STRUCTURALLY STABLE PERTURBATIONS OF POLYNOMIALS IN THE RIEMANN SPHERE

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Abstract. The perturbations of complex polynomials of one variable are considered in a wider class than the holomorphic one. It is proved that under certain conditions on a polynomial p of the plane, the C^r conjugacy class of a map f in a C^1 neighborhood of p depends only on the geometric structure of the critical set of f. This provides the first class of examples of structurally stable maps with critical points in dimension greater than one.

Résumé. Nous considérons les perturbations des polynômes complexes en une variable dans une classe plus vaste que la classe holomorphe. Si f est une application appartenant à un voisinage C^1 d'un polynôme p du plan, nous prouvons, sous certaines conditions sur p, que la classe de conjugaison C^r de f ne dépend que de la structure géométrique de l'ensemble des points critiques de f. Ceci fournit la première classe d'exemples, en dimension supérieure à un, d'applications structurellement stables ayant des points critiques.

1. Introduction

Given a manifold without boundary M, denote by $C^r_W(M)$ the set of C^r endomorphisms of M, considered with the strong (or Whitney) topology. Two maps f and g are *topologically equivalent* if there exists a homeomorphism h such that $hf = gh$. The problem of determining the classes of topological equivalence is central in the theory of dynamical systems. In particular, a great effort has been made to classify those maps that are topologically equivalent to its neighbors. If C is a topological space of self mappings, then f is C structurally stable if there exists a neighborhood of f such that every g in that neighborhood is topologically equivalent to f. Obviously, the concept depends on the space and topology under consideration.

The examples of structurally stable maps on manifolds without boundary that are already known are the following:

(1) A C^1 diffeomorphism of a compact manifold is C^1 structurally stable if and only if it satisfies Axiom A and the strong transversality condition. This theorem is the result of the work of many authors, from the sixties to the nineties. The "only if" part is due to C.Robinson [R] and the other direction was obtained by R.Mañé [Ma], fifteen years later.
It is still not known if there exist C^r structurally stable diffeomorphisms that are not C^1 structurally stable.
(2) Any C^r expanding map of a compact manifold is C^r structurally stable. This was proved by M.Shub [S] in the sixties.

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(3) In the case of one dimensional maps of the circle there are some possible combinations giving conditions for structural stability.
The same occurs for rational maps of the Riemann sphere. This case will be specially considered in the sequel. For example a polynomial map of degree d is stable in the d dimensional space of parameters corresponding to its coefficients, if p is hyperbolic and satisfies the *no critical relations* property: $p^n(S'_p) \cap p^m(S'_p) = \emptyset$ for every $0 \leq n < m$, where S'_p is the set of finite critical points of p. It is not known, however, if the converse of this assertion is true.

Therefore there are no examples of noninvertible nonexpanding structurally stable maps with or without critical points, in dimensions greater than one. In the attempt to construct the simplest possible examples, we consider $C^1_W(\mathbb{C})$ neighborhoods of polynomials and look for $C^r_W(\mathbb{C})$ stable maps there. The theorem of Mañé, Sad and Sullivan of stability of rational mappings [MSS], implies the statement(3) above and also that within the family of degree d polynomials, the stable ones are dense.
It will be clear later that no polynomial can be $C^r_W(\mathbb{C})$ structurally stable, because the critical points of holomorphic maps are nongeneric in those spaces of smooth maps. Indeed. let f and g be topologically equivalent (also called conjugate) and h the *conjugacy* between them, i.e. the homeomorphism such that $h \circ f = g \circ h$; then h carries generic critical points of f to critical point of g and critical values of f to critical values of g. Therefore, some geometric conditions must be imposed on the critical sets of maps f and g in order to obtain the existence of a conjugacy between them. The concept that will be used is the following:

Definition 1. Two maps f and g are *geometrically equivalent* if there exist C^1 diffeomorphisms of M, φ and ψ, such that $\varphi \circ f = g \circ \psi$.

This concept, introduced by R.Thom, is now a central concept in global analysis. It is a concept of geometric nature: it implies, for example, that the set of (generic) critical points and critical values of f and g are diffeomorphic and that the degree of the maps are the same. However, it has no dynamical meaning: for example, two quadratic polynomials of the sphere are always geometrically equivalent. The concept of geometric equivalence has no significance relative to future iterates of the map: the fact that two maps f and g are equivalent in this sense does not imply that its iterates f^2 and g^2 are also equivalent. It is clear, on the other hand, that if two maps are topologically equivalent, then the homeomorphism realizing the conjugacy carries information about the local behavior of the maps; therefore, under generic conditions, topological equivalence implies geometric equivalence. The aim now is to establish conditions implying the converse statement.
Note that if a polynomial p satisfies the no critical relations property (item (3) above) then no critical point of p is periodic or preperiodic.
The main result in this work is the following:

Theorem 1. *Let p be a polynomial that satisfies the no critical relations property. The following conditions are equivalent:*

(1) *The Julia set of p is connected and hyperbolic.*
(2) *There exists a neighborhood $\mathcal{U}$ of p in $C^1_W(\mathbb{C})$, such that, if two maps belonging to $\mathcal{U}$ are geometrically equivalent, then they are topologically equivalent.*

The implication (1) $\Rightarrow$ (2) is the most interesting part of the statement. It contains the proof that, under certain conditions on the polynomial p, it suffices to prove that the sets of critical points and values of two maps C^1 close to p have the same geometry, to obtain that the maps are equivalent from the dynamical point of view.
The dynamical structure of a polynomial p satisfying the hypothesis (1) of the theorem is well known. Recall that the Julia set is connected if and only if every critical point (other than ∞) has bounded orbit. The hyperbolicity of p is equivalent to the fact that every critical point is attracted to a periodic attractor or superattractor, and the hypothesis of no critical relations implies that there are no finite superattractors. Within this context the polynomial is stable under small perturbations of its coefficients. The proof of this fact is based on the construction of conjugacies in the Fatou components of p, that come from the holomorphic local conjugacies at the periodic points (see the theorems of Schröder and Böttcher in the references [St], [Mi]). Then these conjugacies are glue together via the application of the λ lemma [MSS]. When the perturbation is taken in the C^1 Whitney topology, a great number of non holomorphic maps arise, including some with wild critical sets. All the above techniques rely on the conformal structure of the maps in question and therefore cannot be applied in this wider context. To deal with the structure of the nonwandering set one has a basic result, a theorem by F.Przytycki ([P]), that implies that under the hypothesis (1), the polynomial p is C^1 Ω-stable. This means that for a small C^1 perturbation f of p the restrictions of f and p to its nonwandering sets are topologically equivalent. This theorem is used in section 2 to prove that the complement of the nonwandering set of f is the union of the basins of the periodic attractors of f. This is a fundamental step in the proof. In particular, every component of the complement of the nonwandering set of f is periodic or preperiodic. This extends Sullivan's theorem of nonexistence of wandering Fatou components, to Whitney C^1 perturbations of hyperbolic polynomials. It justifies, moreover, the denomination of Fatou component of f for a component of the complement of the nonwandering set of f, and also the concept of *analytic continuation* for Fatou components.
A lot of work is then needed to prove that geometrically equivalent maps f and g are conjugate when restricted to corresponding Fatou components. This is the more technical part of the proof and deserves sections 3 (proof in the unbounded component) and 4 (proof in the bounded components). Then the conjugacies in these components and the conjugacy of the nonwandering sets given by the theorem of Przytycki are glued together using Carathéodory theory in sections 5 and 6.

As a consequence of this part of the theorem the first known examples of C^3-structurally stable maps having critical points are shown:

Corollary 1. *Let p be a polynomial map satisfying the properties of part (1) or (2) of the theorem 1. In each neighborhood $\mathcal{U}$ of p in $C^\infty_W(\mathbb{C})$ there exists some f that is C^3 structurally stable.*

It will become clear in subsequent sections that no polynomial can be C^1 approximated by a C^2 structurally stable map. See remark 2 in section 7.
On the other hand, the converse ((2) $\Rightarrow$ (1)) is easier to prove: it will be shown that if a critical point of p belongs to the Julia set of p, then there exists a C^1 perturbation of p having the same geometrical structure of p, but not topologically equivalent to it. Less evident is the fact that the Julia set of p must be connected

in order to obtain the properties stated in part (2). See section 7. See the remarks at the end of the article concerning some questions about the problem of stability.

2. Whitney perturbations of p

In this section a polynomial p satisfying the hypothesis (1) of theorem 1 is fixed and f is a small C^1 Whitney perturbation of p. The objective is to show that the picture of the dynamics of f is the same as that of p. The following properties are satisfied by a polynomial p verifying the hypothesis (1) of theorem 1:

(1) The point ∞ is an attractor. The basin of ∞, $B_\infty(p)$, is connected and simply connected.
(2) Its boundary, $\partial B_\infty(p)$, is a curve (not necessarily a Jordan curve), and is equal to $\Omega'(p)$, the set of nonwandering points of p that are not periodic attractors. (Clearly $\Omega'(p)$ is the Julia set of p, also denoted J_p).
(3) Every component of the complement of the closure of $B_\infty(p)$ is simply connected and its boundary is a Jordan curve.
(4) The components of the Fatou set of p, are the periodic components and its preimages.

For a proof of this results there are many good references. See for example [St] or [Mi].

Theorem 2. *There exists a neighborhood $\mathcal{U}$ of p in $C^1_W(\mathbb{C})$, such that each $f \in \mathcal{U}$ satisfies conditions 1 to 4 above.*

The remaining of this section is devoted to the proof of this theorem. The first result is trivial and one of the reasons why Whitney topology is considered. See for example reference [H], where the properties of Whitney topology are clearly exposed. If f were a C^r perturbation of p in the topology $C^r(S^2)$, then the intersection of the critical set of f with a neighborhood of ∞ may possibly become a nonconnected set with $d-1$ components, where d is the degree of p, and the analytic continuation of the fixed point at ∞ may not be critical anymore.

Lemma 1. *For every f in a neighborhood of p in $C^0_W(\mathbb{C})$, the point at ∞ is an attractor.*

This means that under these hypothesis, f is a proper map of $\mathbb{C}$ and there exists a disc D with the property that $f(D)$ contains the closure of D and such that the future orbit of any point outside D diverges.
Now consider a C^1_W perturbation f of p. The hypothesis on p imply that the Julia set of p is hyperbolic and hence expanding, in the sense that $|p'(z)| > 1$ for every $z \in J_p$ where the norm is considered with respect to a hyperbolic metric in an open set containing J_p. This implies that p is C^1-Ω stable and then the theorem of Przytycki implies that the restrictions of f and p to its nonwandering set are conjugate. For f close to p define $\Omega'(f) = \Omega(f) \setminus \{\text{periodic attractors}\}$. Obviously periodic attractors of p are carried by the conjugacy h to isolated periodic points of f, so that h must carry J_p onto $\Omega'(f)$.

Lemma 2. *If f is C^1_W close to p, then $\Omega'(f) = \partial B_\infty(f)$.*

Proof : To prove that $\partial B_\infty(f) \subset \Omega'(f)$, observe first that there exists a neighborhood V of $\partial B_\infty(f)$ where Df expands every vector (every point outside a neighborhood of the boundary of $B_\infty(p)$ is contained in the basin of a periodic attractor).

It is claimed now that $f^{-1}(\Omega'(f)) = \Omega'(f)$. Let U be a neighborhood of the critical set of p not intersecting J_p and $\mathcal{U}$ a C^1_W neighborhood of p such that $S_f \cap U$ is empty for every $f \in \mathcal{U}$. Then the restriction of f to the set A, complement of $f^{-1}(f(U))$, is a covering map of degree d onto the complement of $f(U)$. If U is small, then $f(U) \cap \Omega'(f)$ is empty and A contains $\Omega'(f)$. It follows that every point in $\Omega'(f)$ has exactly d preimages, all contained in $\Omega'(f)$, because the same occurs for p and by the theorem of Przytycki. It follows that $f^{-1}(\Omega'(f)) \subset \Omega'(f)$. This implies the claim, because the other inclusion is trivial. It follows that if $x \notin \Omega^{'}(f)$ then $f^n(x) \notin \Omega^{'}(f)$, $\forall n \geq 0$. Then the omega limit set of x, $\omega(x)$, cannot intersect $\Omega^{'}(f)$ because this is an expanding set. It follows that $\omega(x)$ is a periodic attractor, and it follows that $x \notin \partial B_\infty(f)$.
To prove the other inclusion take a point $z \in \Omega'(f)$ and V a neighborhood of z. It is known that the restriction of p to J_p is locally eventually onto; by conjugation, this also holds for the restriction of f to $\Omega'(f)$. Using this and the other inclusion, already proved, there exist $n > 0$ and $x \in V \cap \Omega'(f)$ such that $f^n(x)$ belongs to the boundary of $B_\infty(f)$. Let $U \subset V$ be a neighborhood of z such that $U \cap \Omega'(f) = V \cap \Omega'(f)$ and U does not intersect the set of critical points of f^n. Then $x \in U$ and f^n is open in U, so $f^n(U) \cap B_\infty(f) \neq \emptyset$ and hence U, and also V, intersect $B_\infty(f)$.

□

Proof of theorem 2 : The first assertion of (1) follows from lemma 1. The second one is consequence of the fact that the boundary of $B_\infty(f)$ is connected (by lemma 2 and the theorem of Pryztycki). Also (2) is an immediate consequence of the above arguments.
Let V be a component of the complement of the closure of $B_\infty(f)$. It is clear that the boundary of V is contained in the boundary of $B_\infty(f)$, from which it follows that V is connected and simply connected. Moreover, the boundary of V is a Jordan curve, because the contrary assumption implies that the interior of the closure of V contains points of the boundary of V and this contradicts the fact that the boundary of V is contained in the boundary of $B_\infty(f)$. This proves (3). To prove the remaining statement it is sufficient to show that every point in the complement of the closure of $B_\infty(f)$ is attracted to a periodic attractor. For this an argument similar to that of the proof of lemma 2 works: indeed, if U is a small neighborhood of $B_\infty(p)$, then the complement of U is a compact set contained in the union of the basins of the periodic attractors of p, and the conclusion follows because this condition is open in the topology under consideration.

Now the plan for the proof of the theorem can be delineated: first local conjugacies must be constructed in neighborhoods of ∞ and the other attracting periodic points. Then the local conjugacies must be extended to the whole basins and then it must be proved that these conjugacies can be extended to a global conjugacy that coincides with the Przytycki map in the nonwandering set.

3. Conjugacy in the unbounded domain

In this section, the C^1 Whitney perturbations of a polynomial p with connected Julia set will be considered. Any complex polynomial is holomorphically conjugate to $z \to z^d$ locally at ∞, where d is the degree of the polynomial; moreover, under the hypothesis of part (1) of the theorem (as the Julia set is connected, ∞ is the unique critical point in B_∞), the conjugacy extends to the whole basin of ∞, which implies that this basin is simply connected. By this reason, it can be assumed throughout this section that $p(z) = z^d$. Note that the existence of the conjugacy is not trivial for non holomorphic perturbations of p. It will be shown here that there exist a local conjugacy at ∞ between f and p.

The fundamental step in the proof of the local conjugacy is the existence of an f-invariant foliation each of whose leaves is a C^1 curve homeomorphic to a circle and not homotopically trivial in $B_\infty(f) \setminus \{\infty\}$. This foliation corresponds to the p-invariant circles of $B_\infty(p)$: these are the unique non trivial simple closed curves in $B_\infty(p)$ whose images under p are also simple closed curves. Before proceeding with this construction, it will be found another invariant curve, an immersion of $\mathbb{R}$, that joins the fixed point analytic continuation of 1 to ∞. Note that for any map f there are infinitely many such curves. Indeed, take any $x \in B_\infty(f)$ and choose any curve joining x and $f(x)$. Then the union of images of these curves gives a curve γ such that $f(\gamma) \supset \gamma$. Taking preimages one can also complete the curve to a curve landing at the fixed point of f, continuation of 1, in the boundary of $B_\infty(f)$, because $\partial B_\infty(f)$ is expanding. For the initial map, the polynomial p, there exists only two possibilities: either the curve γ is the real axis from 1 to ∞, or the curve intersects each radius $R_{\theta_0} = \{re^{i\theta} \ : \ \theta = \theta_0 \ , \ r > 1\}$ at infinitely many points. The same is true for f.

Lemma 3. *For every $\epsilon > 0$ there exists a $C^1_W(\mathbb{C})$ neighborhood $\mathcal{U}$ of p such that, for every $f \in \mathcal{U}$, there exists a unique invariant foliation Γ such that each curve $\gamma \in \Gamma$ is ϵ* almost radial*, that is, there exists a C^1 function θ such that $\gamma(r) = re^{2\pi i\theta(r)}$ and $|\theta'(r)| < \epsilon$.*

Proof : It will be convenient to use the coordinate $z \to 1/z$ to treat with 0 instead of ∞. With this new coordinate, f is a small Whitney perturbation of $p(z) = z^d$ in some punctured disc at the origin, so that 0 is an attractor for f. Let $\rho > 0$ be such that the punctured disc $D^* = \{z \,:\, 0 < |z| < \rho\}$ is forward invariant under f. Note that ρ can be made arbitrarily small. Let

$$\Omega = \{(r, \theta) \in \mathbb{R}^2 \,:\, 0 < r < \rho\},$$

and $\pi : \Omega \to D^*$, $\pi(r, \theta) = re^{i\theta}$, the usual polar coordinates covering map. It is clear that if p is lifted to $\tilde{p}(r, \theta) = (r^d, d\theta)$ then f has a lift $\tilde{f} : \Omega \to \Omega$ close to $\tilde{p}$ in $C^1_W(\Omega)$. Note that $\tilde{f}$ is a diffeomorphism onto $\tilde{f}(\Omega)$. Let $\epsilon(r) > 0$ be a continuous function such that $\tilde{f}$ is in the ϵ-C^1 Whitney neighborhood of $\tilde{p}$. Assume that $\epsilon(r) \to 0$ as $r \to 0$ with the order of r^d. Summing up, this means that the map $\tilde{f}$ has a differential of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are functions that evaluated at r and θ satisfy that $|A - dr^{d-1}|$, $|B|$, $|C|$ and $|D - d|$ are all smaller than $\epsilon(r)$. As $r < \rho$ and ρ can be made very small, it follows by simple calculation that there exist invariant unstable cones containing the vertical directions, at any point of a neighborhood of $\{r = 0\}$ in Ω. Indeed, let $\delta > 0$ be a small positive number and let $V = (u, v)$ be a vector tangent to Ω at

(r,θ), such that $|u/v| < \delta$. If $(u_1, v_1) = (D\tilde{f})(V)$, then

$$\left|\frac{u_1}{v_1}\right| = \left|\frac{Au+Bv}{Cu+Dv}\right| \le \frac{A\delta + |B|}{D - |C|\delta} \le \frac{(dr^{d-1}+\epsilon)\delta + \epsilon}{(d-\epsilon) - \epsilon\delta}$$

and this is less than δ if ρ and ϵ are small enough.

This implies by standard arguments the existence of an $\tilde{f}$-invariant normally expanding foliation of Ω formed by C^1 almost horizontal curves. Indeed, if Γ_0 is a foliation by almost horizontal curves of inclination at most δ, then the preimage of this foliation is a foliation of the same type. Then there exists a unique invariant foliation $\tilde{\Gamma} = \lim_n (\tilde{f})^{-n}(\Gamma_0)$. Finally, as $\pi(x) = \pi(y)$ implies that $\pi(\tilde{f}(x)) = \pi(\tilde{f}(y))$, it follows that $\tilde{\Gamma}$ induces an invariant foliation $\Gamma = \Gamma_f$ for f in D^* which is obviously equivalent to have a foliation in a neighborhood of ∞.

□

In particular, there exists a unique curve $\gamma_f \in \Gamma$ which is invariant and almost horizontal. There exist infinitely many f-invariant curves, but all the other are spirals around ∞.

Corollary 2. *If $\gamma = \gamma_f$ is the invariant curve found above, then*

$$\bigcup_{n\ge 0} f^{-n}(\gamma)$$

is dense in a neighborhood of ∞.

Let s be a segment transverse to the foliation $\tilde{\Gamma}$ in Ω. As this foliation was normally expanding it follows that $\tilde{f}^n(s)$ intersects $\tilde{\gamma}$ (the lift of γ) for every n sufficiently large. Therefore, each open set U in D^* contains points of $f^{-n}(\gamma)$ for every n sufficiently large.

□

The fundamental structure of p in B_∞ is the invariant foliation by circles. The next step is to prove that also f has such a foliation.

For the proof of the next lemma it is convenient to return to the map $p(z) = z^d$ defined in a neighborhood $U_\rho = \{|z| > \rho\}$ of ∞, forward invariant under f. Note that ρ can be taken arbitrarily large. Again a covering space (Λ, π), with

$$\Lambda = \Lambda_\rho = \{(r,\theta) \in \mathbb{R}^2 \,:\, r > \rho\},$$

and the same π will be considered. As above, f has a lift $\tilde{f}$ acting in Λ that is Whitney-C^1 close to the lift of p, $\tilde{p}(r,\theta) = (r^d, d\theta)$.

Lemma 4. *There exists a foliation $\mathcal{F}$ of some U_ρ such that:*

1. *Each leave is a C^1 curve homeomorphic to a circle and not homotopically trivial in $B_\infty \setminus \{\infty\}$*
2. *$\mathcal{F}$ is f-invariant; the map f carries a leaf of $\mathcal{F}$, d to 1 to another leaf.*

Proof : It is sufficient to find an $\tilde{f}$-invariant foliation by curves in Λ_ρ that are periodic, or π-invariant. The curve γ of the preceding lemma can be parametrized as the graph of a C^1 function defined in the r-axis.

Given any $\delta > 0$ there exists a neighborhood of p such that the preimage under $\tilde{f}$ of any vertical strip of length δ is contained in a vertical strip of length $2\delta/3$. Let $(r_0, \gamma(r_0))$ be any point in the invariant curve and define r_n so that $\tilde{f}^n(r_0, \gamma(r_0)) = (r_n, \gamma(r_n))$. Let $V_n = V_n(r_0) = \{(r,\theta) \,:\, |r - r_n| < \delta/2\}$.

Given some positive number μ define $\Gamma_{r_0}(\mu)$ as the set of sequences $\{\alpha_n\}_{n\geq 0}$ of curves satisfying the following:

(1) Each α_n is a C^1 curve, $\alpha_n(\theta) = (a_n(\theta), \theta)$ and $|a_n'(\theta)| < \mu$.
(2) The curve α_n is contained in V_n.
(3) Each α_n is periodic in the sense that $\alpha_n(\theta+1) = \alpha_n(\theta)$.
(4) For each n, the curve α_n intersects the invariant curve γ at the point $(r_n, \gamma(r_n))$.

Note that if μ is sufficiently small then α_n is contained in V_n. A distance is defined in $\Gamma_{r_0}(\mu)$ by the formula

$$d(\{\alpha_n\}, \{\beta_n\}) = sup_n ||\alpha_n - \beta_n||_1,$$

where $||.||_1$ stands for the usual C^1 norm. It is claimed now that the operator Φ_{r_0} that assigns to the sequence $\{\alpha_n\}_{n\geq 0}$ the sequence of preimages $\{\tilde{f}^{-1}(\alpha_n)\}_{n\geq 1}$ is an operator in $\Gamma_{r_0}(\mu)$.

Assume that $\tilde{f} = (h, g)$; to find $\tilde{f}^{-1}$ one has to solve $\tilde{f}(x, y) = (a(\theta), \theta)$. It comes that

$$\frac{x'}{y'} = \frac{-\partial_y g a' + \partial_y h}{\partial_x g a' - \partial_x h}$$

From this it follows that an almost vertical curve $|a'| < \mu$ has preimage also vertical, with the inclination:

$$\left|\frac{x'}{y'}\right| \leq \frac{(d+\epsilon)\mu + \epsilon}{dx^{d-1} - \epsilon - \mu\epsilon}$$

which is less than μ if r_0 (and hence x) is large enough. It is clear that the preimage of each α_n is periodic because $\tilde{f}$ is a lift of a map f. The remaining conditions are obviously satisfied by preimages, so the claim is proved. The same estimatives prove that Φ_{r_0} is, in fact, a contraction of $\Gamma_{r_0}(\mu)$, the fact that r_0 is large is used again to prove this (and this is the reason why, instead of using the unit circle as in the previous lemma, it was convenient to use a neighborhood of ∞; otherwise one would have to change the metric).
The fixed point $\{\alpha_n^{r_0}\}$ of Φ_{r_0} is defined for every r_0. Let $\mathcal{F}_0$ be the set of curves $\{\alpha_0^r : r \geq r_0\}$. That this is a foliation of a neighborhood of ∞ is consequence of the following observations:

(1) The curves $\alpha_0^{r_1}$ and $\alpha_0^{r_2}$ do not intersect if $r_1 \neq r_2$. The contrary assumption implies that for every n the intersection of $\alpha_n^{r_1}$ and $\alpha_n^{r_2}$ is not empty. But this is not possible: indeed, given any $r_1 \neq r_2$ there exists a positive n such that the sets $V_n(r_1)$ and $V_n(r_2)$ do not intersect.
(2) The union of the curves $\alpha_0^{r_0}$ covers a neighborhood of ∞. These curves certainly cover the curve γ by construction. Suppose that there exists an open set U that does not intersect any of curves α_0^r; this implies that the union of its images does not intersect any α_0^r. From the corollary it follows that the set $\tilde{f}^{-n}(\pi^{-1}(\pi(\gamma)))$ is dense in Λ, so the periodicity of the curves α implies a contradiction.

This concludes the proof of the lemma, once the foliation $\mathcal{F}$ is defined as the projection of $\mathcal{F}_0$ under π.

□

Observe that this foliation can be extended throughout the whole basin $B_\infty(f)$ if it does not contain critical points of f. This section finishes with the proof of the following:

Theorem 3. *Let p be a complex polynomial of degree $d > 1$ and $\mathcal{U}_0$ a neighborhood of p in $C^1_W(\mathbb{C})$. Suppose that for every $f \in \mathcal{U}_0$ it is known that $S_f \cap B_\infty(f) = \emptyset$. Then there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of p such that for every $f \in \mathcal{U}$ there exists a homeomorphism $h : B_\infty(p) \to B_\infty(f)$ such that $hp = fh$.*

Proof : Let $\mathcal{U}$ be as in the previous lemma, so that the invariant foliations exist. Let C be an element of the foliation $\mathcal{F}$. In C define the map $\hat{f}$ as follows: for $z \in C$, let $\gamma \in \Gamma$ such that $f(z) \in \gamma$ and define $\hat{f}(z)$ as the point of intersection of γ and C. It is claimed that $\hat{f}$ is topologically equivalent to the map $z \to z^d$ acting in the unit circle S^1. The following lemma implies the claim:

Lemma 5. *Let $g : S^1 \to S^1$ a degree d covering of S^1. Assume that g has a fixed point x whose preimages are dense in S^1. Then g is conjugated to $z \to z^d$.*

To prove the lemma define $h_0(x) = 1$ and extend it to the preimages of x in such a way that it is a conjugacy in the set of preimages of x and such that it preserves orientation. The hypothesis imply immediately that h_0 has a unique extension that is a conjugacy.

Following with the proof of the theorem, note that by corollary 2, it follows that the map $\hat{f}$ satisfies the hypothesis of the lemma; the claim is consequence of this. This proceeding gives an application that to each $\gamma \in \Gamma$ associates a ray $R_{\theta(\gamma)}$. Next take a fundamental domain A for $B_\infty(f)$ whose boundary is equal to the union of C and $f(C)$, choose any $r > 1$, and define h from A to the annulus $\{r \leq |z| < r^d\}$ (any $r > 1$), in such a way that it is equal to the h_0 of the previous lemma in C, and carries a leaf of Γ to a ray $R_{\theta(\gamma)}$. To extend h to $f(A)$ proceed as follows: let $z \in f(A)$, and $z_1, \ldots, z_d$ the preimages of z. All of them belong to the same circle $C' \in \mathcal{F}$. By the construction of h in A it follows that all the points $h(z_1), \ldots, h(z_d)$ have the same image under p. Then define $h(z) = p(h(z_1))$. Now, to define h in $f^{-1}(A)$, let $z \in f^{-1}(A)$ and let $\gamma' \in \Gamma$ such that $z \in \gamma'$. Note that $h(f(z))$ has d preimages under p but only one of them belongs to γ': this preimage will be $h(z)$. Proceeding by induction one can define h all over $B_\infty(f)$.

□

4. Conjugacy in bounded domains

The hypothesis of theorem 1 imply that the critical points of p are contained in basins of attraction of finite periodic attractors. If V is a small neighborhood of the set S_p, then there exists a C^1_W neighborhood $\mathcal{U}$ of p, such that for every $f \in \mathcal{U}$, the critical set S_f is contained in V, and so every critical point belongs to the basin of a periodic attractor of f. Assume that f and g are maps C^1 close to p and that they are geometrically equivalent. This means that there exist diffeomorphisms of the plane φ and ψ such that $\varphi f = g\psi$. Begin with a fixed attracting point of p and consider its analytic continuation x_f for $f \in \mathcal{U}$. The immediate basin of x_f is denoted by U_f. Note that theorem 2 implies that U_f is simply connected. The objective throughout this section is to prove that there exists a homeomorphism h realizing the equivalence of $f|_{U_f}$ and $g|_{U_g}$. This map will be produced as an extension of the restrictions of φ to a neighborhood of the set of critical values and of ψ to a neighborhood of the set of critical points of f in U_f.

Lemma 6. *If f and g are geometrically equivalent maps C^1 close to p, then its restrictions to U_f and U_g are topologically equivalent.*

Proof. It will be assumed first that p has only one critical point in the immediate basin of attraction of x_p.

Let V_f be a neighborhood of x_f, such that $f|_{V_f}$ is a diffeomorphism and the annulus $A_f = V_f \setminus f(V_f)$ is a fundamental domain. It is also possible to choose V_f and a topological disc W_f, containing S_f, such that $\overline{f(W_f)}$ is also a topological disc contained in the interior of A_f (see figure 1). For the map g define corresponding V_g, A_g and W_g. Moreover, W_g is chosen so that $\varphi(f(W_f)) = g(\psi(W_f)) = g(W_g)$. In addition a simple curve $\beta_f \subset V_f \setminus \bigcup_{n>0} f^n(W_f)$, joining x_f with some point r in the boundary of $f(W_f)$ and a corresponding curve β_g joining x_g with $\varphi(r)$ will be needed in the sequel.

Under these conditions there exists an orientation preserving homeomorphism h,

$$h : V_f \setminus \bigcup_{n\geq 1} f^n(W_f) \to V_g \setminus \bigcup_{n\geq 1} g^n(W_g), \tag{1}$$

realizing a conjugacy between the restrictions of f and g to the given domains, such that the restriction of h to the boundary of $f(W_f)$ is equal to φ and such that $h(\beta_f) = \beta_g$. Moreover, one can extend h to the whole V_f if it is defined as equal to φ in $f(W_f)$ and then dynamically extended to $f^n(W_f)$, every $n > 0$. It is claimed now that there exists (a unique) extension of h to $U_f \setminus W_f^{'}$, where $W_f^{'} = \bigcup_{m,n\in Z} f^{-m}(f^n(W_f))$ is the grand orbit of W_f. First extend h to the preimage of V_f. Observe that $f : f^{-1}(V_f) \setminus W_f \to V_f \setminus f(W_f)$ is a covering map of degree d, from which it follows that $h \circ f : f^{-1}(V_f) \setminus W_f \to V_g \setminus W_g$ is a degree d covering map. Also the restriction of g to $g^{-1}(V_g) \setminus W_g$ is a degree d covering map onto $V_g \setminus W_g$. Domains and codomains are topological annulus, and the homomorphisms induced by $h \circ f$ and by g in fundamental groups are both conjugated to multiplication by d on $\mathbb{Z}$. Then there exists a unique lift $\tilde{h}$ of $f \circ h$ such that $g \circ \tilde{h} = h \circ f$ and $\tilde{h}(x_f) = x_g$. The uniquenes of $\tilde{h}$ implies that it extends h.

The same argument shows how to extend h to the whole $U_f \setminus W_f'$. Finally one must extend h to U_f.

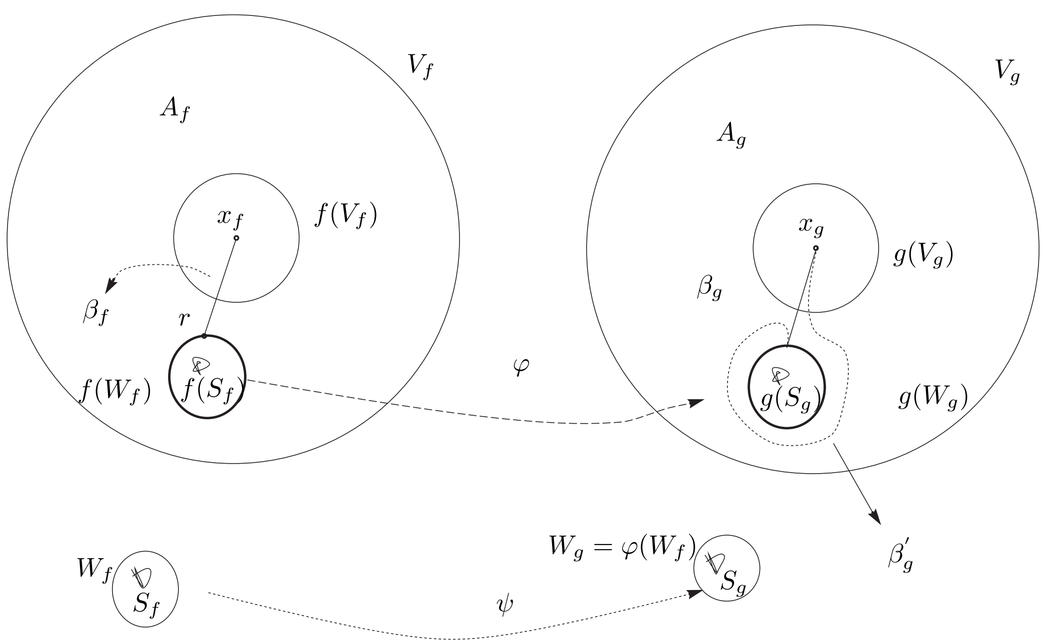
To define h in W_f and its preimages, other details must be taken into account, relative to the fact that the restrictions of h and ψ to the boundary of W_f may be equal or not. In the first case, h can be extended to W_f as equal to ψ and then to the remaining part of U_f dynamically. But in the other case h and ψ differ in the boundary of W_f, so the definition of h started in formula (1) must be changed. Note that the set of points of ∂W_f where h and ψ are equal is open and closed in ∂W_f, so it suffices to find a way of make them coincide at just one point. Recall that the definition of h depends on the choice of a curve β; changing its homotopy class in $V_g \setminus g(W_g)$ that objective will be attained.

Given any element j in the fundamental group of $V_g \setminus g(W_g)$ (say $j \in \mathbb{Z}$), let $\beta_g^j \subset V_g$ be a simple curve joining x_g with $\varphi(r)$ such that the class of the curve $\beta_g^j(\beta_g)^{-1}$

in the fundamental group of $V_g \setminus g(W_g)$ is equal to that element j (see figure 1, where $j = 1$). Then a homeomorphism h^j can be defined as in (1), but changing the curve β_g by β_g^j; then the restrictions of h^j and h to W_f are different. Indeed, more than this can be said: there is a map R that assigns, to each β_g^j, the value of the corresponding h^j at the point r_{-1}, for some fixed $r_{-1} \in f^{-1}(r) \cap \partial W_f$. If ν is the order of the critical point that the polynomial p had close to S_f, then the map R is injective if restricted to the set of curves $\{\beta_g^1, \ldots, \beta_g^\nu\}$. It is thus proved that there exists a curve β_g such that the homeomorphism h determined by this β_g is equal to ψ in the boundary of W_f.
The same reasoning can be applied if there are more critical points of p in U_p.

□

FIGURE 1.



This previous result treated with fixed domains. Suppose now that the polynomial p has an attracting cycle $\alpha_p = \{x_p^1, ..., x_p^n\}$. For every f close to p in C^1 topology, denote by $U_f^1, ..., U_f^n$ the components of the immediate basin of the attractor $\alpha_f = \{x_f^1, ..., x_f^n\}$ analytic continuation of α_p. Define also $U_f = \bigcup U_f^j$. The following is an easy generalization of the previous lemma 6, and its proof is omitted.

Lemma 7. *If f and g are geometrically equivalent maps C^1 close to p, then they are also topologically equivalent when restricted to the grand orbits of U_f and U_g.*

Using that every component of the complement of the set $\Omega'(f)$ is preperiodic and the previous results, it comes at once:

Corollary 3. *If f and g are geometrically equivalent and C^1 close to p, then there exists $h : \mathbb{R}^2 \setminus (\Omega^{'}(f) \cup B_\infty(f)) \to \mathbb{R}^2 \setminus (\Omega^{'}(g) \cup B_\infty(g))$, homeomorphism close to the identity that conjugates f and g.*

5. Extension of the exterior conjugacy

By the results of section 3, it is known that there exists a conjugacy h between the restriction of f to $B_\infty(f)$ and $z \to z^d$ in the complement of the unit disc. If f and g are close to a polynomial p satisfying the hypothesis (1) of the theorem, then there exists a conjugacy (also called h) between the restrictions of these maps to the respective basins of ∞. On the other hand, the theorem of Przytycki provides a conjugacy h_p of these maps in the boundaries of the respective domains. This section is devoted to prove that h can be continuously extended to the closure of $B_\infty(f)$, and that in the boundary is equal to the conjugacy of Przytycki.

Recall from theorem 3 of section 3 that the conjugacy h was constructed taking account of the preimages of the curve γ_f whose endpoint is a fixed point (analytic continuation of the fixed point that the polynomial p has in ∂B_∞). The conjugacy of Przytycki h_p carries this fixed point of f to the corresponding fixed point of g. It follows that the map H defined as h in B_∞ and by h_p in ∂B_∞, is continuous if restricted to the union of the preimages of the closures of γ_f. The difficulty now is to prove the continuity at the other points , mainly because the boundary of the basin of $B_\infty(f)$ may not be a Jordan curve. We begin this exposition with a brief discussion of the elementary definitions of Caratheodry's theory of prime ends.

5.1. **Prime ends.** The works of J.Milnor [Mi] and J.Mather [Mat] are recommended for the proofs of the results here stated.
Let $U \subset S^2$ be a simply connected set such that U^c, the complement of U, contains at least three points. A simple curve Q is a crosscut if it is contained in U except for its extreme points.
Each crosscut Q separates U into two connected components. One of these components is denoted $N(Q)$. A fundamental chain is a sequence of disjoint crosscuts $\{Q_n\}$ such that $N(Q_1) \supset N(Q_2) \supset N(Q_3) \cdots$ and $diam(Q_n) \to 0$. Two fundamental chains $\{Q_n\}$, $\{Q^{'}_n\}$ are equivalent if each $N(Q_i)$ contains some $N(Q^{'}_j)$ and reciprocally. Each equivalent class of fundamental chains is called a **prime end** of U. For a fundamental chain $\{Q_n\}$ defining a prime end $\mathcal{P}$, define the impression of $\mathcal{P}$ as $I(\mathcal{P}) = \bigcap_{n \in N} \overline{N(Q_n)}$.

Theorem 4. *∂U is locally connected if and only if the impression of each prime end has only one point.*

A natural topology was defined by Carathéodory in the union of U and the set of prime ends, denoted from now on by $\widetilde{U}$.

Proposition 1. *The Riemann map $\varphi : \mathbb{D} \to U$ has a unique extension to a homeomorphism $\widetilde{\varphi} : \overline{\mathbb{D}} \to \widetilde{U}$, where $\mathbb{D}$ denotes the closed unit disc.*

Consider a map $f \in C^1(S^2)$, leaving invariant a simply connected set U. Assume that the boundary of U is locally connected and does not intersect the set of critical points of f.

Proposition 2. *Under these conditions, there exists a continuous map $\widetilde{f} : \widetilde{U} \to \widetilde{U}$ such that $\widetilde{f}|_U = f$.*

Proof. Given a prime end $\mathcal{P} \in \widetilde{U}$, consider a fundamental chain $\{Q_n\}$ associated to it. As f is locally a homeomorphism and ∂U is locally connected, it follows that $\{f(Q_n)\}$ is a fundamental chain that defines a prime end $\mathcal{P}^{'}$. Then define $\widetilde{f}(\mathcal{P}) = \mathcal{P}^{'}$. This map is clearly continuous and locally injective at any $\mathcal{P} \in \partial\widetilde{U}$. □

This procedure determines a continuous map (not necessarily a homeomorphism) $F : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $\widetilde{\varphi} \circ F = \widetilde{f} \circ \widetilde{\varphi}$ and verifies the following properties:

- $F|_{\partial\mathbb{D}}$ is continuous and locally injective, because $f|_{\partial U}$ is locally a homeomorphism.
- If $p \in S^1$ is a fixed point of F, then $I(\widetilde{\varphi}(p))$ is a fixed point of f (see theorem 4).
- If J is a connected subset of S^1, then $\cup_{y\in J} I(\tilde{\varphi}(y))$ is connected.
- If every periodic point of f is repelling, then every periodic point of F is repelling.

Proposition 3. Carathéodory *The Riemann map extends to a continuous and surjective map from $\overline{\mathbb{D}}$ to $\overline{U}$ iff ∂U is locally connected and iff $S^2 \backslash U$ is locally connected.*
If the boundary of U is a Jordan curve, then the inverse of the Riemann map extends to a homeomorphism from $\overline{U}$ to the closed disc $\overline{\mathbb{D}}$.

5.2. **Extension to the boundary of B_∞.**

Remark 1. If ∂U is expanding and $C \subset \partial U$ is a nontrivial connected set, then there exists $\delta > 0$ such that $diam(f^n(C)) > \delta$, $\forall n \geq 0$.

Lemma 8. *If $\gamma : [0,1] \to \overline{U}$ is the invariant curve obtained in lemma 3, $\gamma(1) = x \in \partial U$ and $\lim_{t\to 1} \widetilde{\varphi}^{-1}(\gamma(t)) = x_0 \in S^1$, then*

$$\overline{\bigcup_{n\geq 0} F^{-n}(x_0)} = S^1$$

Proof. Supose by contradiction that $\overline{\bigcup_{n\geq 0} F^{-n}(x_0)} \neq S^1$. Note that as $F(x_0) = x_0$ then $\overline{\bigcup_{n\geq 0} F^{-n}(x_0)}$ is invariant under F. If I is a connected component of $S^1 \setminus \overline{\bigcup_{n\geq 0} F^{-n}(x_0)}$ then either I is wandering or there exists $n_0 \in N$ such that $F^{n_0}(I) = I$. In the first case, $diam(F^n(I)) \to 0$ as $n \to +\infty$. Let $z_0 \in S^1$ and $\{n_k\}$ be such that $F^{n_k}(I) \to z_0$; as ∂U is locally connected, the impression of the prime end $\widetilde{\varphi}(z_0)$ consists of a single point. Let $\{Q_i\}$ be a fundamental chain that defines the prime end $\widetilde{\varphi}(z_0)$. For every crosscut Q_i there exists n_{k_i} such that the impression of the prime end $\widetilde{\varphi} \circ F^{n_{k_i}}(I)$ is included in $\overline{N(Q_i)}$, and this implies that $diam(f^{n_{k_i}}(C)) \to 0$, where C is the connected set determined by the impression of the prime end $\widetilde{\varphi}(I)$. This is a contradiction because ∂U is an expanding set (remark 1).
In the remaining case, there is an $n_0 \in N$ such that $F^{n_0}(I) = I$. Hence there exists a periodic point attracting by at least one side. This is a contradiction because every periodic point of F is repelling. □

Let f and g be C^1-close to p. By the theorem of Przytycki, there exists a homeomorphism $h_p : \partial B_\infty(f) \to \partial B_\infty(g)$ close to the identity and such that $h_p f = g h_p$. On the other hand, theorem 3 implies that there exists a homeomorphism $h : B_\infty(f) \to B_\infty(p)$ such that $hf = gh$ and is C^0 closed to the identity.

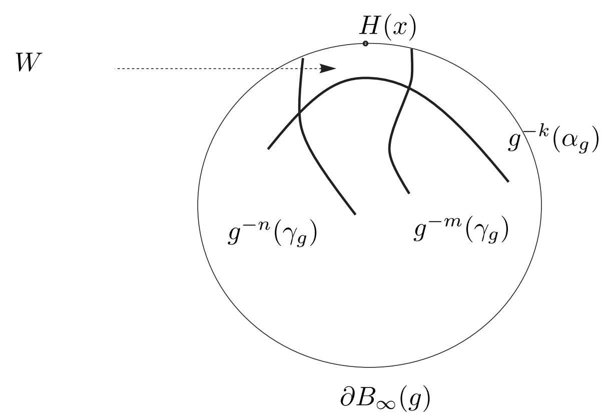
Lemma 9. *Let f and g be C^1-close to p. Then the function*

$$H(x) = \begin{cases} h_p(x) \ si \ x \in \partial B_\infty(f) \\ \\ h(x) \ si \ x \in B_\infty(f) \end{cases}$$

is continuous.

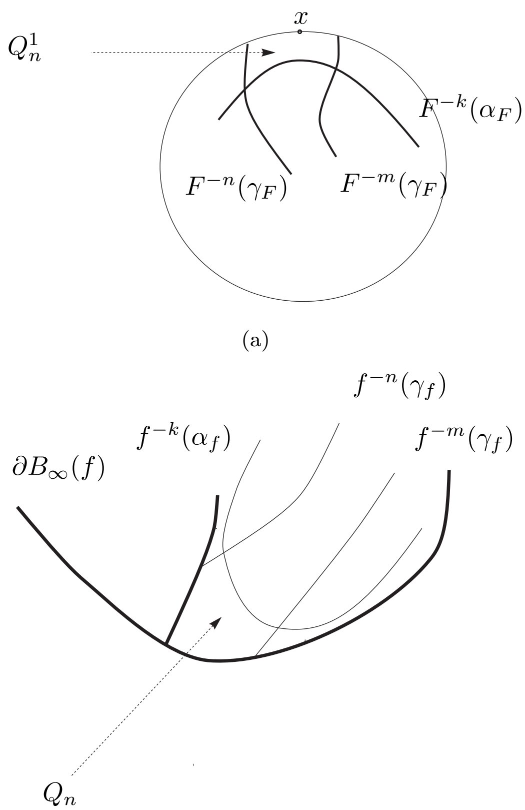
Proof. Note that $\partial B_\infty(f)$ and $\partial B_\infty(g)$ are connected and locally connected. Let $\varphi_1 : \mathbb{D} \to B_\infty(f)$ and $\varphi_2 : \mathbb{D} \to B_\infty(g)$ be the Riemann maps. Denote by $\gamma_F = \varphi_1^{-1}(\gamma_f)$ and $\alpha_F = \varphi_1^{-1}(\alpha_f)$ where γ_f and α_f are the curves given in lemma 3 and 4 respectively. We use corresponding notation for g. Let $\widetilde{\varphi}_1 : \overline{\mathbb{D}} \to \widetilde{B}_\infty(f)$, $\widetilde{\varphi}_2 : \overline{\mathbb{D}} \to \widetilde{B}_\infty(g)$ be the extensions of the Riemann maps. Note that if q_f is the final point of γ_f and q_g is the final point of γ_g, then $h_p(q_f) = q_g$. The same occurs with corresponding preimages.
Now we will divide the proof in two cases, the second one contains the first, but this one will be used in other parts of the work.
Case (a). $B_\infty(f)$ and $B_\infty(g)$ are Jordan curves.
Let $x \in \partial B_\infty(f)$ and $\varepsilon > 0$. Recall that the preimages under g of the curve γ_g are dense in $B_\infty(g)$; note also that $g^{-n}(\alpha_g)$ converges in the Hausdorff topology to $\partial B_\infty(g)$. Let B be the disc of center $H(x)$ and radius ϵ. By the arguments above, one can find positive integers n, m and k, such that the set W whose boundary is formed up by segments of the curves $g^{-n}(\gamma_g)$, $g^{-m}(\gamma_g)$ and $g^{-k}(\alpha_g)$, for some n, m and k, is contained in B as figure 2 shows. By the construction of h made in theorem 3 of section 3, and as was explained at the beginning of this section, the restriction of H to the set of the preimages of the closure of γ_f is continuous. This implies that $H^{-1}(W)$ must be a neighborhood of the point x. The continuity of H follows.

FIGURE 2.



Case (b). $B_\infty(f)$ and $B_\infty(g)$ are arbitrary closed curves.
The idea is to prove first that H induces a function $H^1 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ and then apply case (a) for H^1.
Define first $h^1 : \mathbb{D} \to \mathbb{D}$ by $h^1 = \varphi_2^{-1} \circ h \circ \varphi_1$. Then we will define $h_p^1 : \partial\mathbb{D} \to \partial\mathbb{D}$. Let

$x \in \partial\mathbb{D}$; by lemma 8, and as $F^{-k}(\alpha_F)$ converges to ∂D with the Hausdorff metric of compact subsets, there exists a fundamental chain $\{Q_n^1\}$ that defines the prime end x as in figure 3(a). Then $h^1(\{Q_n^1\})$ is a fundamental chain $\{Q_n\}$. Define $h_p^1(x)$ as the prime end determined by this chain. Note that h_p^1 is well defined, continuous and bijective.



(a)

(b)

FIGURE 3.

Then the function

$$H^1(x) = \begin{cases} h_p^1(x) \text{ if } x \in \partial\mathbb{D} \\ \\ h^1(x) \text{ if } x \in \mathbb{D} \end{cases}$$

is a homeomorphism. This was the crucial step, it implies that for every prime end in $\widetilde{B}_\infty(f)$ there exists a fundamental chain $\{Q_n\}$ whose boundaries are segments of curves α and γ taken from the foliations $\mathcal{F}$ and Γ. It follows that if $\{Q_n\}$ is a fundamental chain whose impression is a point x, then $\{h(Q_n)\}$ is a fundamental chain whose impression is $h_p(x)$. This is almost the continuity of H, indeed it implies the continuity at x "coming from" a fundamental chain. But there can be many of these sets approaching x, so we need to work a little bit more. It follows from the above that if $\rho : [0,1] \to \overline{B_\infty(f)}$ is a curve, then $H \circ \rho$ is also continuous, because each curve landing at a point x in the boundary of $B_\infty(f)$ determines a prime end associated to x. The lemma will be immediately implied by the following general result. □

Lemma 10. *Let U_1 and U_2 be open simply connected subsets of the plane with locally connected boundary. Let $H : \overline{U}_1 \to \overline{U}_2$ be such that the restriction of H to U_1 is a homeomorphism onto U_2. Then H is continuous if and only if H carries curves into curves.*

Proof. Suppose that H is not continuous at a point $x \in \partial U_1$. Then there exists $\varepsilon > 0$ and a sequence (y_n), such that $y_n \in U_1$, $y_n \to x$ and $H(y_n) \notin B(H(x), \varepsilon)$. Let φ be the Riemann map from $\mathbb{D}$ to U_1. Let (z_{n_k}) be a subsequence of $\varphi^{-1}(y_n)$ such that $z_{n_k} \to z \in \partial\mathbb{D}$ and $\gamma : [0,1] \to \overline{\mathbb{D}}$ is a curve such that $\gamma(1) = z$ and $z_{n_k} \in \gamma([0,1])$. As the boundary of U_1 is locally connected, the $\lim_{t\to 1} \varphi \circ \gamma(t)$ exists, and must be equal to x. Therefore $\varphi \circ \gamma$ is a curve, but $H \circ \varphi \circ \gamma$ is not continuous at the point 1. This proves one direction in the statement and the other one is trivial. □

6. Extension of the interior conjugacy

In this section it is shown that the conjugacy constructed in the bounded components can be extended continuously to the boundary of B_∞. The first step is to prove that the conjugacy extends to the boundary of each component. Then it will be proved that the extension is continuous also at the points that are accumulated by different components. Again,the extension is equal to the Przytycki map at the boundary.
Beginning with the first step, we will need the following result.

Lemma 11. *Let U_p be the immediate basin of an attracting periodic orbit $x_p^1, ..., x_p^n$ of p. For every f C^1- close to p there exists a simple curve γ_f that joins x_f^1 with a periodic point z_f in the boundary of U_f^1, and such that $f^n(\gamma_f) = \gamma_f$.*

Proof. We can suppose that U_p is fixed. Let A_f be as in lemma 6. Let x be any point in the exterior boundary of A_f and define a simple curve $\gamma_0 \subset A_f$, joining x with $f(x)$ and disjoint from the grand orbit of S_f. For every $i \in \mathbb{Z}$, define γ^i by $f(\gamma^{i+1}) = \gamma^i$. Note that γ^{i+1} and γ^i have only one common point. It follows that $\gamma_f = \cup\gamma^i$ is a simple curve one of whose extreme points is x_f. It remains to prove that the other extreme point is also a fixed point of f. Denote by z_f this fixed point. This follows from the fact that the boundary of U_f is expanding. Therefore the length of γ^i converges exponentially to zero as i goes to ∞, hence γ_f converges to a fixed point in the boundary. □

Let f and g be as in the final corollary of section 4, and let h be the conjugation between U_f and U_g. Then:
1) $\gamma_g = h(\gamma_f)$ is invariant under g, has a periodic point z_g in the boundary of U_g^1 and $g^n(\gamma_g) = \gamma_g$.
2) The curves γ_f and γ_g are close, because the conjugation h is close to the identity. Recall that the conjugation in the boundaries h_p is also close to the identity, so it follows that the final point of γ_g is equal to the image under h_p of the final point of γ_f. This will be used in the sequel.

Lemma 12. *Given any $\epsilon > 0$ there exist a finite number of connected components of $\overline{B_\infty(f)}^c$ having diameter greater than ϵ.*

Proof. Suppose by contradiction that there exist infinitely many components of diameter at least ϵ, and let $x \in \partial B_\infty$ be an accumulation point of different components. Let V_1 and V_2 be the boundaries of the discs with center x and radius $\epsilon/4$ and $\epsilon/8$ respectively. From the definition of x, there exist infinitely many components intersecting both V_1 and V_2. This contradicts the fact that ∂B_∞ is locally connected.

□

Using lemma 11 and reasoning as in case a) of lemma 9 it follows that the restriction of h to any connected component can be continuously extended to its boundary, and there, it is equal to the Przytycki map. Also denote by h the extended map. To prove that h is continuous, it remains to prove the continuity at the points that are accumulated by different components. To deal with this case proceed as follows:
Let $x \in \partial B_\infty(f)$ be accumulated by points of different components of the complement of $\overline{B_\infty(f)}$. We prove now the continuity of h at x, so give $\epsilon > 0$. The Przytycki map h_p, already known to be continuous, is defined in x, so there exists a $\delta > 0$ such that if $y \in B(x;\delta) \cap \partial B_\infty(f)$ then $h_p(y) \in B(h(x);\epsilon)$. Note also that, by lemma 12, only finitely many components $U_1, \dots, U_N$ having diameter greater than $\delta/4$ intersect $B(x;\delta/2)$. Moreover, it is clear that if a component U having diameter less than $\delta/4$ intersects $B(x;\delta/2)$, then it is contained in $B(x;\delta)$ and then $h(U)$ is contained in $B(h(x);\epsilon)$ because its boundary is contained there by the above observation. If U_i is a component whose boundary does not contain x then we can diminish δ to leave outside this neighborhood. For each i such that x belongs to the boundary of U_i, the continuity of the restriction of h to the closure of U_i implies that there exists $\delta_i > 0$, such that if y belongs to $B(x;\delta_i) \cap U_i$ then $h(y) \in B(h(x);\epsilon)$. These considerations imply the continuity of h.

7. Proof of theorem 1

Proof of $(1) \Rightarrow (2)$
This has already done through the sections 3 to 6. In section 3 it was proved that if f is a Whitney C^1 perturbation of a polynomial p with connected Julia set, then the dynamics of f and p are conjugated in the basins of ∞. To prove this, C^1 perturbations were allowed, and only the connectedness of the Julia set of p was used.
Then lemma 6 of section 4 proves that the restrictions of f and g to the bounded domains of the complement of the nonwandering sets are conjugated whenever f and g are geometrically equivalent. For this part the hyperbolicity of the map p was used.
Afterwards, lemma 9 of section 5 provides the fact that the exterior and Przytycki conjugacies fit together to give a whole conjugacy between the restrictions of f and g to the closure of the basins of ∞.
Finally section 6 gives the proof that the conjugacy in the bounded components of the complement of the closure of the basin of ∞ also extends continuously.

Proof of $(2) \Rightarrow (1)$
The hypothesis gives a C^1_W neighborhood $\mathcal{U}$ of p such that geometric and topological equivalence are the same in $\mathcal{U}$. Maps of class C^3 are dense in $\mathcal{U}$ and its critical points have a generic structure. The proof of the following lemma can be found in [IP].

Lemma 13. *Let c be a simple critical point of p, that is, $p'(c) = 0 \neq p''(0)$. There exist a neighborhood U of c, a C^3 neighborhood $\mathcal{U}_0$ of p and an open and dense subset $\mathcal{G}$ of $\mathcal{U}_0$ such that, for every $f \in \mathcal{G}$, the intersection $S_f \cap U$ is diffeomorphic to a circle.*
Moreover, there exists $f \in \mathcal{G}$ such that the restriction of f to $S_f \cap U$ is injective and $S_f \cap U$ contains exactly three cusp type points.

Remark 2. The study of critical points of differentiable mappings is an interesting subject. Here we use some elementary facts in dimension two, a classical reference is the book by Golubitsky and Guillemin [GG].
We do not know if a neighborhood $\mathcal{U}_0$ can be found such that the restriction of every map $f \in \mathcal{G}$ to $S_f \cap U$ is injective, but only the existence of such a map (as asserted in lemma 13) is needed in forward arguments. It is known, however, that there exists at least one cusp type point in the boundary of the unbounded component of the complement of $S_f \cap U$.
The clasification of critical points for generic maps is very easy in dimension two. Indeed, if c is a critical point of a generic map f, then the kernel of Df_c has dimension one. The critical point c is a fold if the kernel of Df_c is not equal to the tangent space of S_f at c and is a cusp otherwise. Moreover, normal forms are known for both kind of maps:
The normal form of a fold point is the origin for the map $(x, y) \to (x^2, y)$.
The normal form of a cusp point is the origin for the map $(x, y) \to (x^3 - xy, y)$.
A C^3 condition can be imposed to make the cusp type point generic, but no C^2 condition can assure the stability and persistence of such kind of critical point.
As well as maps having critical points cannot be C^1 stucturally stable, it can be concluded now that maps with cusp type points cannot be C^2 structurally stable, because a conjugacy between two maps must carry cusp critical points to critical points of the same type.
As any generic perturbation of a polynomial has a cusp type point, it follows, as asserted in the introduction, that in a small neighborhood of a polynomial no map can be C^2 structurally stable.

The next step is to prove that if $f \in \mathcal{G}$ satisfies that the restriction of f to S_f is injective, then the same holds in a C^3 neighborhood of f. Observe first that given a compact set K contained in the complement of a neighborhood of the cusp type points there exists a C^2 neighborhood of f such that for every g there, the set of cusp type points of g are not in K. As locally in a fold type point the restriction of a map to its critical set is locally injective, it is sufficient to prove that for every g that is C^3 close to f, the restriction of g to its critical set is locally injective at a cusp type point.

Lemma 14. *Let c be a cusp type point of a generic map $f \in \mathcal{G}$. Then there exist neighborhoods U of c and $\mathcal{U}$ of f in C^3 topology such that, for every $g \in \mathcal{U}$, the restriction of g to $S_g \cap U$ is injective.*

Proof : By the aforementioned theorem of Whitney, there exists a neighborhood U of c such that the restriction of f to U is geometrically conjugated to the map $(x, y) \to (x^3 - xy, y)$. Cusps are C^3 stable: is in that sense that g has exactly one cusp point near c. The lemma is first proved for a map $g(x, y) = (h(x, y), y)$ close to f. Then h is C^3 close to $x^3 - xy$, so the critical points of g satisfy the equation $\partial_x h(x, y) = 0$ which has, for x close enough to 0, a unique solution $\tilde{y}(x)$ of class C^2. An easy calculation shows that $\tilde{y}''$ is close to 6, and this implies that $\tilde{y}$ has a unique minimum. To prove that the restriction of g to the intersection of S_g and a neighborhood of c is injective, observe that $g(x, y) = g(x_1, y_1)$ implies $y = y_1$ and $h(x, y) = h(x_1, y)$; if both points are critical, then $\tilde{y}(x) = \tilde{y}(x_1) = y$. But $\tilde{y}$ has a unique minimum, so, for every t in the interval I whose extreme points are x and x_1 it holds that $\partial_x h(t, y) \neq 0$: this implies that $h(x, y) \neq h(x_1, y)$ which is a contradiction.
To prove the lemma for an arbitrary g, C^3 close to $f(x, y) = (x^3 - xy, y)$, let $g(x, y) = (F(x, y), G(x, y))$; as $G_y \approx 1$, the equation $G(x, y) = v$ defines a function y_v such that $G(x, y_v(x)) = v$. Let $\varphi(u, v) = (u, y_v(u))$; to prove that φ is locally a diffeomorphism note that

$$D\varphi_{(u,v)} = \begin{pmatrix} 1 & 0 \\ * & \frac{\partial}{\partial_v} y_v(u) \end{pmatrix}.$$

To obtain that φ is locally invertible it suffices to prove that $\frac{\partial}{\partial_v} y_v(u) \neq 0$. But $G(x, y_v(x)) = v$, hence derivating with respect to v it comes that $G_y(x, y_v(x)) \frac{\partial}{\partial_v} y_v(u) = 1$, which implies that $\frac{\partial}{\partial_v} y_v(u) \neq 0$. It also follows that φ is close to the identity. It comes that

$$g \circ \varphi(u, v) = g(u, y_v(u)) = (\tilde{F}(u, v), G(u, y_v(u))) = (\tilde{F}(u, v), v),$$

which concludes the proof of the lemma, as $\tilde{F}$ is close to $(x, y) \to x^3 - xy$.

□

Remark and notation: Suppose that every critical point of p is simple, and let $S_p = \{c_i \ : \ 1 \leq i \leq d - 1\}$; for each i, let U_i be a small neighborhood of c_i, and $\mathcal{G}_i$ the generic set associated with c_i as in lemma 13.
Recall that p satisfies the non critical relations property, so the degree of p is d and the number of critical values of p is $d - 1$.
Define $\mathcal{G}' \subset \mathcal{U}$ as the set of maps f such that $f|_{S_f}$ is injective and f belongs to every $\mathcal{G}_i$. It follows that S_f has $d - 1$ connected components, each one of them homeomorphic to the circle and such that the restriction of f to S_f is injective. It is left to the last corollary the proof that $\mathcal{G}'$ is nonempty. This, together with the following proposition, will provide the examples of structurally stable maps.

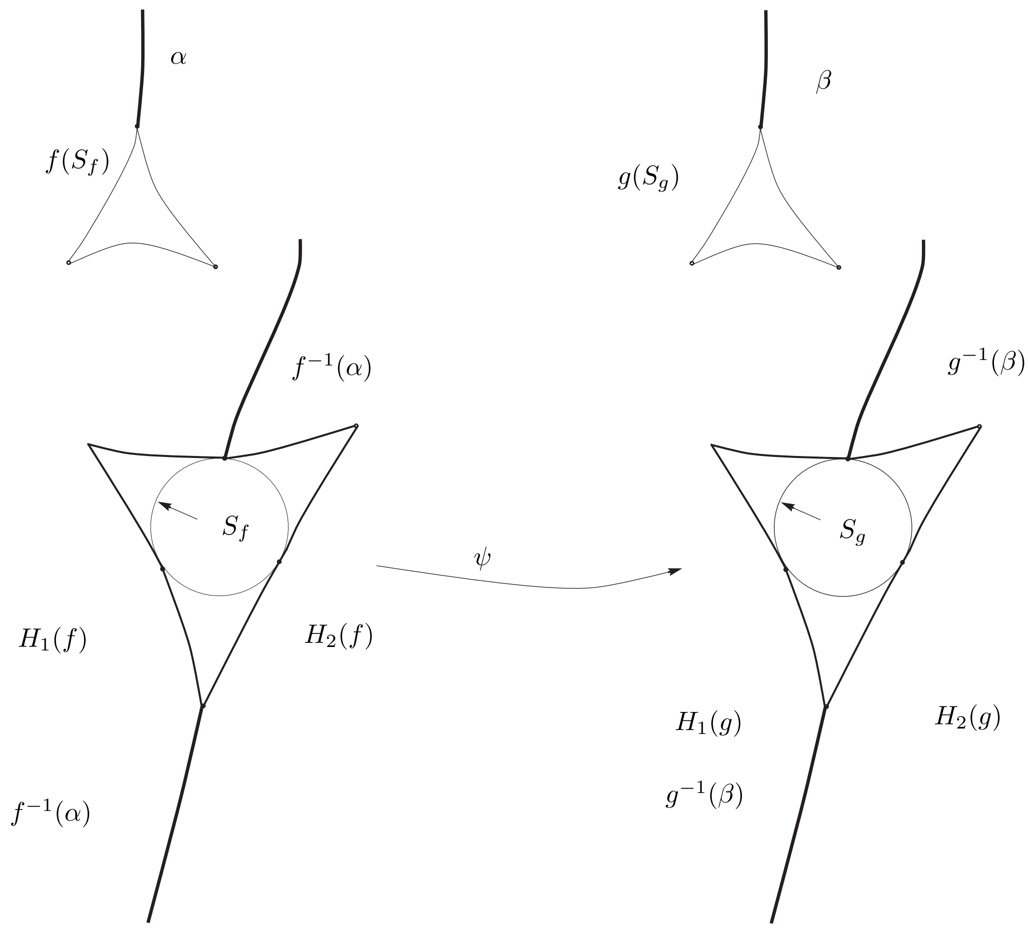
Proposition 4. *If $f \in \mathcal{G}'$, then f is geometrically stable.*

Proof. Let g be a C^3 perturbation of f, let $\{C_1(g), \ldots, C_{d-1}(g)\}$ be the components of the set of critical values of g.
Let φ a diffeomorphism of the plane close to the identity, carrying $C_i(f)$ onto $C_i(g)$ for each i. For each i choose a curve α_i joining the image of a cusp point $z_i \in C_i(f)$ with infinity. This can be done without any intersection, that is, the curves α_i are simple, disjoint and the intersection of α_i with $\cup C_i(f)$ is the set $\{z_i\}$. Let $\beta_i = \varphi(\alpha_i)$ and define $H(f)$ as the complement of the union of $\tilde{S}_f$ with $\cup_i f^{-1}(\alpha_i)$

and $H(g)$ as the union of the unbounded components of the complement of the union of $\tilde{S}_g$ with $\cup_i g^{-1}(\beta_i)$. See the figure 4 below with $d = 2$.

Each component of $H(f)$ corresponds to a unique component of $H(g)$ by proximity. Moreover, these components of $H(f)$ are simply connected, and the restriction of f to each of them is a diffeomorphism onto its image. Therefore, for each component $H_j(f)$ of $H(f)$ there exists a unique diffeomorphism ψ_j that satisfies $\varphi f = g\psi_j$, and whose image is the corresponding component of $H(g)$. These diffeomorphisms can be extended to a unique diffeomrphism ψ of the plane such that $\varphi f = g\psi$. □

Figure 4.



Proof of the connectedness of the Julia set of p.
Assume that J_p is not connected. This implies that there exists a critical point $c = c_1$ of p contained in $B_\infty(p)$ and that c is the critical point of p closest to ∞ (i.e. the circle of the foliation that contains c is the boundary of an open neighborhood of ∞ that does not contain any other finite critical point). Assume first that c is a simple critical point of p. By the proof of proposition 4 two maps f and g

in $\mathcal{G}_1$ that are equal outside the neighborhood U_1 of c, are geometrically equivalent. To arrive to a contradiction it suffices to find f and g as above that are not topologically equivalent. To do this observe first that there exists a neighborhood of ∞ foliated by curves homeomorphic to circles. This foliation is invariant and must be preserved by conjugacies. To find a contradiction, the idea is to take first a perturbation f of p such that the component of S_f in U has two cusps in the preimage of a circle of the foliation. Then perturb f to a map $g \in \mathcal{G}_1$ that is equal to f outside U_1 and such that the previous condition is broken. Thus f and g are geometrically but not topologically equivalent.
Let A be a p-invariant neighborhood of ∞ that contains $p(c)$, does not intersect U_1 and whose boundary is a circle of the foliation $\mathcal{F}_p$. If f is a perturbation of p with support U_1 ($f = p$ outside U_1) then the foliation $\mathcal{F}_f = \mathcal{F}_p$ in A. So perturb p in U_1 such that the perturbation f belongs to $\mathcal{G}_1$ and two cusps in the component of S_f contained in U_1 have image in the same leave of the foliation $\mathcal{F}_f$. This is possible but is not generic; a new perturbation g supported in U_1 and belonging to $\mathcal{G}_1$ can be found such that the image of the three cusps belong to different leaves of the foliation.
To treat the case of c not simple, assume that the order of c is k. Given a neighborhood U_0 of c there exists a C^∞ perturbation q of p such that:

- $q = p$ outside U_0.
- There exists an arbitrary small neighborhood $U_0' \subset U_0$ of c such that q is holomorphic in U_0'.
- q has k critical points in U_0 and they are contained in U_0'.

Once this q was obtained, one can proceed as above.

□

Proof of the hyperbolicity of p.
The first step is to prove that the Julia set cannot have critical points if some type of C^1 stability is required. The proof is very simple, which contrasts with the fact that the problem is open when only holomorphic perturbations are allowed.

Proposition 5. *If p has a critical point in its Julia set, then in every C^1 neighborhood of p there exists an f that is geometrically but not topologically equivalent to p.*

Proof : Let $\mathcal{U}$ be a C^1 neighborhood of p and c be a critical point of p in J_p. This implies that there exist expanding periodic points accumulating at c. An argument based in J.Frank's lemma [F] will imply the existence of a map f in a C^1 neighborhood of p such that f and p have the same sets of critical points but f has a new attracting periodic orbit. Indeed, if ε is such that $f \in \mathcal{U}$ if the C^1 distance between p an f is less than ε, then take a periodic orbit of p contained in J_p and containing a point z close to c in such a way that $|p'(z)| < \varepsilon$. Let $K = |(p^n)'(z)|$, where n is the period of the orbit of z. Note that there exists a neighborhood of the orbit of z such that the restriction of p to this neighborhood is a diffeomorphism onto its image. Under these conditions, Franks' lemma asserts that there exists a map $f \in \mathcal{U}$ such that:

- The orbit of z under f is the same as that of p.
- For every $0 < j < n$, the differential of f at $f^j(z)$ is equal to that of p at the same point. Moreover, f is also conformal at z, and $|f'(z)| < |p^{'}(z)|/K$.

- The support of the perturbation is an arbitrary small neighborhood of the orbit of z not intersecting the critical set of p or the set of periodic attractors of p.
- The perturbation f is a diffeomorphism onto its image when restricted to the support of the perturbation.

The first three items imply that f has a new periodic attractor (the orbit of z) and so it is not topologically equivalent to p. It is geometrically equivalent to p because the support of the perturbation is disjoint with the set of critical points of p.

□

To conclude the proof of the hyperbolicity of p, one has to show that every critical point is attracted to a periodic attractor. First of all note that every periodic point of p must be hyperbolic: under the contrary assumption one can perturb in a neighborhood of the nonhyperbolic orbit to obtain a map that is geometrically but not topologically equivalent to p. This implies that the Fatou set of p does not contain Leau components neither Siegel discs. Hermann rings are forbidden since the Julia set of p is connected. Finally, as the set of critical points do not intersect the Julia set and there are no superattractors, the conclusion is immediate from the classification theorem of Sullivan, see [Mi] or [St].

Proof of corollary 1

For every p in the hypothesis (1) or (2) of theorem 1, and in every C^∞ neighborhood of p there exists a map f that is C^3 structurally stable:

It suffices to show that in every C^∞ neighborhood of p there exists a map $f \in \mathcal{G}'$, because by proposition 4 this map will be geometrically equivalent to every map g in a C^3 neighborhood of it, and then the (1) $\Rightarrow$ (2) ot theorem 1 implies the topological equivalence between f and g. It is very easy to give an example that is generic in the sense of lemma 13 and such that the restriction of f to S_f is injective. It suffices to do it locally, and as the critical points of p are nondegenerate, it suffices to give just an example of a perturbation f of $p(z) = z^2$ such that $f \in \mathcal{G}'$. Generic quadratic polynomials satisfy these conditions ([DRRV]); to give an explicit example: $(x, y) \to (x^2 - y^2 + \lambda y, 2xy)$, $\lambda \neq 0$. So to construct an example of a C^3 structurally stable map, just take $p(z) = z^2 + \epsilon$ (ϵ small so that J_p is connected and hyperbolic) and then perturb in a neighborhood of 0 so that the new map f has the representation above in that neighborhood.

Further considerations.

Throughout this discussion, M is a manifold of dimension at least two, $E^r(M) \subset C^r_W(M)$ denotes the space of noninvertible nonexpanding endomorphisms, and $St^r(M)$ the set of C^r structurally stable maps.

As far as we know, there exist no known examples in $St^1(M) \cap E^1(M)$. Note that $S_f = \emptyset$ is a necessary condition for a map f to be C^1 stable. The theorem of N.Aoki, K.Moriyasu and N.Sumi in [AMS] implies that a map in $St^1(M)$ must satisfy the Axiom A and as is the case for diffeomorphisms, also the strong transversality condition. However, these conditions are not sufficient for stability, as was shown by an example of F.Przytycki in [P]. The problem is the possible existence of a basic piece which is neither attracting nor expanding: indeed, unstable manifolds of a basic piece may have self intersections and can also visit different basic pieces. Then they are too wild to handle and this creates new mechanisms of instability. On the other hand, the arguments in this article seem to be extendable to prove stability in other situations, where the maps have only expanding or attracting basic pieces. The following question is pertinent: There exists an Axiom A map $f \in E^1(M)$ and without critical points such that every basic piece is attracting or expanding? Such an example should be C^1 structurally stable and perhaps the techniques here developed would be of utility to prove that. But if the answer were negative, a possible conjecture would be that there are no new C^1 structurally stable maps.

□

References

[AMS] N.Aoki, K.Moriyasu, N.Sumi. C^1 maps having hyperbolic periodic points. Fundamenta Mathematicae 169 p 1-49. 2001.

[DRRV] J. Delgado, N. Romero, A. Rovella, F. Villamajo. Critical sets of quadratic polynomials. Preprint.

[F] J. Franks. Necessary conditions for stability of diffeomorphisms. Trans. Amer. Math. Soc. 158, 301-308, 1971.

[GG] M. Golubitsky, V. Guillemin. Stable mappings and their singularities. Graduate texts in mathematics 14. Springer, New York, 1973.

[H] M. Hirsch. Differential Topology. Springer-Verlag, New york, Heidelberg, Berlin , 1976.

[IP] J. Iglesias, A. Portela. Real Perturbations of complex polynomials. Bull. Braz. Math. Soc., New Series, 38 (1), 1-27, 2007.

[Ma] R. Mañé. A proof of the C^1 stability conjecture. (English) Publ. Math., Inst. Hautes Étud. Sci. 66, 161-210 (1987).

[Mat] J.Mather. Topological proofs of some purely topological consequences of Carathéodory's theory of prime ends. Th. M. Rassias& G.M. Rassias, eds. *Selected Studies*.North-Holland:Amsterdam, 225-250, 1982.

[Mi] J. Milnor. Dynamics in One Complex Variable. Annals of math studies 160-Princeton Univ. Press 2006.

[MSS] R. Mañé, P. Sad and D. Sullivan. On the dynamics of rational maps. Annales Scientifiques de L'E.N.S. 4^e séries tome 16, N^o 2, 193-217, 1983.

[P] F. Przytycki. On Ω-stability and structural stability of endomorphisms satisfying Axiom A.Studia Mathematica 60, 61-77, 1976.

[R] C. Robinson. Structural stability of C^1 diffeomorphisms. J. Diff. Eq., 22, 28-73, 1976.

[S] M. Shub. Endomorphisms of compact differentiable manifols. Amer. J. Math., 91, 175-199, 1969.

[St] N. Steinmetz. Rational Iteration. de Gruyter Studies in Mathematics 16, 1993.

[W] H. Whitney. On singularities of mappings of Euclidean spaces I: mappings of the plane into the plane. Annals of Mathematics, 62, 374-410, 1955.

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