ROTATION SET AND ENTROPY

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Abstract. In 1991 Llibre and MacKay proved that if f is a 2-torus homeomorphism isotopic to identity and the rotation set of f has a non empty interior then f has positive topological entropy. Here, we give a converselike theorem. We show that the interior of the rotation set of a 2-torus $C^{1+\alpha}$ diffeomorphism isotopic to identity of positive topological entropy is not empty, under the additional hypotheses that f is topologically transitive and irreducible.

1. Introduction

1.1. **History.** The theory of dynamical systems began with Henri Poincaré's approach to studying toral flows. It consists in passing to the first return map on a topological circle. Hence, the initial requirement is replaced by a qualitative study of dynamical properties of a circle map. Let $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a circle homeomorphism and $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ be a lift of f. The Poincaré's rotation number of f is defined as

$$\rho(f) = \lim_{n\to+\infty} \frac{\tilde{f}^n(x) - x}{n} \pmod 1.$$

It's easy to see that this limit exists and depends neither on the point x in $\mathbb{R}$ nor on the lift $\tilde{f}$ of f.

From the definition, the formulas $\rho(f^n) = n\,\rho(f)$ and $\rho(h \circ f \circ h^{-1}) = \rho(f)$ hold for any orientation preserving circle homeomorphism h. If h is orientation reversing then $\rho(h \circ f \circ h^{-1}) = -\rho(f)$.

The rotation number gives rise to a description of the dynamical behavior of circle homeomorphisms. Poincaré proved that:

Poincaré's Theorem. *Let f be an orientation preserving circle homeomorphism with rotation number ρ. Then*

(1) the rotation number ρ is rational if and only if f has a periodic point;

(2) if the rotation number ρ is irrational, then f is semi-conjugate to R_ρ the rotation by ρ, that is there exists a continuous degree one monotone circle map h such that $h \circ f = R_\rho \circ h$.

The most natural generalization of circle homeomorphisms are 2-torus homeomorphisms isotopic to identity.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus and $\Pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ be the natural projection. Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a continuous map and $\widetilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ a lift of f, that is, $f \circ \Pi = \Pi \circ \widetilde{f}$. If $\widetilde{f}_1$ and $\widetilde{f}_2$ are two lifts of f, it holds that there exists $v \in \mathbb{Z}^2$ such that $\widetilde{f}_1(\widetilde{x}) = \widetilde{f}_2(\widetilde{x}) + v$ for every

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$\widetilde{x} \in \mathbb{R}^2$, and if f is isotopic to identity then for every $v \in \mathbb{Z}^2$ one has $\widetilde{f}(\widetilde{x}+v) = \widetilde{f}(\widetilde{x}) + v$.

In order to generalize the rotation number, we can consider the sequences of 2-vectors $\{\frac{\tilde{f}^n(\widetilde{x})-\widetilde{x}}{n}\}_{n\in\mathbb{N}}$. But these sequences may not converge and even in the case that the limit exists, it may depend on the point $\widetilde{x}$. To avoid that difficulty, Misiurevicz and Ziemian ([MZ89]) have proposed to define *a rotation set* as follows.

1.2. **Definitions of the rotation set and the rotation vectors.** Let f be a 2-torus homeomorphism isotopic to identity and $\tilde{f}$ a lift of f, we call $\tilde{f}$**-rotation set** the subset of $\mathbb{R}^2$ defined by

$$\rho(\tilde{f}) = \bigcap_{i=1}^{\infty} \overline{\bigcup_{n\geq i} \left\{ \frac{\tilde{f}^n(\widetilde{x}) - \widetilde{x}}{n}, \quad \widetilde{x} \in \mathbb{R}^2 \right\}}.$$

Equivalently, $(a,b) \in \rho(\tilde{f})$ if and only if there exist sequences $(\widetilde{x}_i)$ with $\widetilde{x}_i \in \mathbb{R}^2$ and $n_i \to \infty$ such that

$$(a,b) = \lim_{i\to\infty} \frac{\tilde{f}^{n_i}(\widetilde{x}_i) - \widetilde{x}_i}{n_i}.$$

Let $\widetilde{x}$ be in $\mathbb{R}^2$, the $\widetilde{f}$**-rotation vector of** $\widetilde{x}$ is the 2-vector defined by $\rho(\widetilde{f},\widetilde{x}) = \lim_{n\to\infty} \dfrac{\tilde{f}^n(\widetilde{x}) - \widetilde{x}}{n} \in \mathbb{R}^2$ if this limit exists.

1.3. **Some classical properties and results on the rotation set.** Let f be a 2-torus homeomorphism isotopic to the identity and $\widetilde{f}$ be a lift of f to $\mathbb{R}^2$.

- Let $\widetilde{x} \in \mathbb{R}^2$ such that $\rho(\widetilde{f},\widetilde{x})$ exists, it holds that $\rho(\widetilde{f},\widetilde{x}) \in \rho(\widetilde{f})$.
- If $\widetilde{f}$ has a fixed point then $(0,0) \in \rho(\widetilde{f})$.
- Misiurewicz and Ziemian (see [MZ89])have proved that:
  (1) $\rho(\tilde{f}^n) = n\rho(\tilde{f})$
  (2) $\rho(\tilde{f} + (p,q)) = \rho(\tilde{f}) + (p,q)$,
  (3) the rotation set is a compact convex subset of $\mathbb{R}^2$.
- Franks (see [Fra89]) proved that any rational point q in $\operatorname{int}\rho(\tilde{f})$ is the rotation vector of a lift of a f-periodic point. That is there exists a f-periodic point $x \in T^2$ and a lift $\widetilde{x}$ of x, such that $\lim_{n\to\infty} \dfrac{\tilde{f}^n(\widetilde{x}) - \widetilde{x}}{n} = q$.
- The rotation set is not a conjugacy invariant. However, if the conjugating homeomorphism h is isotopic to identity, then the homeomorphism $\tilde{f}$ and its conjugate homeomorphism $\tilde{h} \circ \tilde{f} \circ \tilde{h}^{-1}$ have the same rotation set. Anyway, the property of having a lift with a rotation set of non empty interior does not depend on the choice of the lift and it is a conjugacy invariant.

1.4. **Relationship between the rotation set and the entropy.** An important conjugacy invariant is the **topological entropy**, it can be defined for $f\colon X \to X$ as $h_{top}(f) = \lim_{\epsilon \to 0} h_{top}(f, \epsilon)$, where $h_{top}(f, \epsilon) = \limsup_{n \to +\infty} \frac{1}{n} \log S(f, \epsilon, n)$ and $S(f, \epsilon, n)$ is the cardinality of a minimal (n, ϵ) spanning set (i.e a set E such that $X = \bigcup_{x \in E} B_f(x, \epsilon, n)$, where $B_f(x, \epsilon, n)$ are dynamical balls).

A result of Katok ([Kat80]) claims that for $C^{1+\alpha}$ surface diffeomorphisms the topological entropy is majorated by the growth rate of periodic points. Therefore, any $C^{1+\alpha}$ surface diffeomorphism without periodic points has null topological entropy.

In [Lli91], Llibre and MacKay proved that any toral homeomorphism, isotopic to the identity and such that the interior of its rotation set is not empty, has positive topological entropy.

1.5. **Remarks, questions and statement.** The converse of this result by Llibre and MacKay does not hold. We will show examples where the rotation set has empty interior but the topological entropy is positive (see Section 4 -examples 2, 3, 4). So, we are interested in conditions implying that the interior of the rotation set is not empty. In his thesis Kwapish ([Kwa95]) proved that any Pseudo-Anosov homeomorphism relative to a finite set (in the sense of Handel) has rotation set with non empty interior. Our aim is to give dynamical conditions (in addition to positive entropy) to obtain the same conclusion. One of the conditions we will ask for, is that f be topologically transitive.

Definition. A homeomorphism f on M is **topologically transitive** if there exists a point x_0 of M such that the f-orbit of x_0 is dense.

We will prove the following result:

Theorem 1. *Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism isotopic to identity satisfying the following conditions:*
(1) f is of class $C^{1+\alpha}$;
(2) the topological entropy of f is positive;
(3) f is topologically transitive;
(4) f is irreducible (see section 2 for the definition);
then $\mathrm{int}(\rho(\widetilde{f})) \neq \emptyset$*, where $\widetilde{f}$ is a lift of f to $\mathbb{R}^2$.*

Remark 1.1. *The hypotheses 3 and 4 imply that any finite covering of f is topologically transitive. A related fact will be proved in lemma 4.1. Moreover, we prove that the conditions 1, 2 and the condition that any finite covering of f is topologically transitive imply that* $\mathrm{int}(\rho(\widetilde{f})) \neq \emptyset$*.*

Remark 1.2. *Any $C^{1+\alpha}$ Pseudo-Anosov map (in the sense of Handel) f satisfies hypotheses 1, 2 and it is topologically transitive. Since any lift to a finite covering of a Pseudo-Anosov map is also Pseudo-Anosov, f satisfies the hypotheses of the previous remark. As a consequence we get, as it has been already proved by Kwapisz, that* $\mathrm{int}(\rho(\widetilde{f})) \neq \emptyset$*.*

In Section 2 we define the notion of irreducibility and explain how it arises in our context. Roughly speaking, according to a result in [Lli91], the existence of non null homotopic f-invariant circle implies that the rotation set of $\widetilde{f}$ has empty interior. Hence, we have to avoid this case, but not only, as other invariant sets can play a similar role (the pseudo-circles that arise in Anosov-Katok construction (see [Her86]), for example). We prove the main result in Section 3 and in Section 4 we exhibit different examples showing that the hypotheses are necessary.

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2. Irreducibility and invariant circles

2.1. **Irreducibility.** Let M be a manifold and f a homeomorphism of M, it is denoted by (f, M).

Definition 2.1. *A subset K of M is* **essential** *if K is not contained in a disk and there exists a finite covering M_N of M such that $M_N \backslash K_N$ is not connected, where K_N stands for the lift of K to M_N.*
A homeomorphism (f, M) is **irreducible** *if there is no compact, f-invariant, of empty interior set which is essential.*

Remark 2.1. *If f admits a non null homotopic periodic circle C then f is not irreducible. In fact, the orbit of C is a closed invariant subset O_C and it has empty interior. In the double covering of M associated to C, the lift of C and therefore the lift of O_C is a disconnecting set.*

Property 2.1. *If f is an irreducible homeomorphism then any finite covering of f is irreducible.*

Proof. Suppose that there exist (M_0, f) irreducible and (M_1, f_1) a finite lift of (M_0, f) that is not irreducible. So there exists a compact set K_1 that is f_1-invariant, of empty interior and essential. Let $\pi_0 \colon M_1 \to M_0$ be the natural projection. We claim that the set $K_0 \colon = \pi_0(K_1)$ is compact, f-invariant, of empty interior and essential, proving that (M_0, f) is not irreducible.
We first prove that the set K_0 is f-invariant.
Since $\pi_0 \circ f_1 = f \circ \pi_0$, we have $\pi_0 \circ f_1(K_1) = f \circ \pi_0(K_1)$. Thus $\pi_0(K_1) \supseteq f(\pi_0(K_1)$ that is $K_0 \supseteq f(K_0)$.
The set K_0 is clearly compact and of empty interior because π_0 is a local homeomorphism. It remains to prove that it is essential.
Denote by M the finite covering of (M_1, f_1) such that $M \backslash \pi_1^{-1}(K_1)$ is not connected, where $\pi_1 \colon M \to M_1$ is the natural projection. So $M \backslash \pi_1^{-1}(K_1)$ can be written as the union of two disjoint open sets A and B. Denote by $\pi \colon M \to M_0$ the natural projection and $D = \pi^{-1}(K_0)$. We have that $D = \bigcup_{i=1}^{n} \pi_1^{-1}(\gamma_i(K_1))$ where $\{\gamma_i\}$ stands for the finite group consisting in the automorphisms of the covering π_0. It holds that D is closed and has empty interior.

So $M\backslash D = (A \cup B)\backslash D$ since $\pi_1^{-1}(K_1) \subset D$. Consequently, $M\backslash D = (A\backslash D) \cup (B\backslash D)$ where $A\backslash D$ and $B\backslash D$ are non empty open sets. Thus $M\backslash D$ is not connected, it remains to prove that K_0 is not contained in a disk. If K_0 is included in a disk D_0 then K_1 is contained in a finite union of disjoint disks, then we can construct a disk D_1 that contains this union and therefore $K_1 \subset D_1$, this contradicts the irreducibility of f.

□

2.2. **Invariant circles.** Given a homeomorphism f isotopic to the identity on $\mathbb{T}^2$, we are interested in relationships between the existence periodic circles and the interior of the rotation set.

In the case where f admits a homotopically non-trivial invariant single curve, Llibre-Mac Kay (see [Lli91]) proved that the all the rotation vectors of f-periodic points are collinear, therefore the rotation set has empty interior. In the case where f admits a homotopically non-trivial periodic single curve, the rotation set has empty interior since $\rho(\tilde{f}^n) = n\rho(\tilde{f})$.

On the other hand, is there a relationship between the existence of homotopically trivial f-invariant single curve and the interior of the rotation set of $\tilde{f}$?

Let us show two examples:

- The identity map on T^2 fixes every circle and its rotation set is $\{(0,0)\}$.
- In [Lli91], the authors give examples of T^2-homeomorphisms having rotation set of non empty interior. Let us consider the particular example f given by one of its lifts $\widetilde{f}$ to $\mathbb{R}^2$. Let $\mathrm{frac}(x) = x - \lfloor x \rfloor$ be the fractional part of x (where $\lfloor x \rfloor$ is the floor function of x). We define $h, g \colon \mathbb{R}^2 \to \mathbb{R}^2$ as $h(x,y) = (x, y + \mathrm{frac}(2x))$ if $\mathrm{frac}(x) \in [0, \frac{1}{2}]$ and $h(x,y) = (x, y + 2 - \mathrm{frac}(2x))$ if $\mathrm{frac}(x) \in [\frac{1}{2}, 1]$. Analogously $g(x,y) = (x + \mathrm{frac}(2y), y)$ if $\mathrm{frac}(y) \in [0, \frac{1}{2}]$ and $g(x,y) = (x + 2 - \mathrm{frac}(2y), y)$ if $\mathrm{frac}(y) \in [\frac{1}{2}, 1]$. Let us define $\widetilde{f} = g \circ h$. It holds that its rotation set is $[0,1]^2$. Actually, the point $(0,0)$ is fixed by $\widetilde{f}$, its rotation vector is $(0,0)$, also $\widetilde{f}(\frac{1}{2}, 0) = (\frac{1}{2}, 0) + (0,1)$, $\widetilde{f}(0, \frac{1}{2}) = (0, \frac{1}{2}) + (1,0)$ and $\widetilde{f}(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}) + (1,1)$. Then $\rho(\widetilde{f}, (\frac{1}{2}, 0)) = (0,1)$, $\rho(\widetilde{f}, (0, \frac{1}{2})) = (1,0)$, and $\rho(\widetilde{f}, (\frac{1}{2}, \frac{1}{2})) = (1,1))$.

  We are going to modify this example in order that f have an invariant homotopically trivial circle and that $\widetilde{f}$ have still the same rotation set. Let us explain it. The point $(0,0)$ is fixed by f, we replace it by a small disk D by blowing up. This construction does not change the rotation set because of the following facts:
  - the points in D have the same rotation vector than $(0,0)$ (D is $\widetilde{f}$-invariant)
  - the three other vertices of the rotation set are unchanged since they are realized by points for which the blow up did not change the orbits.

  On the other hand, it holds that ∂D is a homotopically trivial single curve which is invariant by this perturbation of f.

Remark 2.2. *Since, there is no relationship between the existence of homotopically trivial f-invariant single curves and the interior of the rotation set of $\widetilde{f}$, we ask for K not to be contained in a disk, in the definition of an essential set.*

Remark 2.3. *There exist compact sets that are f-invariant, with empty interior and essential but that are not circles (they are not even locally connected* [Her86] *and they are* *called pseudo-circles). It is possible to change an invariant non null-homotopic circle by a connected invariant set that disconnects the torus and that is not locally connected. That is the reason why –in the definition of irreducibility– we ask for the non existence of a compact, f-invariant, of empty interior set (instead of a circle) which is essential.*

3. Proof of Theorem 1

In this section we prove Theorem 1.

Proof. Denote by $T_4^2 := \mathbb{R}^2/(2\mathbb{Z})^2$ the 4-1 covering of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and let the natural projections be $\Pi\colon \mathbb{R}^2 \to \mathbb{T}^2$, $\Pi_4\colon \mathbb{R}^2 \to T_4^2$ and $P\colon T_4^2 \to \mathbb{T}^2$. Note that $\Pi = P \circ \Pi_4$. Let $\tilde{f}_4\colon T_4^2 \to T_4^2$ be a lift of f to T_4^2. Let us endow T^2 and T_4^2 with their usual flat Riemanian metrics (inherited of the standard euclididian metric on $\mathbb{R}^2$) and the associated distances.

By hypothesis, f is a $C^{1+\alpha}$ diffeomorphism and $h_{top}(f) > 0$ therefore $\tilde{f}_4$ is also a $C^{1+\alpha}$ diffeomorphism and $h_{top}(\tilde{f}_4) > 0$.

By Katok (see [Kat80]), there exist $n \in \mathbb{N}$ and a hyperbolic periodic point y_0 of $\tilde{f}_4$ with period n such that the intersection of the stable and unstable manifolds of y_0 is transversal. Thus, there exists $k \in \mathbb{N}$ the minimal positive number such that $\tilde{f}_4^{nk}(y_0) = y_0$ and both eigenvalues of the differential $D\tilde{f}_4^{nk}(y_0)$ are positive. In what follows, we denoted $\tilde{f}_4^{nk}$ by f_4. As pointed in Remark 1.1 we will assume that f_4 is topologically transitive.

Let us denote $x_0 = P(y_0)$. Since P is a local diffeomorphism, x_0 is a hyperbolic fixed point of f^{nk} and it has the local type of y_0. Let $\widetilde{x_0} \in [0,1] \times [0,1]$ be a lift of x_0 and F be a lift of f^{nk} to $\mathbb{R}^2$ such that $F(\widetilde{x_0}) = \widetilde{x_0}$. Note that F is a lift of both f^{nk} and f_4.

We define $\widetilde{x}_1 = \widetilde{x_0}$, $\widetilde{x}_2 = \widetilde{x_0} + (1,0)$, $\widetilde{x}_3 = \widetilde{x_0} + (0,1)$ and $\widetilde{x}_4 = \widetilde{x_0} + (1,1)$. Since F is isotopic to identity, then $F(\widetilde{x_0} + (a,b)) = F(\widetilde{x_0}) + (a,b) = \widetilde{x_0} + (a,b)$, for any $(a,b) \in \mathbb{Z}^2$. Then every $\widetilde{x}_i$ is a fixed point F and therefore its projection on T_4^2 denoted by x_i is fixed by f_4. Note that $\exists i \in \{1,2,3,4\}$ such that $y_0 = x_i$.

Proposition 1. *There exists $0 < \epsilon < \frac{1}{2}$ such that for $i \in \{2,3,4\}$ there exist $n_i \in \mathbb{N}$ and non empty compact sets $L_i \subset B_\epsilon(x_i) \subset T_4^2$ and $L_1^i \subset B_\epsilon(x_1)$ such that $L_i = f_4^{n_i}(L_1^i)$ and $P(L_1^i) = P(L_i)$.*

Proof of Proposition 1. By the classical Hartman-Grobman's theorem, there is an open subset U of $\mathbb{T}^2$ containing x_0 such that the restriction of f^{nk} to U is topologically conjugated to its differential $Df^{nk}(x_0)$. By conjugating f^{nk} by a suitable homeomorphism with support in a small compact $K \supset U$, we may suppose that f^{nk} is a linear diagonal map in U with eigenvalues $0 < \lambda_1 < 1 < \lambda_2$.

Fix $\epsilon > 0$ sufficiently small so that the ball $B_\epsilon(x_0) \subset U$ and the lifts by P of it are disjoints ϵ-balls $B_\epsilon(x_i) \subset T_4^2$, $i = 1, ..., 4$ (this fact is realized by taking $\epsilon < \frac{1}{2}$). Restricted to these balls, f_4 is a linear map.

Let x be in T_4^2, we denote by $W^s(x)$ [resp. $W^u(x)$] the stable [resp. unstable] manifold of x for f_4. For any $0 < \delta \leq \epsilon$ and $x \in T_4^2$, let's denote by $W_\delta^s(x)$ [resp. $W_\delta^u(x)$] the connected component of $W^s(x) \cap B_\delta(x)$ [resp. $W^u(x) \cap B_\delta(x)$] containing x.

Since $W^s(x_1)$ and $W^u(x_1)$ have a transverse intersection in some point p, there is:

-$N\in\mathbb{N}$ such that for $n\geq N$, $f^n(p)\in W^s_\epsilon(x_1)$ (these points converge monotonically to x_1 when n goes to $+\infty$) and

-$M\in\mathbb{N}$ such that for $n\geq M$, $f^{-n}(p)\in W^u_\epsilon(x_1)$ (these points converge monotonically to x_1 when n goes to $+\infty$).

Consider a small arc Σ^u of $W^u(x_1)$ containing $f^N(p)$, the segments $f^n(\Sigma^u)$ for $n\geq N$ become larger and more vertical with n, so there is $n\geq N$ minimal such that $d(x_1,f^n(p))\leq\frac{\epsilon}{2}$ and the arc $f^n(\Sigma^u)$ intersects transversally the boundary of $B_\epsilon(x_1)$ in two points.

Analogously, consider an arc Σ^s of $W^s(x_1)$ containing $f^{-M}(p)$, there is $n'\geq M$ minimal such that $d(x_1,f^{-n'}(p))\leq\frac{\epsilon}{2}$ and the arc $f^{-n'}(\Sigma^s)$ (almost horizontal) intersects transversally the boundary of $B_\epsilon(x_1)$ in two points.

The arcs $f^n(\Sigma^u)\cap B_\epsilon(x_1)$ and $f^{-n'}(\Sigma^s)\cap B_\epsilon(x_1)$ intersect transversally.

Finally, we define a rectangle R_1 in T^2_4 whose boundary is the union of arcs $C^j_{1,s}$ for $j=1,2$ included in $W^s(x_1)$ and arcs $C^j_{1,u}$ for $j=1,2$ included in $W^u(x_1)$. In fact x_1 is a corner of R_1 and it is the intersection of the sides $C^1_{1,s}$ and $C^1_{1,u}$ which are included in $W^s_{\frac{\epsilon}{2}}(x_1)$ and $W^u_{\frac{\epsilon}{2}}(x_1)$ respectively, the two other sides are $C^2_{1,s}\subset f^n(\Sigma^s)\cap B_\epsilon(x_1)$ and $C^2_{1,u}\subset f^{-n'}(\Sigma^u)\cap B_\epsilon(x_1)$. By definition, the diameter of R_1 is less than ϵ.

Let $R_0=P(R_1)$ be a rectangle in $\mathbb{T}^2$. For $i=1\ldots4$, denote by γ_i the automorphism of the finite covering P such that $\gamma_i(x_1)=x_i$, and let $R_i=\gamma_i(R_1)$ be a rectangle in T^2_4 and for $j=1,2$ we set $C^j_{i,u}=\gamma_i(C^j_{1,u})$, $C^j_{i,s}=\gamma_i(C^j_{1,s})$ (see Figure 1).

From now on we assume that f_4 is topologically transitive (see corollary of lemma 4.1 for the proof) and fix $i\in\{2,3,4\}$.

It follows that there exists $m_i\in I\!N$ such that $f_4^{m_i}(R_1)\cap\mathrm{int}(R_i)\neq\emptyset$.

It is not possible that $f_4^{m_i}(R_1)\supset R_i$. In fact, projecting via P on $\mathbb{T}^2$, we obtain that $f^{nkm_i}(R_0)\supset R_0$ so there exists a repelling fixed point of f^{nk} in $P(R_0)$ which is a contradiction with the Hartman-Grobman Theorem. Furthermore, it is not possible that x_i belongs to $f_4^{m_i}(R_1)$ since x_i is a f_4-fixed point and R_i and R_1 are disjoint.

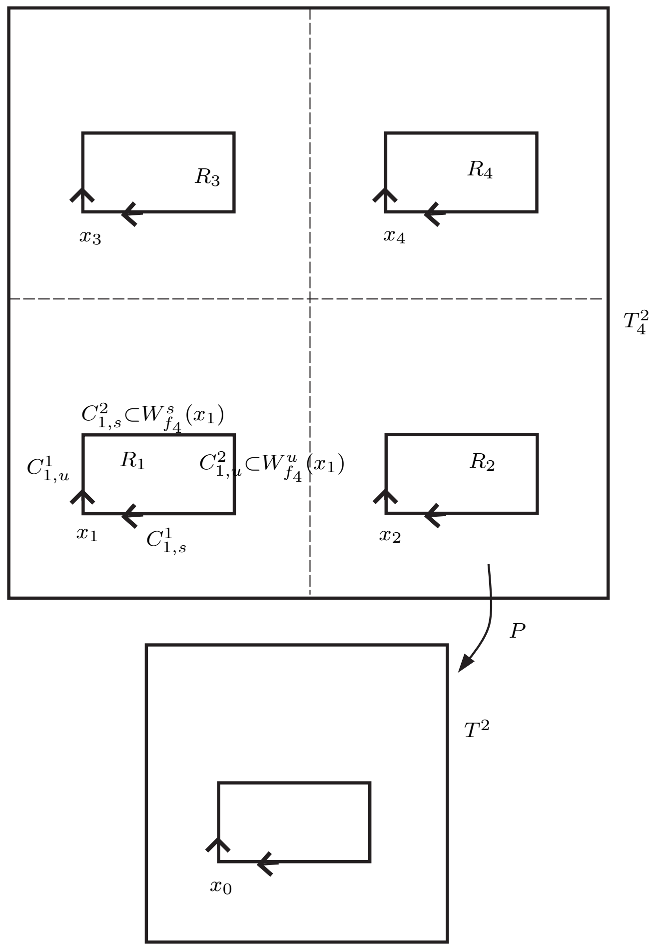
As $W^u(x_1)\cap W^u(x_i)=\emptyset$ for $i\neq1$ we have that there exists $l_i\geq m_i$ such that $f_4^{l_i}(C^1_{1,u})\cap(C^1_{i,s}\cup C^2_{i,s})\neq\emptyset$ and this intersection is topologically transversal.

There is no loss of generality if we suppose that $f_4^{l_i}(C^1_{1,u})\cap C^1_{i,s}\neq\emptyset$, i.e $W^u(x_1)\cap W^s_{\epsilon/2}(x_i)\neq\emptyset$.

Since $W^u(x_1)$ is topologically transversal to $W^s_{\epsilon/2}(x_i)$ we can assert that there exists N_i such that at least one connected component of $f_4^m(C^1_{1,u})\cap R_i$ has one end point in $C^1_{i,s}$ and another one in $C^2_{i,s}$ for all $m\geq N_i$.

Let us define B_i as a small subrectangle in R_i whose boundary contains $C^1_{i,u}$ and stable and unstable arcs. We take the stable sides of B_1 contained in the stable arcs of the boundary of R_1. We choose the other unstable side of B_1, L_u close enough to $C^1_{1,u}$ so that a connected component of $f_4^m(L_u)\cap R_i$ has one end point in $C^1_{i,s}$ and another one in $C^2_{i,s}$ for all $m\geq N_i$. Hence, one connected component of $f_4^m(B_1)\cap R_i$ is a compact set with nonempty intersection with $C^1_{i,s}$ and with $C^2_{i,s}$, for $m\geq N_i$. Let $B_i\subset R_i$ verifying $P(B_i)=P(B_1)$.

FIGURE 1. Stable and unstable sides of R_i.



There exists $n_i \geq N_i$ such that for all $m \geq n_i$ one connected component denoted by D_i of $f_4^m(B_1) \cap B_i$ is a compact set included in B_i with nonempty intersection with $C_{i,s}^1$ and with $C_{i,s}^2$; and with empty intersection with the unstable sides of B_i. One can show that the set $f_4^{-m}(D_i)$ is connected, compact and contained in B_1. It intersects the unstable sides of B_1 and it does not intersect the stable sides of B_1 (see Fig 2).

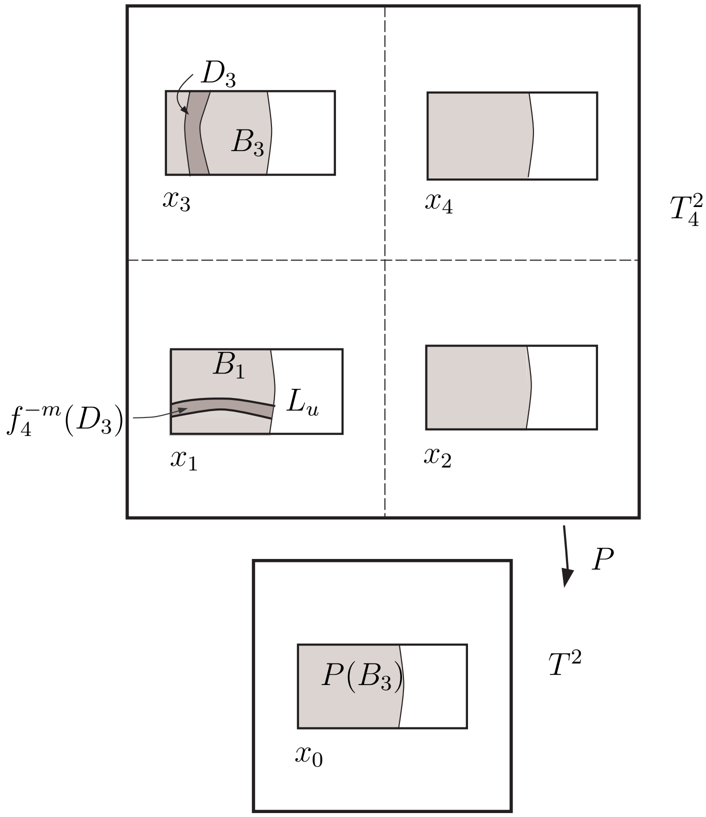
It follows that $\bigcap_{j=-N}^{N} f^{nkmj}(P(D_i))$ has the finite intersection property.

Consequently, the compact set (depending on i and m) in $\mathbb{T}^2$ defined by:

$$L = \bigcap_{j=-\infty}^{\infty} f^{nkmj}(P(D_i))$$

is non empty, $f^{mnk}-$invariant and it's contained in $R_0 = P(R_i) \subset \mathbb{T}^2$.

FIGURE 2. The sets B_i and D_i.



In what follows, we argue for $m = n_i$. For $j = 1, ..., 4$, let $L_j^i = P^{-1}(L) \cap R_j$. It holds that $\bigcup_{j=1}^{4} L_j^i$ is $f_4^{n_i}$-invariant. Moreover, since f_4 is surjective and $L_1^i \subset f_4^{-n_i}(D_i)$, we have that $f_4^{n_i}(L_1^i) = L_i^i$. Therefore, we have proved that there exists an integer n_i and compact sets L_1^i and $L_i := L_i^i$ such that $P(L_i) = P(L_1^i)$ and $f_4^{n_i}(L_1^i) = L_i$. We get the proposition 1. ◀

Proof of the theorem. We prove that the proposition 1 implies the theorem.

Since $\widetilde{x_0}$ is a fixed point of F it follows that $\rho(F, \widetilde{x_0}) = (0, 0)$.

By proposition 1 for $i = 2$, there exists n_2 and non empty compact sets $L_2 \subset R_2$ and $L_1^2 \subset R_1$ such that $P(L_2) = P(L_1^2)$ and $f_4^{n_2}(L_1^2) = L_2$.

Let us denote $\mathcal{L}_2$ [resp. $\mathcal{L}_1^2$] a lift of L_2 [resp. L_1^2] to $\mathbb{R}^2$. Then there exist $\mathbf{k_2} \in (2\mathbb{Z})^2$ such that $F^{n_2}(\mathcal{L}_1^2) = \mathcal{L}_2 + \mathbf{k_2}$. Since $P(L_2) = P(L_1^2)$ and $L_2 \subset B_{x_2}(\varepsilon)$, we have that necessary $\mathcal{L}_2 = \mathcal{L}_1^2 + (1, 0)$. Therefore

$$F^{n_2}(\mathcal{L}_1^2) = \mathcal{L}_1^2 + \mathbf{k_2} + (1, 0).$$

It holds that $F^{n_2}(\mathcal{L}_1^2+\mathbf{k_2}+(1,0)) = F^{n_2}(\mathcal{L}_1^2)+(\mathbf{k_2}+(1,0)) = \mathcal{L}_1^2+2(\mathbf{k_2}+(1,0))$ and for every $k\in\mathbb{N}$

$$F^{kn_2}(\mathcal{L}_1^2) = \mathcal{L}_1^2 + k(\mathbf{k}_2+(1,0)).$$

Let $\widetilde{x}\in\mathcal{L}_1^2$. For every k, there exists $\widetilde{y_k}\in\mathcal{L}_1^2$ such that $F^{kn_2}(\widetilde{x}) = \widetilde{y_k}+k(\mathbf{k_2}+(1,0))$. It follows that

$$\frac{F^{kn_2}(\widetilde{x})-\widetilde{x}}{kn_2} = \frac{\widetilde{y_k}-\widetilde{x}}{kn_2} + \frac{k(\mathbf{k}_2+(1,0))}{kn_2}$$

Then

$$\lim_{k\to\infty}\frac{F^{kn_2}(\widetilde{x})-\widetilde{x}}{kn_2} = \frac{\mathbf{k_2}+(1,0)}{n_2}.$$

Hence,

$$\frac{\mathbf{k_2}+(1,0)}{n_2}\in\rho(F).$$

Analogously, for $i=3$ there exist integers n_3, k_3 and a compact set $\mathcal{L}_1^3$ in $\mathbb{R}^2$ such that

$$F^{n_3}(\mathcal{L}_1^3) = \mathcal{L}_1^3+\mathbf{k_3}+(0,1).$$

It comes that

$$\frac{\mathbf{k_3}+(0,1)}{n_3}\in\rho(F).$$

Finally, it holds that $(0,0)\in\rho(F)$ and the vectors $\frac{\mathbf{k_2}+(1,0)}{n_2}$ and $\frac{\mathbf{k_3}+(0,1)}{n_3}$ are linearly independent.

Actually, for $i=2,3$ let us write $\mathbf{k_i}=(2p_i,2q_i)$ and compute the determinant:

$$\det\left(n_2\frac{\mathbf{k_2}+(1,0)}{n_2}, n_3\frac{\mathbf{k_3}+(0,1)}{n_3}\right) = \begin{vmatrix} 2p_2+1 & 2p_3 \\ 2q_2 & 2q_3+1 \end{vmatrix} \neq 0,$$

since it is the difference between an even number and an odd number.

Then, it follows that $\rho(F)$ has 3 non colinear points. By convexity (see [MZ91]) of $\rho(F)$, we have that $\operatorname{int}(\rho(F))\neq\varnothing$, for a lift F of f^{nk} to $\mathbb{R}^2$. Thus, this property holds for any lift of f^{nk} and therefore for any lift of f to $\mathbb{R}^2$. $\square$

4. Proof of the topological transitivity of f_4.

Let $T_h^2 := \mathbb{R}^2/(2\mathbb{Z}\times\mathbb{Z})$ (resp. $T_v^2 := \mathbb{R}^2/(\mathbb{Z}\times 2\mathbb{Z})$) be a 2-1 covering of $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$, and let the natural projection be $\Pi_h\colon T_h^2\to T^2$ (resp. $\Pi_v\colon T_v^2\to\mathbb{T}^2$). Let $f_h\colon T_h^2\to T_h^2$ be the lifting of f to T_h^2 (resp. $f_v\colon T_v^2\to T_v^2$ be the lifting of f to T_v^2)

Lemma 4.1. *Let $f\colon\mathbb{T}^2\to\mathbb{T}^2$ be a torus homeomorphism satisfying*

(a) f is topologically transitive;
(b) f is irreducible;

then f and the 2-1 coverings f_h and f_v are topologically transitive.

Corollary 4.1. *Let $f\colon\mathbb{T}^2\to\mathbb{T}^2$ be a torus homeomorphism topologically transitive and irreducible. Let f_4 be the lift of f defined in the proof of the theorem. Then f_4 is topologically transitive*

Proof of the Corollary. By definition, $f_4 = (f_h)_v = (f_v)_h$. According to the previous lemma f_h is topologically transitive and by the property 2.1 it is irreducible, so we can apply once again this lemma to f_h and obtain that $(f_h)_v$ is topologically transitive. □

Proof of lemma 4.1.

We will argue by absurd for f_h. We suppose that f is transitive but not f_h. Since f is transitive, there exists x_0 such that $O_f(x_0) := \{f^n(x_0) : n \in \mathbb{Z}\}$ is a dense set in $\mathbb{T}^2$. Let $\{x_1, x_2\}$ be the lifts of x_0 by π_h^{-1}. Let $\mathcal{O}(x_i)$ be the f_h-orbit of x_i. We have that $\mathcal{O}(x_1) \cup \mathcal{O}(x_2) = \pi_h^{-1}(O_f(x_0))$ is a dense set in T_h^2 since π_h is a local homeomorphism. Since f_h is not transitive, neither $\mathcal{O}(x_1)$ nor $\mathcal{O}(x_2)$ is dense. We claim that $\mathrm{int}(\overline{\mathcal{O}(x_2)}) \cap \overline{\mathcal{O}(x_1)} = \varnothing$. Actually if there was a point y in the intersection, there exists $m \in \mathbb{Z}$ such that $f_h^m(x_1)$ belongs to $\overline{\mathcal{O}(x_2)}$ then $\mathcal{O}(x_1) \subset \overline{\mathcal{O}(x_2)}$. Thus $\overline{\mathcal{O}(x_1)} \subset \overline{\mathcal{O}(x_2)}$ so we have $T_h^2 = \overline{\mathcal{O}(x_1)} \cup \overline{\mathcal{O}(x_2)} = \overline{\mathcal{O}(x_2)}$: a contradiction.
Analogously, the symetric holds and these two equalities imply that $\partial\overline{\mathcal{O}(x_1)} = \partial\overline{\mathcal{O}(x_2)}$ and $T_h^2 = \mathrm{int}(\overline{\mathcal{O}(x_1)}) \sqcup \mathrm{int}(\overline{\mathcal{O}(x_2)}) \sqcup \partial\overline{\mathcal{O}(x_1)}$, where $\sqcup$ denotes disjoint union. The set $\partial\overline{\mathcal{O}(x_1)}$ is a closed invariant of empty interior subset that disconnects T_h^2.

We are going to prove that it can not be contained in a disk $D \subset T_h$.
Suppose, by absurd that $\partial\overline{\mathcal{O}(x_1)} \subset D$. First, we prove that $\mathrm{int}\,\overline{\mathcal{O}(x_1)}$ or $\mathrm{int}\,\overline{\mathcal{O}(x_2)}$ is included in D. In fact, if both of them intersect the complement D^c of D, we can take a path in D^c joining a point of $\mathrm{int}\,\overline{\mathcal{O}(x_1)}$ and a point of $\mathrm{int}\,\overline{\mathcal{O}(x_2)}$. By connexity, this path must contain a point of the boundary $\partial\overline{\mathcal{O}(x_1)}$, which contradicts the fact that $\partial\overline{\mathcal{O}(x_1)} \subset D$.

Finally, suppose that $\mathrm{int}\,\overline{\mathcal{O}(x_1)} \subset D$ then $\mathrm{int}\,\overline{\mathcal{O}(x_2)} \supset D^c$, but this is not possible since $\mathrm{int}\,\overline{\mathcal{O}(x_1)}$ and $\mathrm{int}\,\overline{\mathcal{O}(x_2)}$ are homeomorphic.

We have proved that $\partial\overline{\mathcal{O}(x_1)}$ is a closed invariant of empty interior subset that disconnects T_h^2 and that is not contained in a disk. But this is a contradiction with the fact that f is irreducible. □

Remark 4.1. *The topological transitivity of f is not enough to guarantee that f_h (or f_v) is topological transitive. For example, consider a diffeomorphism f of $\mathbb{A} = [0,1] \times S^1$ obtained by the Katok-Anosov process (see* [AK70]*) in such a way that:*

- *f is topologically transitive in* $\mathrm{int}(\mathbb{A}) = (0,1) \times S^1$, *and*
- *$f(0,x) = f(1,x)$ for all $x \in S^1$.*

We collapse the circles $\{0\} \times S^1$ and $\{1\} \times S^1$ by identifying $(0,x)$ with $(1,x)$ for $x \in S^1$. Then, we have a diffeomorphism of the torus, $\widehat{f}$, verifying that $\widehat{f}$ is topologically transitive on T^2 and $C = \{0\} \times S^1$ is a circle invariant of $\widehat{f}$. Let us consider the finite covering T_h^2 of the torus T^2 and let $\widehat{f_h}$ be the lifting of $\widehat{f}$ to T_h^2. The lifting of C is the union of two circles C_1 and C_2 that disconnect T_h^2, and the $T_h^2 \backslash (C_1 \cup C_2)$ is the disjoint union of two cylinders. The orbits of $\widehat{f_h}$ are dense in each cylinder but $\widehat{f_h}$ is not topologically transitive.

5. Examples

In this section, we give examples in order to show that each hypothesis of the theorem is necessary.

1) Missing hypothesis 2.

Let $R_{\alpha,\beta}$ be the rotation of vector (α, β) with α and β irrational, that is, the projection to T^2 of the translation of vector (α, β) in R^2. It is a well known fact that $R_{\alpha,\beta}$ is topologically transitive, it is differentiable and it is irreducible, but its rotation set is $\{(\alpha, \beta)\}$. This example shows that conditions 1, 3, and 4 of the theorem 1 do not ensure that the interior of the rotation set is not empty.

2) Missing hypothesis 3 and 4.

Let $f_D\colon \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ be a diffeomorphism such that there exists a horseshoe in the interior of $\mathbb{D}^2$ and such that f_D is the identity on $\partial\overline{\mathbb{D}^2}$. It follows that f_D has positive entropy. Let us embed $\overline{\mathbb{D}^2}$ in $\mathbb{T}^2$ and then extend f_D to f by the identity on $\mathbb{T}^2\backslash\overline{\mathbb{D}^2}$. It holds that f has positive entropy and the rotation set has empty interior(because there exist invariant circles homotopically non trivial). This example show that conditions 1 and 2 do not ensure that the conclusion of the theorem is verified.

3) Missing hypothesis 3.

We start with an irrational flow ϕ_0^t on T^2. By making an appropriate smooth time change vanishing at one point x_0 (we replace the vector field X by $g.X$ where $g(x_0) = 0$ and $Dg(x_0) = 0$), we get a new smooth topologically transitive flow ϕ_g^t with a fixed point x_0. Consider the time one map of this flow, f, and replace x_0 by a small closed disk D_0 by blowing up. The dynamic of the blow up of f on ∂D_0 is of the type north-south. We have that $D_0\backslash\{N, S\}$ is foliated by meridians $\{M_t\}_{t\in[-1,1]}$ and $\partial D_0 = M_{-1} \cup M_1 \cup \{N, S\}$. Let $\gamma\colon D_0 \to D_0$ a differentiable map such that $\gamma|M_i = f|M_i$, for $i = -1, 1$, $\gamma(N) = N$, $\gamma(S) = S$, for all $t \in [-1, 1]$ M_t is γ-invariant and $\gamma|M_t = Id$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$. In the blow up manifold, T^2 we define the differentiable map Γ as $\Gamma(x) = f(x)$ if $x \in T^2/D_0$ and $\Gamma(x) = \gamma(x)$ if $x \in D_0$. Let $D_1 = \cup_{t\in[-\frac{1}{2},\frac{1}{2}]} M_t \cup \{N, S\}$, it holds that $\Gamma|D_1 = Id$.

As in the previous example, we can put a horseshoe in the interior of D_1. The resulting diffeomorphism satisfies trivially the conditions 1, 2.

It also verifies the condition 4. Moreover, an invariant compact set K of Γ is included either:

- in D_0, in this case K is not essential or
- in the complement D_0^c of D_0, in this case K coincides with D_0^c (since each orbit in D_0^c is dense in it) hence its interior is not empty.

But it does not satisfy the condition 3, since it has an invariant disk.

Finally, its rotation set has empty interior. In fact, before the blowing up, the map f is the time one map of a flow with a fixed point x_0 so according to Franks and Misiurewicz's result (see [FM90]) its rotation set is a line segment containing $(0, 0)$. The blowing up does not change the rotation set because of the following facts:

- the points in D_0 have the same rotation vector as x_0 which is $(0, 0)$ (D_0 is Γ-invariant),
- for the points out of D_0, the blowing up does not change the orbits so it does not change their rotation vectors.

4) Missing hypothesis 1.

According to [Ree81] there exists a torus homeomorphism f_0 isotopic to the identity such that it is minimal and it has positive entropy. Since f_0 is minimal, all its orbits are dense so it has no periodic points. By [Fra89] we know that if the interior of the rotation set is not empty, then each vector with rational coordinates in the interior of the rotation set is realized as the rotation vector of a periodic point. It follows that f_0 verifies that $\text{int}(R(f_0)) = \varnothing$. This example shows that conditions 2, 3 and 4 are not enough to guarantee that the interior of the rotation set is not empty.

5) Missing hypothesis 4

According to [Kat79] there exists a topologically transitive C^∞ Bernoulli diffeomorphism $f_0 \colon S^2 \to S^2$ which preserves a smooth positive measure on S^2. Since f_0 (or f_0^2) preserves orientation then it is isotopic to the identity.

As in the construction of [Kat79], there exist x_1, x_2, two fixed points of f_0 (or f_0^k) such that $Df_0(x_i) = Id,\ i = 1, 2$. We can replace x_1 and x_2 by small closed disks D_1 and D_2, respectively, by blowing up. The dynamic of the blow up of f_0 on ∂D_1 and ∂D_2 is the identity. By gluing ∂D_1 and ∂D_2 we have a smooth map $f \colon T^2 \to T^2$ which is topologically transitive and it has positive entropy but there exists a compact f-invariant of empty interior set (∂D_1) which is essential. This example fails to be irreducible because of the existence of a non null homotopic invariant circle, then its rotation set has empty interior because of Llibre and Mac Kay's result (see [Lli91]).

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