Entropy-expansiveness and domination

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Abstract

Let $f : M \to M$ be a C^r-diffeomorphism, $r \geq 1$, defined on a compact boundary-less manifold M. We prove that C^1-generically if $H(p)$, the f-homoclinic class of a hyperbolic periodic point p, has a dominated splitting then $f/H(p)$ is entropy-expansive. Conversely, if there exists a C^1 neighborhood $\mathcal{U}$ of a diffeomorphism f defined on a compact surface and a homoclinic class $H(p)$ of an f- hyperbolic periodic point p, such that for every $g \in \mathcal{U}$ the continuation $H(p_g)$ of $H(p)$ is entropy-expansive then there is a dominated splitting for $H(p)$.

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1 Introduction

Since the seminal work of Smale [Sm] establishing the main goals to describe the long term evolution of a discrete or continuous time dynamical system, the strategy has been to prescribe some property at the infinitesimal level of the system that implies a definite behavior for the underlined dynamics. Examples include the concepts of hyperbolicity, partial hyperbolicity and dominated splitting. On the other hand one may ask what are the consequences at the infinitesimal level from a known behavior of the evolution system at the ambient manifold. But rarely a property displayed by a system solely implies an interesting behavior of the differential map acting at the tangent bundle. For instance, in [Ge, GK] it is proved that a (generalized) pseudo-Anosov map f is ergodic and even Bernoulli. For those maps there is at least one point p where the derivative $Df(p)$ is the identity map and so the dynamics at the tangent bundle cannot be characterized in terms of hyperbolicity or even dominance Example 1 in this article is a generalized pseudo-Anosov map illustrating such a behavior. So, it is natural to ask which robust properties satisfied by the underlined systems has dynamical consequences at the tangent bundle level and vice versa. Several authors have worked in this line of ideas (see for instance [Ma2, Ma3, DPU, PPV, SV]). Here by a robust property we mean a property shared by all system in a neighborhood of the original one. In this paper we study what are the consequences at the dynamical behavior of the tangent map Df of a diffeomorphism $f : M \to M$, assuming that f is robustly entropy expansive. In this direction we obtain that the tangent bundle has a Df-invariant dominated splitting. Reciprocally, we show, in the case of surfaces, that the existence of a dominated splitting for the tangent bundle implies robust entropy expansiveness for the diffeomorphism f. Thus robust entropy expansiveness is equivalent to the existence of a dominated spitting for surface diffeomorphisms.

We also give an example of a diffeomorphism that is not entropy expansive. This example is of class C^∞ and so it is asymptotically entropy expansive by a result of Buzzi [Bu]. The first example of a diffeomorphism that is not entropy expansive neither asymptotically entropy expansive was given by Misiurewicz in [Mi] answering a question posed by Bowen [Bo]. Nevertheless we add our example because of its nice properties: (1) it is defined on the sphere S^2, (2) it has no dominated splitting, (3) it is ergodic and even Bernoulli, (4) it admits analytic models. Moreover, a straightforward modification of this example shows that there are diffeomorphisms defined on manifolds of dimension greater than 2 that has a dominated splitting defined on a homoclinic class but that are not entropy expansive.

Let us now give precise definitions. Let M be a compact connected boundary-less Riemannian d-dimensional manifold and $f : M \to M$ a homeomorphism. Let K be a compact invariant subset of M and $\mathrm{dist} : M \times M \to I\!R^+$ a distance in M compatible with its Riemannian structure. For $E, F \subset K$, $n \in I\!N$ and $\delta > 0$ we say that E (n, δ)-spans F with respect to f if for each $y \in F$ there is $x \in E$ such that $\mathrm{dist}(f^j(x), f^j(y)) \le \delta$ for all $j = 0, \dots, n-1$. Let $r_n(\delta, F)$ denote the minimum cardinality of a set that (n, δ)-spans F. Since K is compact $r_n(\delta, F) < \infty$. We define

$$h(f, F, \delta) \equiv \lim\sup_{n\to\infty} \frac{1}{n} \log(r_n(\delta, F))$$

and the topological entropy of f restricted to F as

$$h(f, F) \equiv \lim_{\delta\to 0} h(f, F, \delta)\,.$$

The last limit exists since $h(f, F, \delta)$ increases as δ decreases to zero.

For $x \in K$ let us define

$$\Gamma_\epsilon(x, f) \equiv \{y \in M \,/\, d(f^n(x), f^n(y)) \le \epsilon,\ n \in Z\!\!\!Z\}\,.$$

We will simply write $\Gamma_\epsilon(x)$ instead of $\Gamma_\epsilon(x, f)$ when it is understood which f we refer to.

Following Bowen (see [Bo]) we say that f/K is **entropy-expansive** or **h-expansive** for short, if and only if there exists $\epsilon > 0$ such that

$$h^*_f(\epsilon) \equiv \sup_{x\in K} h(f, \Gamma_\epsilon(x)) = 0\,.$$

The importance of f being h-expansive is that the topological entropy can be derived from its ϵ-estimate $h(f, K, \epsilon)$, as showed by [Bo, Theorem 2.4].

A similar notion to h-expansiveness, albeit weaker, is the notion of *asymptotically h-expansiveness* [Mi]: let K be a compact metric space and $f : K \to K$ an homeomorphism. We say that f is asymptotically h-expansive if and only if

$$\lim_{\epsilon \to 0} h_f^*(\epsilon) = 0 .$$

Thus we do not require that for a certain $\epsilon > 0$ $h_f^*(\epsilon) = 0$ but that $h_f^*(\epsilon) \to 0$ when $\epsilon \to 0$. It has been proved by Buzzi that any C^∞ diffeomorphism defined on a compact manifold is asymptotically h-expansive.

Next we recall the notion of dominated splitting.

Definition 1.1. *We say that a compact f-invariant set $\Lambda \subset M$ admits a dominated splitting if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$, such that*

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n \ \forall x \in \Lambda,\ n \geq 0. \tag{1}$$

Our main results are the following:

Theorem A. *Let M be a compact boundaryless C^∞ surface and $f : M \to M$ be a C^r diffeomorphism such that $K \subset M$ is a compact f-invariant subset with a dominated splitting $E \oplus F$. Then f/K is h-expansive.*

Since the property of having a dominated splitting is open we may conclude that any g C^1 close to f is such that g/K_g is h-expansive where K_g is a continuation of $K = K_f$.

In case M is a d-dimensional manifold with $d \geq 3$ the existence of a dominated splitting is not enough to guarantee h-expansiveness as it is shown in Example 2 presented below. Nevertheless a weaker result can be achieved:

Theorem B. *Let M be a compact boundaryless C^∞ d-dimensional manifold and $f : M \to M$ be a C^r diffeomorphism. Let $H(p)$ be an isolated f-homoclinic class associated to the f-hyperbolic periodic point p. Assume that $H(p)$ admits a dominated splitting. Then there is a C^1 neighborhood $\mathcal{U}$ of f such that for a residual subset $\mathcal{R} \subset \mathcal{U}$ any $g \in \mathcal{R}$ is h-expansive.*

Observe that if the topological entropy of a map $f : M \to M$ vanishes, $h(f) = 0$, then f is h-expansive. For instance the identity map $id : M \to M$ is h-expansive. Nevertheless, robustness of h-expansiveness has a dynamical meaning as shows the following theorem.

Theorem C. *Let M be a compact boundaryless C^∞ surface and $f : M \to M$ be a C^r diffeomorphism. Let $H(p)$ be an f-homoclinic class associated to the f-hyperbolic periodic point p. Assume that there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ it holds that the continuation $H(p_g)$ of $H(p)$ is h-expansive. Then $H(p)$ has a dominated splitting.*

A natural question that arises is if Theorem C holds not only for surfaces but also for compact manifolds of any finite dimension. We believe that this is the case and it will be the subject of a forthcoming paper. This would imply that C^1 generically h-expansiveness of an isolated $H(p,f)$ is equivalent to the existence of a dominated splitting for $H(p,f)$.

1.1 Idea of the proofs

To prove Theorem A we proceed as follows. First observe that there is a compact neighborhood $U(K)$ of K such that we may extend the cones defining the dominated splitting $E \oplus F$ to $U(K)$ in a continuous way. If a point $y \in M$ is such that its f-orbit $\mathrm{orb}(y) \subset U(K)$ then they are defined local center stable manifolds and local center unstable manifolds $W^{cs}_{loc}(f^n(y))$, $W^{cu}_{loc}(f^n(y))$ for any $n \in \mathbb{Z}$. We choose $\epsilon > 0$ such that if $x \in K$ and $\mathrm{dist}(x,y) \leq \epsilon$ then $y \in U(K)$. Assuming $y \in \Gamma_\epsilon(x)$ and that $y \in W^{cu}_{loc}(x)$ give us that the center-unstable arc $[x,y]^{cu} \subset W^{cu}_{loc}(x)$ is a (ϵ, E)-interval, [PS1], and therefore by domination $W^{cs}_{loc}(z)$ is a true stable manifold for all $z \in [x,y]^{cu}$, that is, $W^{cs}_{loc}([x,y]^{cu})$ contains a neighborhood in M. Either $\ell(f^n([x,y]^{cu})) \to 0$ when $n \to +\infty$ or the ω-limit of $[x,y]^{cu}$ is contained in a periodic arc or circle. Hence either $f^n(W^{cs}_{loc}([x,y]^{cu}))$ shrinks to a point when $n \to +\infty$ or the ω-limit of $W^{cs}_{loc}([x,y]^{cu})$ is contained in a periodic arc or circle. In any case we derive that $h(W^{cs}_{loc}([x,y]^{cu}), f) = 0$. The same holds in case $y \in W^{cs}_{loc}(x)$ taking limits for $n \to -\infty$ and arguing with the α-limit set. In the case when $y \notin W^{cu}_{loc}(x) \cup W^{cs}_{loc}(x)$ we project along $W^{cs}_{loc}(y)$ into $W^{cu}(x)$ obtaining a point y_F such that, due to the fact that the angle between $E(z)$ and $F(z)$ is bounded away from zero for any point $z \in U(K)$, is contained in $\Gamma_{L\cdot\epsilon}(x)$ for some constant $L > 0$. If $\epsilon > 0$ is sufficiently small we may repeat with y_F the arguments used when supposed that $y \in W^{cu}_{loc}(x)$. So in any case we conclude that $\Gamma_\epsilon(x)$ is contained in the local stable manifold of some $(L \cdot \epsilon, E)$-interval or in the local unstable manifold of some $(L \cdot \epsilon, F)$-interval and therefore $h(\Gamma_\epsilon(x), f) = 0$. Since this last equality holds for all $x \in K$ Theorem A follows.

To prove Theorem B we use that the finest dominated splitting (see Definition 4.1) of a homoclinic class $H(p,f)$ of a generic diffeomorphisms has the form

$$T_{H(p,f)}M = E \oplus F_1 \oplus \cdots \oplus F_{j-i} \oplus G$$

with $\dim(E) = i$ and $\dim(F_h) = 1$ for all h and $\dim(G) = \dim(M) - j$ and the sub-bundles F_h are not hyperbolic, [ABCDW, Go]. Moreover, E is contracting and G is expanding when $H(p,f)$ is an isolated homoclinic class [BDPR].
Again we choose $\epsilon > 0$ such that the dominated splitting extends to any point whose orbit is at a distance less than ϵ from the orbit of a point in $H(p,f)$. So if for some $x \in H(p,f)$ a point $y \in \Gamma_\epsilon(x)$, that point cannot be in the unstable manifold of x (tangent to G) neither in the stable manifold (tangent to E). Moreover, such a point cannot have a projection $y_G \neq x$ along its center-stable manifold into the unstable manifold of x because in that case, taking into account that the angles between the different sub-bundles is bounded away from

zero, $\forall n \in \mathbb{Z} : f^n(y_G) \in \Gamma_{L\cdot\epsilon}(f^n(x))$ where $L > 0$ is some constant. Reducing ϵ if it were necessary we would have that if $\mathrm{dist}(f^n(y_G), f^n(x)) \leq L \cdot \epsilon$ then $f^n(y_G) \in W^u_{loc}(f^n(x))$. But by forward iteration by f the distance between $f^n(x)$ and $f^n(y_G)$ growths exponentially till $f^n(y_G)$ leaves $W^u_{loc}(f^n(x))$ hence $y_G \neq x$ leads to a contradiction.
Thus y lies in a center manifold. But the tangent bundle to this center manifold splits into one-dimensional ones and taking into account this fact we may repeat the arguments employed for surfaces in Theorem A.

The proof of Theorem C introduces new concepts such as those of symbolic extension and principal symbolic extension (see Definition 5.1). It is proved in [BFF] that for a homeomorphism defined on a compact metric space it is equivalent to have a principal symbolic extension to be asymptotically entropy expansive. We profit from a result of Downarowicz and Newhouse, [DN], that shows that if we have a Hènon like tangency between the stable manifold and the unstable manifold of a periodic point then f cannot have a principal symbolic extension.
Thus, under the assumption that we do not have a dominated splitting, we can perturb f to create a tangency between the stable manifold and the unstable manifold of the continuation p_g of the f-periodic point p. Then we may assume that the perturbed diffeomorphism is C^2 and apply the results of [DN] obtaining that $H(p_g, g)$ cannot have a principal symbolic extension and therefore that it cannot be entropy expansive.

2 Examples

In [Bo] Bowen asked for examples of diffeomorphisms which are not h-expansive. The first giving such an example was Misiuriewicz, [Mi]. Nevertheless we give here a C^∞ example in S^2 which illustrates the fact that we should have " arbitrarily small " horseshoes to brake h-expansiveness. By "arbitrarilly small horseshoe" we mean a horseshoe contained in the intersection of the ϵ-stable and unstable manifolds for any positive ϵ. A modification of such example gives a 3-dimensional one which does have a dominated splitting but is not h-expansive illustrating that in the general case dominance is not enough to guarantee h-expansiveness. As Theorem C shows dominance implies h-expansiveness generically.
Example 1: *There is a C^∞ diffeomorphism of S^2 that is not h-expansive.*
We consider in $\mathbb{R}^2$ the action given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Since the entries of A are integers and $\det(A) = 1$, the lattice $\mathbb{Z}^2$ is preserved by this action and therefore it passes to the quotient $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This gives us a very well known linear Anosov diffeomorphism $a : \mathbb{T}^2 \to \mathbb{T}^2$.
Let $[x, y]$ represent the equivalence class of $(x, y) \in \mathbb{R}^2$ in $\mathbb{T}^2$. We define in $\mathbb{T}^2$ the relation $[x, y] \sim [-x, -y] = -[x, y]$. The quotient $\mathbb{T}^2/\sim$ gives the sphere S^2.
In order to see this let us take the square in $\mathbb{R}^2$ limited by the straight lines $x = -\frac{1}{2}$, $x = \frac{1}{2}$,

$y = -\frac{1}{2}$, $y = \frac{1}{2}$. We obtain a fundamental domain for the torus and we identify it with $\mathbb{T}^2$.

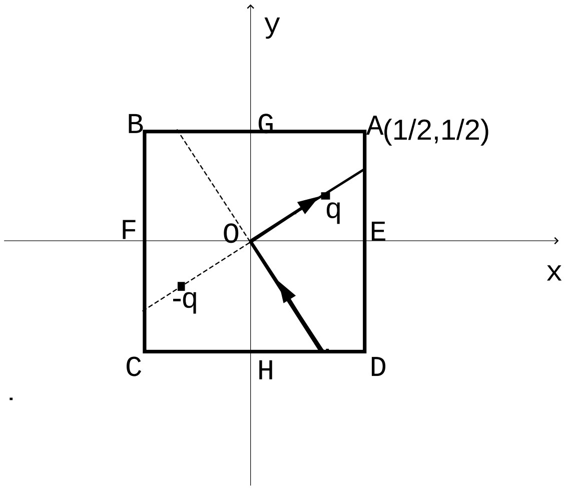

Figure 1: Fundamental domain for $\mathbb{T}^2$

In the quotient $\mathbb{T}^2$ the vertices A $(1/2, 1/2)$,B $(-1/2, 1/2)$,C $(-1/2, -1/2)$, D$(1/2, -1/2)$, of the square are all identified. Let us call E to the point $(1/2, 0)$, F to the point $(-1/2, 0)$, G to the point $(0, 1/2)$ and H to the point $(0, -1/2)$. Observe that E is identified with F and G is identified with H in $\mathbb{T}^2$. Now observe that the boundary of the square OEAG is identified with the boundary of the square OEDH (by the relations $(x, y) \sim -(x, y)$ and $(x, y) \sim (x', y')$ if $(x - x', y - y') \in \mathbb{Z}^2$). Hence both squares are two different disks glued in their boundaries by this identification. It is not difficult to see that the quotient topology coincides in the interior of the squares OEAG and OEDH with the topology of $\mathbb{R}^2$ and that the common boundary of both disks is a circle separating $\mathbb{T}^2/\sim$ Moreover, the rest of the square ABCD doesn't add more points to the quotient because the squares OEAG and OFCH, and OEDH and OFBG, are identified by the relation $(x, y) \sim -(x, y)$. Hence we obtain that $\mathbb{T}^2/_\sim \cong S^2$. See Figure 1 where we have marked two points, q and $-q$ which are identified by the relation $\sim$.

On the other hand $a([x, y]) \sim -a([x, y]) = a(-[x, y])$ by linearity, and therefore projects to S^2 as a map $g : S^2 \to S^2$, known as a generalized pseudo-Anosov map which is shown to be Bernoulli with respect to Lebesgue [Ge]. If $\Pi : \mathbb{T}^2 \to S^2$ is the projection defined by the relation $\sim$, we may write $g(x) = \Pi(a(\Pi^{-1}(x)))$. Observe that the projection $\Pi : \mathbb{T}^2 \to S^2$ is a branched covering and that the definition of g doesn't depend on the pre-image of x by Π^{-1}. Therefore periodic points of a projects into periodic points of g and dense orbits of a projects into dense orbits of g. For g there are singular points P where the local ϵ-stable and ϵ-unstable sets are arcs with the point P as an end-point. This local stable (unstable) sets are called 1-prongs (see figures 1 and 2 where O is a point with 1-prongs).

Let $O \in S^2$ be the image by Π of $[0,0]$. Then O is a fixed point of g. The point O is singular because the local stable and unstable manifolds of $[0,0]$ in $\mathbb{T}^2$ project into S^2 as arcs ending at O (because $[x,y] \sim -[x,y]$). The local stable and unstable manifolds of the points in $\mathbb{T}^2$ near $[0,0]$ project onto arcs contained in the stable and unstable sets respectively of points in S^2 near O like in Figure 2. Note that we do not speak of stable (unstable) manifolds but of stable (unstable) sets because neither $W^s_{loc}(x)$ nor $W^u_{loc}(x)$ are locally connected (see [PPV]).

The intersection of the stable and unstable manifolds of the points $[0,x]$ and $[0,-x]$ of $\mathbb{T}^2$ consists of four points identified by pairs by the relation $[x,y] \sim -[x,y]$. If $[x,y] \in \mathbb{T}^2$ projects to $X \in S^2$, let us call s_X and u_X to the projections of the ϵ-local stable and ϵ-local unstable manifolds respectively of the point $[x,y]$. Hence if a point X is very near to a singular point like O, s_X and u_x will intersect twice. Points in s_X are in the ϵ-local stable set of X and points in u_X are in the ϵ-local unstable set of X. Moreover, if $Y \in s_X$ then $\text{dist}(g^n(Y), g^n(X)) \to 0$ when $n \to +\infty$. Similarly for points in u_X replacing $n \to +\infty$ by $n \to -\infty$.

Given $\epsilon' > 0$ choose $p \in \mathbb{T}^2$ periodic so close to $[0,0]$ that $\Pi(p) = P$ is a periodic point satisfying $\text{dist}(P, 0) < \epsilon'$. Such a point exists since periodic points are dense for the Anosov diffeomorphism a defined on $\mathbb{T}^2$ and projects on S^2 as periodic points for g by Π that is a continuous surjection with respect to the quotient topology.

Let $\{P, P'\} = s_P \cap u_P$. Then it is not difficult to see that given $\epsilon > 0$ there is $\epsilon' > 0$ small enough such that $P' \in W^u_\epsilon(P) \cap W^s_\epsilon(P)$. Thus we have a homoclinic intersection between ϵ-local stable and ϵ-local unstable arcs of the periodic point P, P' being a homoclinic point such that its orbit is always at a distance less than ϵ from the orbit of P. It follows that for all $\epsilon > 0$ there are points P such that $\Gamma_\epsilon(P)$ contains a small horseshoe. Thus $g : S^2 \to S^2$ is not h-expansive since the topological entropy of those horseshoes is positive. Moreover, this example is transitive and they are not only C^∞ but real analytic (see [Ge], and [LL]).

Clearly the example is a homoclinic class which has no dominated splitting.

Example 2 *Let us show that property (1) sole does not imply h-expansiveness in dimension 3 or more.*
Consider the 3-manifold $S^2 \times S^1$ with $g : S^2 \to S^2$ as in the example above, and put in S^1 a diffeomorphism $h : S^1 \to S^1$ with a North-South dynamics, say, $N \in S^1$ is a source and $S \in S^1$ is a sink and the ω-limit of any point in S^1 is S and the α-limit of every point in S^1 is N. We may assume that $|Dh_N| > 2k$ where $k = \sup\{\|Dg(x)\|, x \in S^2\}$. Let us define $f : S^2 \times S^1 \to S^2 \times S^1$ by $f(x,y) = (g(x), h(y))$. Then if $K = S^2 \times \{N\}$, K is compact invariant and there is a dominated splitting for K, $E \oplus F$, where $E = T_x S^2$, $F = T_N S^1$. By the previous example f is not h-expansive.

This example shows what is the problem: the strongly expanding direction F along S^1

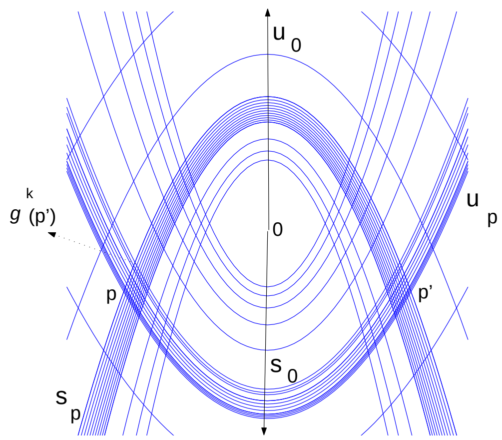

Figure 2: Singularity of a generalized pseudo-Anosov

does not interferes on the dynamics of f/S^2. Thus property (1) holds for f defined on $S^2 \times S^1$ albeit does not for the projection $g = \Pi_{S^2} f$.

3 Dominated splittings versus h-expansiveness on surfaces

Here we shall prove Theorem A. Let us begin stating the following lemma.

Lemma 3.1 (Pliss). *Let $0 < \lambda_1 < \lambda_2 < 1$ and assume that there exists $n > 0$ arbitrarily large such that*

$$\prod_{j=1}^{n} \|Df/E(f^j(x))\| \le \lambda_1^n \,.$$

Then there exist a positive integer $N = N(\lambda_1, \lambda_2, f)$, $c = c(\lambda_1, \lambda_2, f) > 0$ such that if $n \ge N$ then there exist numbers

$$0 \le n_1 \le n_2 \le \cdots \le n_l \le n$$

such that

$$\prod_{j=n_r}^{h} \|Df/E(f^j(x))\| \le \lambda_2^{h-n_r} \,,$$

for all $r = 1, 2, \ldots, l$, with $l \ge cn$, and for all h with $n_r \le h \le n$.

Proof. The proof of this lemma can be found in [Pl1]. $\square$

Proof of Theorem A. Let M be a surface and $K \subset M$ a compact and f-invariant subset such that $T_K(M)$ can be written as a dominated splitting $E \oplus F$. By continuity of f and Df there is $\delta_0 > 0$ such that we may extend the cones defining equation (1) to the closed δ_0-neighborhood of K, $U(K) = \{y \in M \,/\, \mathrm{dist}(y, K) \le \delta_0\}$. In this neighborhood there exists a continuous splitting $T_{U(K)}(M) = \hat{E} \oplus \hat{F}$ extending the splitting $T_K(M) = E \oplus F$ ([Ma1]). If the orbit of a point y, $orb(y)$, is contained in $U(K)$ then for that point there are defined local center-stable and center-unstable manifolds $W^{cs}_{loc}(y)$ and $W^{cu}_{loc}(y)$ where $loc > 0$ stands for a small real number. By Peano Theorem on the existence of solutions of an ordinary differential equation given by a continuous field, $W^{cs}_{loc}(y)$ can be obtained as a solution to the ODE

$$\begin{cases} Y(0) = y \\ Y'(u) = \hat{E}(u),\ u \in \mathbb{R}^2 \end{cases}$$

where we have identified $B(y, \delta_0)$ with $\mathbb{R}^2$. Such a solution is tangent to $\hat{E}$ and therefore, by domination, it is a local center-stable manifold for y. Similarly for $\hat{F}$ we obtain $W^{cu}_{loc}(y)$ as a solution of the ODE obtained replacing $\hat{E}$ by $\hat{F}$. We may also assume that for all $x \in M$ the δ_0-neighborhood of x, $B(x, \delta_0)$, is contained in a local chart that we can identify with $\mathbb{R}^2$. Moreover, there is δ_1, $0 < \delta_1 \le \delta_0$ such that if $\mathrm{dist}(f^j(y), f^j(z)) \le \delta_1$ for all $j = 0, \ldots, n$ and $z \in W^{cs}_{loc}(y)$ then $f^j(z) \in W^{cs}_{loc}(f^j(y))$ for all $j = 0, \ldots, n$. Similarly for the local center-unstable manifold (see [PS1, Lemma 3.0.4 and Corollary 3.2]).

Let us denote by $\hat{K}$ the maximal f-invariant subset of $U(K)$:

$$\hat{K} = \bigcap_{j \in \mathbb{Z}} f^j(U(K)),$$

and by

$$\hat{K}^+ = \bigcap_{j=0}^{\infty} f^{-j}(U(K)), \quad \hat{K}^- = \bigcap_{j=0}^{\infty} f^{j}(U(K)),$$

the forward and backward maximal invariant subsets.
The following lemma relates the length of a stable (unstable) arc joining two points with the distance between those points.

Lemma 3.2. *Given $y \in \hat{K}^+$ there is δ_2, $0 < \delta_2 \le \delta_1$ such that if the length of the arc $[y, z]^{cs} \subset W^{cs}_{loc}(y)$ is greater than $\delta > 0$ for $0 < \delta \le \delta_2$, $\ell([y, z]^{cs}) > \delta$, then $\mathrm{dist}(y, z) > \delta/2$. Moreover, we may choose δ_2 such that if $\mathrm{dist}(y, z) \le \delta \le \delta_2$ then $\ell([y, z]^{cs}) \le 2 \cdot \delta$. Similarly for an arc $[y, z]^{cu} \subset W^{cu}_{loc}(y)$, $y \in \hat{K}^-$.*

Proof. Since $\hat{E}$ is a continuous sub-bundle of $T_{U(K)}M$ we may find δ_2, $0 < \delta_2 \le \delta_1$ such that given $\pi/8 \ge \eta > 0$ then the angle $\angle(E(y), E(w)) < \eta$ for all $w \in B(y, \delta_2) \cap U(K)$. Thus if we parameterize $[y, z]^{cs}$ by arc-length $\beta : [0, l] \to M$, $\beta(s) = (\beta_1(s), \beta_2(s))$, with $\beta(0) = y$, $\beta(l) = z$, then $\beta'(s) = (\beta_1'(s), \beta_2'(s))$ is parallel to $E(\beta(s))$, here we have put

$l = \text{length}([y,z]^{cs})$. Therefore, since $(\beta_1'(s))^2 + (\beta_2'(s))^2 = 1$, we have by the Mean Value Theorem

$$\text{dist}(y,z) = \|\beta(l) - \beta(0)\| =$$

$$= \sqrt{(\beta_1(l) - \beta_1(0))^2 + (\beta_2(l) - \beta_2(0))^2} = \sqrt{((\beta_1'(s_1))^2 + (\beta_2'(s_2))^2} \cdot l =$$

$$= l\left(1 - (\sqrt{((\beta_1'(0))^2 + (\beta_2'(0))^2} - \sqrt{((\beta_1'(s_1))^2 + (\beta_2'(s_2))^2)}\right) =$$

$$= l\left(1 - \frac{(\beta_1'(0))^2 - (\beta_1'(s_1))^2 + (\beta_2'(0))^2 - (\beta_2'(s_2))^2}{1 + \sqrt{((\beta_1'(s_1))^2 + (\beta_2'(s_2))^2)}}\right) \geq$$

$$\geq l\left(1 - |\beta_1'(0) - \beta_1'(s_1)|\,|\beta_1'(0) + \beta_1'(s_1)| + |\beta_2'(0) - \beta_2'(s_2)|\,|\beta_2'(0) + \beta_2'(s_2)|\right)\ .$$

But, since $\angle(\hat{E}(\beta(s)), \hat{E}(\beta(0))) < \eta$,

$$\|(\beta_1'(s) - \beta_1'(0), \beta_2'(s) - \beta_2'(0))\| \leq 2\sin(\eta/2) < \eta\ ,$$

(see Figure 3). Therefore, taking into account that $|\beta_1'(0) + \beta_1'(s_1)| \leq |\beta_1'(0)| + |\beta_1'(s_1)| \leq 2$ and that the same is true with respect to β_2' we have

$$\text{dist}(y,z) \geq l(1 - 4\eta) > l/2 > \delta/2$$

if $\eta > 0$ is sufficiently small.

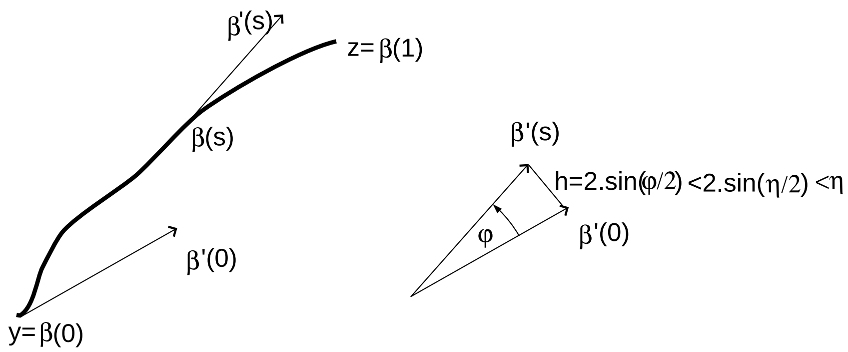

Figure 3: Bounds for small angles

To prove that if $\operatorname{dist}(y,z) \le \delta$ then $\ell([y,z]^{cs}) \le 2\delta$ let us assume that y is the origin O of coordinates in $I\!R^2$ and that the Ox_1 axis is in the direction of $\hat{E}(y)$. Since $\angle(\hat{E}(y), \hat{E}(w)) < \eta \le \pi/8$ for all $w \in B(y, \delta_2)$, all the solutions starting at y are contained in the cone of center $y = O$, axis Ox_1 and angle with Ox_1 equal to η. It follows that the arc $[y,z]^{cs}$ of the local center stable manifold of y is contained in that cone and that the local center stable manifold of y can be written as the graph of a C^1 function $x_2 = h(x_1)$. Moreover $|h'(x_1)| \le \tan(2\eta) \le 4\eta$, whenever $0 \le \eta \le \pi/8 \approx 0.3927$. By our choice of η we have $\sqrt{1+16\eta^2} < \sqrt{1+16\times(0.4)^2} < 2$. Thus, if $x_1(z)$ denotes the abscissa of z, since $|x_1(z)| \le \operatorname{dist}(y,z) \le \delta$, then

$$\ell([y,z]^{cs}) = \int_0^{x_1(z)} \sqrt{1+h'^2(x_1)}\, dx_1 \le \int_0^{\delta} \sqrt{1+16\eta^2}\, dx_1 = \delta \sqrt{1+16\eta^2} \le 2\cdot\delta\,.$$

$\square$

Continuing with the proof of Theorem A we first observe that taking an iterate f^m of f we may assume $C = 1$ at equation (1) defining domination in order to simplify calculations. For, we have for all $n \ge 1$

$$\begin{aligned} &\|\|D(f^m)^n|E(x)\| \cdot \|D(f^m)^{-n}|F(f^{mn}(x))\| = \\ &= \|Df^{mn}|E(x)\| \cdot \|Df^{-mn}|F(f^{mn}(x))\| \le \\ &\le C\lambda^{mn} = C(\lambda^m)^n \le \lambda'^n\,, \end{aligned}$$

if we choose $1 > \lambda' > \lambda$ and $m > 0$ such that $C \le \lambda'/\lambda^m$. Since for a compact invariant set X we have that the topological entropy $h(f^m/X) = m \cdot h(f/X)$, if we prove that for some $\epsilon > 0$, $h(f^m/\Gamma_\epsilon(x,f)) = 0$ then the same is true for f. Thus we assume that for f itself $C = 1$ and $\lambda = \lambda'$. Let $\sqrt{\lambda} < \lambda_1 < \lambda_2 < \lambda_3 < 1$. We find δ_3, $0 < \delta_3 \le \delta_2$, such that if $\operatorname{dist}(z,w) \le \delta_3$, $z, w \in U(K)$, then

$$1 - c < \frac{\|Df/\hat{E}(z)\|}{\|Df/\hat{E}(w)\|} < 1 + c \quad \text{and} \quad 1 - c < \frac{\|Df^{-1}/\hat{F}(z)\|}{\|Df^{-1}/\hat{F}(w)\|} < 1 + c\,,$$

where $c > 0$ is such that $(1+c)\lambda_2 \le \lambda_3$.

Since $U(K)$ is a compact neighborhood of K and $T_{U(K)}M = \hat{E} \oplus \hat{F}$ is a dominated splitting we may find $\gamma > 0$ such that for all $y \in U(K)$ it holds $\angle(\hat{E}(y), \hat{F}(y)) \ge \gamma$. Let us pick a point $x \in K$ and, identifying $I\!R^2$ with a coordinate neighborhood around x, let $l_E(x)$ be the straight line at x tangent to $E(x)$ and $l_F(x)$ the straight line tangent to $F(x)$. From a point $y_F \in l_F(x)$, $y_F \ne x$, we consider the straight line $y_F + l_E(x)$ parallel to $E(x)$. Then for any point y in $y_F + l_E(x)$ we have that the distance between y and x is greater than the distance between y_F and x multiplied by $\sin\gamma$, that is

$$\operatorname{dist}(y,x) \ge \operatorname{dist}(y_F,x)\sin\gamma \implies \operatorname{dist}(y_F,x) \le \frac{\operatorname{dist}(y,x)}{\sin\gamma}\,, \text{ (see Figure 4).}$$

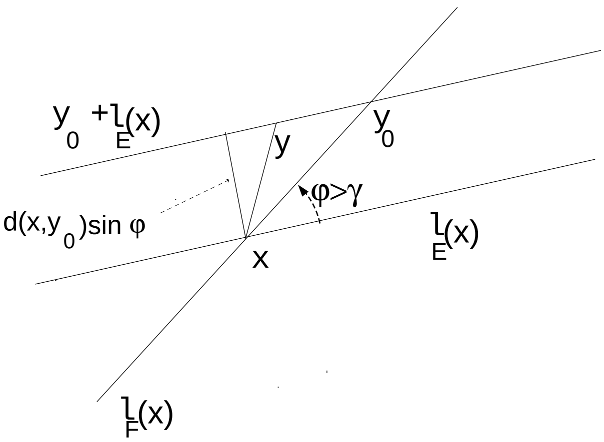

Figure 4: Bounds for the distance between x and $y \in y_0 + l_E(x)$

When we substitute the linear model by that given by the local center-stable and center-unstable manifolds, since the local center-unstable manifold is tangent to $\hat{F}$ and the local center-stable manifold is tangent to $\hat{E}$ we may assume that δ_3 is so small that

$$\operatorname{dist}(y,x) \geq \operatorname{dist}(y_F,x)(\frac{\sin\gamma}{3}) \Longrightarrow \operatorname{dist}(y_F,x) \leq \frac{3\operatorname{dist}(y,x)}{\sin\gamma} \tag{2}$$

for $y_F \in W^{cu}_{loc}(x) \cap B(x,\delta_3)$, $y \in W^{cs}_{loc}(y_F) \cap B(x,\delta_3)$.

Now let $\epsilon > 0$ be such that

$$\epsilon < \frac{\delta_3 \sin\gamma}{6}\,. \tag{3}$$

We will prove that for all $x \in K$, $h(f/\Gamma_\epsilon(x)) = 0$. This will prove that f/K is entropy-expansive.

Let us first assume that $y \in W^{cu}_{loc}(x) \cap \Gamma_\epsilon(x)$ and that $y \neq x$. Since $y \in \Gamma_\epsilon(x)$ we have that $orb(y) \subset U(K)$ and so $y \in \hat{K}$. Therefore for all $j \in \mathbb{Z}$ it holds that

$$\|Df/E(f^{j-1}(y))\| \cdot \|Df^{-1}/F(f^j(y))\| < \lambda$$

and so[1]

$$\prod_{j=1}^{n} \|Df/E(f^{j-1}(y))\| \cdot \|Df^{-1}/F(f^j(y))\| < \lambda^n,\ \forall\, n \geq 1\,.$$

If it were the case that

$$\prod_{j=1}^{n} \|Df^{-1}/F(f^j(y))\| \leq \lambda_1^n$$

for arbitrarily large $n > 0$ then by Lemma 3.1 there are $N = N(\lambda_1, \lambda_2) \in I\!N$ and $c = c(\lambda_1, \lambda_2) > 0$ such that if $n \geq N$ there exists $1 \leq n_1 < n_2 < \ldots < n_k \leq n$ with $k > c \cdot n$ and

$$\prod_{j=h}^{n_i} \|Df^{-1}/F(f^j(y))\| \leq \lambda_2^{n_i - h}\,,$$

for $n_i \geq h \geq 1$; $i = 1, \ldots, k$. Observe in particular that $n_k > c \cdot n$ otherwise we cannot have $k > c \cdot n$. By our choice of δ_3 we then have that

$$\prod_{j=h}^{n_k} \|Df^{-1}/F(f^j(z))\| \leq \lambda_3^{n_1 - h}\,,$$

for all $h : n_k \geq h \geq 1$ if $\mathrm{dist}(f^j(z), f^j(y)) \leq \delta_3$ for all $j : h \leq j \leq n_k$.

By the choice of ϵ we have that $f^j(y) \in W^{cu}_{loc}(f^j(x))$ for all $j \geq 0$ and moreover $f^j([x,y]^{cu}) \subset W^{cu}_{loc}(f^j(x))$.

If $\rho = \mathrm{dist}(x, y) > 0$, we have, taking $h = 1$, that

$$\rho \leq \ell([x,y]^{cu}) \leq \ell([f^{n_k}(x), f^{n_k}(y)]^{cu})\lambda_3^{n_k - 1}\,.$$

Since $[f^{n_k}(x), f^{n_k}(y)]^{cu}$ is tangent to F and $\mathrm{dist}(f^{n_k}(x), f^{n_k}(y)) \leq \epsilon$, by Lemma 3.2 we have that $\ell([f^{n_k}(x), f^{n_k}(y)]^{cu}) \leq 2\epsilon$. Thus

$$\rho \leq \ell([x,y]^{cu}) \leq 2\epsilon \cdot \lambda_3^{n_k - 1}$$

and since $0 < \lambda_3 < 1$ and $n_k > c \cdot n \to \infty$ when $n \to \infty$ we conclude that $\rho = 0$ which contradicts that $x \neq y$.

Hence we have that it is not true that for arbitrarily large $n > 0$

$$\prod_{j=1}^{n} \|Df^{-1}/F(f^j(y))\| \leq \lambda_1^n\,,$$

[1] In order to simplify notation we will put $\hat{E} = E$ and $\hat{F} = F$ in the sequel.

and since

$$\prod_{j=1}^{n} \|Df/E(f^{j-1}(y))\| \, \|Df^{-1}/F(f^{j}(y))\| < \lambda^n ,$$

we may conclude, taking into account that $\lambda_1^2 > \lambda$, that there is n_0 such that

$$\prod_{j=1}^{n} \|Df/E(f^{j-1}(y))\| \leq \lambda_1^n ,$$

for all $n \geq n_0$. Thus, in the notation of [PS1], $I = [f^{n_0}(x), f^{n_0}(y)]^{cu}$ is an (ϵ, E, λ_1)-interval. Let us assume, without loss of generality, that $n_0 = 0$ and so $I = [x, y]^{cu}$.

There are two cases: either $\ell(f^n(I)) \to 0$ when $n \to \infty$ or $\ell(f^n(I)) \not\to 0$. In any case we may assume that for all point $z \in I$ we have that $W^{cs}_{loc}(z)$ is a stable manifold (see [PS1, Corollary 3.3]) and so $W^{cs}_{loc}(I)$ contains a neighborhood in M.

Let us assume first that $\ell(f^n(I)) \to 0$ when $n \to \infty$. Choose $\zeta > 0$ and let us find bounds for $r_n(\zeta, W^{cs}_{loc}(I))$ where $r_n(\zeta, W^{cs}_{loc}(I)))$ is the minimum cardinality of a set that (n, ζ)-spans $W^{cs}_{loc}(I))$. Since $\ell(f^n(I)) \to 0$ there is $n_0 > 0$ such that $\mathrm{diam}(f^n(W^{cs}_{loc}(I))) \leq \zeta$ for all $n \geq n_0$. Then we may find a finite subset E such that (ζ, n_0)-spans $W^{cs}_{loc}(I)$ and this set also (ζ, n)-spans $W^{cs}_{loc}(I)$ for all $n \geq 0$. It follows readily that

$$h(f, W^{cs}_{loc}(I), \zeta) = \limsup_{n\to\infty} \frac{1}{n} \log(r_n(\zeta, W^{cs}_{loc}(I)) = 0$$

and therefore $h(f, W^{cs}_{loc}(I)) = 0$.

On the other hand, if $\ell(f^n(I)) \not\to 0$ then by [PS1, Proposition 3.1] we have that for all $z \in I$, the omega -limit set of z, $\omega(z)$, is a periodic orbit or lies in a periodic circle[2].

In case of $\omega(x)$ being included in a periodic circle $\mathcal{C}$ this circle is normally hyperbolic attracting a neighborhood V of $\mathcal{C}$ and points in V converge exponentially fast to $\mathcal{C}$. If f is C^2 then as in [PS1] we conclude that the dynamics by f^τ (τ being the period of $\mathcal{C}$) in $\mathcal{C}$ is conjugate to an irrational rotation while if f is just C^1 we only have semi-conjugacy (we may have a Cantor set in $\mathcal{C}$ and wandering intervals). In any case (conjugacy or semi-conjugacy with an irrational rotation R_α) we profit from the fact that $h(R_\alpha) = 0$. This implies that if $f^\tau/\mathcal{C}$ is conjugate or semi-conjugate to R_α then $h(f^\tau/\mathcal{C}) = 0$.

On the other hand if $\omega(x)$ is a periodic orbit, say of a point q, since $\ell(f^n(I)) < \delta$ for all $n \geq 0$ we have that there is a periodic point q' in $W^{cu}_{loc}(q)$ such that attracts points in $f^n(I \backslash \{x\})$ (for instance the other end-point of $f^n(I)$ different from $f^n(x)$), see [PS1, Lemma 3.3.1]. Note than since $W^{cu}_{loc}(q)$ is an arc, the period of q' is the same of that of q, or the double of

[2]In the proof of that proposition Pujals and Sambarino use that f is of class C^2. But this is used in the case when $\ell(f^n(I)) \to 0$ when $n \to \infty$ in order to argue as in Schwartz's proof of the Denjoy property ([Sc]). If we already know that $\ell(f^n(I)) \not\to 0$ then it is enough to assume f of class C^1 to ensure that the ω-limit of I is contained in a periodic arc or circle and this is implicit in the proof of [PS1, Proposition 3.1].

it. Let P be the set of periodic points of f in $W^{cu}_{loc}(q)\backslash\{q\}$. Then all of them have the same period, say τ. The set P divides $W^{cu}_{loc}(q)$ in arcs on which the dynamics by f^τ is monotone. It follows that the topological entropy of $f^\tau/W^{cu}_{loc}(q)$ is zero.
So in both cases, periodic orbit or periodic circle, $f^{\tau n}(W^{cs}_{loc}(I))$ approaches an f^τ invariant one-dimensional manifold $\mathcal{L}$ such that the topological entropy $h(f^\tau, \mathcal{L}) = 0$. Let $\zeta > 0$ and $m \in I\!\!N$ large be given an find $S' \subset \mathcal{L}$, (m, ζ) spanning $\mathcal{L}$. We may find n_0 and a subset S of $f^n(I)$ for $n \geq n_0$, such that (m, ζ)-spans $f^n(I)$ with respect to f^τ. Projecting along the fibers of the local center-stable manifolds which, by equation (1), are dynamically defined ($W^{cs}_{loc}(z)$ is strong stable for all $z \in \mathcal{L}$) we know that there is $n_1 > 0$ such that for any point $z \in I$, $\ell(f^n(W^{cs}_{loc}(z))) < \zeta$. We add points to S in order to ensure that we do have a (m, ζ) spanning set for $f^m(W^{cs}_{loc}(I))$ for $m = 0, 1, \ldots, n_1 - 1$. We conclude that $h(f, W^{cs}_{loc}(I), \zeta) = 0$. Since $\zeta > 0$ is arbitrary we obtain that $h(f, W^{cs}_{loc}(I)) = 0$.
By [Bo, Corollary 2.3] we have that if there is an ϵ-E-interval I such that $\Gamma_\epsilon(x) \subset W^{cs}_{loc}(I)$ then $h(\Gamma_\epsilon(x), f) = 0$.

Similarly if $y \in W^{cs}_{loc}(x)$ then $J = [x, y]^{cs}$ is an ϵ-F-interval and reasoning with the α-limit of J we obtain that $h(f, W^{cu}_{loc}(J)) = 0$.

Assume now that $y \notin W^{cs}_{loc}(x)$, $y \notin W^{cu}_{loc}(x)$. Since $y \in \hat{K}$ we have that $W^{cs}_{loc}(y)$ and $W^{cu}_{loc}(y)$ are well defined and are embedded arcs. Since for all $z \in U(K)$ the angle between $E(z)$ and $F(z)$ is bounded from below by $\gamma > 0$, shrinking ϵ if necessary, from $\mathrm{dist}(z, w) \leq \epsilon$, $z, w \in \hat{K}$, we may assume first that $W^{cs}_{loc}(w) \cap W^{cu}_{loc}(z) = w_F$ and $W^{cs}_{loc}(z) \cap W^{cu}_{loc}(w) = w_E$, and secondly that

$$f(w_F) = W^{cs}_{loc}(f(w)) \cap W^{cu}_{loc}(f(z)),\ f(w_E) = W^{cu}_{loc}(f(w)) \cap W^{cs}_{loc}(f(z))\,,$$

$$f^{-1}(w_F) = W^{cs}_{loc}(f^{-1}(w)) \cap W^{cu}_{loc}(f^{-1}(z)),\ f^{-1}(w_E) = W^{cu}_{loc}(f^{-1}(w)) \cap W^{cs}_{loc}(f^{-1}(z))\,.$$

If $z = x$, $w = y$ we obtain points $y_F = W^{cs}_{loc}(y)) \cap W^{cu}_{loc}(x)$ and $y_E = W^{cs}_{loc}(x) \cap W^{cu}_{loc}(y)$. By our assumption $y_E \neq x$ and $y_F \neq x$. See Figure 5.

Since $\mathrm{dist}(f^n(x), f^n(y)) \leq \epsilon$ for all $n \in Z\!\!\!Z$, by induction we have that

$$f^n(y_F) = W^{cs}_{loc}(f^n(y)) \cap W^{cu}_{loc}(f^n(x)) \text{ for all } n \in Z\!\!\!Z. \tag{4}$$

Moreover, we have that

$$\mathrm{dist}(f^n(x), f^n(y_F)) < \frac{3\,\mathrm{dist}(f^n(x), f^n(y))}{\sin\gamma} < \frac{3\epsilon}{\sin\gamma} < \frac{\delta_3}{2}$$

which implies by Lemma 3.2 that $\ell(f^n([x, y_F]^{cu}) = \ell([f^n(x), f^n(y_F)]^{cu}) < \delta_3$ which in turn implies that $\mathrm{dist}(f^n(x), f^n(z)) < \delta_3$ for all $z \in [x, y_F]^{cu}$ and for all $n \geq 0$.
Thus $[x, y_F]^{cu} = I$ is a (δ_3, E)-interval ([PS1]) and therefore we have that $W^{cs}_{loc}(I)$ is a stable

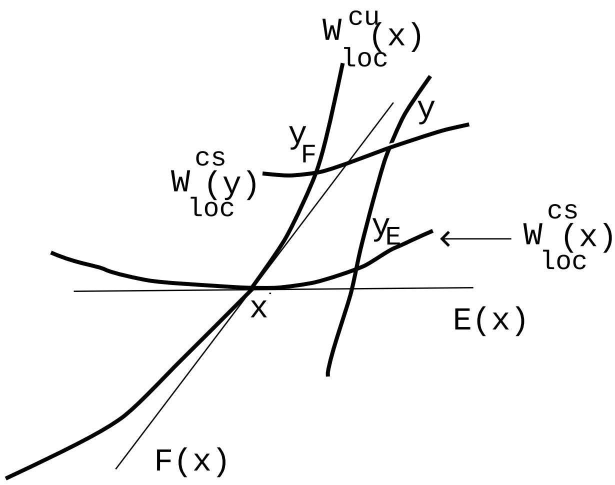

Figure 5: Case $y \notin W^{cs}_{loc}(x)$, $y \notin W^{cu}_{loc}(x)$.

manifold which implies that $\text{dist}(f^n(y_F), f^n(y)) \to 0$ when $n \to \infty$. Reasoning as in the case in which $y \in W^{cu}_{loc}(x)$ we obtain

$$h(f, W^{cs}_{loc}(I), \zeta) = \limsup_{n\to\infty} \frac{1}{n} \log(r_n(\zeta, W^{cs}_{loc}(I)) = 0.$$

Hence $h(f, W^{cs}_{loc}(I)) = 0$ and from $\Gamma_\epsilon(x) \subset W^{cs}_{loc}(I))$ we conclude that $h(f, \Gamma_\epsilon(x)) = 0$. Since this last inequality holds for all $x \in K$ we have that f/K is h-expansive, finishing the proof of Theorem A. □

4 Generalization to higher dimensions.

Theorem B generalizes Theorem A and as Example 2 shows, to obtain its proof, we have to impose certain restrictions on the nature of the splitting. In order to do so, we recall the concept of *finest dominated splitting* introduced in [BDP].

Definition 4.1. *Let $\Lambda \subset M$ be a compact f-invariant subset such that $TM/\Lambda = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ with E_j Df invariant, $j = 1, \ldots, k$. We say that $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is dominated if for all $1 \le j \le k-1$*

$$(E_1 \oplus \cdots E_j) \oplus (E_{j+1} \oplus \cdots \oplus E_k)$$

has a dominated splitting We say that $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is the finest dominated splitting when for all $j = 1, \ldots, k$ there is no possible decomposition of E_j as two invariant sub-bundles having domination.

C^1-generically the finest dominated splitting has a very special form. Let us first put us in that generic situation which are proved elsewhere (see [ABCDW, §2.1]).

Generic assumptions. There exists a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that if $f : M \to M$ is a diffeomorphisms belonging to $\mathcal{G}$ then

1. f is Kupka-Smale, (i.e.: all periodic points are hyperbolic and their stable and unstable manifolds intersect transversally)
2. for any pair of saddles p, q, either $H(p, f) = H(q, f)$ or $H(p, f) \cap H(q, f) = \emptyset$.
3. for any saddle p of f, $H(p, f)$ depends continuously on $g \in \mathcal{G}$.
4. The periodic points of f are dense in $\Omega(f)$.
5. The chain recurrent classes of f form a partition of the chain recurrent set of f.
6. every chain recurrent class containing a periodic point p is the homoclinic class associated to that point.

Theorem 4.1. *There is a residual subset $\mathcal{I} \subset \mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{I}$ has a homoclinic class $H(p, f)$ which contains hyperbolic saddles of indices $i < j$ then either*

1. *For any neighborhood U of $H(p, f)$ and any C^1-neighborhood $\mathcal{U}$ of f there is a diffeomorphism $g \in \mathcal{U}$ with a homoclinic tangency associated to a saddle of the homoclinic class $H(p_g, g)$, where p_g is the continuation of p.*
2. *There is a dominated splitting*

$$T_{H(p,f)}M = E \oplus F_1 \oplus \cdots \oplus F_{j-i} \oplus G$$

with $\dim(E) = i$ and $\dim(F_h) = 1$ for all h and $\dim(G) = \dim(M) - j$. Moreover, the sub-bundles F_h are not hyperbolic.

Proof. This is [ABCDW, Corollary 3] taking into account that a result by Gourmelon guarantees that the homoclinic tangency can be associated to a saddle inside the homoclinic class (see [Go, Corollary, 6.6.2, Theorem 6.6.8]). □

Remark 4.2. *In Theorem 4.1 we cannot assure that E is contracting and G is expanding unless the homoclinic class is isolated (see [ABCDW, BDPR]).*

Sketch of the proof of Theorem B. Observe first that by assumption we are C^1-far from homoclinic tangencies. Therefore taking into account Theorem 4.1 and assuming that $H(p, f)$ is isolated, we have that for all $x \in H(p, f)$ it holds that

$$T_xM = E_0(x) \oplus E_1(x) \oplus \cdots \oplus E_k(x) \oplus E_{k+1}(x),$$

with $E_0(x)$ contracting, $E_{k+1}(x)$ expanding and all $E_j(x)$, $j = 1, \dots, k$, one dimensional and not hyperbolic. From this we have that the proof of Theorem B is similar to the proof of Theorem A: as in that proof let $y \in \Gamma_\epsilon(x)$ where $x \in H(p, f)$. We cannot have $y \in W^u_{loc}(x)$, where $W^u_{loc}(x)$ is the local unstable manifold tangent to E_{k+1}. Otherwise, since on $W^u_{loc}(x)$ there is a hyperbolic Df-expansion at the tangent level, taking $\epsilon > 0$ less than the minimum of the diameters of the local unstable manifolds of points in $H(p, f)$, we will have that for some $n_0 > 0$ it holds that $\text{dist}(f^{n_0}(x), f^{n_0}(y)) > \epsilon$, a contradiction. Similarly we cannot have that there is $y \in \Gamma_\epsilon(x) \cap W^s_{loc}(x)$ where $W^s_{loc}(x)$ is the local stable manifold tangent to E_0. Suppose now that $y \notin W^s_{loc}(x) \cup W^u_{loc}(x)$. If $y \in \Gamma_\epsilon(x)$ we may project $f^n(y)$ along a center stable manifold tangent to $E_0(f^n(y)) \oplus E_1(f^n(y)) \oplus \cdots \oplus E_k(f^n(y))$ into the unstable manifold of $f^n(x)$ obtaining a point $f^n(y') \in W^u_{loc}(f^n(x))$[3]. Since the angles given by the dominated splitting are bounded away from zero the diameter of this projection goes to zero when ϵ goes to zero. So this projection $f^n(y')$ belongs to $\Gamma_{L\epsilon}(f^n(x))$ for some constant $L > 0$ which depends on the lower bound for the angles between the different sub-bundles of the splitting and therefore, as in the proof of Theorem A, taking into account that along E_{k+1} we have uniform expansion, we obtain that $y' = x$. Similarly the projection of y along a center unstable manifold tangent to $E_1(y) \oplus \cdots \oplus E_{k+1}(y)$ into the stable manifold of x (tangent to $E_0(x)$) has to coincide with x for sufficiently small $\epsilon > 0$. Therefore $\Gamma_\epsilon(x)$ is included in a center manifold of x, $W^c_{loc}(x)$. Either $W^c_{loc}(x)$ is one-dimensional ($k = 1$) and then we may argue as in the first part of Theorem A, or there is some surface S tangent to $E_j(x), E_{j+1}(x)$ for some $j \in \{1, \dots, k-1\}$ in which there is a projection $y' \neq x$ of y along a center unstable manifold tangent to $E_{j+2}(y), \dots E_{k+1}(y)$ such that $y' \in \Gamma_{K\epsilon}(x)$. In the former case the existence of y' implies readily that there is $n_0 \geq 0$ such that $E_{j+2}(f^n(y')) \oplus \cdots \oplus E_{k+1}(f^n(y'))$ is uniformly expanding for all $n \geq n_0$ and that $E_0(f^n(y')) \oplus \cdots \oplus E_{j-1}(f^n(y'))$ is uniformly contracting for $n \leq -n_0$. Therefore points in $\Gamma_\epsilon(x)$ have to be in S. Hence we may repeat the second part of the proof of Theorem A taking into account that E_i is one dimensional for all $i = 1, \dots, k$. In any case we obtain that $h_{top}(\Gamma_\epsilon(x)) = 0$. Therefore $f/H(p, f)$ is entropy expansive. This finishes the sketch of the proof. $\square$

5 Robust h-expansiveness implies domination on surfaces.

In this section we prove Theorem C. In order to do that we will argue by contradiction assuming that we do not have a dominated splitting. This will allow us to create a tangency between the stable and unstable manifolds of p. Using results of Downarowicz and Newhouse (see [DN] and [Nh2]) we will see that this is not possible when $f/H(p)$ is h-expansive in a robust way.

[3] These center-stable manifolds are just locally defined but since $\text{dist}(f^n(x), f^n(y)) \leq \epsilon \;\; \forall n \in \mathbb{Z}$, by [PS1, Lemma 3.0.4, Corollary 3.2], as in equation (4) we have that $\forall n \in \mathbb{Z} : f^n(y') = W^{cs}_{loc}(f^n(y)) \cap W^u_{loc}(f^n(x))$.

Recall that a subshift (g, Y) is the restriction of the full shift in a finite alphabet to a closed invariant subsystem.

Definition 5.1. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. A symbolic extension of the pair (f, X) is a pair (g, Y), where (g, Y) is a subshift with a continuous surjection $\pi : Y \to X$ such that $f\pi = \pi g$. A symbolic extension is principal if the topological entropy of the extension coincides with that of the original system, that is, $h(g, Y) = h(f, X)$.*

Theorem 5.1 (Downarowicz, Newhouse)**.** *Fix $2 \leq r < \infty$. There is a residual subset $\mathcal{R}$ of the space* Diff $^r(M)$ *of C^r-diffeomorphisms of a closed surface M such that if $f \in \mathcal{R}$ and f has a homoclinic tangency, then f has no principal symbolic extension.*

Proof. See [DN, Theorem 1.4]. □

Moreover, if f has no principal symbolic extension then f cannot be asymptotically h-expansive as has been proved by M. Boyle, D. Fiebig and U. Fiebig (see [BFF]).

Proof of Theorem C. Let $H(p)$ be an f-homoclinic class associated to the hyperbolic periodic point p. Assume that there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ it holds that there is a continuation $H(p_g)$ of $H(p)$ such that $H(p_g)$ is h-expansive. Let $x \in W^s(p) \cap W^u(p)$ be a transverse homoclinic point associated to the periodic point p. We define $E(x) \equiv T_xW^s(p)$ and $F(x) \equiv T_xW^u(p)$. Since p is hyperbolic we have that $E(x) \oplus F(x) = T_xM$. Moreover, $E(x)$ and $F(x)$ are Df-invariant, i.e.: $Df(E(x)) = E(f(x))$ and $Df(F(x)) = F(f(x))$.

By definition $H(p) \equiv \text{clos}(\text{hom}(p))$ where $\text{hom}(p)$ is the set of transverse homoclinic points associated to p so if we prove that there is a dominated splitting for $\text{hom}(p)$ we are done since then we can extend by continuity the splitting to the closure $H(p)$. Moreover, since C^2-diffeomorphisms are dense in the C^1-neighbourhood $\mathcal{U}$ we may assume that f is of class C^2.

So let us assume that f is of class C^2 and prove that there is a dominated splitting for $\text{hom}(p)$. To do so it is enough to prove that there exists $m > 0$ such that for some $k :\ 0 \leq k \leq m$ it holds for all $x \in \text{hom}(p)$ that

$$\|Df^k/E(x)\|\, \|Df^{-k}/F(f^k(x))\| \leq \frac{1}{2}\,.$$

Hence arguing by contradiction let us assume that for all $m > 0$ there is $x_m \in \text{hom}(p)$ such that for all $k :\ 0 \leq k \leq m$ we have

$$\|Df^k/E(x_m)\|\, \|Df^{-k}/F(f^k(x_m))\| > \frac{1}{2}\,.$$

Hence as in [Ma1, SV] given any $\gamma > 0$ and $\epsilon > 0$ we may find $m > 0$, depending on ϵ and γ, such that with an ϵ-C^1-perturbation we obtain a C^2 diffeomorphism g' with a homoclinic point $x_{g'}$ associated to $p_{g'}$ such that the angle at $x_{g'}$ between $W^s_{loc}(x_{g'}, g')$ and $W^u_{loc}(x_{g'}, g')$ is less than γ. Since γ is arbitrarily small we may C^1-perturb g' obtaining g'' of class C^2 with a tangency at $x_{g''}$ between $W^s_{loc}(x_{g''})$ and $W^u_{loc}(x_{g''})$. As in [Nh1] we may C^2-perturb g'' obtaining g in $\mathrm{Diff}^2(M)$ with a C^2 robust tangency of Hènon-like type. By the results of [DN] and [Nh2] we conclude that there is no symbolic extension for $g/H(p_g)$. Therefore, by [BFF], $g/H(p_g)$ is not asymptotically h-expansive and *a fortiori* it is not h-expansive contradicting our hypotheses. This finishes the proof of Theorem C. $\square$

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