice that $\|\Gamma^t \frac{\partial B}{\partial y_1}\| \le \frac{\epsilon}{4} \frac{2}{\epsilon'} \epsilon' = \frac{\epsilon}{2}$ and $\|BD\Gamma\| < \frac{\epsilon}{2}$. Hence $\|Dh - Id\| < \epsilon$. Therefore h is ϵ-C^1-close to the identity (and so, if ϵ is small), h is a diffeomorphism).

Finally, if $\|y_1\| < \frac{\epsilon'}{4}$ then $h(y_1, \Gamma(y_1)) = (y_1, 0)$. Taking $\delta < \frac{\epsilon'}{4}$ the lemma is proved.

□

Lemma 2.3. *Let p be a hyperbolic fixed point for f of index k and $H_T(p)$, its transverse homoclinic class, persistently expansive. If $x \in W^s(p) \cap W^u(p)$ then $W^s(p)$ intersects transversely $W^u(p)$ at x.*

Proof. By our assumptions $W^s(p)$ is an Euclidean k dimensional hyperplane and $W^u(p)$ an Euclidean $(d-k)$-dimensional hyperplane both immersed in M. If the intersection at x of $W^s(p)$ and $W^u(p)$ is not transversal we should have a vector $u \neq 0$ in $T_xW^u(p) \cap T_xW^s(p)$. Using Lemma 2.1 we may assume that the subspace generated by u is the unique in common between $T_xW^u(p)$ and $T_xW^s(p)$, that is $T_xW^u(p) + T_xW^s(p)$ has dimension $d-1$. Moreover, we also may assume that $k \geq d-k$.

Let ϵ be small enough such that $B(x,\epsilon)$ is contained in a local chart and $W^s_\epsilon(x)$ is contained in a fundamental domain of $W^s(p)$. Moreover, we may assume that $f^n(W^s_\epsilon(x)) \cap B(x,\epsilon) = \emptyset,\ n \geq 1$. We require a similar statement for $W^u_\epsilon(x)$. Through the local chart we identify $B(x,\epsilon)$ with an open set in $\mathbb{R}^d$ and x with $0 \in \mathbb{R}^d$.

We apply Lemma 2.2 for $W = W^s_\epsilon(x)$ obtaining h_s and for $W = W^u_\epsilon(x)$ obtaining h_u. Thus, we have that $h_s(W^s_\epsilon(x))$ contains a neighbourhood B_s of 0 in $T_xW^s_\epsilon(x)$ and $h_u(W^u_\epsilon(x))$ contains a neighbourhood B_u of 0 in $T_xW^u_\epsilon(x)$.

Let us define $g : M \to M$ by $g(y) = h_u \circ f \circ h_s^{-1}(y)$ if $y \in B(x,\epsilon)$ and $g = f$ otherwise. Then g is C^1-close to f and $B_s \subset W^s(p,g)$ and $B_u \subset W^u(p,g)$. Indeed, if $y \in B_s$ then, for $n \geq 1$, $g^n(y) = f^n(h_s^{-1}(y)) \to p$ as $n \to \infty$. A similar argument works for B_u.

Summarizing, the intersection between $W^u(p,g)$ and $W^s(p,g)$, in the local chart, contains a segment $\{su : -\delta \leq s \leq \delta\}$ where $T_xW^s_\epsilon(x) \cap T_xW^u_\epsilon(x) = \langle u \rangle$.

Let w be a unit vector in T_xM not contained in $T_xW^s(x,g) + T_xW^u(x,g)$. Let $b(s) \in [0,\epsilon']$ be a C^∞-bump function such that $b(0) = \epsilon'$ and such that all its derivatives, of any order, vanish at 0. We also require that $\text{supp}(b) \subset [-\epsilon', \epsilon']$. In the plane given by the point 0 and the vectors u and w we consider the graph of the function $l : \gamma \to \mathbb{R}$ given by $l(s) = b(s)s^3 \sin\frac{1}{s}$ if $s \neq 0$, $l(0) = 0$.

We put coordinates in the local chart $Y = (S, U, s, t)$ where we identify$(S, 0, s, 0)$ with B_s and $(0, U, s, 0)$ with B_u Let $h : M \to M$ be given by

$$(S, U, s, t) \mapsto (S, U, s, (t + l(s))B(Y))$$

and $h = id$ outside $B(x,\epsilon)$, where B is the bump function defined on the proof of Lemma 2.2. Define $\hat{g} : M \to M$ by $\hat{g} = h \circ g$. As before, it is not difficult to see that $B_s \subset W^s(p,\hat{g})$ and $(0, U, s, l(s)) \subset W^u(p,\hat{g})$. Furthermore, it is straightforward

to show that $W^s(p,\hat{g})$ and $W^u(p,\hat{g})$ intersects transversely at the points $x_n = (0,0,\frac{1}{n\pi}, l(\frac{1}{n\pi}))$ for n large enough. Thus $x \in H(p,\hat{g})$ although the intersection at x is not transverse.
Observe that in the construction above we may assume that f and $\hat{g}$ are as close as we wish.

Let us prove that $\hat{g}|_{H(p,\hat{g})}$ is not α- expansive for any $\alpha > 0$.
For all $\alpha > 0$ there exists $N = N(\alpha) > 0$ such that $f^N(W^s_\epsilon(x)) \subset W^s_{\alpha/4}(p)$ and $f^{-N}(W^u_\epsilon(x)) \subset W^u_{\alpha/4}(p)$. As $\hat{g}$ is near f and coincides with f outside a small neighborhood of x we may assume that the same holds for $\hat{g}$. But $x_n \to x$ and therefore we may find x_n near x such that $\mathrm{dist}(\hat{g}^j(x), \hat{g}^j(x_n)) < \alpha$ for all $j \in [-N, N]$. For $j \geq N$ or $j \leq -N$ we also have

$$\mathrm{dist}(\hat{g}^j(x), \hat{g}^j(x_n)) \leq \mathrm{dist}(\hat{g}^j(x), p) + \mathrm{dist}(\hat{g}(x_n), p) < \alpha\,.$$

Therefore $\hat{g}$ is not α-expansive for any positive α, contradicting the persistence of expansiveness. □

A hyperbolic periodic point p is C^1-*far* from homoclinic tangency if every diffeomorphism g C^1-close to f does not have homoclinic tangency associated to p_g, the continuation of p.

Corollary 2.4. *Under the hypothesis of Lemma 2.3 we have that p is C^1-far from homoclinic tangency.*

Corollary 2.5. *Let $H(p,f)$ be persistently expansive and let $\mathcal{U}(f)$ be such that for $g \in \mathcal{U}$ there is a continuation of p such that $H(p_g, g)$ is expansive. Then, every homoclinic point in $H(p_g, g)$ is transversal.*

Corollary 2.6. *Under the hypothesis of Lemma 2.3 we have that $H_T(p) = H(p)$, i.e., $H(p)$ is the closure of the transversal homoclinic points.*

Let us define $E^s(x)$ and $E^u(x)$ as the subspaces of T_xM tangent respectively to $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ at the point x, where x is a (transversal) homoclinic point associated to p.

Let E and F be two subspaces of $\mathbb{R}^d$ such that $E \oplus F = \mathbb{R}^d$. Hence $\dim(F) = \dim(E^\perp)$ and F is the graph of the linear map $L : E^\perp \to E$ defined as follows: given $v \in E^\perp$ there exist a unique pair of vectors $u \in E$, $w \in F$, such that $v + u = w$. Define $L(v) = u$ obtaining that L is linear and $\mathrm{graph}(L) = F$. We may define,as it is done in [Ma3], the angle $\angle(E,F)$ between E and F as $\|L\|^{-1}$ In particular $\angle(E, E^\perp) = +\infty$.

Proposition 2.7. *Assume that $H(p, f)$ is persistently expansive on $\mathcal{U}(f)$. Then, there exist $\gamma > 0$ and $\mathcal{U}_1(f) \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{U}_1(f)$ and any homoclinic point $x \in H(p_g, g)$ we have that the angle between $E^s(x, g)$ and $E^u(x, g)$, $\angle(E^s(x, g), E^u(x, g))$, is greater than γ.*

Proof. Let $\mathcal{U}_1(f) = \mathcal{U}_0(f)$ and δ be as in Lemma 2.1 corresponding to $\mathcal{U}(f)$. We may assume without loss of generality that there is some $C > 0$ such that $\sup\{\|D_x g\| : g \in \mathcal{U}_1(f)\} \leq C$.

Arguing by contradiction, assume that no γ satisfies the conclusion of the proposition. Then there exist $g_n \in \mathcal{U}_1(f)$ and $x_n \in H(p_{g_n})$ such that

$$\angle(E^s(x_n, g_n), E^u(x_n, g_n)) < 1/n.$$

Take n such that $1/n < \delta/C$. For the sake of simplicity, set $g = g_n, x = x_n$. Since $\angle(E^s(x, g), E^u(x, g)) < 1/n$ then there exist $v \in E^{s\perp}$ and $w \in E^s$ such that $v + w \in E^u$, $\|w\| = 1, \|v\| < 1/n$. Let $T : T_xM \to T_xM$ be such that $T_{/E^{s\perp}} = 0, T(w) = -v$ and $\|T\| < \delta/C$. Let $L : T_{g^{-1}(x)}M \to T_xM$ be defined by $L = (I + T) \circ D_{g^{-1}(x)}g$. It is not difficult to see that $\|L - D_{g^{-1}(x)}g\| < \delta$ and that $w \in L(E^u(g^{-1}(x)))$. Take a neighborhood U of $g^{-1}(x)$ such that $\mathcal{O}_g(x) \cap U = g^{-1}(x)$. Using lemma 2.1 we find $\tilde{g} \in \mathcal{U}(f)$ such that $g^j(x) = \tilde{g}^j(x)\ \forall j,\ \tilde{g} = g$ outside U and $D_{g^{-1}(x)}\tilde{g} = L$. Hence $x \in H(p_{\tilde{g}}, \tilde{g})$ and $w \in E^s(x, \tilde{g}) \cap E^u(x, \tilde{g})$ and so the intersection of $W^s(p_{\tilde{g}})$ and $W^u(p_{\tilde{g}})$ is not transverse at x. This contradicts corollary 2.5. □

3 Proof of Theorem A

In this section we will see that C^1-persistently expansive homoclinic classes $H(p)$ have a dominated splitting $E \oplus F$, $\dim(E) = \mathrm{index}(p)$.

Recall that we have defined $E(x)$ as the tangent subspace to $W^s_\epsilon(x)$ at x and $F(x)$ as the tangent subspace to $W^u_\epsilon(x)$ at x where x is a (transversal) homoclinic point associated to p. Denote by $\mathrm{hom}(p)$ the set of transversal homoclinic points associated to p.

Proposition 3.1. *There exists $m > 0$ such that for every $x \in hom(p)$ there exists m_1, $0 < m_1 \leq m$, such that*

$$\|Df^{m_1}|E(x)\| \cdot \|Df^{-m_1}|F(f^{m_1}(x))\| \leq 1/2\ .$$

Proof. Arguing by contradiction we suppose that for all $m \in \mathbb{Z}^+$ there exists x_m such that

$$\|Df^n|E(x_m)\| \cdot \|Df^{-n}|F(f^n(x_m))\| > 1/2\ ,$$

for all $0 \leq n \leq m$. Hence there exists vectors $v_m \in F(x_m)$ and $w_m \in E(x_m)$ with $\|v_m\| = \|w_m\| = 1$ and such that for them

$$\frac{\|Df^m(w_m)\|}{\|Df^m(v_m)\|} > \frac{1}{2}\,.$$

Let δ be as in lemma 2.1 corresponding to $\mathcal{U}(f) = \mathcal{U}_1(f)$ where $\mathcal{U}$ is as in proposition 2.7.

Let us define $T_j : T_{f^j(x_m)}M \to T_{f^j(x_m)}$ a linear map such that $T_j|_{E(f^j(x_m))} = (1+\delta)id$ and $T_j|_{F(f^j(x_m))} = id$, $j = 0, \ldots, m$. Notice that T_j stretches $E = T_{x_m}W^s_\epsilon(x_m, f)$ and left unchanged $F = T_{x_m}W^u_\epsilon(x_m, f)$. Let $P : T_{x_m}M \to T_{x_m}M$ be a linear map satisfying, $P = id$ in $E(x_m)$ and $P = id + L$ in $F(x_m)$ where $L : F(x_m) \to E(x_m)$ is a linear map such that $L(v_m) = \delta w_m$ and $\|L\| = \delta$. Finally define $G_0 = T_1 \cdot Df_{x_m} \cdot P$, $G_j = T_{j+1} \cdot Df_{f^j(x_m)}$ for $j = 1, \ldots, m-1$. By lemma 2.1 there exists a diffeomorphism $g : M \to M$ such that $\mathrm{dist}(g, f) < \epsilon$ and keeps the orbit of x_m unchanged for $j = 0, 1, \ldots, m$. We may assume (and do) that the support of the perturbation does not cut a small neighborhood of p. It follows that x_m continues to be a homoclinic point of g. Moreover, we do not change $E(f^j(x_m))$, $j \in \mathbb{Z}$ and $F(f^j)(x_m)$ is changed only for $j \geq 0$. Thus they are the stable and unstable directions of a homoclinic point of a diffeomorphism $g \in \mathcal{U}_1$. We obtain that $v_m \mapsto v_m + \delta w_m = u$ and after m iterates we have $u_m = Dg^m(u) = Dg^m(v_m + \delta w_m) = Df^m(v_m) + (1+\delta)^m Df^m(\delta w_m)$.
Given $\delta > 0$ we may find $m > 0$ so big that $\delta(1+\delta)^m \geq 4 + 2/\gamma$ where $\gamma > 0$, given by proposition 2.7, is such that $\angle(E(x), F(x)) > \gamma$ for all $x \in \hom(p_g)$, $g \in \mathcal{U}_0$. By [Ma3, Lemma II.10] we have that if $v \in E(x)$, $u \in F(x)$ then

$$\|v - u\| \geq \frac{\angle(E(x), F(x))}{1 + \angle(E(x), F(x))}\|v\|\,.$$

Therefore, with this choice of m,

$$\|Df^m(v_m)\| = \|u_m - (1+\delta)^m Df^m(\delta w_m)\| \geq$$

$$\frac{\gamma}{1+\gamma}\|u_m\| \geq \frac{\gamma}{1+\gamma}\,|\|\delta(1+\delta)^m Df^m(w_m)\| - \|Df^m(v_m)\||$$

Dividing both terms of the inequality by $\frac{\gamma}{1+\gamma}\|Df^m(v_m)\|$ and taking into account that by hypothesis of absurd

$$\frac{\|Df^m(w_m)\|}{\|Df^m(v_m)\|} > \frac{1}{2}$$

we find that

$$\frac{1+\gamma}{\gamma} > \frac{\delta(1+\delta)^m}{2} - 1 = \frac{\delta(1+\delta)^m - 2}{2} \geq \frac{2\gamma + 2}{2\gamma}$$

which is a contradiction. $\square$

End of proof of Theorem A: By proposition 3.1 we have that there exists $m > 0$ such that for all homoclinic point $x \in W^s(p) \pitchfork W^u(p)$ it holds that

$$\|Df^{m_1}|E(x)\| \cdot \|Df^{-m_1}|F(f^{m_1}(x))\| \leq 1/2 \text{ for some } 0 < m_1 = m_1(x) \leq m\,.$$

This property clearly implies the existence of a dominated splitting for points in hom(p) (see [Ma3]. Set $m_2(x) = m_1(f^{m_1(x)}(x)), m_k(x) = m_1(f^{m_1(x)+\cdots+m_{k-1}(x)}(x))$. Let $n \geq m$ and find $0 < r < m$ and $l > 0$ such that

$$n = m_1(x) + \cdots + m_l(x) + r.$$

Since $m_1(y) < m$ for all homoclinic points y, we have that $l \geq \left[\frac{n}{m}\right.$

Let $C_1 > 0$ be a bound in $H(p)$ for

$$\|Df\|, \|Df^2\|, \ldots, \|Df^{m-1}\|$$

then

$$\|Df^n\| \leq C_1 \left(\frac{1}{2}\right)^{[\frac{n}{m}]} \leq C \left(\frac{1}{2}\right)^{\frac{n}{m}}\,.$$

Choosing $\lambda = \left(\frac{1}{2}\right)^{\frac{1}{m}}$ we prove domination for homoclinic points.
Since homoclinic points are dense in $H(p)$ and the dominated splitting can be extended by continuity to the closure of hom(p) we have finished the proof.

4 Proof of Theorem B

For a periodic point q of period $\pi(q)$ the normalized eigenvalues of $Df_q^{\pi(q)}$ are $\{\lambda^{1/\pi(q)} : \lambda \text{ eigenvalue of } Df_q^{\pi(q)}\}$. Let q be a hyperbolic periodic point and let $0 < |\lambda_1| \leq ... \leq |\lambda_k| < 1$ be the modula of the normalized eigenvalues of $Df^{\pi(q)}_{/E^s(q)}$. Assume that λ_k is real and has multiplicity one and let $\eta > 0$ be such that for $j = 1, ..., k-1$ we have $|\lambda_j| < |\lambda_k| - \eta$. It is well known (see [HPS]) that there is a (local) strong stable manifold W^{ss} of codimension one in $W^s(q)$ and for some $\epsilon > 0$ we have that $W^s_\epsilon(q) \backslash W^{ss}_\epsilon(q)$ has two components. We shall call such a periodic point a η-simple periodic point, or just simple periodic point if it is η-simple for some $\eta > 0$. Moreover, we shall call $1 - |\lambda_k|$ the stable gap of the periodic point.

Definition 4.1. *Let q be a hyperbolic periodic point, $q \sim p$. We say that q is a non-ss boundary point provided*

1. *q is a η-simple periodic point for some $\eta > 0$.*

2. *$W^u(p)$ intersects both components of $W^s_\epsilon(q)\backslash W^{ss}_\epsilon(q)$.*

Proposition 4.2. *Let $\mathcal{U}(f)$ be a neighborhood of f and let $\rho > 0$ be given. Then there is $\delta = \delta(\rho)$ such that if $g \in \mathcal{U}(f)$ and $q_g \sim p_g$ is a non-ss boundary point whose stable gap is less than δ then the following holds:*

1. *There is $\tilde{g}$ ρ-C^1 close to g and q_1, q_2 two periodic point of $\tilde{g}$ such that $q_i \sim p_{\tilde{g}}, i = 1, 2,$ $\pi(q_1) = \pi(q_2)$ and $\mathrm{dist}(\tilde{g}^j(q_1), \tilde{g}^j(q_2)) \leq \delta$ for $0 \leq j \leq \pi(q_1)$. Moreover, $\tilde{g} = g$ outside an arbitrarily small neighborhood of q_g.*

2. *There exists a periodic point $\tilde{q}$ of $\tilde{g}$ such that $\tilde{q} \sim p_{\tilde{g}}$ and it is a non-ss boundary point of $H(p_{\tilde{g}}, \tilde{g})$. whose stable gap is less than $\delta/2$.*

Proof. The idea is to "bifurcate" q such that splits into two hyperbolic periodic points. These perturbation is done along a small neighborhood of the orbit of q. And moreover, the size of the perturbation is of order the perturbation of the linear part along q. Let us be more precise. Fix small neighborhoods $U_i = U(g^i(q)), i = 0, 1, ..., \pi(q) - 1$, with one disjoint of each other. We shall use local charts so we identify each U_i with a ball around 0 in $\mathbb{R}^d$ and $g^i(q)$ with 0. Also, we shall identify $W^s_\epsilon(g^i(q))$ with a ball (say of radius ϵ) in $\mathbb{R}^k \subset \mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^{d-k}$ and the eigenvector associated to λ_k with e_1. Let z_1, z_2 be two points of $W^u(p)$ in different components of $W^s_\epsilon(q)\backslash W^{ss}_\epsilon(q)$ and let $\gamma > 0$ be an arbitrarily small number (satisfying some condition to be explained later on).

Now, let $\sigma : \mathbb{R}^d \to \mathbb{R}^+$ be a C^∞ bump function, $0 \leq \sigma(x) \leq 1,$ $\sigma(x) = 1$ if $\|x\| \leq \gamma/4; \sigma(x) = 0$ if $\|x\| \geq \gamma$ and $\|\nabla\sigma\| < 2/\gamma$.

Let A be the matrix in $\mathbb{R}^d$ such that $A_{/\mathbb{R}^k \times \{0\}} = \lambda_k^{-1} Id$ and $A_{/\{0\}\times\mathbb{R}^{d-k}} = id$. Now, we are ready to perform our first perturbation. Let g_1 be defined as follows: if $x \in U_i$ then $g_1(x) = \sigma(x) A \circ Dg_{g^i(q)} x + (1 - \sigma(x)) g(x)$. Let C be such that $\|Dg(x)\| \leq C$ for any x and any $g \in \mathcal{U}(f)$ and let δ be such that $C\delta < \rho/4$. Then, it is not difficult to see that for γ arbitrarily small g_1 is $\rho/2$ close to g. Moreover, if we set $I = \{te_1 : -\gamma/4 \leq t \leq \gamma/4\}$ then we have that $g_1^{\pi(q)}$ is $\pm Id$ on I and any point in $W^s_\epsilon(q)$ converge, under forward iterations of $g_1^{\pi(q)}$ to a point in I. Furthermore, if γ is small enough, then the points z_1, z_2 still belong to $W^u(p)$.

Next, we perform the second perturbation. Let q_1, q_2 be the endpoint in I. We perturb g_1 to obtain $\tilde{g}$ arbitrarily close to g_1 such that q_1, q_2 and $0 = q$ are the only periodic points (and hyperbolic) in I and the index of $q_i, i = 1, 2$ is k. The periods of q_1 and q_2 are either $\pi(q)$ or $2\pi(q)$ (the latter corresponds to $g_1^{\pi(q)} = -Id$ on I, in this case $g^{\pi(q)}(q_1) = q_2$ and vice-versa). It follows that $\mathrm{dist}(\tilde{g}^j(q_1), \tilde{g}^j(q_2)) \leq \delta$ for $0 \leq j \leq \pi(q_1)$. Moreover, $z_i \in W^s(q_i)$. On the other hand, $W^u(q_i) \cap W^s_{loc}(p) \neq \emptyset$ by continuation of $W^u(q, g)$ on compact subsets, hence $q_i \sim p_{\tilde{g}}$. Notice also that the points q_i are simple whose stable gap is arbitrarily small. This proves item 1.

To prove item 2 of the proposition, observe that q_1 and $p_{\tilde{g}}$ belong to a hyperbolic set (since they are homoclinically related) having the shadowing property. As in [PPV, Propositions 4.3 and 4.4] it follows that we may take a periodic point in this set that spends most of the time near (the orbit of) q_1 and which is a non-ss boundary point of $H(p_{\tilde{g}}, \tilde{g})$. This point will be simple and the stable gap arbitrarily small. $\square$

In order to prove Theorem B, following [Ma3, Proposition II.3], it is enough to show that the family of periodic sequences of linear isomorphisms of $\mathbb{R}^d$ generated by Df along the hyperbolic periodic points $q \in H(p, f), q \sim p,\ index(q) = index(p)$ is uniformly hyperbolic.

Arguing by contradiction, assume that this does not hold. This means that for any $\epsilon > 0$ we may find $q \in H(p), q \sim p,\ index(q) = index(p)$ and linear maps $L_i : T_{f^i(q)}M \to T_{f^{i+1}(q)}M, i = 0, 1, ..., \pi(q) - 1,\ \|L_i - Df_{f^i(q)}\| < \epsilon$ and such that $\Pi_{i=0}^{\pi(q)-1} L_i$ has an eigenvalue of modulus 1. Let $L_i(t), 0 \leq t \leq 1$ be an isotopy between $Df_{f^i(p)}$ and L_i such that $\|L_i(t) - Df_{f^i(p)}\| < \epsilon,\ 0 \leq t \leq 1$. Since $\Pi_{i=0}^{\pi(q)-1} L_i(0) = Df_q^{\pi(q)}$ is hyperbolic there is some t_0 such that $\Pi_{i=0}^{\pi(q)-1} L_i(t_0)$ has an eigenvalue of modulus one and $\Pi_{i=0}^{\pi(q)-1} L_i(t)$ is hyperbolic for $t < t_0$. Thus, for $t < t_0$ but arbitrarily close to it we have that the index of $\Pi_{i=0}^{\pi(q)-1} L_i(t)$ is the same as q and have an eigenvalue, say λ_k with modulus close to one. We may assume without loss of generality that $|\lambda_k| < 1$.

From lemma 2.1 we find $g \in \mathcal{U}(f)$ such that q is a periodic point of g and $Dg_{g^i(q)} = L_i(t)$. Hence, q is a hyperbolic periodic point of g with $index(q) = index(p_g)$, and $Dg_q^{\pi(q)}$ has an eigenvalue λ_k with modulus close to one. Moreover, as in Proposition 4.2, the perturbation can be made so that $q \sim p_g$. Notice also that we may take g as close as we wish to f by taking ϵ arbitrarily small.

We have two possibilities: either λ_k is complex or real. We shall get to a contradiction in either case. Let us assume first that λ_k is complex. Making another arbitrarily small perturbation if necessary we may assume that λ_k and its conjugate are the eigenvalues whose modulus are closest to one. Let $g_s, -1 \leq s \leq 1$ be a generic one parameter family in $\mathcal{U}(f),\ g_{-1} = g$ that goes under a Hopf Bifurcation at $s = 0$ (see [HK, Chapter 11])). It follows, that for some $s > 0$ (and close to $s = 0$) we have an invariant (periodic) circle $\mathcal{C}$, normally hyperbolic, and such that $g_s^{\pi(q)}/\mathcal{C}$ is conjugated to an irrational rotation. We have also that $W^u(\mathcal{C}) \cap W^s(p_{g_s}) \neq \emptyset$ and $W^s(\mathcal{C}) \cap W^u(p_{g_s}) \neq \emptyset$. It follows that $\mathcal{C} \subset H(p_{g_s}, g_s)$ which is imposible since $H(p_{g_s})$ is expansive.

It remains to arrive to a contradiction in case λ_k is real. For this, we shall use Proposition 4.2. Fix β such that any g which is β close to f is in $\mathcal{U}(f)$. Consider $\rho_0 = \beta/2$ and $\delta_0 = \delta(\rho_0)$ from proposition 4.2. We may take the above g having the hyperbolic periodic point q such that its stable gap is less than δ_0. Morevoer, we may assume (making a small perturbation if necessary) that q is simple. If q is not a non-ss boundary point, using the fact that $q \sim p_g$ we may find another hyperbolic periodic point, that for the sake of simplicity we still denote it by q, such that it is a non-ss boundary point whose stable gap is less than δ_0 (see [PPV, Section 4.1]. Now consider $\rho_i = \beta/2^{i+1}$, $i = 1, 2, ...$ and $\delta_i = \delta(\rho_i)$ from the above proposition. Set $g_0 = g$. Therefore, we can construct a sequence g_n of diffeomorphisms such that:

1. g_n is $\beta/2^n$ close to g_{n-1}.

2. g_n has a finite sequence $q_{1,i}, q_{2,i}, i = 1, 2, .., n$ of periodic points in $H(p_{g_n}, g_n)$ such that $dist(g_n^k(q_{1,i}), g_n^k(q_{2,i})) \leq \delta_i,\ \forall k \in \mathbb{Z},\ i = 1, 2, ..., n.$

Therefore, g_n is a Cauchy sequence in $Diff^1(M)$ and hence converge to some diffeomorphism, say $\tilde{g}$. It follows that $\tilde{g}$ is β close to f and so $\tilde{g} \in \mathcal{U}(f)$. Moreover, there are sequences of periodic points $q_{1,i}, q_{2,i}, i = 1, 2, ...$ in $H(p_{\tilde{g}}, \tilde{g})$ such that

$$\mathrm{dist}(\tilde{g}^k(q_{1,i}), \tilde{g}^k(q_{2,i})) \leq \delta_i,\ \forall k \in \mathbb{Z},\ i = 1, 2, ...$$

This contradicts the expansivity of $H(p_{\tilde{g}}, \tilde{g})$ and we have finished the proof of Theorem B.

5 Shadowing properties

In this section we derive from persistence of expansiveness and germ expansiveness that center-stable and center-unstable manifolds of $f/H(p)$ are true stable and unstable manifolds. Moreover, $f/H(p)$ has a local product structure (see definition 5.1 below) and every f-pseudo-orbit in $H(p)$ is shadowed by a true f-orbit also in $H(p)$. This will be used in the next section to prove hyperbolicity of $H(p)$.

We begin observing that it is easy to see that if $f/H(p)$ is C^1-persistently expansive and, moreover, the $H(p)$-germ of f is expansive then the same holds for f^m, $m \in \mathbb{N}^+$. Moreover, if the f-periodic point q is homoclinically related to p, $p \sim q$, the same holds with respect to f^m. Thus we may assume for the sake of simplicity that $m = 1$ in Theorem B. Therefore, with this observation in mind, we can restate Theorem B as follows: there exist $C > 0$ and $0 < \lambda < 1$ such that for q a periodic point of f homoclinically related to p of period k it holds

$$\prod_{i=0}^{k-1} \|Df_{/E^s(f^i(q))}\| < C\lambda^k \text{ and } \prod_{i=0}^{k-1} \|Df^{-1}_{/E^u(f^{-i}(q))}\| < C\lambda^k .$$

In a set having dominated splitting there are always locally invariant manifolds tangent to the invariant subspaces of the splitting. To be more precise, denote by $s = dimE$ and by $u = dimF$, and let $D^j_r = \{z \in \mathbb{R} : \|x\| \leq r\}$, $j = s, u$. Denote by $Emb_\Lambda(D^j_1, M)$ the space of C^1 embeddings $\beta : D^j_1 \to M$ such that $\beta(0) \in \Lambda$ endowed with the C^1 topology. The following proposition can be found in [HPS] (see also [Ma2, Proposition 2.3]).

Proposition 5.1. *Let $\Lambda = H(p, f)$ having dominated splitting $E \oplus F$. Then there exist $\phi^s : \Lambda \to Emb_\Lambda(D^s_1, M)$ and $\phi^u : \Lambda \to Emb_\Lambda(D^u_1, M)$ such that defining $W^{cs}_\epsilon(x) = \phi^s(x)D^s_\epsilon$ and $W^{cu}_\epsilon(x) = \phi^u(x)D^u_\epsilon$ the following hold:*

1. *$T_xW^{cs}(x) = E(x)$ and $T_xW^{cu}(x) = F(x)$.*
2. *For every $0 < \epsilon_1 < 1$ there exists $0 < \epsilon_2 < 1$ such that $f(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f(x))$ and $f^{-1}(W^{cu}_{\epsilon_2}(x)) \subset W^{cu}_{\epsilon_1}(f^{-1}(x))$.*

We shall call W^{cs}_ϵ and W^{cu}_ϵ the local center-stable and center-unstable manifolds respectively. Observe that for any $\epsilon > 0$ there exists $\rho(\epsilon) > 0$ such that for all $x \in H(p)$, W^{cs}_ϵ contains a ball of radius $\rho(\epsilon)$ inside the local center stable manifold (with respect to the Riemannian metric inherited from M), the same for $W^{cu}_\epsilon(x)$. For the sake of simplicity we shall assume $\rho(\epsilon) = \epsilon$. Also, for $y \in W^{cs}_\epsilon(x)$ we shall denote $E(y) = T_yW^{cs}_\epsilon(x)$, and for $y \in W^{cu}_\epsilon(x)$ we shall denote $F(y) = T_yW^{cu}_\epsilon(x)$,

Lemma 5.2. *Let C, λ be as in Theorem B and let $\delta > 0$ be such that $\lambda' = \lambda(1+\delta) < 1$ and let $q \sim p$. Then, there exists $0 < \epsilon_1 < \epsilon$ such that if for all $0 \leq n \leq \pi(q)$ it holds that for some $\epsilon_2 > 0$, $f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q))$ then*

$$f^{\pi(q)}(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{C\lambda'^{\pi(q)}\epsilon_2}(q)\,.$$

Similarly, if $f^{-n}(W^{cu}_{\epsilon_2}(q)) \subset W^{cu}_{\epsilon_1}(f^{-n}(q))$ then

$$f^{-\pi(q)}(W^{cu}_{\epsilon_2}(q)) \subset W^{cu}_{C\lambda'^{\pi(q)}\epsilon_2}(q)\,.$$

Proof. Let $\epsilon_1 > 0$ be such that if $\operatorname{dist}(x, y) < \epsilon_1$ then

$$(1-\delta)\|Df_x|_{E^s}\| \leq \|Df_y|_{E^s}\| \leq (1+\delta)\|Df_x|_{E^s}\|,$$

and

$$(1-\delta)\|Df^{-1}_x|_{E^u}\| \leq \|Df^{-1}_y|_{E^u}\| \leq (1+\delta)\|Df^{-1}_x|_{E^u}\|.$$

Thus for all $x \in W^{cs}_{\epsilon_2}(q)$, $0 \leq n \leq \pi(q)$ it holds that

$$\prod_{j=1}^{n} \|Df_{f^j(x)}|_{E^s}\| \leq \prod_{j=1}^{n}(1+\delta)\|Df_{f^j(q)}|_{E^s}\| \leq C(\lambda(1+\delta))^n = C\lambda'^n.$$

In particular

$$\prod_{j=1}^{\pi(q)} \|Df_{f^j(x)}|_{E^s}\| \leq \prod_{j=1}^{\pi(q)} (1+\delta)\|Df_{f^j(q)}|_{E^s}\| \leq C(\lambda(1+\delta))^{\pi(q)} = C{\lambda'}^{\pi(q)}.$$

Hence for all arc γ joining q with a point $y \in W^{cs}_{\epsilon_2}(q)$ we have, applying the mean value theorem, that the length of $f^n \circ \gamma$ verifies $\ell(f^n \cdot \gamma) \leq C{\lambda'}^n \ell(\gamma)$ from which the result follows.
Analogously with the center-unstable manifolds. □

Lemma 5.3. *The following statements hold:*
(a) For all $q \sim p$, $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that for all $n \geq 0$: $f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q))$. Similarly for the center-unstable manifolds.
(b) If $\epsilon_1 < \alpha$ where α is an expansivity constant for the f-germ of $H(p)$, then for all $y \in W^{cs}_{\epsilon_2}(q)$, $\lim_{n\to+\infty} \operatorname{dist}(f^n(q), f^n(y)) = 0$.

Proof. To prove (a) let us begin defining

$$\varepsilon(q) = \sup\{\epsilon > 0 \,/\, f^n(W^{cs}_{\epsilon}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q)),\, \forall n \geq 0\}.$$

On account of the periodicity of $q \sim p$ and by Proposition 5.1 and Lemma 5.2 $\varepsilon(q) > 0$. If we prove that the infimum ϵ_2 in $q \in S$ of $\varepsilon(q)$ is positive then we are done. Suppose on the contrary that for some sequence $\{q_n\} \subset S$ we have that $\varepsilon(q_n) \to 0$ when $n \to +\infty$. Let $m_n > 0$ and $y_n \in W^{cs}_{\varepsilon(q_n)}(q_n)$ be such that $\operatorname{dist}(f^{m_n}(q_n), f^{m_n}(y_n)) = \epsilon_1$. By Lemma 5.2

$$f^{\pi(q_n)} W^{cs}_{\varepsilon(q_n)}(q_n) \subset W^{cs}_{C'\lambda^{\pi(q_n)}\varepsilon(q_n)}(q_n).$$

Therefore, without loss of generality, we may assume that $0 < m_n < \pi(q_n)$. It follows that $m_n \to +\infty$ and $\pi(q_n) - m_n \to +\infty$ when $n \to +\infty$. Taking limit points x and y from $f^{m_n}(q_n)$ and $f^{m_n}(y_n)$ respectively, we have that $x \in H(p)$ and on account of the expansivity of the f-germ of $H(p)$ we obtain, for $\epsilon_1 > 0$ small, a contradiction because for all $k \in \mathbb{Z}$ we have $\operatorname{dist}(f^k(x), f^k(y)) \leq \epsilon_1$ which would imply that $x = y$ contradicting that $\operatorname{dist}(x, y) = \epsilon_1$.

The proof of (b) is again a consequence of the expansive properties of the f-germ of $H(p)$ and it is left to the reader.

As we have said above, similar results hold for the center-unstable manifolds. □

Taking into account Lemma 5.3 we see that for $x \sim p$ we have that the local center stable and unstable manifolds are true stable and unstable ones. This allows us to write $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ instead of $W^{cs}_\epsilon(x)$ and $W^{cu}_\epsilon(x)$.

Definition 5.1. *We say that a compact f-invariant set Λ has a local product structure if given $\epsilon > 0$ there exists $\delta > 0$ such that if* $\text{dist}(x, y) < \delta$ *and $x, y \in \Lambda$ then*

$$\emptyset \neq W^s_\epsilon(x) \cap W^u_\epsilon(y) \subset \Lambda.$$

Assuming domination, the manifolds $W^{cs}_\epsilon(x)$ and $W^{cu}_\epsilon(x)$ have not, in general, any dynamic meaning. But in our case where $H(p)$ is persistently expansive and germ-expansive we have that center-stable manifolds and center-unstable manifolds are indeed stable and unstable ones.

Lemma 5.4. *For all $x \in H(p)$ it holds that $W^{cs}_\epsilon(x) = W^s_\epsilon(x)$ and $W^{cu}_\epsilon(x) = W^u_\epsilon(x)$. More precisely: for all $x \in H(p)$, $\epsilon_1 > 0$ there exists $\epsilon > 0$ such that for all $n \geq 0$: $f^n(W^{cs}_\epsilon(x)) \subset W^{cs}_{\epsilon_1}(f^n(x))$. Moreover if $\epsilon_1 < \alpha$ where $\alpha > 0$ is an expansivity constant for the f-germ of $H(p)$, then for all $y \in W^{cs}_{\epsilon_2}(x)$,* $\lim_{n\to+\infty} \text{dist}(f^n(x), f^n(y)) = 0$. *Similarly for the center-unstable manifolds. Moreover, there exists a local product structure for $H(p)$.*

Proof. First of all if $x \sim p, y \sim p$, then choosing $\epsilon = \epsilon_2$ as in Lemma 5.3 and using the fact that $W^s_\epsilon(x)$ is tangent to E and $W^u_\epsilon(x)$ is tangent to F and $\angle(E, F) \geq \gamma_0 > 0$ and $E_x \oplus F_x = T_xM$ we can see that there exists $\delta > 0$ such that $W^s_\epsilon(x) \cap W^u_\epsilon(y) \neq \emptyset$ whenever $\text{dist}(x, y) < \delta$. As x, y are homoclinically related with p we also have that $W^s_\epsilon(x)$ is accumulated by $W^s(p)$ and $W^u_\epsilon(x)$ is accumulated by $W^u(p)$ by the λ-lemma. Thus if $z \in W^s_\epsilon(x) \cap W^u_\epsilon(y)$ then $z \in H(p)$. Therefore if $\epsilon < \alpha/2$ with $\alpha > 0$ an expansivity constant then z is unique.

In the general case let $x, y \in H(p)$ and take ϵ and $\delta' = \delta/2$ where $\epsilon > 0$ and $\delta > 0$ are the same as above. Take sequences $\{x_n\}$ and $\{y_n\}$ of periodic points homoclinically related to p converging to x and y respectively. As periodic points homoclinically related to p are dense in $H(p)$ (see [Sm1]) such sequences exist. Then $W^s_\epsilon(x_n)$ converges to $D(x) \subset W^{cs}_\epsilon(x)$ and $W^u_\epsilon(y_n)$ converges to $C(x) \subset W^{cu}_\epsilon(x)$ in the Hausdorff sense and are tangent respectively to E and F. As $W^s_\epsilon(x_n) \cap W^u_\epsilon(y_n) = \{z_n\}$ we have that $z_n \to z \in D(x) \cap C(y)$. As $z_n \in H(p)$ and $H(p)$ is closed we conclude that $z \in H(p)$. We remark that in fact $D(x) \subset W^s_\epsilon(x)$ and $C(y) \subset W^u_\epsilon(y)$. $\square$

Recall that a sequence $\{x_n : n \in \mathbb{Z}\}$ is called an ϵ-pseudo orbit provided $\text{dist}(f(x_n), x_{n+1}) < \epsilon \; \forall n \in \mathbb{Z}$. It is said periodic if there exists m such that $x_{m+n} = x_n \; \forall n \in \mathbb{Z}$. And it is said to be δ shadowed if there exits x such that $\text{dist}(f^n(x), x_n) < \delta \; \forall n \in \mathbb{Z}$.

Lemma 5.5. *Let $H(p)$ be C^1-persistently expansive and germ-expansive. Given $\delta > 0$ there exists $\epsilon > 0$ such that any ϵ-pseudo-orbit $\{x_n\} \subset H(p)$ is δ-shadowed* *by an orbit in $H(p)$. Moreover, if ϵ, δ are less than half the expansivity constant then the orbit is unique (and periodic if the pseudo orbit is periodic).*

Proof. Fathi has proved in [Ft] that if a compact f-invariant set Λ is expansive for f there exists a distance D, defining the topology of Λ, such that with respect to that distance f behaves hyperbolically. More exactly, there exist $r > 0$ and $0 < \lambda < 1$ such that if $x, y \in \Lambda$ are at D-distance less than r then if $y \in W^s_\epsilon(x)$ then $D(f(x), f(y)) \leq \lambda D(x, y)$ and if $y \in W^u_\epsilon(x)$ then $D(f^{-1}(x), f^{-1}(y)) \leq \lambda D(x, y)$. By Lemma 5.4 there exists a local product structure in $H(p)$ and $H(p)$ is expansive for f. Thus the proof is similar to that given in [Bo, Proposition 3.6]. □

Proposition 5.6. *Let $H(p)$ be persistently expansive and germ-expansive and let $q \in H(p)$ be a periodic point. Then we have that*

1. *$W^s(q) \cap W^u(p) \neq \emptyset$, $W^s(p) \cap W^u(q) \neq \emptyset$,*
2. *q is hyperbolic and index(p) = index(q),*

Proof. By Lemma 5.4 we have that if x is sufficiently close to y then $W^s_\epsilon(x)$ cuts $W^u_\epsilon(y)$ and $W^u_\epsilon(x)$ cuts $W^s_\epsilon(y)$. Since homoclinically related periodic points are dense in $H(p)$ we have a hyperbolic periodic point q_1, homoclinically related to p as close as we wish to q. Therefore $W^s_\epsilon(q_1)$ cuts $W^u_\epsilon(q)$ and $W^u_\epsilon(q_1)$ cuts $W^s_\epsilon(q)$ (we are <u>not</u> assuming that q is hyperbolic but merely using Lemma 5.4). It follows that $\dim(W^s(q)) = \dim(W^s(q_1)) = \dim(W^s(p))$. By the C^0 adaptation of the λ-lemma, $W^u(p)$ accumulates in $W^u_\epsilon(q_1)$ and $W^s(p)$ accumulates in $W^s_\epsilon(q_1)$. Thus, by domination, $W^u(p)$ cuts $W^s_\epsilon(q)$ and $W^s(p)$ cuts $W^u_\epsilon(q)$.

If we show that q is hyperbolic then index(p) = index(q). Otherwise we can slightly perturb f to obtain g such that q is hyperbolic and $q \sim p_g$ and having a eigenvalue arbitrarily close to one: this contradicts Theorem B. □

As a consequence of the previous lemma and Theorem B we have:

Corollary 5.7. *If $H(p)$ is persistently expansive and germ expansive then there exist $C > 0$ and $0 < \lambda < 1$ such that if q is a periodic point in $H(p)$ of period $\pi(q)$ then*

$$\prod_{i=0}^{\pi(q)-1} \|Df_{/E^s(f^i(q))}\| < C\lambda^{\pi(q)} \quad \text{and} \quad \prod_{i=0}^{\pi(q)-1} \|Df^{-1}_{/E^u(f^{-i}(q))}\| < C\lambda^{\pi(q)}.$$

6 Proof of Theorem C

In this section we prove that if $H(p)$ is persistently expansive and has the shadowing property then it is hyperbolic. As we have mentioned in the introduction we first show that if $H(p)$ is an attractor (repeller) or maximal invariant for U open, then it is germ-expansive.

Definition 6.1. *Let $f: M \to M$ be a C^1-diffeomorphism. We say that a compact invariant set $\Lambda \subset M$ is an attractor (resp.: repeller) if there exists a neighbourhood $U \supset \Lambda$ such that $\Lambda = \cap_{n=0}^{+\infty} f^n(U)$ (resp.: $\Lambda = \cap_{n=0}^{-\infty} f^n(U)$). We say that Λ is the maximal invariant set of the open subset $U \supset \Lambda$ if $\Lambda = \cap_{n \in \mathbb{Z}} f^n(U)$.*

It is proven in [Sm2, Lemma 4.2] that if Λ is an attractor (repeller) we may assume that for U it holds that $\overline{f(U)} \subset U$ (resp.: $\overline{f^{-1}(U)} \subset U$).

Lemma 6.1. *If $H(p)$ is an attractor (repeller or maximal invariant for U open) and $f/H(p)$ is expansive then the $H(p)$-germ of f is expansive.*

Proof. We give the proof for the case when Λ is an attractor. Let U be a neighborhood of $H(p)$ as in definition 6.1. As $f/H(p)$ is expansive with a constant of expansiveness $\alpha > 0$ if $x, y \in H(p)$ and $\text{dist}(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{Z}$ then $x = y$. Let $\delta > 0$ be such that the δ-neighborhood of $H(p)$ is contained in U. Assume without loss of generality that $\delta < \alpha$. If now $x \in H(p)$ and $y \in M$ are such that $\text{dist}(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ then, in particular, $f^{-j}(y) \in U$ for all $j \geq 0$ and hence $y \in \cap_{j \geq 0} f^j(U) = H(p)$. Therefore $x = y$. □

Let us restate corollary 5.7 without the assumption $m = 1$. The proof is immediate from the results of section 5.

Corollary 6.2. *If $H(p)$ is persistently expansive and germ expansive then there exist $C > 0$, $0 < \lambda < 1$ and $m > 0$ such that if q is a periodic point in $H(p)$ of period $\pi(q)$ then*

$$\prod_{i=0}^{k-1} \|Df^m_{/E^s(f^{im}(q))}\| < C\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \|Df^{-m}_{/E^u(f^{-im}(q))}\| < C\lambda^k$$

where $k = [\pi(q)/m]$

Now, we shall conclude the proof of Theorem C. Let λ and m as above, and take $\gamma > 0$ such that $\lambda(1 + \gamma) < 1$. It is not difficult to prove that there exists δ such that if $dist(x, y) \leq \delta, x, y \in H(p)$ then

$$1 - \gamma \leq \frac{\|Df^m_{/E(x)}\|}{\|Df^m_{/E(y)}\|} \leq 1 + \gamma.$$

Let $K = inf\{\|Df^m_{/E(x)}\| : x \in H(p)\}$ and let $\epsilon = \epsilon(\delta)$ from lemma 5.5.

We have to prove that if $H(p)$ is persistently expansive and germ expansive then it is hyperbolic. In other words we need to prove that $\|Df^n_{/E(x)}\| \to 0$ as $n \to \infty$ and $\|Df^{-n}_{/F(x)}\| \to 0$ as $n \to \infty$ for any $x \in H(p)$. Let us show only that $\|Df^n_{/E(x)}\| \to 0$ as $n \to \infty$, the other one being similar. For this, it is enough to show that for any $x \in H(p)$ there exists $k = k(x)$ such that

$$\prod_{i=0}^{k} \|Df^m_{/E(f^{im}(x))}\| < \frac{1}{2}.$$

Arguing by contradiction, assume this does not hold. Then, there exist sequences $x_n \in H(p)$ and $k_n \to \infty$ such that

$$\prod_{i=0}^{k} \|Df^m_{/E(f^{im}(x_n))}\| \geq \frac{1}{2},\ 0 \leq k \leq k_n.$$

Let z be an accumulation point of x_n. It follows that

$$\prod_{i=0}^{k} \|Df^m_{/E(f^{im}(z))}\| \geq \frac{1}{2} \quad \forall k \geq 0.$$

Let w be an accumulation point of the sequence $f^{im}(z)$, i. e., there exits $i_k \to \infty$ such that $f^{i_k m}(z) \to w$. Since $f/H(p)$ is topologically mixing, there exist $z_1 \in H(p)$ and n_0 such that $dist(z_1, w) < \epsilon/2$ and $dist(f^{n_0 m}(z_1), z) < \epsilon$.

Take j_0 such that

$$C((1+\gamma)\lambda)^{j+n_0} < \frac{1}{2}K^{n_0} \quad \forall j \geq j_0.$$

Finally, choose $i_k > j_0$ such that $dist(f^{i_k m}(z), w) < \epsilon/2$. Consider now the ϵ periodic pseudo orbit defined by $\{z, ..., f^{i_k m}(z), z_1, ..., f^{n_0 m}(z_1), z\}$. By lemma 5.5 there exists a periodic point q that δ-shadows the pseudo orbit and $(i_k + n_0)m$ is (a multiple of) $\pi(q)$. Therefore

$$\begin{aligned}
\frac{1}{2}K^{n_0} &\leq \prod_{i=0}^{i_k-1} \|Df^m_{/E(f^{im}(z))}\| \prod_{i=0}^{n_0-1} \|Df^m_{/E(f^{im}(z_1))}\| \\
&\leq \prod_{i=0}^{i_k+n_0-1} (1+\gamma)\|Df^m_{/E^s(f^{im}(q))}\| \\
&\leq C((1+\gamma)\lambda)^{i_k+n_0} < \frac{1}{2}K^{n_0},
\end{aligned}$$

a contradiction. This completes the proof of Theorem C.

References

[Ao] N. Aoki, *The set of axiom A diffeomorphisms with no cycles.,* Bol. Soc. Bras. Mat., Nova Sér., **23, No.1-2,** (1992), p. 21-65.

[Bo] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Springer Lecture Notes in Math, **470,** 1975.

[Fr] J. Franks, *Necessary conditions for stability if diffeomorphisms,* Trans. Amer. Math. Soc., **158** (1971), p. 301-308.

[Ft] Fathi A., *Expansiveness, Hyperbolicity and Hausdorff Dimension,* Communications in Mathematical Physics, **126** (1989), p. 249-262.

[Ha] Hayashi, S., *Diffeomorphisms in $\mathcal{F}^\infty(\mathcal{M})$ satisfy Axiom A.,* Ergodic Theory Dyn. Syst., **12, No.2** (1992), p. 233-253.

[HK] Hale J., Koçak H., *Dynamics and Bifurcation,* Springer-Verlag, **Texts in applied Mathematics 3** 1991.

[HPS] Hirsch, M.W.; Pugh, C.C.; Shub, M., *Invariant manifolds,* Lecture Notes in Mathematics. 583, **Springer-Verlag** 1977.

[Ma1] R. Mañé, *Expansive diffeomorphisms,* Lectures Notes in Mathematics, Springer, Berlin, **468** (1975), p. 162-174.

[Ma2] R. Mañé, *Contributions to the Stability Conjecture,* Topology, **17** (1978), p. 383-396.

[Ma3] R. Mañé, *An ergodic closing lemma,* Annals of Mathematics, **116** (1982), p. 503-540.

[Nh] S. E. Newhouse, *Hyperbolic Limit Sets,* Trans. of the Amer. Math. Soc., **Vol. 167** (May 1972), p. 125-150.

[PPV] M. J. Pacifico, E. R. Pujals, J. L. Vieitez, *Robustly expansive homoclinic classes,* Accepted for publication in Erg Th & Dyn Sys, (2004), p. -.

[PS1] E. Pujals, M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms,* Annals of Mathematics, **151** (2000), p. 961-1023.

[PS2] E. Pujals, M. Sambarino, *On the Dynamics of Dominated Splitting,* Preprint, (2002), p. -.

[PT] Palis, J; Takens, F., *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations,* Cambridge Studies in Advanced Mathematics. 35, **Cambridge University Press** 1993.

[Sm1] S. Smale, *Diffeomorphisms with many periodic points,* Differential and Combinatorial Topology, **Princeton University Press** 1965.

[Sm2] S. Smale, *The* Ω*-stability theorem,* Proc. Sympos. Pure Math. AMS, **vol. 14** (1970), p. 289-297.

[We] L. Wen, *Homoclinic Tangencies and Dominated Splittings,* Nonlinearity, **15** (No.5), p. 2002-1445.1469

M. Sambarino
Centro de Matematica,
Facultad de Ciencias,
Igua 4225 esq. Mataojo
Universidad de la Republica,
Montevideo, Uruguay
samba@cmat.edu.uy,

J. L. Vieitez,
Instituto de Matematica,
Facultad de Ingenieria,
Herrera y Reissig 565,
Universidad de la Republica,
Montevideo, Uruguay
jvieitez@fing.edu.uy