AXIOM A DIFFEOMORPHISMS WHICH ARE DERIVED FROM ANOSOV FLOWS.

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Abstract. Let M be a closed 3-manifold, and X_t be a transitive Anosov flow. We build a diffeomorphism of the form $f(p) = Y_{t(p)}(p)$, where Y is an Anosov flow equivalent to X. The diffeomorphism f is structurally stable (i.e. satisfies the Axiom A and the strong transversality condition); the non-wandering set of f is the union of a transitive attractor and a transitive repeller; finally f is also partially hyperbolic (the direction $\mathbb{R}.Y$ is the central bundle).

1. Introduction

Let X be a transitive Anosov vector field on a 3-manifold M, and $X(.,t)\colon M\times\mathbb{R}\to M$ its flow. In 1975, Palis and Pugh ([14])asked if the time one map $p\mapsto X(p,1)$ of X may be C^1 approximated by Axiom A diffeomorphisms; as they noticed, the answer is positive when X is the suspension of an Anosov diffeomorphism. It was only by the beginning of this century that [11] and [2] give a partial negative answer to this question: a transitive Anosov flow which is not topologically equivalent to a suspension cannot be approximated by Axiom A diffeomorphisms having more than one attractor.

A flow is equivalent to a suspension if and only if it admits a closed embedded global cross-section. Fried noticed that transitive Anosov flows on 3-manifolds *"almost admit global cross-sections"*. More precisely, a *Birkhoff section* is an embedded surface with boundary $B\hookrightarrow M$ such that:

- the interior $B\setminus\partial B$ of B is transverse to the vector field X;
- the boundary ∂B is the union of finitely many periodic orbits of X;
- there is $T>0$ such that for every point $x\in M$ there is $t\in(0,T]$ with $X(x,t)\in B$.

In [7] Fried built (infinitely many) Birkhoff sections for any transitive Anosov flow on a 3-manifold M. He also proved that the first return map P defined on the interior of B induces a pseudo-Anosov diffeomorphism $\tilde{P}$ on the closed surface $\tilde{B}$ obtained from B by replacing each boundary component by a point. In that meaning, Fried noticed that X looks like the suspension of a pseudo-Anosov homeomorphisms and he described a simple surgery on the suspension of $\tilde{P}$ reconstructing the flow X.

Hence, it is tempting to try to build an Axiom A diffeomorphism close to the time one map of the flow of X, in the same way that it has been done in the case of a suspension flow. Indeed, it is what we propose in this paper. However, it cannot be done in a naive way, because a Birkhoff section is not transverse to the flow on its boundary. For this

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reason, our main difficulty is the local situation on a neighborhood of the periodic orbits, boundary components of a Birkhoff section. Let us now state precisely our result.

In this paper we consider (not necessarily C^1-small) perturbations of the time one of the flow of X, "in the direction of the flow": in other words we consider diffeomorphisms of M which maps each point of M on a point of its X-orbit. Let us be more precise. We denote by $\tilde{\mathcal{E}}(X)$ the set of diffeomorphisms $f: M \to M$ such that:

- there is a C^1 map $t: M \to (0, +\infty)$ such that
$$f(x) = X(x, t(x)), \forall x \in M.$$
In particular, f leaves invariant every leaf of the 1-foliation generated by X.
- f is partially hyperbolic and its central bundle is directed by X.

We define $\mathcal{E}(X)$ as the set of diffeomorphisms f such that there is an Anosov flow Y topologically equivalent to X with $f \in \tilde{\mathcal{E}}(Y)$. In [2], we proved that an Axiom A diffeomorphism in $\mathcal{E}(X)$ contains at most one transitive attractor (and at most one transitive repeller). Our main result is :

Theorem 1. *Let X be a transitive Anosov flow of a compact 3-manifold M.*
Then there exists a diffeomorphism $f \in \mathcal{E}(X)$ such that:

- *f satifies the Axiom A and the strong transversality condition (i.e. f is structurally stable)*
- *$\Omega(f)$ is the union of just a transitive attractor and a transitive repeller.*

We also describe the dynamics on the attractor and on the repeller, which are *derived from pseudo-Anosov diffeomorphisms* (see Section 8).

This result may be seen as a partial answer to Palis Pugh question in [14]. Our construction depends on the choice of a Birkhoff section of the vector field X; given a Birkhoff section, our construction cannot lead to a C^1-small perturbation of the time one map of the flow (for this reason, at different steps of our construction, we did not care of choosing large perturbations even if this step could be done by C^1-small perturbations). However, we don't know if, given a size of perturbation, one can choose the Birkhoff section so that our procedure would lead to a C^1-small perturbation.

Our result is also interesting by itself as being a new class of examples of structurally stable diffeomorphisms. Let us be a little bit more precise. One of the main success of the hyperbolic theory is the characterization of the structural stability as being equivalent to the Axiom A and the *strong transversality condition* (i.e. the transversality of all the invariant manifolds associated to non-wandering points). The strong transversality is a very restrictive condition, in particular when the diffeomorphism admits a non-trivial hyperbolic attractor (i.e. not reduced to a sink) or a non-trivial repeller, because their basins are open set foliated by invariant manifolds. For instance, if f is a structurally stable diffeomorphisms of compact surface, [3, Théorème 2.3.4] states that the basin of a non-trivial hyperbolic attractor cannot meet the unstable manifold of any non-trivial hyperbolic set (see also [9]). In dimension ≥ 3, [10] gives a complete classification (up to topological conjugacy) of structurally stable diffeomorphisms with a codimension 1 attractor (that is, the unstable manifold of every point of the attractor is of codimension 1).

In dimension 3 there are very few examples of known structurally stable diffeomorphisms having non-trivial codimension 2 attractors:

- diffeomorphisms on a fiber bundle over the circle S^1 whose fibers are tori T^2, having the following dynamical behavior: the non-wandering set consists on topological tori, isotopic to a fiber, and which are alternately attracting and repelling tori[1]; moreover, the dynamics on each of these tori is conjugated to a linear Anosov diffeomorphism of the torus T^2.
- more generally, diffeomorphisms obtained (up to topological conjugacy) by completing the dynamics of structurally stable diffeomorphisms of a normally hyperbolic surface[2].

These comments show that our examples are really different from the known examples and therefore important for the topological classification of structurally stable diffeomorphisms. Furthermore, they present two additional important properties: they are partially hyperbolic and their non-wandering sets are the union of just one attractor and one repeller. These two properties define two classes of structurally stable diffeomorphisms for which it seems reasonable to get a complete classification.

Problem 1. *Give a classification (up to topological conjugacy) of structurally stable diffeomorphisms on closed* 3*-manifolds whose non-wandering set consists of one attracting basic set and one repelling basic set.*

(Our examples show that this problem is far to be trivial).

Problem 2. *Let* $f\colon M \to M$ *be an "Axiom A plus strong transversality" diffeomorphism of a closed orientable* 3*-manifold* M*. Assume furthermore that* f *leaves invariant a partially hyperbolic splitting* $TM = E^s \oplus E^c \oplus E^u$*, where* $dimE^s = dimE^c = dimE^u = 1$

Is it true that f *satisfies one of the following possibilities?:*

- *either* $M = T^3$ *(up to a finite covering) and* E^c *is tangent to an invariant circle foliation; furthermore any attractor and any repeller is homeomorphic to a torus* T^2*;*
- *or* f *is an Anosov diffeomorphism;*
- *or else* E^c *is tangent to an invariant foliation* $\mathcal{F}^c$ *which is topologically equivalent to the* 1*-foliation directed by an Anosov vector field* X*; furthermore there is* $n > 0$ *such that* f^n *leaves invariant every leaf of* $\mathcal{F}^c$*.*

(See [4] for results on the classification of partially hyperbolic diffeomorphisms on 3-manifolds).

Let us formulate a more specific question. Consider an Anosov vector field X on a closed 3-manifold and a structurally stable diffeomorphism $f \in \mathcal{E}(X)$ given by Theorem 1; we denote by A and R the attractor and the repeller of f. The open set $\tilde{M} = M \setminus (A \cup R)$

[1] see Section 2.5 where we recall the construction of such examples by doing a small perturbation of the suspension flow of an Anosov diffeomorphisms

[2] In fact, Grines asked to the first author if there are structurally stable diffeomorphisms on 3-manifolds having a hyperbolic attractor which is not conjugated to a hyperbolic attractor on a compact surface. Unluckily, the attractors in our examples are conjugated to surface attractors, hence these examples do not answer to Grines's question.

is invariant by f. The orbits space $V_f = \tilde{M}/f$ of f in $\tilde{M}$ is a closed 3-manifold, and the central foliation $\mathcal{F}^c_f$ of f restricted to $\tilde{M}$ induces on V_f a 1-dimensional foliation directed by some unit vector field Y_f.

Problem 3. *What is the dynamics of Y_f? How is it related with X and f?*

This paper is organized as follows:

- Section 2 recalls basic definitions and properties of hyperbolicity and partial hyperbolicity that we will use in the paper. We end this section by recalling the construction of the suspension of an Anosov diffeomorphism, and by showing how one can perturb the time one map of its flow for getting a structurally stable diffeomorphism.
- Given a Birkhoff section B of an Anosov vectorfield X, Section 3 and 4 show how one can change slightly the vectorfield and the Birkhoff section, in a neighborhood of the boundary ∂B, in order to get an explicit local model for the Birkhoff section.
- Section 5 proves the main part of Theorem 1, building the announced structurally stable diffeomorphism f assuming the existence of a local model $f_{mod}\colon \mathbb{R}^2\times S^1 \to \mathbb{R}^2\times S^1$ corresponding to the local dynamics in the neighborhood of each boundary component of the Birkhoff section.
- Sections 6 and 7 are devoted to the construction of the local model f_{mod}. It is the technical heart and certainly the most difficult part of the paper.
- Section 8 describes dynamical properties of the diffeomorphism f which use the explicit construction of the local model: one shows that the non-wandering set of f consists in the union of an attractor and a repeller, and that the dynamics in restriction to the attractor or the repeller is "derived from the pseudo Anosov diffeomorphisms associated to the Birkhoff section".

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2. Basic definitions and notations

The aim of this section is to recall briefly the basic definitions and properties of hyperbolicity and partial hyperbolicity. We also recall the construction of a structurally stable diffeomorphism by perturbating the time 1 map of the suspension flow of an Anosov diffeomorphism.

Let $f\colon M \to M$ be a diffeomorphism on a compact Riemannian manifold M and $K \subset M$ an invariant compact set.

2.1. Hyperbolic dynamics.

One says that K is *hyperbolic* if there is an integer $n > 0$ and a Df-invariant splitting $TM|_K = E^s \oplus E^u$ of the tangent bundle of M over K in direct sum of subbundles E^s and E^u such that, for any point $x \in K$, any vectors $u \in E^s(x)$, and $v \in E^u(x)$ one has:

$$\|Df^n(u)\| \leq \frac{1}{2}\|u\| \text{ and } \|Df^n(v)\| \geq 2\|v\|.$$

One says that Df *contracts uniformly the vectors in* E^s and *expands uniformly the vectors in* E^u. The bundle E^s and E^u are called the *stable* and *unstable* bundles of K, respectively. We refer to [12] for a nice survey on the elementary properties of hyperbolic sets.

One of the main properties of a hyperbolic set K is the existence of stable and unstable manifolds through each point of K. The *stable manifold* $W^s(x)$ of a point $x \in K$ is the set of points y for which the distance $d(f^n(y), f^n(x))$ tends to 0 for $n \to +\infty$; for $\varepsilon > 0$, the *local stable manifold* $W^s_\varepsilon(x)$ is the set of points whose distance $d(f^n(y), f^n(x))$ remains smaller than ε for $n \geq 0$. For $\varepsilon > 0$ small enough, the local stable manifolds $\{W^s_\varepsilon(x)\}_{x\in K}$ form a continuous family of disjoint embedded C^1-discs centered at x and tangent at x to the stable space $E^s(x)$. Furthermore $W^s_\varepsilon(x) \subset W^s(x)$; one deduces that $W^s(x)$ is a C^1 injective immersion of $E^s(x)$. One defines the unstable (and local unstable) manifold of x as its stable (local stable) manifold for f^{-1}. The stable (resp. unstable) manifold $W^s(K)$ (resp. $W^u(K)$) of the hyperbolic set K is the union of the stable (resp. unstable) manifold of the points $x \in K$, and (according to the shadowing lemma) is the set of point whose ω-limit set (resp.α-limit set) is contained in K.

A *basic set* is a transitive hyperbolic set K admiting an *isolating neighborhood* U; this means that K is the maximal invariant set in U (in formula $K = \bigcap_{n\in\mathbb{Z}} f^n(U)$). A basic set K is a *hyperbolic attractor* if it admits an isolating neighborhood U which is *strictly positively invariant*: the image of the closure of U is contained in the interior of U. In this case, $W^u(K) = K$. A hyperbolic repeller is a hyperbolic attractor for f^{-1} and it holds that $W^s(K) = K$.

A diffeomorphism f satisfies *the Axiom A* if the non-wandering set $\Omega(f)$ is the closure of the periodic points of f and it is hyperbolic. One important property of an Axiom A diffeomorphism f is that the non-wandering set is the disjoint union of finitely many basic sets called the *basic pieces of f*.

A diffeomorphism f satisfies *the strong transversality condition* if the stable manifold $W^s(x)$ and the unstable manifold $W^u(y)$ are transverse for any pair $x, y \in \Omega(f)$. One of the main theorems on hyperbolic dynamics (proved by Robbin, Robinson and Mañé) states that f satisfies the Axiom A and the strong transversality condition if and only if f is *structurally stable*: every diffeomorphism in a small C^1 neighborhood of f is conjugated to f.

A *cycle* of an Axiom A diffeomorphisms f is a sequence $K_1, ..K_\ell$ of distinct basic pieces such that $W^u(K_i) \cap W^s(K_{i+1}) \neq \emptyset$ and $W^u(K_\ell) \cap W^s(K_1) \neq \emptyset$. An Axiom A diffeomorphism satisfies the *no cycle condition* if it has no cycles. According to [13], the hyperbolicity of the chain recurrent set is equivalent to Axiom A plus the no cycle condition.

2.2. **Partial hyperbolicity.** One says that K is *partially hyperbolic*[3], if there is a Df-invariant splitting $TM|_K = E^s \oplus E^c \oplus E^u$ of the tangent bundle of M over K in direct sum of subbundles E^s, E^c and E^u such that the vectors of E^s are uniformly contracted, the vectors in E^u are uniformly expanded, and the splitting $E^s \oplus E^c \oplus E^u$ is a *dominated splitting* : there is $n > 0$ such that for any point $x \in K$ and any unit vector $u \in E^s(x)$,

[3]According to the different authors, one requires that either both bundles or at least one of the bundles E^s and E^u are not trivial (i.e. of dimension > 0). Here we will always assume that both bundles E^s and E^u are not trivial.

$v \in E^c(x)$, and $w \in E^u(x)$ one has $2\|Df^n(u)\| \leq \|Df^n(v)\| \leq \frac{1}{2}\|Df^n(w)\|$. The bundles E^s, E^c, and E^u are called the *strong stable*, the *central* and the *unstable bundles* of K, respectively. (We refer to [1, Appendix B] for more precise definitions and basic properties of partial hyperbolicity and dominated splitting).

One says that f is *partially hyperbolic* when the whole manifold M is a partially hyperbolic set . The partial hyperbolicity is an open property for the C^r-topology, $r \geq 1$, on the space of diffeomorphisms.

If f is a partially hyperbolic diffeomorphism then there are foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, called *strong stable* and *strong unstable foliations*, tangent to the bundles E^s and E^u, respectively. These foliations are unique, hence are f-invariant. The existence of a foliation tangent to E^c remains an open question even when $dim E^c = 1$.

2.3. **Hyperbolic set of partially hyperbolic diffeomorphisms.** Assume that f is a partially hyperbolic diffeomorphism such that the dimension of the central bundle is 1: $TM = E^s \oplus E^c \oplus E^u$ and $dim E^c = 1$.

Consider now a hyperbolic set K of f. This means that the central bundle, E^c is either expanding or it is contracting, hence the stable manifold (resp. unstable manifold) of the points $x \in K$ are everywhere tangent either to E^s or to $E^s \oplus E^c$ (resp. either to $E^c \oplus E^u$ or to E^u).

As a consequence one gets that if f satisfies the Axiom A, then f satisfies the strong transversality condition if and only if for every pair K_1, K_2 of basic sets one has

$$W^s(K_1) \cap W^u(K_2) \neq \emptyset \Longrightarrow dim(W^s(K_1)) + dim(W^u(K_2)) \geq dim(M).$$

Let us state a direct consequence which will be usefull in our case:

Remark 1. *Let f be a partially hyperbolic diffeomorphism which satisfies the Axiom A. Assume that every basic piece of f is either an attractor or a repeller. Assume finally that the central bundle is contracted on the attractors and expanded on the repellers.*

Then f is structurally stable.

Lemma 2.1. *Let X be a transitive Anosov flow of a compact 3-manifold. Let $f \in \mathcal{E}(X)$ be an Axiom A diffeomorphism such that every basic piece of f is an attractor or a repeller. Then f satisfies the strong transversality condition, and therefore is structurally stable.*

Proof : Let $\mathcal{F}^c$ and $E^c = T\mathcal{F}^c$ denote the central foliation and bundle of f. Consider a basic piece Λ of f which is an attractor, and a periodic point $x \in \Lambda$. Each leaf of $\mathcal{F}^c$ is f-invariant. As a consequence, the central leaf $F^c_f(x)$ of x is closed.

If $E^c(x)$ is contained in the unstable space of x, then the local central leaf through x is contained in $W^u(x)$, hence in Λ. The point x is a repeller for the restriction of f to the closed central leaf $F^c_f(x)$. A point y in the local central leaf of x belongs to Λ and its positive iterates converge to an attracting point x' of $f|_{F^c_f(x)}$. By compactness of Λ one has $x' \in \Lambda$. However $E^c(x')$ is contained in the stable space of x', contradicting the fact that $W^s(x)$ and $W^s(x')$ have the same dimension.

This argument shows that E^c is uniformly contracted on Λ. Analogously, E^c is uniformly expanded on any repelling basic piece. We conclude the proof by Remark 1 $\square$

2.4. **Anosov vector fields.** Two vector fields X and Y on a closed manifold M are topologically equivalent if there is a homeomorphism $h : M \to M$ such that the image by h of any dynamically oriented orbit of X is a dynamically oriented orbit of Y.

Consider now a C^1-vector field X on M and let $\{X_t\}_{t\in\mathbb{R}}$ denote its flow. The vector field X is an Anosov vector field if there is a splitting $TM = E^s \oplus E^c \oplus E^u$ which is invariant by the flow and such that E^c is the line bundle $\mathbb{R}X$ directed by X, and such that vectors in E^s and E^u are uniformly contracted and expanded, respectively, by the flow of X. Anosov flows are structurally stable: there is a C^1-neighborhood $\mathcal{U}$ of X such that every vector field $Y \in \mathcal{U}$ is topologically equivalent to X (furthermore Y is Anosov too). Notice that, for every $t > 0$ the diffeomorphism X_t is partially hyperbolic, with the same bundles as X.

The vector field X is transitive if there is a point x whose orbit is dense in M. Recall that [8] builds examples of non-transitive Anosov flow on closed 3-manifolds. Here we will always consider transitive Anosov flows on 3-manifolds. Notice that, for a transitive Anosov flow, the periodic orbits are dense in M.

2.5. **Case of the suspension.** We recall the construction of an Axiom A diffeomorphism obtained as a perturbation of the time one map of an Anosov flow X, in the case where X is a suspension.

2.5.1. *Definition of the suspension flow.* Consider $A\colon T^2 \to T^2$ be an Anosov diffeomorphism. Consider now $T^2\times\mathbb{R}$ endowed with the vector field $\frac{\partial}{\partial t}$ and let $F_A\colon T^2\times\mathbb{R} \to T^2\times\mathbb{R}$ be the diffeomorphism defined by $F_A(p,t) = (A^{-1}(p), t+1)$. Notice that F_A preserves the vector field $\frac{\partial}{\partial t}$: $(F_A)_*(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t}$. One denotes by M_A the orbit space of F_A that is the quotient of $T^2 \times \mathbb{R}$ by the equivalent relation generated by $(p,t) \sim (A^{-1}(p), t+1)$. The quotient space M_A is a closed 3-manifold. The vector field $\frac{\partial}{\partial t}$ induces on M_A a vector field X whose flow preserves the natural fibration $\pi\colon M_A \to S^1 = \mathbb{R}/\mathbb{Z}$ induced by the projection $\Pi\colon T^2 \times \mathbb{R} \to \mathbb{R}$ on the second factor.

As A is a transitive Anosov diffeomorphism, one deduces that X is a transitive Anosov flow on M_A.

2.5.2. *Perturbation of the time one map of the suspension flow.* Consider now a Morse-Smale diffeomorphism h of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ having exactly two fixed points, a sink at 0 and a source at $\frac{1}{2}$. One may write h as $t \mapsto h(t) = t + \varphi(t)$ where $\varphi\colon S^1 \to \mathbb{R}$ is a smooth map whose derivative is strictly larger than -1, such that $\varphi(t) < 0$ for $t \in (0, \frac{1}{2})$ and $\varphi(t) > 0$ for $t \in (\frac{1}{2}, 1)$.

Consider now the diffeomorphism $f\colon M_A \to M_A$ defined as $f(p) = X(p, 1 + \varphi(\pi(p)))$. One has the following commutative diagramm

$$\begin{array}{ccc} M_A & \xrightarrow{f} & M_A \\ \pi \downarrow & & \downarrow \pi \\ S^1 & \xrightarrow[h]{} & S^1 \end{array}$$

Notice that the fibers $T^2 \times \{0\}$ and $T^2 \times \{\frac{1}{2}\}$ are invariant tori. Furthermore the restriction of f to these fibers is smoothly conjugated to the Anosov diffeomorphism A.

As 0 is a hyperbolic sink of h and $\frac{1}{2}$ is a hyperbolic source, one deduces that $T^2 \times \{0\}$ is a hyperbolic attractor and $T^2 \times \{\frac{1}{2}\}$ is a hyperbolic repeller. Furthermore, any point (x,t) with $t \notin \{0, \frac{1}{2}\}$ has its α-limit set contained in $T^2 \times \{\frac{1}{2}\}$ and its ω-limit set contained in $T^2 \times \{0\}$. One deduces that the non-wandering set of f is $\Omega(f) = T^2 \times \{0, \frac{1}{2}\}$. One easily deduces that f is an Axiom A diffeomorphism satisfying the strong transversality condition. Finally, f is C^1-close to the time one map of the flow X whenever φ is C^1-close to the constant map 0.

3. Topological equivalence of Birkhoff sections

3.1. **Birkhoff section of an Anosov flow.** Let X be a transitive Anosov vector field on a closed 3-manifold M. According to [7], a Birkhoff section B of X is a compact surface with boundary, embedded in M, with the following properties:

- every connected component of the boundary ∂B is a periodic orbit of X,
- the interior of B is transverse to X
- there is $T > 0$ such that every segment of orbit of X of time length T meets B

In [7] Fried proves:

Theorem 2. *Every transitive Anosov flow on a closed* 3*-manifold has a Birkhoff section.*

By construction, the Birkhoff sections built by Fried satisfy an additional property which we will use in our construction. Let γ denote a connected component of the boundary ∂B where B is a Birkhoff section built by Fried. Then the bundles E^s and E^u are orientable along γ; in other words, the eigenvalues of the derivative of the Poincaré return map associated to γ are positive. In what follows, when we speak on a *Birkhoff section* B, we will always assume that the restriction to ∂B of the bundles E^s and E^u are orientable.

3.2. **The tame property.** The Birkhoff sections B built by Fried satisfy another property that we will use in our construction and we will called *tame property.*

Let γ be a connected component of ∂B and $W^s_{loc}(\gamma)$ and $W^u_{loc}(\gamma)$ denote the local stable and unstable manifolds of the periodic orbit γ.

Definition 3.1. *We say that B verifies the* tame property at γ *if there is a neighborhood B_γ of γ in B such that $B_\gamma \cap W^s_{loc}(\gamma)$ and $B_\gamma \cap W^u_{loc}(\gamma)$ consist in the union of γ with finitely many compact segments each of them intersecting γ exactly at one of its extremities.*

We say that B is a tame Birkhoff section *if it is tame at each of its boundary components.*

3.3. **Topological and local Birkhoff sections.** In this work we will use generalizations of the notion of Birkhoff sections:

A *topogical Birkhoff section* is a compact surface with boundary B topologically embedded in M such that

- the embedding of B in M is *regular*, that is B is a submanifold of M: at each point $x \in B$ there are local coordinates $\varphi\colon U \to V \subset \mathbb{R}^3$ of M centered at x such that $\varphi(B \cap U) = V \cap [0, +\infty[\times \mathbb{R} \times \{0\}$.
- every connected component of the boundary ∂B is a periodic orbit of X,

- the interior of B is topologically transverse to X
- there is $T > 0$ such that every segment of orbit of X of time length T meets B.
- the restrictions to ∂B of the bundles E^s and E^u are orientable.
- B verifies the tame property at each of its boundary components (the *tame property* is defined exactly in the same way as for smooth Birkhoff sections). .

Let γ be a periodic orbit of X such that the restriction of E^s and E^u along γ are orientable. A *local Birkhoff section at* γ (resp. *topological local Birkhoff section at* γ) is an embedding (resp. regular topological embedding) of the annulus $[0,1] \times S^1$ in M such that

(1) γ is the image of $\{0\} \times S^1$,
(2) the image of $]0,1] \times S^1$ is transverse (resp. topologically transverse) to X
(3) there is $T > 0$ and a neighborhood U of γ such that every segment of orbit of X of time length T contained in U meets B.
(4) f verifies tame property at γ (where the tame property is defined exactly as in Section 3.2).

3.4. **Linking number of a Birkhoff section at a boundary component.** Let $B\colon [0,1] \times S^1 \to M$ be a local Birkhoff section at a periodic orbit γ. As, by definition, the eigenvalues associated to γ are positive, the normal bundle $N|_\gamma \subset T_\gamma M$ (consisting of all the vectors in TM at a point of γ which are orthogonal to the vectorfield X) is orientable. We fix an orientation of the normal bundle N_γ.

Consider the unit normal bundle $N_{1,\gamma} \subset N|_\gamma$ of γ: at each point p of γ we consider the circle of the unit vectors $v \in T_p(M)$ which are orthogonal to $X(p)$. The unit normal bundle $N_{1,\gamma}$ is a torus T^2, endowed with a projection on the circle γ. The fibers of the projection are circles called *meridians*. The chosen orientation on $N|_\gamma$ induces an orientation of the meridians. We denote by $a \in H_1(N_{1,\gamma}, \mathbb{Z})$ the homology class of the meridians.

As we assumed that the eigenvalues associated to γ are positive, the unit tangent vectors at $W^s(\gamma)$ induce two disjoint circles on $N_{1,\gamma}$, each of them cutting each meridian in exactly one point. These curves are naturally oriented by γ, and are homotopic, hence they define the same homology class $b \in H_1(N_{1,\gamma}, \mathbb{Z})$ called the *longitude*. Notice that we get the same homology class b if we consider the unstable manifold of γ instead of the stable one. We endow the homology space $H_1(N_{1,\gamma}, \mathbb{Z})$ with the basis (a, b).

At each point p of γ there is exactly one unit normal vector $v(p)$ tangent to B and entering in B (because γ is a boundary component of B). We consider the induced curve (oriented by γ) on $N_{1,\gamma}$, and we denote by γ_B its homology class. There is $n(\gamma, B) \in \mathbb{Z}$ such that $\gamma_B = n(\gamma, B)a + b$.

Definition 3.2. *The number $n(\gamma, B)$ defined above is called the linking number of B at γ.*

Notice that the definition of the linking number $n(\gamma, B)$ depends on the choice of the orientation of the normal bundle along γ. Changing this orientation will change $n(\gamma, B)$ by $-n(\gamma, B)$.

Remark 2. *As any orbit cuts B in a bounded time, and as $B \setminus \partial B$ is transverse to the flow, one can verify:*
$$n(\gamma, B) \neq 0.$$

The definition 3.2 holds for smooth local Birkhoff section. As we will deal with topological Birkhoff section we give now a topological version of this definition.

Let Γ be a tubular neighborhood of γ. Then $H_1(\Gamma \setminus \gamma, \mathbb{Z})$ is canonically identified to $H_1(N_{1,\gamma}, \mathbb{Z})$, hence is isomorphic to $\mathbb{Z}^2$ and endowed with the basis (a, b) where a is the meridian and b the longitude.

Let B be a topological local Birkhoff section at γ. Consider a close curve $\sigma \subset B \cap (\Gamma \setminus \gamma)$ which is isotopic to γ in $B \cap \Gamma$. The homology class of σ in $H_1(\Gamma \setminus \gamma, \mathbb{Z})$ does not depend on the choice of σ and it is of the form $n(\gamma, B)a + b$; the integer $n(\gamma, B)$ is called the *linking number of B at γ*. This definition coincides with the above definition if B is a smooth local Birkhoff section.

We will see in the next sections that the linking number is the unique invariant of a local Birkhoff section, up to isotopies obtained by pushing the Birkhoff section along the orbits of the flow.

3.5. **Homological intersection in a neighborhood of γ.** Let γ be a normally oriented periodic orbit with positive eigenvalues, and Γ be a tubular neighborhood of γ. As in the previous section we endow $H_1(\Gamma \setminus \gamma, \mathbb{Z})$ with a basis $\{a, b\}$ where a is a meridian and b is a longitude. The choice of the basis $\{a, b\}$ induces an isomorphism of $H_1(\Gamma \setminus \gamma, \mathbb{Z})$ onto $\mathbb{Z}^2$. We can also identify $H_1(\Gamma \setminus \gamma, \mathbb{Z})$ with the homology group of a torus (boundary of a tubular neighborhood of γ). This identification allows us to endow $H_1(\Gamma \setminus \gamma, \mathbb{Z})$ with the intersection quadratic form. Evaluated on the basis $\{a, b\}$, this intersection form is given by $a \cdot a = b \cdot b = 0$, $a \cdot b = 1$, and $b \cdot a = -1$. In other word, the intersection form is the bilinear antisymetric form on $\mathbb{Z}^2$ associated to the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $B\colon [0, 1] \times S^1 \to M$ be a local topological Birkhoff section at $\gamma = B(\{0\} \times S^1)$ such that the boundary component $B(\{1\} \times S^1)$ is disjoint from Γ. We endow B of an orientation in such a way that the vectorfield followed by this orientation endow the orientation of the manifold (or the local orientation we have chosen in a neighborhood of γ). Hence, the orbits intersect B with positive intersection number.

Let $\sigma \subset \Gamma \setminus \gamma$ be a simple closed curve and $[\sigma] = ia + jb$ be its homology class; in other words $[\sigma] = (i, j) \in \mathbb{Z}^2 \simeq H_1(\Gamma \setminus \gamma, \mathbb{Z})$.

Lemma 3.1. *If σ is a longitude, that is $[\sigma] = (0, 1)$ then the algebraic intersection number $\sigma \cdot B$ is $|n(\gamma, B)|$.*

Proof : Consider an essential curve in B on the boundary of a tubular neighborhood of γ. By definition of the linking number, this curve is in the homology class $(n(\gamma, B), 1)$. Hence the intersection number with a curve in the homology class $(0, 1)$ is $n(\gamma, B)$. Furthermore the curve in this class cuts B always with the same orientation. For this reason we get the announced equality, up to the signal: $|\sigma \cdot B| = |n(\gamma, B)|$. It remains to see that $\sigma \cdot B > 0$. For that we realize the class $(0, 1)$ by the concatenation of an orbit segment in $W^s_{loc}(\gamma)$ and a segment in $B \cap W^s_{loc}(\gamma)$ (that is possible by the tame hypothesis), proving that the intersection is positive because the chosen orientation of B implies that the orbits intersect B with positive intersection number. □

Notice that the intersection number of the homology class $(n(\gamma,B),1)$ with B is zero (because it can be realized by a curve on B and B is normally oriented, so that one can push this curve as a curve disjoint from B). So, we have that

$$\left\{\begin{array}{ccc}(0,1)\cdot B & = & |n(\gamma,B)| \\ (n(\gamma,B),1)\cdot B & = & 0\end{array}\right.$$

One deduces

Corollary 3.1. *Let* $[\sigma]=(i,j)\in H_1(\Gamma\setminus\gamma,\mathbb{Z})$. *Then*

- *if* $n(\gamma,B)>0$ *then*
$$[\sigma]\cdot B=-(i,j)\cdot(n(\gamma,B),1)=-i+n(\gamma,B)j;$$
- *if* $n(\gamma,B)<0$ *then*
$$[\sigma]\cdot B=(i,j)\cdot(n(\gamma,B),1)=i-n(\gamma,B)j.$$

3.6. **Quadrants of a local Birkhoff section.** Consider a periodic orbit γ with positive eigenvalues. Let Γ, $W^s_{loc}(\gamma)$, and $W^u_{loc}(\gamma)$ be a small tubular neighborhood, the stable and the unstable manifolds of γ such that $\Gamma\setminus W^s_{loc}(\gamma)\cup W^s_{loc}(\gamma)$ has exactly 4 connected components. The choice of a transverse orientation of γ induces a cyclic order on these components. We call *quadrants* of Γ and we denote by Γ_i, $i\in\mathbb{Z}/4\mathbb{Z}$ the closure of these connected components.

Let $D\subset\Gamma$ be a small disk transverse to X and cutting γ in a point x_D. One chooses D in such a way that $D\setminus W^s_{loc}(\gamma)\cup W^s_{loc}(\gamma)$ has exactly 4 connected components, each of them contained in one of the quadrants Γ_i. We call *quadrant of* D the closure of these components and we denote $D_i=D\cap\Gamma_i$.

We denote by P_D the first return map on D defined in a neighborhood of x_D. Notice that P_D respects the quadrants: it induces a homeomorphism from a neighborhood of x_D in D_i on another neighborhood of x_D in D_i.

Let $B\subset\Gamma$ be a small local topological Birkhoff section at γ (satisfying the tame property at γ). Consider the connected components of $B\setminus(W^s_{loc}(\gamma)\cup W^u_{loc}(\gamma))$ containing a point of γ in their closure. The closure of these connected components are called *quadrants* of B. Any quadrant of B is bounded by a segment of γ and by two segments in $B\cap(W^s_{loc}(\gamma)\cup W^u_{loc}(\gamma))$; (by the tame property, this intersection consists in finitely many segments having one extremity in γ).

Lemma 3.2. *Let* $\sigma\subset B\cap W^s_{loc}(\gamma)$ *be a connected component of* $B\cap W^s_{loc}(\gamma)\setminus\gamma$ *which is a segment having an extremity in* γ. *Let* B_σ *denote the square obtained by cutting the annulus* B *along* σ. *There is a neighborhood* V *of* γ *in* B_σ *and a continuous and bounded function* $t_\sigma\colon V\to\mathbb{R}$ *such that the map* Π_σ *defined by* $x\mapsto X(t_\sigma(x),x)$ *is a continuous map from* V *to* D.

Proof : Cutting the annulus $B\simeq[0,1]\times\mathbb{R}/\mathbb{Z}$ by σ , one gets an immersion of the square $B_\sigma\simeq[0,1]\times[0,1]$ in $B\subset M$ such that the image of $\{0\}\times[0,1]$ is precisely γ. Consider a tubular neighborhood $\Gamma\simeq D^2\times S^1$ of γ such that $D=D^2\times\{0\}$ and $\gamma=\{0\}\times S^1$. Let $\tilde\Gamma\simeq D^2\times\mathbb{R}$ be the universal cover of Γ. Then B_σ admits a lift $\tilde B_\sigma$ on $\tilde\Gamma$, which is an embedded compact square. In particular, it is contained in a compact cylinder of the form $D \times [-T,T]$. Let $D_r \subset D^2$ denote the disc of radius r centered at 0. There is r such that the orbit of every point in $D_r \times [-T,T]$ cuts D in exactly one point. This induces a continuous projection of $\tilde{B}_\sigma \cap D_r \times [-T,T]$ to D. $\square$

Corollary 3.2. *Given $i \in \{1,\dots,,4\}$ and a quadrant $Q \subset B \cap \Gamma_i$. Then, there are a neighborhood U_Q of x in D_i, a neighborhood V_Q of γ in Q and a continuous and bounded map $t_Q : U_Q \setminus \{x_D\} \to \mathbb{R}$ such that $X(z, t_Q(z)) \in Q$ for all $z \in U_Q \setminus \{x\})$ and such that the map $P_Q(z)$ defined as $z \mapsto X(z, t_Q(z))$ is a homeomorphism from $U_Q \setminus \{x_D\}$ onto $V_Q \setminus \gamma$.*

Proof : Consider the map Π_σ defined in Lemma 3.2. As $B \setminus \gamma$ is transverse to the orbits, one gets that this projection is a local homeomorphism out of γ and of σ. The projection along the flow preserves each stable and unstable separatrix of γ. Hence Π_σ sends quadrants of B on quadrants of D.

Keeping the notation of the proof of Lemma 3.2, we fix $r' > 0$ such that the disc $D_{r'} \subset D$ is disjoint from $\tilde{B}_\sigma \cap ((\partial D_r) \times [-T,T])$. Then for every $i \in \{1,\dots,,4\}$ and every quadrant $Q \subset B \cap \Gamma_i$, there is a neighborhood V_Q of γ in Q such that Π_σ induces an homeomorphism from $V_Q \setminus \gamma$ onto $D_{r'} \cap \Gamma_i \setminus \{x_D\}$. The announced map P_Q is the inverse of the restriction of Π_σ to $D_{r'} \cap \Gamma_i \setminus \{x_D\}$. $\square$

3.7. **The quadrants and the first return map.** The items 2) and 3) of the definition of a local Birkhoff section implies that the first return map P_B of the orbits of X on $B \setminus \gamma$ is well defined and continuous. More precisely there are neighborhood U_B and V_B of γ in B and a continuous and bounded function $t_B\colon U_B \setminus \gamma \to \mathbb{R}$ such that the map $P_B\colon x \mapsto X(t_B(x), x)$ induces a homeomorphisms from $U_B \setminus \gamma \to V_B \setminus \gamma$. Furthermore the orbit segment joining x to $P_B(x)$ meets B exactly at its extremities x and $P_B(x)$.

The first return map P_B preserves the stable and the unstable manifolds of γ, hence permutes the quadrants of B: for every quadrant Q of B there is a quadrant Q' of B such that P_B induces a homeomorphism from a neighborhood of γ in Q, minus γ, to a neighborhood of γ in Q', minus γ. This induces a permutation on the set of quadrants.

Lemma 3.3.
- *Each quadrant Γ_i of Γ contains exactly $|n(\gamma,B)|$ quadrants of B*
- *$P_B^{|n(\gamma,B)|}$ preserves each quadrant of B;*
- *if $z \in Q \setminus \gamma$ is a point in a quadrant Q of B close enough to γ, then there is a path $\sigma \subset Q \setminus \gamma$ joining $P_B^{|n(\gamma,B)|}(z)$ to z. Furthermore, the closed curve obtained by adding to σ the orbit segment joining z to $P_B^{|n(\gamma,B)|}(z)$ is homotopic to γ in Γ_i*

Proof : Let $Q \subset \Gamma_i$ be a quadrant of B and $P_Q\colon D_i \setminus \{x_D\} \to Q \setminus \gamma$ be the homeomorphism (defined in a neighborhood of x_D) built in Corollary 3.2. Consider $z \in Q$ close enough to γ. So z belongs to the image of P_Q, that is $z = P_Q(x)$ for $x \in D_i \setminus \{x_D\}$. Now we consider a small arc $\sigma_0 \subset D_i \setminus \{x_D\}$ joining $P_D(x)$ to x. If z is very close to γ then x and $P_D(x)$ are very close to x_D and one can choose σ_0 contained in the domain of definition of P_Q. Furthermore, the closed curve γ_1 obtained by adding to σ_0 the orbit segment joining x to $P_D(x)$ is homotopic to γ in $\Gamma_i \setminus \gamma$.

Let denote $\sigma = P_Q(\sigma_0)$. It is a segment in Q joining z to $P_Q(P_D(P_Q^{-1}(z)))$. Notice that there are segments of orbits joining z to x , x to $P_D(x)$ and finally $P_D(x)$ to $P_Q(P_D(x))$. So there is a segment of orbit joining z to $P_Q(P_D(P_Q^{-1}(z)))$. We consider the closed curve

γ_2 obtained by adding σ to this segment. The curve γ_2 is by construction homotopic to γ_1 hence to γ in $\Gamma_i \setminus \gamma$. Lemma 3.1 implies that its algebraic intersection number with B is $|n(\gamma, B)|$. Furthermore it consists in a segment of orbit and σ which is contained in $Q \subset B$. The segment of orbit cuts B always with the same orientation. One deduces that the segment of orbit joining z to $P_Q(P_D(x))$ meets B in exactly $|n(\gamma, B)| - 1$ points. As a consequence, $P_Q(P_D(x)) = P_B^{|n(\gamma,B)|}(z)$, this implies that $P_B^{|n(\gamma,B)|}$ preserves the quadrant Q.

We proved the second and third items of the lemma; notice that the other quadrants in Γ_i coincide with $P_B(Q), \dots, P_B^{|n(\gamma,B)|-1}(Q)$ in a small neighborhood of γ, ending the proof. □

We denote by B_j, $j \in \mathbb{Z}/4|n(\gamma, B)|\mathbb{Z}$ the quadrants of B in such a way that:

- $B_j \subset \Gamma_i$ if $j \equiv i$ modulo 4
- write $j = i + 4k$; then B_{j+1} is the quadrant of B in Γ_{i+1} which is adjacent to B_j, that is $B_j \cap B_{j+1} \setminus \gamma \neq \emptyset$

As the first return map P_B is a local homeomorphism of $B \setminus \gamma$ preserving the orientation and inducing a permutation of the quadrants B_j, one gets that there is k such that, for every $j \in \mathbb{Z}/4|n(\gamma, B)|\mathbb{Z}$, P_B maps the quadrants B_j on the quadrants B_{j+4k}.

As any simple essential curve in B, disjoint from γ, induces in homology the class $n(\gamma, B)a + b$ in $H_1(\Gamma \setminus \gamma, \mathbb{Z}) = H_1(N_{1,\gamma}, \mathbb{Z})$ one deduces:

Lemma 3.4. *P_B maps B_j onto B_{j+4} if $n(\gamma, B) > 0$ and onto B_{j-4} if $n(\gamma, B) < 0$, for every $j \in \mathbb{Z}/4|n(\gamma, B)|\mathbb{Z}$.*

Proof : Let denote $n = |n(\gamma, B)|$. First notice that the statement of the lemma is trivial if $n = 1$; so we assume $n \geq 2$. Consider a point $x \in B_1$ and $P_B^n(x) \in B_1$, and let σ be the orbit segment joining x to $P_B^n(x)$. One has seen that σ cuts every quadrant, in particular the quadrant B_5 at a point $P_B^\ell(x)$, $\ell \in \{1, \dots, n-1\}$. We want to prove that $\ell = 1$ if $n(\gamma, B) > 0$ and $\ell = n - 1$ otherwise.

Let α be a simple path on B joining the point x to $P_B^\ell(x)$ and contained in $(B_1 \cup \dots \cup B_5) \setminus \gamma$. Let β be a simple path contained in $B_1 \setminus \gamma$ joining x to $P_B^n(x)$.

We denote by σ_0 the closed path obtained by concatenation of α with the orbits segment of X joining $P_B^\ell(x)$ to x. We denote by σ_1 the closed path obtained by concatenation of α with the orbit segment joining $P_B^\ell(x)$ to $P_B^n(x)$ and with β.

By construction, the intersection number of σ_0 with B is $-\ell$ (because the orbit segment joining $P_B^\ell(x)$ to x is negatively oriented and its interior cuts B in $\ell - 1$ points) and the intersection number of σ_1 with B is $n - \ell$.

Let $[\sigma_0]$ and $[\sigma_1]$ denote the class of σ_0 and σ_1 in $H_1(\Gamma \setminus \gamma, \mathbb{Z})$. Using the notation in Section 3.5, we identify $H_1(\Gamma \setminus \gamma, \mathbb{Z})$ with $\mathbb{Z}^2$ where $(1, 0)$ is the meridian and $(0, 1)$ is the homology class of an essential curve contained in one of the quadrants Γ_i.

By construction there is s such that $[\sigma_0] = (1, s)$; notice that the homology class $[\sigma_1] - [\sigma_0]$ is the homology class of the closed path obtained by concatenation of the orbit segment joining x to $P_B^n(x)$ and β; this closed path is homotopic to a longitude i.e. $(0, 1)$. Hence $[\sigma_1] = (1, s + 1)$. Now, according to Corollary 3.1 one gets:

- If $n(\gamma, B) > 0$ then
$$[\sigma_0] \cdot B = -(1, s) \cdot (n(\gamma, B), 1) = s \cdot n(\gamma, B) - 1 = -\ell.$$
As $\ell \in \{1, \dots, , n-1\}$, and $n(\gamma, B) > 0$, the unique possibility is $\ell = 1$ and $s = 0$.
- If $n(\gamma, B) < 0$, then
$$[\sigma_1] \cdot B = (1, s+1) \cdot (n(\gamma, B), 1) = 1 - (s+1) \cdot n(\gamma, B) = |n(\gamma, B)| - \ell = -n(\gamma, B) - \ell.$$
That is $\ell + 1 = sn(\gamma, B)$. Since $\ell \in \{1, \dots, , n-1\}$, and $n(\gamma, B) < 0$, the unique possibility is $s = -1$ and $\ell = n - 1$.

This ends the proof of the lemma.

□

3.8. **Projection of a local Birkhoff section along the flow.** As in the previous section we consider a local topological Birkhoff section B at a (normally oriented) periodic orbit γ and a disk D transverse to X and cutting γ at a point x_D. Using the notation of the previous section, we divide B in $4n = 4|n(\gamma, B)|$ quadrants and D in 4 quadrants.

As B is not homotopic to a transverse disk, we cannot project it continuously on D along the orbits of X: if one chooses a projection at a point and one tries to extend it by continuity, one gets another value of the projection when one follows an essential curve. More precisely, consider the segment $\sigma \subset B \cap W^s_{loc}(\gamma)$ which is the intersection of the quadrants B_1 with $B_{4|n(\gamma,B)|}$. Let B_σ be the square obtained by cutting the annulus B along σ. We have seen at Lemma 3.2 that there is a continuous projection Π_σ of B_σ to D, defined in the neighborhood of γ, of the form $\Pi_\sigma(x) = X(t_\sigma(x), x)$ where the time projection t_σ is continuous and bounded.

The aim of this section is to estimate the continuity defect of Π_σ at σ. More precisely, B_σ contains two copies of σ so that Π_σ is bivaluate on σ. We will calculate here the difference of this two functions.

Let denote $n = |n(\gamma, B)|$. For every $k \in \{1, \dots, 4n\}$, $k = 4j + i$ with $i \in \{1, \dots, 4\}$ and $j \in \{0, \dots, n-1\}$, there is a neighborhood U_k of γ in B_k such that the restrictions $\Pi_k \colon U_k \to D_i$ of Π_σ induces an homeomorphism from $U_k \setminus \gamma$ onto its image which is a punctured neighborhood of x_D in D_i (in the previous notations this homeomorphism is the inverse of P_{B_k}); By construction, one has $\Pi_k(x) = \Pi_{k+1}(x)$ for $x \in U_k \cap U_{k+1}$ and for every $k \in \{1, \dots, 4n-1\}$.

Lemma 3.5. (1) *for $x \in U_{4n} \cap U_1$ one has*
$$\Pi_{4n}(x) = P_D(\Pi_1(x)) \text{ if } n(\gamma, B) < 0 \text{ and } \Pi_{4n}(x) = P_D^{-1}(\Pi_1(x)) \text{ if } n(\gamma, B) > 0;$$
(2) *Π_k induces a conjugacy between P^n_B and P_D: if $x \in U_k$ and $P^n_B(x)$ is defined and belongs to U_k then $\Pi_k(P^n_B(x)) = P_D(\Pi_k(x))$.*

Proof :

Let us prove now item (1). Let x be a point of $B_{4n} \cap B_1 \setminus \gamma$. Let $\alpha \colon [0, 4n] \to B$ be an essential closed curve obtained by concatenation of curves $\alpha_k \colon [k-1, k] \to B_k$, $k \in \{1, \dots, 4n\}$ with $\alpha_1(0) = x = \alpha_{4n}(1)$ and $\alpha_k(1) = \alpha_{k+1}(0) \in B_k \cap B_{k+1}$.

Now the projection $\Pi_k \circ \alpha_k$ induces a continuous projection of α on $D \setminus \{x_D\}$ along the orbits of X, whose image is a continuous path $\Pi\alpha \colon [0, 4n] \to D \setminus \{x_D\}$ joining $\Pi_1(x)$

to $\Pi_{4n}(x)$. The closed path obtained by concatenation of the orbit segment joining x to $\Pi_1(x)$, the segment $\Pi\alpha$, and the orbit segment joining $\Pi_{4n}(x)$ to x is homotopic to α in $\Gamma \setminus \gamma$. In particular its homology class in $H_1(\Gamma \setminus \gamma)$ is $(n, 1)$ if $n(\gamma, B) > 0$ and $(n, -1)$ is $n(\gamma, B) < 0$.

Let $\beta \subset D_1 \cap D_4$ be a segment joining $\Pi\alpha(4n) = \Pi_{4n}(x)$ to $\Pi\alpha(0) = \Pi_1(x)$. The closed path obtained by concatenation of $\Pi\alpha$ and β is homologous to $(n, 0)$ in $H_1(\Gamma \setminus \gamma, \mathbb{Z})$.

As a consequence the closed path obtained by concatenation of β and the X-orbit segment joining $\Pi_1(x)$ to $\Pi_{4n}(x)$ is homologous to $(0, -1)$ if $n(\gamma, B) > 0$ and $(0, 1)$ is $n(\gamma, B) < 0$. This proves the item (1).

For proving item (2), one considers a segment δ joining $P_B^n(x)$ to x in U_x. Then the closed path obtained by concatenation of the orbit segment joining x to $P_B^n(x)$ with δ is isotopic to γ in the tubular neighborhood of γ. Now this segment is isotopic (along the orbits of the flow) to the closed path obtained by concatenation of the orbit segment joining $\Pi_k(x)$ to $\Pi_k(P_B^n(x))$ and the segment $\Pi_k(\delta)$ (which is joining $\Pi_k(P_B^n(x))$ to $\Pi_k(x)$ in D). This closed path is therefore homotopic to γ in the tubular neighborhood of γ: this implies that $\Pi_k(P_B^n(x))$ is the first return on D of the orbit starting at $\Pi_k(x)$, proving item (2).

□

3.9. **Equivalent Birkhoff sections.** We say that two Birkhoff sections S_0 and S_1 are *X-isotopic* if there is a continuous and bounded function $t\colon S_0 \setminus \partial S_0 \to \mathbb{R}$ such that $p \mapsto X(p, t(p))$ induces a homeomorphism τ from $S_0 \setminus \partial S_0$ to $S_1 \setminus \partial S_1$.

Remark 3. *If S_0 and S_1 are X-isotopic, then $\partial S_0 = \partial S_1$; furthermore, for every connected component γ of the boundary, the linking numbers $n(\gamma, S_0)$ and $n(\gamma, S_1)$ are equal.*

Let γ be a periodic orbit. We say that two local Birkhoff sections B_0 and B_1 at γ are *X-isotopic* if there are neighborhoods U_0 and U_1 of γ in B_0 and B_1, respectively, and a continuous and bounded function $t\colon U_0 \setminus \gamma \to \mathbb{R}$, such that $p \mapsto X(p, t(p))$ induces a homeomorphism τ from $U_0 \setminus \gamma$ to $U_1 \setminus \gamma$.

Lemma 3.6. *Two topological local Birkhoff sections B_0 and B_1 at γ (satisfying the tame hypothesis) are X-isotopic if and only if the linking numbers at γ are equal:*

$$n(\gamma, B_0) = n(\gamma, B_1).$$

Furthermore if $\tau\colon B_0 \setminus \gamma \to B_1 \setminus \gamma$ is a homeomorphism realizing the X-isotopy, then it induces a conjugacy between the first return maps on B_0 and B_1 (on a neighborhood of γ).

Proof : Let $\Pi_\sigma^0\colon B_0 \setminus \gamma \to D$ and $\Pi_\sigma^1\colon B_1 \setminus \gamma \to D$ be the projections defined at Lemma 3.2. We define $\tau = (\Pi_\sigma^1)^{-1} \circ \Pi_\sigma^0\colon B_0 \setminus \gamma \to B_1 \setminus \gamma$. Then τ is a homeomorphism from $B_0 \setminus (\gamma \cup \sigma)$ and item (1) of Lemma 3.5 ensures that it is a homeomorphism $\tau\colon B_0 \setminus \gamma \to B_1 \setminus \gamma$. By construction τ is obtained by isotopy along the orbits; in other words there is a continuous function $t\colon B_0 \setminus \gamma \to \mathbb{R}$ such $\tau(x) = X(t(x), x)$ for $x \in B_0 \setminus \gamma$. Furthermore t is bounded, because the orbits segments joining a point $x \in U_0$ (reps. $x \in U_1$) to $\Pi_\sigma^0(x)$ (resp. $\Pi_\sigma^1(x)$) are uniformly bounded. Hence we proved that B_0 and B_1 are X-isotopic.

It remains to prove that τ induces a conjugacy of the return maps. We write the proof assuming the $n(\gamma, B_0) > 0$, the proof in the case $n(\gamma, B_0) < 0$ is identical.

Let us denote $P_0 = P_{B_0}$ and $P_1 = P_{B_1}$ the first return maps on B_0 and B_1, and $t_0 \colon B_0 \setminus \gamma \to \mathbb{R}$ and $t_1 \colon B_1 \setminus \gamma \to \mathbb{R}$ the corresponding return times. Consider now a point $x \in B_{0,k}$. Then we have seen that $P_0(x)$ belongs to $B_{0,k+4}$. Let α be a segment joining $P_0(x)$ to x and contained in $B_{0,k} \cup B_{0,k+1} \cup \cdots \cup B_{0,k+4}$ (in the case that $n(B_0, \gamma) = 1$ this is not enough to fix the homotopy class of α; in that case, we consider a lift of the $B_{0,k}$ on the universal cover of a tubular neighborhood of γ). Consider the closed path β_0 obtained by concatenation of the orbit segment $X([0, t_0(x)], x)$ with α. Then the homology class of β_0 is $(-1, 0)$.

Consider now the segment $\tau(\alpha)$ joining $\tau(x) \in B_{1,k}$ to $\tau(P_0(x)) \in B_{1,k+4}$ and contained in $B_{1,k} \cup B_{1,k+1} \cup \cdots \cup B_{1,k+4}$ (once more if $n(B_0, \gamma) = 1$ we need to pass to the universal cover of a neighborhood of γ). The orbit segment $X([0, -t(x) + t(P_0(x)) + t_0(x)], \tau(x))$ is joining $\tau(x)$ to $\tau(P_0(x))$

Consider the closed path β_1 obtained by concatenation of the segment $X([0, -t(x) + t(P_0(x)) + t_0(x)], \tau(x))$ and $\tau(\alpha)$. The segment β_1 is isotopic to β_0 along the orbits so that its homology class is $(-1, 0)$. This gives that the intersection number $\beta_1 \cdot B_1$ is 1 implying that $\tau(P_{B_0}(x))$ is the first return of $\tau(x)$ on B_1. This ends the proof.

$\square$

Lemma 3.6 allow us to prove:

Lemma 3.7. *Let B be a topological Birkhoff section of X and $\widetilde{B}$ be a local topological section at a component γ of ∂B. Assume that $n(\gamma, B) = n(\gamma, \widetilde{B})$. Then for any neighborhood O of γ there is a topological Birkhoff section of X which is X-isotopic to B, coincides with B out of O and coincides with $\widetilde{B}$ in a small neighborhood of γ.*

Sketch of proof: : According to lemma 3.6 we can push B on $\widetilde{B}$ along the orbits of X in a small neighborhood of γ; the time function of this projection is bounded in absolute value by some constant K. Multiplying this time function by a bump function φ, we get a surface B_φ immersed in M, transverse to the orbits of the flow, which coincides with B out of an arbitrary small neighborhood of γ (contained in O) and with $\widetilde{B}$ in a smaller neighborhood of γ.

However, the surface B_φ may not be embedded in M: it may have self intersections. The surface coincides with B out of an arbitrary small neighborhood of γ and with $\tilde{B}$ in a smaller neighborhood of γ; moreover this surface is obtained by pushing B along the flow with a time bounded by K. Hence we can assume that this surface is embedded out of O and in a small neighborhood of γ.

Claim 1. *One can choose φ such that B_φ has no self intersection point in $W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma)$*

Proof : Consider the intersection of B with $W^s_{loc}(\gamma)$. By the tame property, it consists in $2|n(B, \gamma)|$ segments ($|n(B, \gamma)|$ in each separatrix) having exactly one extremity on γ, and their interiors are pairwise disjoint: hence the segments contained in one separatrix have a natural cyclic order: the first return map on B sends a segment on the next segment for this order. The same happens for the segments of $\tilde{B} \cap W^s_{loc}(\gamma)$. Furthermore, as τ is conjugating the first return maps on B and $\tilde{B}$ one gets that τ preserves the cyclic order on the set of segments. We need something more precise. Consider the universal cover of a tubular neighborhood of γ. As τ is isotopic to the identity along the orbits, this isotopy defines a lift of τ sending a lift of B over a lift on $\tilde{B}$. The fact that τ conjugates the first return maps on B and $\tilde{B}$ implies that the lift of τ preserves the natural order on the components of intersection of the lifts of B and $\tilde{B}$ with the lifts of the separatrices of $W^s_{loc}(\gamma)$ and $W^u_{loc}(\gamma)$.

This allows us to choose the bump function in such a way that B_φ has no self-intersection on $W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma)$. □

A small perturbation of φ (obtained by pushing B_φ along the orbits) in the neighborhood of the intersection points allows us to assume that the self intersection of B_φ are all transversal, hence are finitely many compact curves. As these curves are disjoint from $W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma)$ each of them is contained in a quadrant.

Consider the lift of B_φ on the universal cover $\tilde{\Gamma}$ of the tubular neighborhood $\Gamma \simeq D^2 \times S^1$ of γ. A quadrant of B_φ is obtained by pushing a quadrant of B along the orbits by a bounded time, on an arbitrarily small neighborhood of γ; furthermore, on the lift, the orbits of the flow intersect a quadrant of B in at most one point. As a consequence, one gets that each lift of each quadrant of B_φ is embedded in the universal cover.

In each quadrant $\tilde{\Gamma}_j$, $j \in \{1,2,3,4\}$ of $\tilde{\Gamma}$, the quadrants of the lift of B_φ are naturally ordered. Let us index them by $B^n_{\varphi,j}$, $n \in \mathbb{Z}$; with this notation $B^n_{\varphi,j}$ and $B^{n+kn(\gamma,B)}_{\varphi,j}$, $k \in \mathbb{Z}$, are lifts of the same quadrant of B_φ.

Let k_φ be the largest integer such that there are $m, n = m + k$ such that $B^n_{\varphi,j} \cap B^m_{\varphi,j} \neq \emptyset$. Let ℓ_φ be the sum of number of connected component of $B^{m+k_\varphi}_{\varphi,j} \cap B^m_{\varphi,j}$ for $m = 1, \dots n(\gamma, B)$. Let us assume that φ has been chosen in such a way that k_φ is the minimum possible on all the φ; let k_0 denote this minimum. We assume also that ℓ_φ is the minimum possible for all φ with $k_\varphi = k_0$.

One concludes the proof of Lemma 3.7 by proving:

Claim 2. *Assume that $k_\varphi > 0$ and $\ell_\varphi > 0$. Then there is $\tilde{\varphi}$ with $k_{\tilde{\varphi}} < k_\varphi$ or $k_{\tilde{\varphi}} = k_\varphi$ and $\ell_{\tilde{\varphi}} < \ell_\varphi$.*

Proof : By definition of k_φ there is m and $n = m + k_\varphi$ such that there is a point $x \in B^m_{\varphi,j}$ and $t > 0$ such that $X(-t,x) \in B^n_{\varphi,j}$. On considers the set $\Delta = \{x \in B^m_{\varphi,j} | \exists t(x) \geq 0, X(-t(x),x) \in B^n_{\varphi,j}\}$. Its is a compact set in $B^m_{\varphi,j}$ bounded by $B^m_{\varphi,j} \cap B^n_{\varphi,j}$, which consist in finitely many circles. Notice that $t(x)$ is unique (because the orbits intersects the quadrants in at most one point. Let D_m be a connected component of this set, and $D_n = \{X(-t(x),x), x \in D_m\}$. Notice that the segments of orbits joining $X(-t(x),x)$ to x, for $x \in D_m$ form a continuous family of segment. This allows to make an isotopy of $B^m_{\varphi,j}$ by pushing the points in a small neighborhood of D_m in $B^m_{\varphi,j}$, along the negative orbits; by this isotopy a point in D_m is transformed in a point $y = X(-t,x)$ with $t > t(x)$. Hence the negative orbit of the new point (in the new $B^m_{\varphi,j}$), is now disjoint from $B^n_{\varphi,j}$: we removed the component D_m of Δ. There are two difficulties to be solved: first, one needs to show that this isotopy can be obtained by a choice of φ; second, one needs to show that this isotopy either decreases k_φ or keep k_φ identical but decreases ℓ_φ.

For solving the first difficulty, one needs to show that one can change φ on D_m without changing φ on D_n. If $k_\varphi = n - m$ is not a multiple of $n(\gamma, B)$, $B^n_{\varphi,j}$ $B^m_{\varphi,j}$ are obtained by pushing (along the orbits) different quadrants of B. Hence one can change the value of φ independently on these two quadrants. If k_φ is a multiple of $n(\gamma, B)$ we have to show that D_n and D_m are coming from disjoint regions of the same quadrant. Arguing by contradiction, we assume that there is a point y in D_n which is the image by the cover automorphism $h_{k_\varphi}\colon D^2\times\mathbb{R}\to D^2\times\mathbb{R}, (p,t)\mapsto(p,t+k_\varphi)$ of a point $x\in D_m$. In that case there is $z\in D_m$ such that $X(-t(z),z)=y$. But $X(-t(x),y)$ is the image by h_{k_φ} of $X(-t(x),x)\in B^n_{\varphi,j}$, hence $X(-t(x),y)\in B^{n+k_\varphi}_{\varphi,j}$. As a consequence $X(-t(x)-t(z),z)\in B^{n+k_\varphi}_{\varphi,j}=B^{m+2k_\varphi}_{\varphi,j}$. Therefore $B^{m+2k_\varphi}_{\varphi,j}\cap B^m_{\varphi,j}\neq\emptyset$ leading to a contradiction.

For solving the second difficulty, it is enough to see that if $x\in D_m$ and the orbit interval $X((-t(x),0],x)$ cuts $B^i_{\varphi,j}$ then $|i-m|<k_\varphi$: these indices are the possible new intersection of $B^m_{\varphi,j}$ with the other quadrants after modification. First notice that one has $i\leq m+k_\varphi=n$: in fact, if $i>n$ and there is $x\in B^m_{\varphi,j}$ and $t>0$ such that $X(-t,x)\in B^i_{\varphi,j}$ then $B^m_{\varphi,j}\cap B^i_{\varphi,j}\neq\emptyset$; the conclusion follows now from the definition of k_φ. On the other hand $i\neq n$ because the orbit segment $X([-t(x),0],x)$ would cut $B^n_{\varphi,j}$ twice, which is impossible. So $i-m<k_\varphi$. Analogously, notice that a negative orbit starting at $y=X(-t,x)\in B^i_{\varphi,j}$ cut $B^n_{\varphi,j}$. As $i<n$, this implies that $B^i_{\varphi,j}\cap B^n_{\varphi,j}\neq\emptyset$. Hence $i\geq m$, which concludes.

□

□

As a consequence of Lemma 3.7 we get:

Lemma 3.8. *Let X be a transitive Anosov flow with oriented center stable and center unstable foliations on a closed 3-manifold M. Every topological Birkhoff section $\tilde B$ is X-isotopic to a (smooth) Birkhoff section B.*

Proof :

Just notice that by lemma 3.7 the topological Birkhoff section $\tilde B$ is X-isotopic to a (smooth) Birkhoff section B in a neighborhood of γ. Out of the neighborhood of the boundary we have that $\tilde B$ is topologically transversal to X then locally we can perturb $\tilde B$ to have a smooth Birkhoff section. □

The following straightforward corollary is the main goal of this section:

Corollary 3.3. *Let X and Y be two transitive Anosov flows with oriented foliations defined on closed 3 manifolds M and N respectively. Assume that X and Y are topologically equivalent by a homeomorphism $h\colon M\to N$. Let B be a Birkhoff section of X. Then there is a Birkhoff section B' of Y which is Y-equivalent to $h(B)$. In particular:*

- $\partial B'=h(\partial B)$
- *for every component γ of ∂B one has $n(\gamma,B)=n(h(\gamma),B')$.*

3.10. **Foliations induced on a Birkhoff section.** Let X be an Anosov vector field on a closed 3-manifold and γ be a periodic orbit of X such that the stable and unstable bundles are oriented along γ. Let B be a tame local topological Birkhoff section at γ. Let B_0 denote $B\setminus\gamma$. Then B_0 is homeomorphic to a punctured disc; furthermore it is transverse to X so that the weak stable and unstable foliations of X induce on B_0 a pair of transverse foliations F^s and F^u. Let Δ be the disk obtained by compactifying B_0 by adding a point 0 (in other words Δ is the quotient of the annulus B by the boundary component γ). We endow Δ with the foliations F^s and F^u, 0 being the unique singular point.

Exactly as Fried noticed in the case of a global Birkhoff section, there is a neighborhood of 0 in Δ such that, on this neighborhood, the pair of foliations F^s and F^u are conjugated to a pseudo-Anosov type of singular foliation: each foliation has a saddle type singularity with $2|n(\gamma, B)|$ separatrices (also called *prongs* in the usual terminology for Pseudo-Anosov maps). The stable prongs (or separatrices) of the singularity corresponds to the connected component of the intersection of B_0 with $W^s(\gamma)$.

More precisely, in the neighborhood of 0; the pair (F^s, F^u) is conjugated to the pair of foliation obtained by endowing the unit disc of $\mathbb{R}^2$ with the trivial horizontal and vertical foliation and by considering a ramified cover of this disc, with $|n(\gamma, B)|$ folds, having a unique ramification at 0.

4. Normal position of a Birkhoff section

4.1. **Model of local Birkhoff sections: regular helicoid transverse to a hyperbolic periodic orbit .** In this section we choose a simple linear model for a vector field in a neighborhood a periodic orbit. Then we show that regular helicoids are local Birkhoff sections for this model vector field. This will provide us model Birkhoff sections of any linking number for the model vector field.

4.1.1. *The model vector field at a periodic orbit.* We denote by S^1 the circle $\mathbb{R}/\mathbb{Z}$.

We consider the vector field on $\mathbb{R}^2 \times S^1$ whose expression in the canonical coordinates is

$$X_{mod}(x, y, z) = (\log 2)x\frac{\partial}{\partial x} - (\log 2)y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

We denote by $\tilde{X}_{mod}$ the lift of X_{mod} to $\mathbb{R}^3$. Notice that the time one map $X_{mod1} \colon \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$ is $X_{mod1}(x, y, z) = (2x, \frac{1}{2}y, z)$. We fix a Riemaniann metric on $\mathbb{R}^2 \times S^1$ such that $||X_{mod}|| = 1$.

4.1.2. *The half helicoid.* We consider two surfaces with boundary $\tilde{S}_0$ and $\tilde{S}_1$, diffeomorphic to the half plane $\mathbb{R} \times [0, +\infty)$ and properly embedded in $\mathbb{R}^3$ and defined as follows:

$$\begin{array}{rcl} \tilde{S}_0 & = & \{(x, y, z) \in \mathbb{R}^3 | \exists r \geq 0, (x, y) = (r\cos 2\pi z, r\sin 2\pi z)\} \\ \tilde{S}_1 & = & \{(x, y, z) \in \mathbb{R}^3 | \exists r \geq 0 (x, y) = -(r\cos 2\pi z, r\sin 2\pi z\} \\ & = & \{(x, y, z) \in \mathbb{R}^3 | \exists r \geq 0 (x, y) = (r\cos 2\pi(z - \frac{1}{2}), r\sin 2\pi(z - \frac{1}{2}))\} \end{array}$$

The surfaces $\tilde{S}_0$ and $\tilde{S}_1$ are two half helicoid. The union $\tilde{S}_0 \cup \tilde{S}_1$ is a whole helicoid. The intersection $\tilde{S}_0 \cap \tilde{S}_1$ is the z axis which is the boundary $\partial\tilde{S}_0 = \partial\tilde{S}_1$. We denote $Int(\tilde{S}_i) = \tilde{S}_i \setminus \partial\tilde{S}_i$.

Notice that $\tilde{S}_0$ and $\tilde{S}_1$ are invariant by the transformation $(x, y, z) \mapsto (x, y, z + 1)$. Hence they induce two surfaces S_0 and S_1 properly embedded in $\mathbb{R}^2 \times S^1$.

Finally notice that the transformation $(x, y, z) \mapsto (x, y, z + \frac{1}{2})$ induces a diffeomorphism from $\tilde{S}_0$ to $\tilde{S}_1$ and from $\tilde{S}_1$ to $\tilde{S}_0$

4.1.3. *The half helicoids are local Birkhoff sections of the model vector field.* Notice that the surfaces S_0, S_1 and the vector field X_{mod} are invariant for the natural action of the maps $h_\alpha\colon \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$ defined as $(x, y, z) \mapsto (\alpha x, \alpha y, z)$, for every $\alpha > 0$.

Lemma 4.1. *For $i \in \{0, 1\}$, the surface $Int(S_i)$ is transverse to the vector field X_{mod}. Furthermore, let γ be any orbit segment with length $\ell(\gamma) \geq \frac{5}{4}$ then γ meets S_i, that is $\gamma \cap S_i \neq \emptyset$.*

Finally any orbit segment σ of X_{mod} with length $\ell(\sigma) \leq \frac{3}{4}$ meets S_i in at most 1 point.

Proof : As the surfaces S_0, S_1 and the vector field X_{mod} are invariant for the natural action of the maps $h_\alpha\colon \mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$ defined as $(x, y, z) \mapsto (\alpha x, \alpha y, z)$, for every $\alpha > 0$, it is enough to show the transversality at any point (x, y, z) with $x^2 + y^2 = 1$. At this point the horizontal component of $X_{mod}(x, y, z)$ is $log 2$ and it is strictly less than the vertical component of $X_{mod}(x, y, z)$ that is 1. The tangent space at such a point is generated by the radial vector field $\partial/\partial r$ and $\partial/\partial z + \partial/\partial\theta$. As a consequence, the horizontal component of any vector tangent to S_i at such a point is larger than the vertical component: this proves the transversality.

Consider now the intersection $\tilde{S}_0 \cap \{y \leq 0\}$. Notice that the half space $y \leq 0$ is invariant by the flow of $\tilde{X}_{mod}$. The connected components of $\tilde{S}_0 \cap \{y \leq 0\}$ are half planes which disconnect this half space. Each of these components is included in $\{(x, y, z) \in \mathbb{R}^3 |\ y \leq 0,\ z \in [i - \frac{1}{2}, i]\}$ for $i \in \mathbb{Z}$. Hence every segment in the half space $y \leq 0$ whose starting point has its z-coordinates less that $i - \frac{1}{2}$ and its end point has its z-coordinates larger than i cuts $\tilde{S}_0$. Since any orbit segments of length larger than $\frac{5}{4}$ verifies that the difference between the z coordinates of its end points are greater than $\frac{5}{4}$ it follows that any $\tilde{X}_{mod}$ orbit segment contained in the half space $y \leq 0$ verifying that its length is larger than $\frac{5}{4}$ cuts $\tilde{S}_0$.

As each connected component of $\tilde{S}_0 \cap \{y \leq 0\}$ disconnects the half space, the transversality implies that every orbit of $\tilde{X}_{mod}$ in $y \leq 0$ (and different from $x = y = 0$) cuts every of this connected component in exactly one point. As each component of $\tilde{S}_0 \cap \{y \leq 0\}$ is included in a region $z \in [i - \frac{1}{2}, i]$ for $i \in \mathbb{Z}$, a segment σ of orbit of length $\ell(\sigma) \leq \frac{3}{4}$ can intersect at most one of these components. As a consequence, $\sigma \cap \tilde{S}_0$ contains at most one point.

Analogous arguments hold for $\tilde{S}_0$ in the half space $y \geq 0$, and for $\tilde{S}_1$ in these two half spaces.

□

4.1.4. *Lower bound for the orbit time from S_0 to S_1.*

Lemma 4.2. *There is $\delta_0 > 0$ such that every segment σ of orbit of X_{mod}, disjoint from the z axis $x = y = 0$, and such that*

$$\sigma \cap S_0 \neq \emptyset \text{ and } \sigma \cap S_1 \neq \emptyset \Rightarrow \ell(\sigma) > \delta_0$$

Proof : Using the invariance of S_0, S_1 and X_{mod} under the action of the maps h_α, $\alpha > 0$, one can see that it is enough to prove the lemma assuming that σ starts at a point (x, y, z) of S_0 (or S_1) with $x^2 + y^2 = 1$.

Now the lemma follows from the fact that $S_0 \cap \{x^2+y^2=1\}$ is a closed curve disjoint from the closed subset $S_1 \subset \mathbb{R}^2 \times S^1$ □

4.1.5. *Model for local Birkhoff section with an arbitrary linking number.* By construction, the linking number of the local Birkhoff sections S_0 and S_1 at the periodic orbit of X_{mod} is equal to 1. One gets models for Birkhoff section with arbitrary linking numbers by considering a finite covering of our model.

For every $n > 0$, we consider the n-folds covering $\mathbb{R}^2 \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$. We denote by $X_{mod,n}$ the vector field on $\mathbb{R}^2 \times \mathbb{R}/n\mathbb{Z}$ define by

$$X_{mod,n}(x,y,z) = (\log 2)x\frac{\partial}{\partial x} - (\log 2)y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

In other words, $X_{mod,n}$ is the lift of X_{mod} to $\mathbb{R}/n\mathbb{Z}$.

We denote by $\gamma_{mod,n}$ the periodic orbit of $X_{mod,n}$.

We denote by $S_{0,n}$ and $S_{1,n}$ the lifts to $\mathbb{R}^2 \times \mathbb{R}/n\mathbb{Z}$ of S_0 and S_1 (or equivalently $S_{i,n}$ are the quotient of $\tilde{S}_i$ on $\mathbb{R}^3$ by the map $(x,y,z) \mapsto (x,y,z+n)$). One easily verifies

Lemma 4.3. *The surfaces $S_{0,n}$ and $S_{1,n}$ are Birkoff sections of $X_{mod,n}$ at $\gamma_{mod,n}$ whose linking number is*

$$n(\gamma_{mod,n}, S_{0,n}) = n(\gamma_{mod,n}, S_{1,n}) = n.$$

We have now a model vector field and a model Birkhoff section corresponding to any positive linking number n. For getting Birkhoff section with linking number equal to $-n$ it is enough to consider the vector field $X_{mod,n}$ and the image of $S_{0,n}$ by the symmetry $(x,y,z) \mapsto (x,-y,z)$.

4.2. **Normal forms of local Birkhoff section.** In this section, $\mathbb{D}^2$ denotes the unit disk of $\mathbb{R}^2$. The solid torus $\mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$ is endowed with the coordinates (x,y,z).

Definition 4.1. *We say that an Anosov flow X on a 3 manifold and a Birkhoff section B of X are* in normal form *if , for every boundary component γ of ∂B there is an integer $n > 0$, a neighborhood O_γ of γ, and a diffeomorphism $\Gamma_\gamma\colon O_\gamma \to \mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$ such that :*

- *the vector field ${\Gamma_\gamma}^*(X)$ is the restriction of $X_{mod,n}$ to $\mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$.*
- *the image by the map Γ_γ of the intersection $B \cap O_\gamma$ is the half helicoid $S_{0,n} \cap \mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$.*

Notice that in the definition above, the image $\Gamma_\gamma(\gamma)$ is the circle $\gamma_{mod,n} = \{x = y = 0\}$.

Proposition 4.1 asserts that that every pair (X,B) where X is an Anosov vector field and B is a Birkhoff section of X, can be put in a normal form.

Proposition 4.1. *Given any pair (X,B), where X is a transitive Anosov flow on a closed 3-manifold with oriented foliations and B is a Birkhoff section of X, there is an Anosov vector field Y endowed with a Birkhoff section $\widetilde{B}$ such that:*

- *Y is topologically equivalent to X by a homeomorphism $h : M \to M$ (which maps the oriented orbits of X to the oriented orbits of Y),*
- *the Birkhoff section $\widetilde{B}$ is Y-equivalent to the topological Birkhoff section $h(B)$. In particular $\partial\widetilde{B} = h(\partial B)$.*

- $(Y, \widetilde{B})$ *are in normal form.*

Proof : For any component γ_i of ∂B we consider the strictly positive integer $n_i = |n(\gamma_i, B)|$, modulus of the linking number of B at γ_i. Then by an isotopy of X preserving the hyperbolicity, we can get an Anosov vector field Y topologically equivalent to X and such that, for any component γ_i of ∂B the vector field Y coincides with X_{mod,n_i} in a tubular neighborhood O_{γ_i} of $h(\gamma_i)$ where h is a homeomorphism realizing the equivalence between X and Y. Then by Lemma 3.8 and Corollary 3.3, there is a Birkhoff section $\widetilde{B}_0$ of Y which is Y-equivalent to $h(B)$.

If $n(\gamma_i, B) > 0$ using Lemma 3.7 applied to $\widetilde{B}_0$ and the local section S_{0,n_i} with $n_i = n(\gamma_i, B)$ we conclude that there is a Birkhoff section $\widetilde{B}_i$ Y equivalent to $\widetilde{B}_0$ and coinciding with $\widetilde{B}_0$ out of a small neighborhood of γ_i and coinciding with S_{0,n_i} in a small neighborhood V_i of γ_i. Up to shrinking V_i we may assume that V_i is of the form $\{(x,y,z) \in \mathbb{D}^2 \times S^1_{n_i}, \sqrt{x^2+y^2} \leq r_i\}$ for some $r_i > 0$. The change of coordinates $(x,y,z) \mapsto (\frac{x}{r_i}, \frac{y}{r_i}, z)$ preserves the expression of the vector field X_{mod,n_i} and the equation of the local Birkhoff section S_{0,n_i}: hence the new Birkhoff section is in normal form at γ_i. We may apply inductively this argument to all the boundary components with positive linking number. Let now explain how to deal with components with negative linking number:

The change of coordinates $(x,y,z) \mapsto (-x,y,z)$ preserves the vector field $X_{mod,n}$ but change the section (the new section is symmetric with respect to $x = 0$) so it changes the signal of the linking number of a local Birkhoff section at the circle $S^1_{n_i}$. Hence, up to considering this (orientation reversing) change of coordinates, we may also put the Birkhoff section in normal form at the component with negative linking numbers. □

Remark 4. *We have seen in Section 4.1 that if $p \notin \gamma$ then every segment of orbit $X_{mod}(p,[0,t])$ contained in $\mathbb{D}^2 \times S^1$ cuts the local Birkhoff section in at most 1 point if $t \leq \frac{3}{4}$ and in at least 1 point if $t \geq \frac{5}{4}$. Therefore, if $p \notin \gamma$ then every segment of orbit $X_{mod,n}(p,[0,t])$ contained in $\mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$ cuts the local Birkhoff section in at most 1 point if $t \leq \frac{3}{4}$ and in at least 1 point if $t \geq \frac{5}{4}$.*

As a consequence of the remark 4 and Proposition 4.1 one proves:

Corollary 4.1. *Given any pair (X, B), where X is a transitive Anosov flow on a closed 3-manifold, with oriented foliations and B is a Birkhoff section of X, there is an Anosov vector field Y endowed with a Birkhoff section $\widetilde{B}$ such that:*

- *Y is topologically equivalent to X by a homeomorphism $h : M \to M$ (which maps the oriented orbits of X on the oriented orbits of Y),*
- *the Birkhoff section $\widetilde{B}$ is Y-equivalent to the topological Birkhoff section $h(B)$. In particular $\partial\widetilde{B} = h(\partial B)$.*
- $(Y, \widetilde{B})$ *are in normal form.*
- *the return times $r(p)$ of a point $p \in \widetilde{B} \setminus \gamma$ to $\widetilde{B}$ belongs to $(\frac{3}{4}, \frac{5}{4})$*

Proof : The proof just consists in a time reparametrization of the orbit of the vector field Y given by Proposition 4.1, by multiplying Y by a smooth function which is equal to 1 in a small neighborhood of the boundary components γ_i of $\partial\widetilde{B}$.

□

4.3. **Pair of parallel Birkhoff sections in normal position.** We say that two Birkhoff sections B_0 and B_1 of the same Anosov flow X are *parallel* if $B_0 \cap B_1 = \partial B_0 = \partial B_1$. Notice that parallel Birkhoff sections are X-isotopic. Furthermore, for every $p \in B_0 \setminus \partial B_0$, the X-orbit segment joining p to the first return of the orbit of p on B_0 meets B_1 in exactly one point.

Definition 4.2. *Let us denote by X an Anosov flow and by (B_0, B_1) a pair of parallel Birhoff sections. We say that the triple (X, B_0, B_1) is* in normal form *if for every boundary component γ of $\partial B_0 = \partial B_1$ there is an integer $n > 0$, a neighborhood O_γ of γ, and a diffeomorphism $\Gamma_\gamma \colon O\gamma \to \mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$ such that :*

- *the vector field ${\Gamma_\gamma}^*(X)$ is the vector field $X_{mod,n}$*
- *the image $\Gamma_\gamma(B_0 \cap O_\gamma)$ is the helicoid $S_{0,n} \cap \mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$.*
- *the image $\Gamma_\gamma(B_1 \cap O_\gamma)$ is the helicoid $S_{1,n} \cap \mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$.*
- *the return times $r_i(p)$ of a point $p \in (B_i \setminus \partial B_i)$ to $B_i \setminus \partial B_i$ belongs to $(\frac{3}{4}, \frac{5}{4})$, for $i \in \{0, 1\}$;*
- *the return time $r(p)$ of a point $p \in (B_0 \cup B_1) \setminus \partial B_0$ to $B_0 \cup B_1 \setminus \partial B_0$ belongs to $(\frac{1}{4}, \frac{3}{4})$.*

This definition means that both sections B_0 and B_1 are in normal form, and furthermore the local normalizing coordinates at $\gamma \in \partial B_0 = \partial B_1$ for B_0 and for B_1 differ by the translation map $(x, y, z) \mapsto (x, y, z + \frac{1}{2})$.

The aim of Section 4 is to show

Corollary 4.2. *Given any pair (X, B), where X is a transitive Anosov flow on a closed 3-manifold and with oriented foliations and B is a Birkhoff section of X, there is an Anosov vector field Y endowed with two parallel Birkhoff sections $\widetilde{B}_0$ and $\widetilde{B}_1$ such that:*

- *Y is topologically equivalent to X by a homeomorphism $h : M \to M$ (which maps the oriented orbits of X on the oriented orbits of Y),*
- *the Birkhoff sections $\widetilde{B}_i$ are Y-equivalent to the topological Birkhoff section $h(B)$. In particular $\partial\widetilde{B}_0 = \partial\widetilde{B}_1 = h(\partial B)$.*
- *$(Y, \widetilde{B}_0, \widetilde{B}_1)$ is in normal form.*

Sketch the proof : we consider $Y, \widetilde{B}_0$ in normal form, given by Corollary 4.1 such that the return time belongs to $(\frac{3}{4}, \frac{5}{4})$. We denote $\Sigma_1 = Y(\widetilde{B}_0, \frac{1}{2})$. Then Σ_1 is a Birkhoff section parallel to $\widetilde{B}_0$. The Birkhoff section $\widetilde{B}_1$ is obtained from Σ_1 by pushing Σ_1 on $S_{1,n}$ along the orbits of Y, in the neighborhood of any boundary component γ. For every p far from $\partial(\widetilde{B}_0) = \partial(\widetilde{B}_1)$ we have that $t(p) = 1/2$. A smooth time rescalling out of a small neighborhood of the boundary and not so far of $\partial(\widetilde{B}_0) = \partial(\widetilde{B}_1)$ allows us to get the return time conditions. □

5. Reduction of the proof to a construction of a local model

The aim of Section 5 is to give a proof of Theorem 1, assuming the existence of a local model of diffeomorphism associated to local Birkhoff sections and a local model of vector field (X_{mod}, S_0, S_1) built in Section 4.

Corollary 4.2 allows us to start with an Anosov vector field X and two Birkhoff sections B_0 and B_1 such that the triple (X, B_0, B_1) is in normal form. We will associate to (X, B_0, B_1) an Axiom A diffeomorphism $f \in \tilde{\mathcal{E}}(X)$ whose non-wandering set consists in only one hyperbolic attractor and one hyperbolic repeller. The construction of f will be done in different regions of the manifold. In each of these regions we will consider transverse sections to the vector field, cutting the orbits in compact segments. We consider the orbits segments with their natural parametrization by the flow, so that we get a continuous family of segments of $\mathbb{R}$. The diffeomorphism f will be defined segment by segment, using a smooth family of diffeomorphisms from a segment $[0, r]$ to a segment $[0, s]$, depending on r, s.

5.1. **Building the diffeomorphism, far from the boundary of the Birkhoff section.** In this section we will build f out of a small neighborhood of the periodic orbits in ∂B_0. In this region, $B_i \setminus \partial B_i$ are complete sections cutting the orbits in segments, and we will define f on these segments.

5.1.1. *Cutting the orbits in segments.* We fix $\lambda > 1$ $C > 0$, such that for any $p \in M$ any unit vectors $u \in E^{ss}(p)$, $v \in E^{uu}(p)$ and any $t > 0$ one has:

$$\|DX_t(u)\| \leq \frac{1}{C}\lambda^{-t} \text{ and } \|DX_t(v)\| \geq C\lambda^t.$$

We denote by $P_{B_0}\colon B_0 \setminus \partial B_0 \to B_0 \setminus \partial B_0$, $P_{B_1}\colon B_1 \setminus \partial B_1 \to B_1 \setminus \partial B_1$ and $P_{B_0 \cup B_1}\colon B_0 \cup B_1 \setminus \partial B_0 \to B_0 \cup B_1 \setminus \partial B_0$ the first return maps of X to the interior of the Birkhoff sections B_0, B_1 and on the union of these interior, respectively.

Every point $p \in M \setminus (B_0 \cup B_1)$ belongs to exactly one orbit segment of X with extreme points q and $P_{B_0 \cup B_1}(q)$; we denote this orbit segment by $[q, P_{B_0 \cup B_1}(q)]^c$.

Since (X, B_0, B_1) is in normal form, the return time $r_i(p)$ of a point $p \in B_i \setminus \partial B_i$ to $B_i \setminus \partial B_i$ belongs to $(\frac{3}{4}, \frac{5}{4})$, for $i \in \{0, 1\}$, and the return time $r(p)$ of a point $p \in (B_0 \cup B_1) \setminus \partial B_0$ to $B_0 \cup B_1 \setminus \partial B_0$ belongs to $(\frac{1}{4}, \frac{3}{4})$.

Let $\alpha \in (1, \lambda^{\frac{1}{100}}]$. All our construction will depend on a number $\delta > 0$ whose value will be fixed at the end. During the construction, our notations will often omit this dependence on α and δ .

5.1.2. *A smooth family of diffeomorphisms $\Theta_{r,s}\colon [0, r] \to [0, s]$.* The proof of the following lemma is left to the reader.

Lemma 5.1. *For any $\delta > 0$, there exists a smooth family of diffeomorphism $\Theta_{r,s}\colon [0, r] \to [0, s]$, where the parameters r, s belong to $[3\delta, +\infty)$ with the following properties:*

(1) $\Theta_{r,s}(t) = \alpha^{-1}t$ *for* $t \in [0, \delta]$
(2) $\Theta_{r,s}(t) = s - \alpha(r - t)$ *for* $t \in [r - \alpha^{-1}\delta, r]$
(3) $\Theta_{r,s}(t) \in [\alpha^{-1}\delta, \delta]$ *for every* $t \in [\delta, r - \delta]$

(See Figure 1)

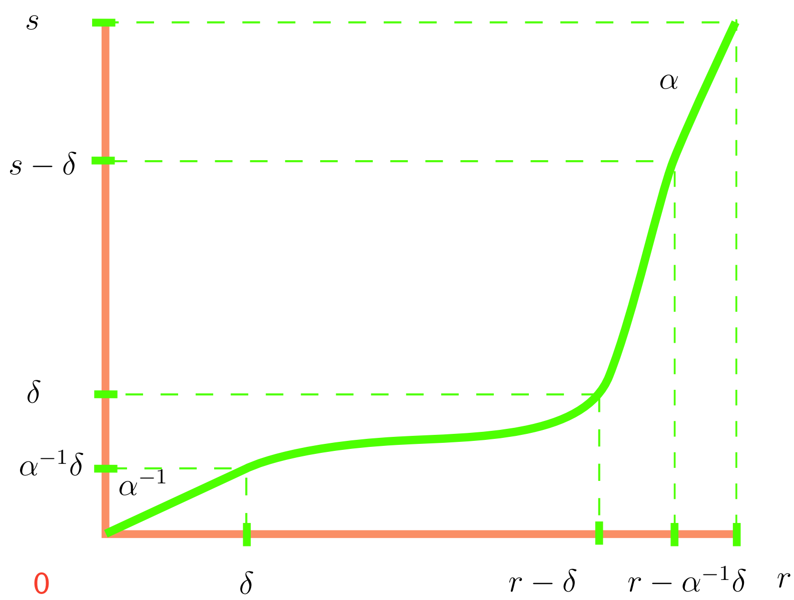

FIGURE 1. The map $\Theta_{r,s}$.

5.1.3. *A diffeomorphism f_{ext} defined on $M \setminus \partial B_0$.* We denote by U_{ext} and V_{ext} the union of the orbit segments of X with length 2δ centered at the points of $B_0 \setminus \partial B_0$ and $B_1 \setminus \partial B_1$, respectively:

$$\begin{aligned} U_{ext} &= \{p \in M \setminus \partial B_0 | \exists t \in [-\delta, \delta], X(p,t) \in B_0\} \quad \text{and} \\ V_{ext} &= \{p \in M \setminus \partial B_1 | \exists t \in [-\delta, \delta], X(p,t) \in B_1\}. \end{aligned}$$

The sets U_{ext} and V_{ext} are disjoint for $0 < \delta < \frac{1}{8}$.

Definition 5.1. *We denote by $f_{ext}\colon M \setminus \partial B_0 \to M \setminus \partial B_0$ defined as follows:*

- *If $p \in B_i \setminus \partial B_i$ then $f_{ext}(p) = P_{B_i}(p)$.*
- *Assume now that p belongs to an orbit segment $[q, P_{B_0 \cup B_1}(q)]^c$ with $q \in B_0$. Hence $p = X(q,s)$ with $s \in [0, r(q)]$. Then*

$$f_{ext}(p) = X(P_{B_0}(q), \Theta_{r(q), r(P_{B_0}(q))}(s))$$

- *Finally, assume that* p *belongs to an orbit segment* $[q, P_{B_0\cup B_1}(q)]^c$ *with* $q \in B_1$*. We denote* $\tilde q = P_{B_0\cup B_1}(q) \in B_0$*. Hence* $p = X(\tilde q, -s)$ *with* $s \in [0, r(q)]$*. Then*
$$f_{ext}(p) = X(P_{B_0}(\tilde q), -\Theta_{r(q),r(P_{B_1}(q))}(s))$$

Lemma 5.2. *The map* f_{ext} *defined above is a diffeomorphism of* $M \setminus \partial B_0$*. Furthermore,* U_{ext} *and* V_{ext} *are attracting and repelling regions of* f_{ext}*, respectively:* $f_{ext}(U_{ext})$ *and* $f_{ext}^{-1}(V_{ext})$ *are contained in the interior of* U_{ext} *and* V_{ext} *respectively. Furthermore, for any* $p \in M \setminus (U_{ext} \cup V_{ext})$ *one has:* $f_{ext}(p) \in U_{ext}$ *and* $f_{ext}^{-1}(p) \in V_{ext}$*.*

Proof : One first verifies that f_{ext} is well defined. The unique difficulty is at the points in B_1. For that we use that for $q \in B_0$ one has:

$$\begin{aligned} X(P_{B_0}(q), \Theta_{r(q),r(P_{B_0}(q))}(r(q))) &= X(P_{B_0}(q), r(P_{B_0}(q))) \\ &= P_{B_0\cup B_1}(P_{B_0}(q)) \\ &= P_{B_1}(P_{B_0\cup B_1}(q)). \end{aligned}$$

That shows that the expression of f_{ext} given by the second item sends $p = P_{B_0\cup B_1}(q)$ on $P_{B_1}(p)$ and hence coincides on B_1 with the expression given by the first item. An analogous argument shows that the expression given by the third item also coincides with this expression.

Now one verifies that for $p \in B_0$ the map f_{ext} sends the orbit segment of length 2δ centered at p to the orbit segment of length $2\alpha^{-1}\delta$ centered at $P_{B_0}(p)$, and the expression of f_{ext} in the time parametrization is the homothety of ration α^{-1}. This show that f_{ext} induces a diffeomorphism from U_{ext} to its image, which is contained in the interior of U_{ext}.

In the same way for $p \in B_1$ the map f sends the orbit segment of length $2\alpha^{-1}\delta$ centered at p on the orbit segment of length 2δ centered at $P_{B_1}(p)$, and the expression of f_{ext} in the time parametrization is the homothety of ration α. This show that f^{-1} induces a diffeomorphism from V_{ext} to its image, which is contained in the interior of V_{ext}.

So, we have proven the differentiability of f_{ext} in a neighborhood of B_0 and B_1. In the complement this property is a consequence of differentiability of P_{B_0}, P_{B_1}, Θ and the flow X. Finally, if $p \in M \setminus (U_{ext} \cup V_{ext})$ belongs to a segment $[q, P_{B_0\cup B_1}(q)]^c$ with $q \in B_0$ then $p = X(q, s)$ with $s \in [\delta, r(q) - \delta]$. Then item 3 of the definition of the map Θ implies that $\Theta_{r(q),r(P_{B_0}(q))}(s) \in [0, \delta]$ implying that $f_{ext}(p) \in U_{ext}$. An analogous argument holds if $p \in M \setminus (U_{ext} \cup V_{ext})$ belongs to a segment $[q, P_{B_0\cup B_1}(q)]^c$ with $q \in B_1$, proving that $f_{ext}(p) \in U_{ext}$ also in this case.

□

For every $p \in M$ there is $q \in B_0 \cup B_1$ such that p belongs to the segment $[q, P_{B_0\cup B_1}(q)]^c$. Then $f_{ext}(p)$ belongs to either the orbit segment joining $P_{B_0}(q)$ to $P_{B_1}(P_{B_0\cup B_1}(q))$ (if $q \in B_0$) or the orbit segment joining $P_{B_1}(q)$ to $P_{B_0}(P_{B_0\cup B_1}(q))$ if $q \in B_1$. As the triple (X, B_0, B_1) is in normal form, the length of $[q, P_{B_0\cup B_1}(q)]^c$ belongs to $[\frac{1}{4}, \frac{3}{4}]$ and the length of $[P_{B_0\cup B_1}(q), P_{B_i}(P_{B_0\cup B_1}(q))]^c$ belongs to $[\frac{3}{4}, \frac{5}{4}]$. One deduces that:

Remark 5. *For every* $p \in M$ *there is* $t \in [\frac{1}{4}, 2]$ *such that* $f_{ext}(p) = X(p, t)$*.*

Finally notice that the definitions of $\Theta_{r,s}, U_{ext}, V_{ext}$ and f_{ext} depend on the choice of α and δ. For simplifying notation we omitted the dependence on α and δ.

5.2. **The model diffeomorphism.** Our main technical lemma is the construction of a diffeomorphism f_{mod} defined in the neighborhood of the periodic orbit of the vector field X on $\mathbb{R}^2 \times S^1$. This diffeomorphism will be used as a model for the announced diffeomorphism f in the neighborhood of the boundary components of the Birkhoff section.

We will use the following definition.

Definition 5.2. *Let f be a diffeomorphism leaving invariant each orbit of a vector field X. The central derivative of f at the point (x,y,z) is $D^c f = \|Df_{(x,y,z)}(X(x,y,z))\|$.*

Lemma 5.3. *There is δ_0 such that for every $\delta \in (0,\delta_0]$ there is a diffeomorphism f_{mod} of $\mathbb{R}^2 \times S^1$, and two closed subset U_{mod} $V_{mod} \subset \mathbb{R}^2 \times S^1$ with the following properties*

1. *for every point (x,y,z) there is $t \in [\frac{1}{5}, 3]$ such that $f_{mod}((x,y,z)) = X_t(x,y,z)$;*
2. *U_{mod} is strictly invariant by f_{mod} : $f_{mod}(U_{mod}) \subset Int(U_{mod})$; V_{mod} is strictly invariant by f_{mod}^{-1} : $f_{mod}^{-1}(V_{mod}) \subset Int(V_{mod})$;*
3. *$U_{mod} \cap \{(x,y,z), \sqrt{x^2+y^2} \geq 2\}$ and $V_{mod} \cap \{(x,y,z), \sqrt{x^2+y^2} \geq 2\}$ coincide with the orbit segment of X of length 2δ centered at the half helicoid S_0 and S_1, respectively;*
4. *the restriction of f_{mod} to the periodic orbit $(0,0) \times S^1$ is a Morse-Smale diffeomorphism of the circle having exactly four fixed points, two of them are in U_{mod} and two in V_{mod};*
5. *Let $p = (x,y,z) \in S_0$ be a point such that $\sqrt{x^2+y^2} \geq 100$. Let I_p be the orbit segment of X joining (x,y,z) to its first return $P_{S_0}(x,y,z)$ on S_0. Then $f_{mod} I_p = I_{P_{S_0}(p)}$. Furthermore the expression of f_{mod} in restriction to the segment I_p is the same as the expression of f_{ext}. More precisely the segment contains a unique point $q \in S_1$. Let $r,s,r',s' > 0$ such that $q = X_r(p)$, $P_{S_0}(p) = X_s(q)$ $P_{S_1}(q) = X_{r'}(P_{S_0}(p))$ and $P^2_{S_0}(p) = X_{s'}(P_{S_1}(q))$. Then*
   - *for $t \in [0,r]$ $f_{mod}(X_t(p) = X_{t'}(P_{S_0}(p))$ with $t' = \Theta_{r,r'}(t)$*
   - *for $t \in [0,s]$ $f_{mod}(X_{-t}(P_{S_0}(p)) = X_{-t'}(P^2_{S_0}(p))$ with $t' = \Theta_{s,s'}(t)$*
6. *the central derivatives $D^c f_{mod}$ and $D^c f_{mod}^{-1}$ are less than or equal to α^{-1} in U_{mod} and V_{mod} respectively;*
7. *the central derivatives $D^c f_{mod}$ and $D^c f_{mod}^{-1}$ are precisely α^{-1} in $U_{mod} \cap \{(x,y,z), \sqrt{x^2+y^2} \geq 3\}$ and in $V_{mod} \cap \{(x,y,z), \sqrt{x^2+y^2} \geq 3\}$ respectively. More precisely, every orbit segment I of length 2δ, centered at a point $(x,y,z) \in S_0$ (resp. $(x,y,z) \in S_1$) such that $\sqrt{x^2+y^2} \geq 3$, is mapped by f_{mod} (resp. by f_{mod}^{-1}) in an affine way on the orbit segment of length $\alpha^{-1}\delta$ centered at $P_{S_0}(x,y,z)$ (resp. $P_{S_1}^{-1}(x,y,z)$).*
8. *For all $\alpha' > \alpha$, there is $N > 0$ such that for any $n \geq N$, for any point $(x,y,z) \in U_{mod}$ and $(x',y',z') \in V_{mod}$ and one has*
$$D^c f^n_{mod}(x,y,z) \in (\alpha'^{-n}, \alpha^{-n}]$$
$$D^c f^{-n}_{mod}(x',y',z') \in (\alpha'^{-n}, \alpha^{-n}].$$

Definition 5.3. *For every $n \in \mathbb{N}\backslash 0$ we denote by $f_{mod,n}$ the diffeomorphism of $\mathbb{R}^2 \times \mathbb{R}/n\mathbb{Z}$ obtained as follows:*
Let $\mathbb{R}^2 \times \mathbb{R}/n\mathbb{Z} \to \mathbb{R}^2 \times S^1$ be the canonical covering. Then $f_{mod,n}$ is the unique lift of f_{mod} which preserve each orbit of the vector field X.

5.3. **gluing the local models with f_{ext}.** Let (X, B_0, B_1) be a transitive Anosov vector field endowed with a pair of parallel Birkhoff sections in normal form (and verifying the condition on the time return given by Corollary 4.2).

For every periodic orbit in ∂B_0 we fix normalizing coordinates $\Gamma\colon O_{0,\gamma} \to \mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$ (where n is the absolute value of the linking number of γ). We denote $O_\gamma = \Gamma^{-1}(D_{\frac{1}{8}} \times \mathbb{R}/n\mathbb{Z})$, where $D_{\frac{1}{8}} \subset \mathbb{D}^2$ is the disk of radius $\frac{1}{8}$ centered at 0.

For every periodic orbit $\gamma \subset \partial B_0$ and every $r > 1$ we denote by $f_{r,\gamma}\colon O_\gamma \to O_{0,\gamma}$ the diffeomorphism $\Gamma^{-1} \circ h_r^{-1} \circ f_{mod,n} \circ h_r \circ \Gamma$ where $h_r\colon (x,y,z) \mapsto (rx, ry, z)$.

Remark 6. *The map $f_{r,\gamma}$ is well defined because, if $(x,y,z) \in \mathbb{R}^2 \times \mathbb{R}/n\mathbb{Z}$ satisfies $\sqrt{x^2+y^2} \leq \frac{1}{8}$ then $h_r^{-1} \circ f_{mod,n} \circ h_r(x,y,z)$ belongs to $\mathbb{D}^2 \times \mathbb{R}/n\mathbb{Z}$.*

Proof : Just notice that the expansion of the flow X_{mod} on the x,y-coordinates at time t is bounded by 2^t. □

For every point $p \in B_0$ we denote by I_p the orbit segment of X joining p to its first return $P_{B_0}(p)$ on B_0. By construction, $f_{ext}(I_p) = I_{P_{B_0}(p)}$, for every p.

Remark 7. *Consider $p \in B_0 \cap O_{0,\gamma}$ and $(x,y,z) = \Gamma(p)$. Since every point in I_p is of the form $X_t(p)$ with $0 \leq t \leq \frac{5}{4} < 2$, then :*

1. *if $\sqrt{x^2+y^2} \leq \frac{1}{8}$ (that is $p \in O_\gamma$) then $I_p \subset O_{0,\gamma}$;*
2. *if $\sqrt{x^2+y^2} \leq \frac{1}{32}$ then $I_p \subset O_\gamma$:*
3. *as a consequence , if $\sqrt{x^2+y^2} \leq \frac{1}{32}$ then $I_p \subset O_\gamma$ and $I_{P_{B_0}(p)} \subset O_{0,\gamma}$*

Lemma 5.4. *There is $r_0 > 1$ such that for every $r \geq r_0$, for every $p \in B_0 \cap O_{0,\gamma}$ and $(x,y,z) = \Gamma(p)$ such that $\sqrt{x^2+y^2} \in [\frac{1}{1000}\frac{1}{32}]$ one has: $f_{r,\gamma}(I_p) = f_{ext}(I_p) = I_{P_{B_0}(p)}$ and the restriction of $f_{r,\gamma}$ and f_{ext} to this segment are equal.*

Proof : Just take r_0 such that $r_0\frac{1}{1000} \gg 100$ and see definitions of $f_{r,\gamma}$ and f_{ext}. □

Let $O_2(\gamma)$ be the union of the orbit segments I_p for $p \in B_0$ such that $\sqrt{x^2+y^2} \leq \frac{1}{16}$, where $(x,y,z) = \Gamma_\gamma(p)$.

Corollary 5.1. *For $r > r_0$ there is a diffeomorphism $f_r\colon M \to M$ which coincides with f_{ext} out of the union of the $O_2(\gamma)$ for all connected component γ of ∂B_0 and it coincides with $f_{r,\gamma}$ for $x \in O_2(\gamma)$.*

Theorem 1 is a direct consequence of next proposition

Proposition 5.1. *There is $r_1 \geq r_0$ such that for every $r \geq r_1$ the diffeomorphism f_r satisfies the following properties:*

1. *f_r is of the form $p \mapsto X_{t(p)}(p)$ where $t(p)$ is a smooth function with values in $[1/5, 3]$.*
2. *f_r satisfies the Axiom A and the strong transversality condition*
3. *f_r is partially hyperbolic (its central bundle is directed by X)*
4. *f_r has exactly two basic pieces: one of them is a (connected) attractor and the other is a repeller.*

Proof :

(1) For every $r \geq r_0$, the diffeomorphism f_r is of the form $p \mapsto X_t(p)$ with $t \in [\frac{1}{5}, 3]$: if $p \notin \bigcup_\gamma O_2(\gamma)$ then $f_r(p) = f_{ext}$ which is in that form by construction; if $p \in O_2(\gamma)$ then $f_r(p) = f_{r,\gamma}(p)$. Notice that $f_{r,\gamma}$ is conjugated to $f_{mod,n}$ by $h_r \circ \Gamma_\gamma$, which maps the Anosov vector field X in M to the linear vector field X_{mod} in $\mathbb{R}^2 \times \mathbb{R}/n\mathbb{Z}$. Now the claim comes from the fact that $f_{mod,n}(h_r(\Gamma_\gamma(p)) = X_{mod t}(h_r(\Gamma_\gamma(p))$ for some $t \in [\frac{1}{5}, 3]$. This concludes the proof of item 1).

(2) Let denote by U_r and V_r the subsets of M define as

$$U_r = \bigcup_\gamma \left((h_r\Gamma_\gamma)^{-1}(U_{mod} \cap \{(x,y,z), \sqrt{x^2+y^2} \leq \frac{r}{8}\})\right) \cup \left(U_{ext} \setminus \bigcup_\gamma Int(O_\gamma)\right)$$

$$V_r = \bigcup_\gamma \left((h_r\Gamma_\gamma)^{-1}(V_{mod} \cap \{(x,y,z), \sqrt{x^2+y^2} \leq \frac{r}{8}\})\right) \cup \left(V_{ext} \setminus \bigcup_\gamma Int(O_\gamma)\right)$$

Then U_r and V_r are disjoint compact sets. Moreover, $f_r(U_r) \subset Int(U_r)$ (we see that independently in the parts where f_r coincide with f_{ext} or with $f_{r,\gamma}$ using the fact that U_{ext} and $(h_r\Gamma_\gamma)^{-1}\left(U_{mod} \cap \{(x,y,z), \sqrt{x^2+y^2} \leq \frac{r}{8}\}\right)$ coincides on $O_{0,\gamma} \setminus \Gamma_\gamma^{-1}(\{\sqrt{x^2+y^2} \leq \frac{2}{r}\})$.

In the same way $f_r^{-1}(V_r) \subset Int(V_r)$.

We denote by $A_r = \bigcap_{n\in\mathbb{Z}} f_r^n(U_r)$ and $R_r = \bigcap_{n\in\mathbb{Z}} f_r^n(V_r)$ the maximal invariant sets in U_r and V_r, respectively.

Claim 3. *The chain recurrent set $\mathcal{R}(f_r)$ is contained in $A_r \cup R_r$.*

Proof : As $f_r(U_r) \subset Int(U_r)$, we get that there is $\eta > 0$ such that every η-pseudo orbit $(p_i)_{i\in\mathbb{Z}}$ verifies that

$$p_i \in U_r \Rightarrow p_j \in U_r, \forall j \geq i.$$

In the same way,

$$p_i \in V_r \Rightarrow p_k \in V_r, \forall k \leq i.$$

One deduces that $\mathcal{R}(f_r) \cap (U_r \cup V_r) \subset A_r \cup R_r$ (see [5]).

Recall that, by construction, every point $p \notin U_{ext} \cup V_{ext}$ verifies $f_{ext}(p) \in U_{ext}$. One deduces that, for $\eta > 0$ small enough, every η-pseudo orbit meeting $M \setminus \bigcup_\gamma O_\gamma$ meet either U_r or V_r. One deduces that every point $p \in \mathcal{R}(f_r)$ is either contained in $A_r \cup R_r$ or there is a boundary component γ of ∂S_0 such that the orbit of p is contained in O_γ. However, the maximal invariant set of O_γ is γ. So, if $p \in \mathcal{R}(f) \setminus (A_r \cup R_r)$ then p belongs to some component γ. However, for every point $p \in \gamma$ the ω-limit set of p is a periodic point of the restriction of f_r to γ, and this periodic point belongs, by construction, to $U_r \cap V_r$, contradicting the fact that $p \notin A_r \cup R_r$. $\square$

Claim 4. *The invariant compact sets A_r and R_r are hyperbolic. Furthermore, the stable spaces of the points of A_r (resp R_r) have dimension 2 (resp. dimension 1).*

Proof : This is a simple consequence of the fact that f_r is of the form $p \mapsto X_t(p)$ with $t \geq \frac{1}{5}$, and of the fact that the central derivative Df_r^c is less or equal to α^{-1} on U_r. Hence every vector in the center-stable direction of the vector field X at a point $p \in U_r$ is (uniformly) contracted by Df_r^n for large $n > 0$. Analogously, every vector in the center-unstable direction of X at a point $q \in V^r$ is uniformly contracted by Df_r^{-n} for large $n > 0$. □

So we proved that the chain recurrent set $\mathcal{R}(f_r)$ is hyperbolic: according to [13] this implies that f_r satisfies the Axiom A and has no cycles. Furthermore, if K and L are two hyperbolic sets of f_r such that $W^u(K) \cap W^s(L) \neq \emptyset$ then either K and L are both contained in U_r or both in V_r or $K \subset V_r$ and $L \subset U_r$. As a consequence one gets that $dimW^u(K) + \dim W^s(L) \geq 3 = dimM$. As we noticed in section 2.3 this implies that f_r satisfies the strong transversality condition. This ends the proof of item 2

(3) For all $x \in M$, there exist n_0, n_1 positive numbers such that $f_r^{n_0}(x) \in U_r$ and $f_r^{-n_1}(x) \in V_r$. Let us fix $\alpha < \alpha' < \lambda^{\frac{1}{50}}$. According to item (8) of Lemma 5.3, there is $N > 0$ such that, in U_r and $f_r^{-1}(V_r)$, the central derivative of f^n, for $n \geq N$ and its inverse are bounded by $\alpha'^n < \lambda^{\frac{n}{50}}$.

On the other hand, f_r is on the form $p \mapsto X_t(p)$ with $t \geq \frac{1}{5}$ so that $\|Df_r^n\|$ contract the vectors on E^s by a factor smaller that $C^{-1}\lambda^{-\frac{n}{5}}$ and expands the vectors in E^u by a factor larger that $C\lambda^{\frac{n}{5}}$. This proves that the contraction of $\|Df_r^n\|$ (resp. $\|Df_r^{-n}\|$) restricted to E^{ss} (resp. to E^{uu}) is uniformly stronger than the contraction in the direction of the flow, that is, f_r is partially hyperbolic and $\mathbb{R}X$ is the central bundle.

(4) The fact that f_r has exactly two basic pieces is a direct consequence of Lemma 5.5 below with $\Omega(f_r) \subset U_r \cup V_r$ ending the proof.

□

Lemma 5.5. *For r large enough, the maximal invariant sets in U_r and V_r are transitive.*

A precise analysis of the dynamics of f_r is the aim of Section 8, and this analysis will provide a detailed proof of Lemma 5.5. However we can give a direct proof now.

Proof : All the diffeomorphisms f_r are Axiom A diffeomorphisms and satisfy the strong transversality condition. Hence they are all structurally stable. As f_r is a continuous family of structurally stable diffeomorphisms, there are all conjugated. Furthermore, the conjugacy homeomorphisms preserves the central foliation (directed by X). Finally, every periodic orbit in U_r, not contained in a boundary component of B_0, meets $U_{ext} \setminus \bigcup_{\gamma \subset \partial B_0} \mathcal{O}_\gamma$. Fix some s, and let $x_s \in U_{ext} \setminus \bigcup_\gamma \mathcal{O}_\gamma$ be a periodic point. So x_s belongs to an orbit segment of length 2δ centered at a point $x \in B_0$ (let us call *x the projection of x_s on B_0*). Let x_r be the continuation of x_s for f_r. A continuity argument proves that x_r belongs to the same orbit segment of length 2δ centered at a point $x \in B_0$.

For proving the transitivity of the maximal invariant set in U_r we will show

Claim 5. *Any two periodic points $x_s, y_s \in U_{ext} \setminus \bigcup_\gamma \mathcal{O}_\gamma$ are homoclinically related.*

Proof : It is enough to prove that x_r and y_r are homoclinically related for r large enough.

Recall that the first return map P_{B_0} is a pseudo-Anosov diffeomorphisms. In particular there is an hyperbolic basic set K of P_{B_0} contained in $B_0 \setminus \partial B_0$ which contains the projection x, y of x_s, y_s on B_0.

For r large enough, f_r coincides with f_{ext} hence with P_{B_0} in the complement of an arbitrarily small neighborhood of ∂B_0 in B_0, in particular on K. One deduces that, for r large enough, K is a basic set of f_r. As x_r is a periodic point in the segment of size 2δ centered at x and using that f_r is uniformly contracting in the orbits segment of size 2δ at the points of K, one deduces $x_r = x$ and $y_r = y$. So x_r and y_r are homoclinically related.

□

□

6. In the neighborhood of the boundary of a Birkhoff section

6.1. **General presentation of our construction.** In this section we start the construction of the model diffeomorphism f_{mod} of $\mathbb{R}^2 \times S^1$ announced in Lemma 5.3. For that, we consider $\mathbb{R}^2 \times S^1$ endowed with the model vector field $X = X_{mod}$ and the model Birkhoff section S_0 and S_1.

We will divide $\mathbb{R}^2 \times S^1$ in regions having global sections which cut the orbits in compact segments, in order to define the diffeomorphism segment by segment. The first regions we consider are the quadrant associated to the periodic orbit. As the vector field is the model vector field, the quadrants can be expressed in formula by:

$$\begin{array}{rclcrcl} C^{++} & = & \{(x,y,z), x \geq 0 \text{ and } y \geq 0\}, & & C^{+-} & = & \{(x,y,z), x \geq 0 \text{ and } y \leq 0\}, \\ C^{-+} & = & \{(x,y,z), x \leq 0 \text{ and } y \geq 0\}, & \text{and} & C^{--} & = & \{(x,y,z), x \leq 0 \text{ and } y \leq 0\}. \end{array}$$

In order to glue the diffeomorphisms in a quadrant with f_{ext}, we will consider transverse sections in each quadrant which coincide with S_0 and S_1 out of a neighborhood of the periodic orbit $(0,0) \times S^1$. We cannot use S_0 and S_1 because these two Birkhoff sections are not disjoint. Section 6.2 will provide us these sections, called Σ_0 and Σ_1, obtained by pushing S_0 and S_1 along the orbits, in different way depending on the quadrants. In each quadrant $C^{\pm\pm}$, the section Σ_0 and Σ_1 induce disjoint smooth surfaces whose boundaries are contained in the boundary of the quadrants. Exactly as we have build f_{ext}, we will build diffeomorphisms $f^{\pm\pm}$ of the quadrants, admitting a tubular neighborhood of Σ_0 as an attracting region, and a tubular neighborhood of Σ_1 as a repelling region.

Far from the periodic orbit $(0,0) \times S^1$ of X, all the $f^{\pm\pm}$ have the same expression (analogous to those announced in Lemma 5.3). But this diffeomorphisms $f^{\pm\pm}$ do not coincide neither on $(0,0) \times S^1$ nor on their invariant manifolds, which are the intersections of the quadrants.

For solving this difficulty, Section 6.7 defines a diffeomorphism f_0 in the neighborhood of the periodic orbit (once more, f_0 is defined on orbit segments obtained by cutting the orbits along local sections through the periodic orbit). A bump function will allow us to glue f_0 to the $f^{\pm\pm}$ obtaining new diffeomorphisms $f_0^{\pm\pm}$ equal to $f^{\pm\pm}$ far from the periodic orbit and equal to f_0 in the neighborhood of the periodic orbit.

The most difficult part will consist in gluing the diffeomorphisms $f_0^{\pm\pm}$ in the intersections of the quadrants. It will be done in Section 7.

Let us start by introducing a bump function that we will use many times during this construction.

We denote by $\psi\colon [0,+\infty) \to [0,1]$ a smooth map such that

- the derivative $\psi'(x)$ is strictly negative for $x \in (\frac{1}{3}, \frac{2}{3})$
- $\psi(t) = 1$ for $t \in [0, \frac{1}{3}]$
- $\psi(t) = 0$ for $t \geq \frac{2}{3}$.

6.2. Surgery on the helicoids S_0 and S_1 for getting disjoint sections of X.

The intersection $S_0^- = S_0 \cap \{y \leq 0\}$ is not only homeomorphic to a half plane but diffeomorphic to $[0,1] \times [0,+\infty)$ as well: the intersection with the plane $y = 0$ is composed of the half line $x \leq 0, y = 0, z = -\frac{1}{2}$ the segment $x = y = 0$, $z \in [-\frac{1}{2}, 0]$ and the half line $x \geq 0, y = z = 0$.

Consider $(x,y,z) \in S_0^-$. The coordinates z belongs to $\mathbb{R}/\mathbb{Z}$; however, for a point in S_0^- one has $y = r\sin 2\pi z < 0$, with $r > 0$. Hence, one may choose a representative $z \in [-\frac{1}{2}, 0]$. Now, the map π_0^- defined by $(x,y,z) \mapsto \pi_0^-(x,y,z) = X_{-z-\frac{1}{4}}(x,y,z)$ is a projection of S_0^- on the half plane $\{y \leq 0, z = -\frac{1}{4}\}$, which is a diffeomorphism in restriction to $S_0^- \setminus \{x = y = 0\}$.

We will push S_0^- along the orbit of X in the direction of the projection π_0^-, in order to get a smooth surface which coincides with S_0^- for large radius, i.e. $r = \sqrt{x^2+y^2}$, and which coincides with the half plane $y \leq 0$, $z = -\frac{1}{4}$ for small radius (see Figure 2).

For that we consider a barycenter between $(x,y,z) \in S_0^-$ and $\pi_0^-(x,y,z)$ in the segment of orbit of X joining this two points, and whose coefficient is given by the function ψ defined in Section 6.1.

We denote

$$\Sigma_0^- = \left\{ X_t(x,y,z), \text{ where } (x,y,z) \in S_0, y \leq 0, z \in [-\frac{1}{2}, 0], t = \psi(\sqrt{x^2+y^2}) \cdot (-z - \frac{1}{4}) \right\}$$

Notice that Σ_0^- is a smooth surface with boundary which coincides with S_0^- for $\sqrt{x^2+y^2} \geq \frac{2}{3}$ and which coincides with the half plane $\{y \leq 0, z = -\frac{1}{4}\}$ for $\sqrt{x^2+y^2} \leq \frac{1}{3}$; more precisely, the half disk $\{y \leq 0, z = -\frac{1}{4}, \sqrt{x^2+y^2} \leq \frac{1}{3}\}$ is contained in Σ_0^-.

In the same way we push $S_0^+ = S_0 \cap \{y \geq 0\}$ along the orbit of X in the direction of the half plane $y \geq 0$, $z = \frac{1}{4}$. One defines:

$$\Sigma_0^+ = \left\{ X_t(x,y,z), \text{ where } (x,y,z) \in S_0, y \geq 0, z \in [0, \frac{1}{2}], t = \psi(\sqrt{x^2+y^2}) \cdot (-z + \frac{1}{4}) \right\}$$

One denote $\Sigma_0 = \Sigma_0^- \cup \Sigma_0^+$ (see Figure 3). Notice that $\Sigma_0^- \cap \Sigma_0^+ = \{y = 0, z = -\frac{1}{2} = \frac{1}{2} \in S^1, x \leq -\frac{2}{3}\} \cup \{y = 0 = z, x \geq \frac{2}{3}\}$. Hence Σ_0 is a surface with boundary and corners, whose boundary is contained in $\{y = 0, x \in [-\frac{2}{3}, \frac{2}{3}]\}$ and it is composed of the segment in $\partial\Sigma_0^- \subset \{y = 0\}$ joining the points $(2/3, 0, 0)$ and $(-2/3, 0, -1/2)$ and the segment in $\partial\Sigma_0^+$ joining the points $(2/3, 0, 0)$ and $(-2/3, 0, 1/2) = (-2/3, 0, -1/2)$.

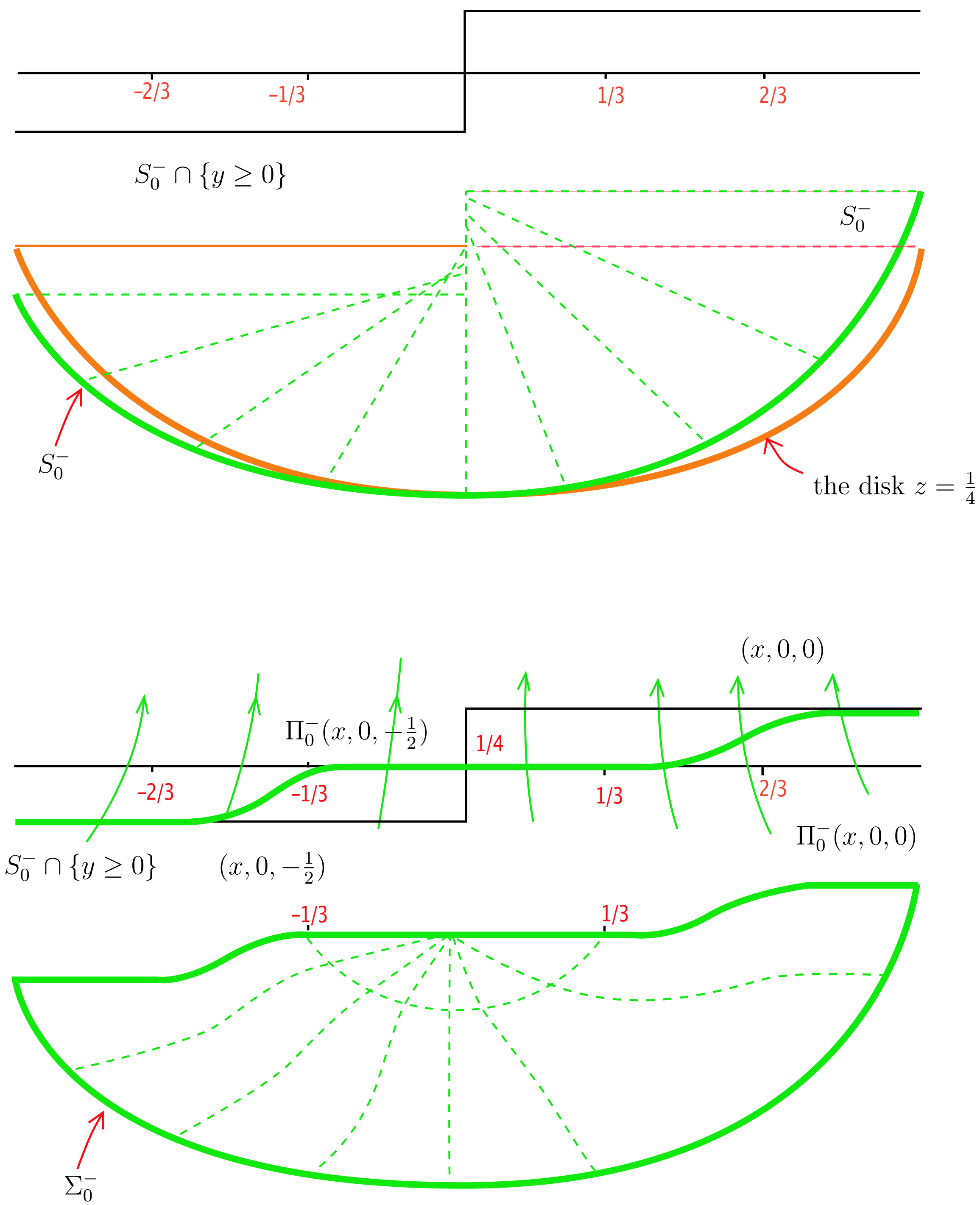

FIGURE 2. Pushing S_0^- in direction of the plane $z = -\frac{1}{4}$.

In the same way, we define a surface Σ_1 by pushing $S_1^- = S_1 \cap \{x \leq 0\}$ and $S_1^+ = S_1 \cap \{x \geq 0\}$ along the orbits of X in the direction of the planes $\{z = 0\}$ and $\{z = 1/2\}$ respectively. More precisely:

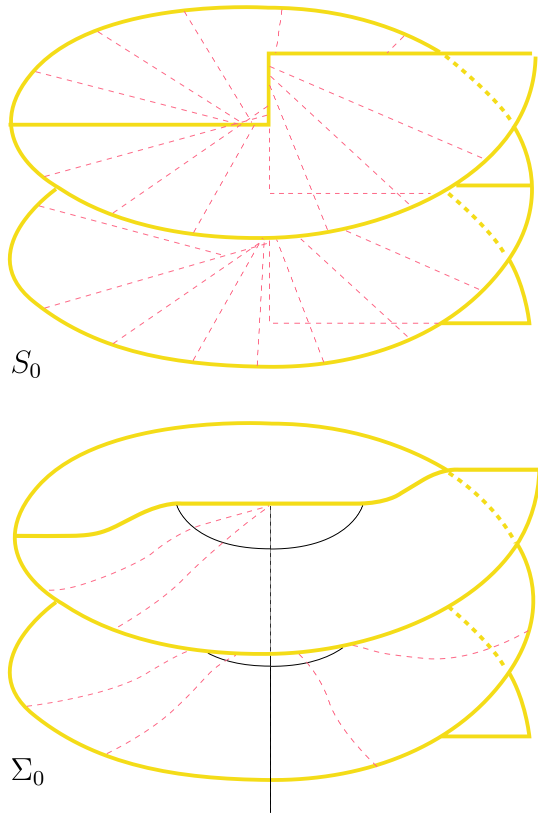

FIGURE 3. The section S_0 and Σ_0.

For $(x,y,z) \in S_1^-$ we can choose $z \in [-\frac{1}{4}, \frac{1}{4}]$. We define:

$$\Sigma_1^- = \left\{ X_t(x,y,z), \text{ where } (x,y,z) \in S_1, x \leq 0, z \in [-\frac{1}{4}, \frac{1}{4}], t = \psi(\sqrt{x^2+y^2}) \cdot (-z) \right\}$$

For $(x,y,z) \in S_1^+$ we can choose $z \in [\frac{1}{4}, \frac{3}{4}]$. We define:

$$\Sigma_1^+ = \left\{ X_t(x,y,z), \text{ where } (x,y,z) \in S_1, x \geq 0, z \in [\frac{1}{4}, \frac{3}{4}], t = \psi(\sqrt{x^2+y^2}) \cdot (-z + \frac{1}{2}) \right\}$$

One denote $\Sigma_1 = \Sigma_1^- \cup \Sigma_1^+$. Notice that $\Sigma_1^- \cap \Sigma_1^+ = \{x = 0, z = \frac{1}{4} \in S^1, y \leq -\frac{2}{3}\} \cup \{x = 0, z = \frac{3}{4} = -\frac{1}{4}, y \geq \frac{2}{3}\}$. Hence Σ_1 is a surface with boundary and corners, whose boundary is contained in $\{x = 0, y \in [-\frac{2}{3}, \frac{2}{3}]\}$. Furthermore $\partial\Sigma_1$ is composed of the segment in $\partial\Sigma_1^+ \subset \{x = 0\}$ joining the points $(0, -2/3, 1/4)$ and $(0, 2/3, 3/4)$ and the segment in $\partial\Sigma_1^-$ joining the points $(0, 2/3, -1/4 = 3/4)$ and $(0, -2/3, 1/4)$.

Lemma 6.1. *The surfaces Σ_0 and Σ_1 are disjoint.*

Figure 4 shows how the surfaces Σ_0 and Σ_1 fit together, and this may help the reader to follow the proof below.

Proof : The proof consists in looking at the z-coordinates of the points in Σ_0 and Σ_1 according to the signs of the two other coordinates.

The Figures 5 and 6 may help the reader to follow the proof below.

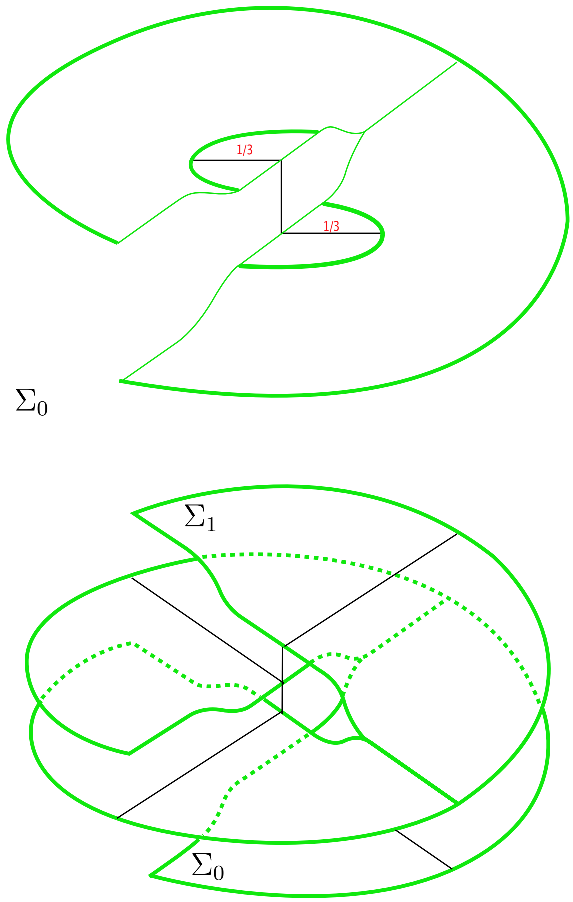

FIGURE 4. Σ_0 and Σ_1 are disjoint.

(1) Notice that in the quadrant $C^{++} = \{x \geq 0, y \geq 0\}$ it holds that
   (a) $(x, y, z) \in \Sigma_0^+$ implies that $z \in [0, \frac{1}{4}]$. Furthermore $z = 0 \Rightarrow y = 0$ and $x \geq 2/3$.
   (b) $(x, y, z) \in \Sigma_0^-$ implies that $y = 0$ and $z \in [-\frac{1}{4}, 0]$. Furthermore $z = 0 \Rightarrow x \geq 2/3$.
   (c) $(x, y, z) \in \Sigma_1^+$ implies that $z \in [\frac{1}{2}, \frac{3}{4}]$. Furthermore $z = 3/4 = -1/4 \Rightarrow x = 0$ and $y \geq 2/3$.
   (d) $(x, y, z) \in \Sigma_1^-$ implies that $x = 0$ and $z \in [-\frac{1}{4}, 0]$. Furthermore $z = -\frac{1}{4} \Rightarrow y \geq 2/3$.

   One gets directly that, in this quadrant, $\Sigma_0^+ \cap \Sigma_1^+ = \emptyset$. Notice that $\Sigma_1^+ \cap \{x \geq 0, y = 0\} \cap \{z \in [-1/4, 0]\} = \emptyset$ hence $C^{++} \cap \Sigma_1^+ \cap \Sigma_0^- = \emptyset$.

   Analogously, $C^{++} \cap \Sigma_0^+ \cap \{x = 0\} \cap \{z \in [-\frac{1}{4}, 0]\} = \emptyset$ so that $C^{++} \cap \Sigma_0^+ \cap \Sigma_1^- = \emptyset$. In the case that $x = 0$ and $y = 0$ the point $(0, 0, z) \in \Sigma_0^-$ if and only if $z = -1/4$ and $(0, 0, z) \in \Sigma_1^-$ if and only if $z = 0$, then $C^{++} \cap \Sigma_1^- \cap \Sigma_0^- = \emptyset$. We have proved that $C^{++} \cap \Sigma_1 \cap \Sigma_0 = \emptyset$.
(2) Notice that in the quadrant $C^{-+} = \{x \leq 0, y \geq 0\}$ it holds that

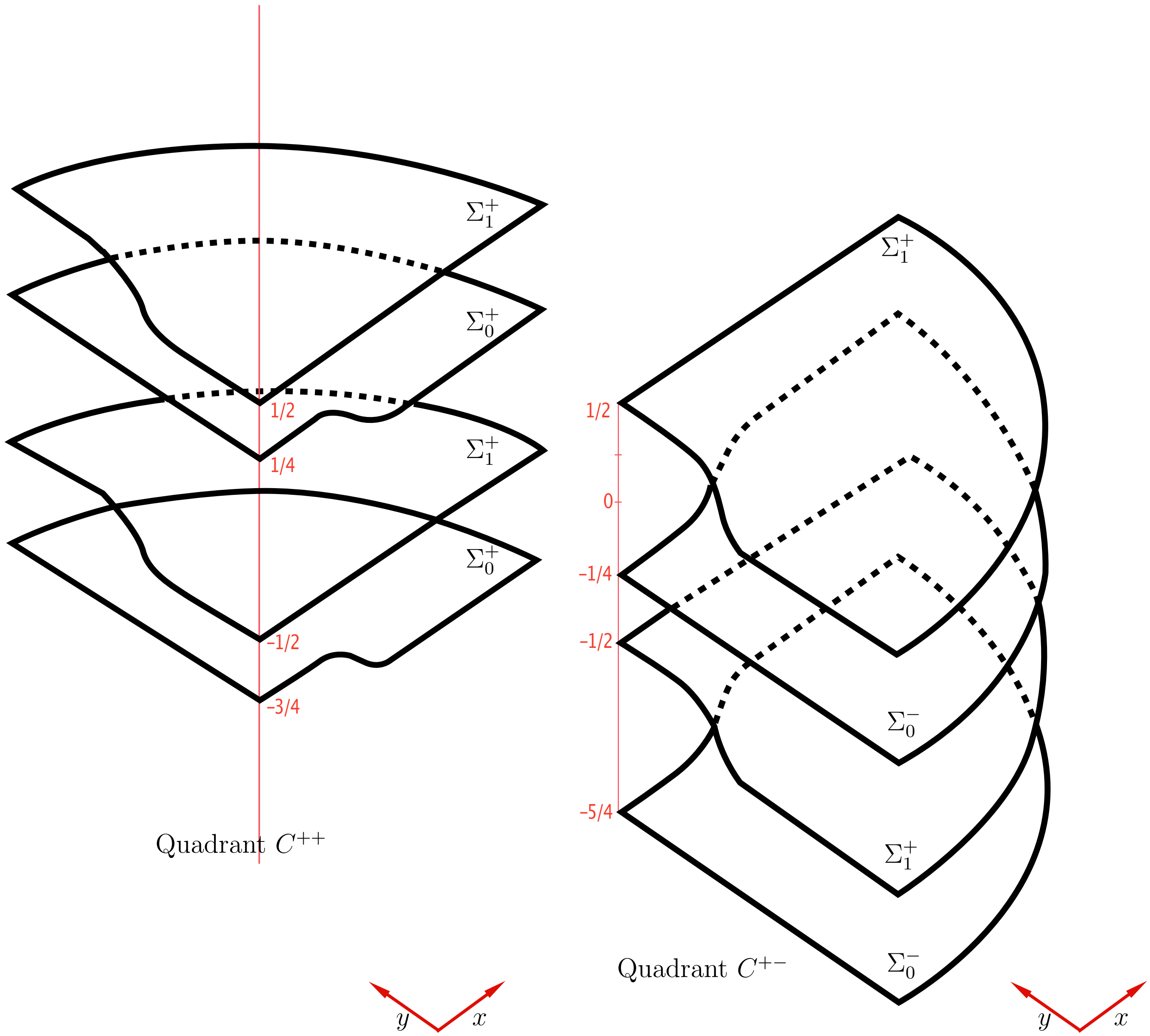

FIGURE 5. The surfaces Σ_0 and Σ_1 in the quadrants C^{++} and C^{+-}.

(a) $(x, y, z) \in \Sigma_0^+$ implies that $z \in [\frac{1}{4}, \frac{1}{2}]$. Furthermore $z = \frac{1}{2} \Rightarrow y = 0$ and $x \leq -2/3$.
(b) $(x, y, z) \in \Sigma_0^-$ implies that $y = 0$ and $z \in [-\frac{1}{2}, -\frac{1}{4}] = [\frac{1}{2}, \frac{3}{4}]$. Furthermore $z = \frac{1}{2} \Rightarrow x \leq -2/3$.
(c) $(x, y, z) \in \Sigma_1^+$ implies that $x = 0$ and $z \in [\frac{1}{2}, \frac{3}{4}]$. Furthermore $z = \frac{3}{4} = -\frac{1}{4} \Rightarrow y \geq 2/3$.
(d) $(x, y, z) \in \Sigma_1^-$ implies that $z \in [-\frac{1}{4}, 0] = [\frac{3}{4}, 1]$. Furthermore $z = 3/4 = -1/4 \Rightarrow x = 0$ and $y \geq 2/3$.

One gets directly that, in this quadrant, $\Sigma_0^+ \cap \Sigma_1^- = \emptyset$. Notice that $\Sigma_1^- \cap \{x = 0, y = 0\} \cap \{z \in [1/2, 3/4]\} = \emptyset$ hence $C^{-+} \cap \Sigma_1^- \cap \Sigma_0^- = \emptyset$.

Analogously, $C^{-+} \cap \Sigma_0^+ \cap \{x = 0\} \cap \{z \in [\frac{1}{2}, \frac{3}{4}]\} = \emptyset$ so that $C^{-+} \cap \Sigma_0^+ \cap \Sigma_1^+ = \emptyset$. In the case that $x = 0$ and $y = 0$ the point $(0, 0, z) \in \Sigma_0^-$ if and only if $z = -1/4$ and $(0, 0, z) \in \Sigma_1^+$ if and only if $z = 1/2$, then $C^{-+} \cap \Sigma_1^+ \cap \Sigma_0^- = \emptyset$. We have proved that $C^{-+} \cap \Sigma_1 \cap \Sigma_0 = \emptyset$.

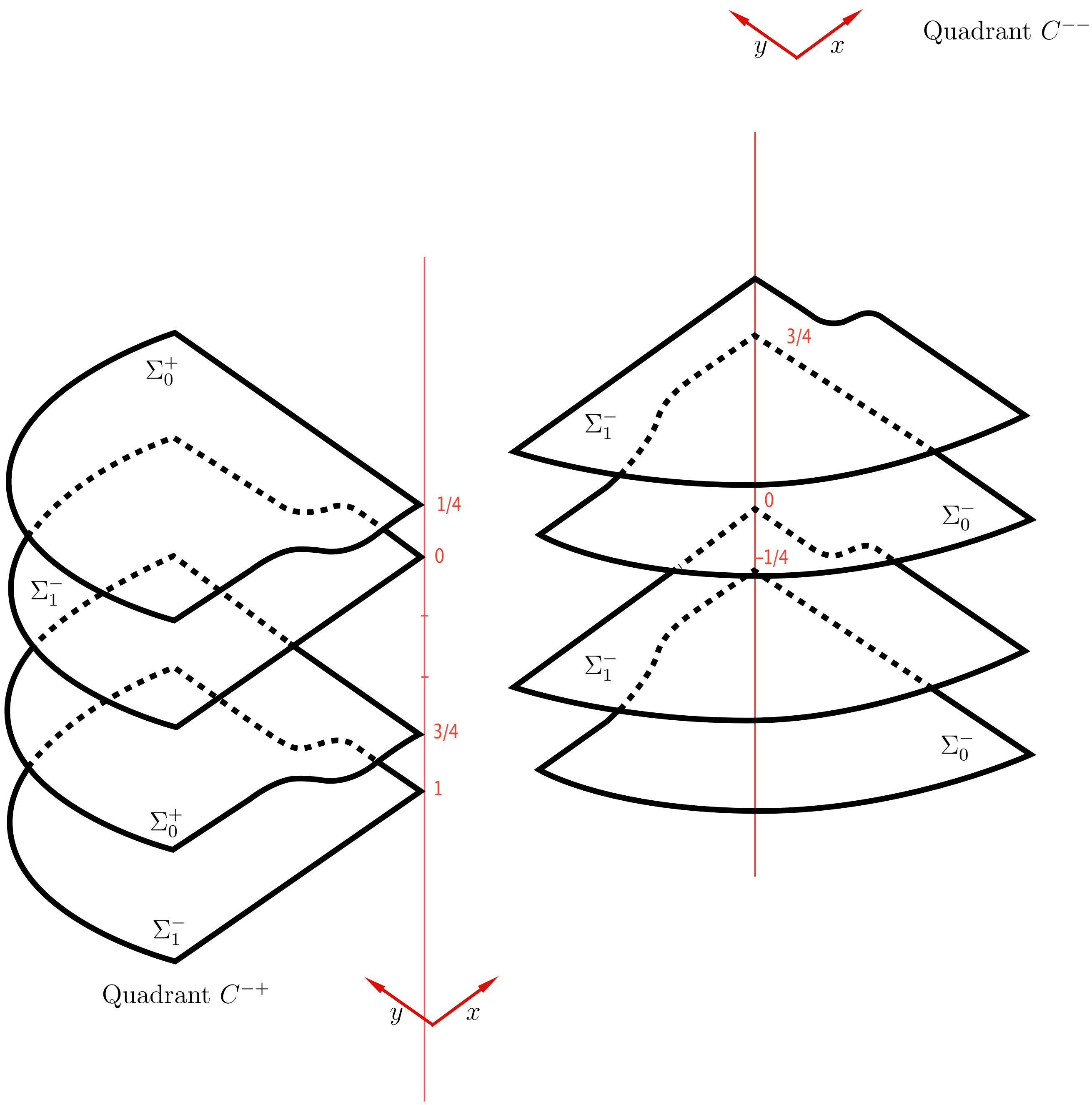

FIGURE 6. The surfaces Σ_0 and Σ_1 in the quadrants C^{-+} and C^{--}.

(3) Notice that in the quadrant $C^{--} = \{x \leq 0, y \leq 0\}$ it holds that
   (a) $(x,y,z) \in \Sigma_0^-$ implies that $z \in [-\frac{1}{2}, -\frac{1}{4}] = [\frac{1}{2}, \frac{3}{4}]$. Furthermore $z = \frac{1}{2} \Rightarrow y = 0$ and $x \leq -2/3$.
   (b) $(x,y,z) \in \Sigma_0^+$ implies that $y = 0$ and $z \in [\frac{1}{4}, \frac{1}{2}]$. Furthermore $z = \frac{1}{2} \Rightarrow x \leq -2/3$.
   (c) $(x,y,z) \in \Sigma_1^+$ implies that $x = 0$ and $z \in [\frac{1}{4}, \frac{1}{2}]$. Furthermore $z = \frac{1}{4} \Rightarrow y \leq -2/3$.
   (d) $(x,y,z) \in \Sigma_1^-$ implies that $z \in [0, \frac{1}{4}]$. Furthermore $z = 1/4 \Rightarrow x = 0$ and $y \leq -2/3$.

   One gets directly that, in this quadrant, $\Sigma_0^- \cap \Sigma_1^- = \emptyset$. Notice that $\Sigma_1^- \cap \{x \leq 0, y = 0\} \cap \{z \in [1/4, 1/2]\} = \emptyset$ hence $C^{--} \cap \Sigma_1^- \cap \Sigma_0^+ = \emptyset$.

   Analogously, $C^{--} \cap \Sigma_0^- \cap \{x = 0\} \cap \{z \in [\frac{1}{4}, \frac{1}{2}]\} = \emptyset$ so that $C^{--} \cap \Sigma_0^- \cap \Sigma_1^+ = \emptyset$. In the case that $x = 0$ and $y = 0$ the point $(0,0,z) \in \Sigma_0^+$ if and only if $z = 1/4$ and $(0,0,z) \in \Sigma_1^+$ if and only if $z = 1/2$, then $C^{--} \cap \Sigma_1^+ \cap \Sigma_0^+ = \emptyset$. We have proved that $C^{--} \cap \Sigma_1 \cap \Sigma_0 = \emptyset$.
(4) This case is analogous to the previous cases.

□

6.3. **First return maps.**

Notice that Σ_0^- is a global section of the vector field X on the (X-invariant) half space $\{y \leq 0\}$. Hence the first return map $P_0^- : \Sigma_0^- \to \Sigma_0^-$ of the orbits of X on the section Σ_0^- is well defined and it is a diffeomorphism of Σ_0^-.

Lemma 6.2. *For every $p \in \Sigma_0^-$ the orbit segment joining p to $P_0^-(p)$ have its length larger than* $1/2$.

Proof : That is just because Σ_0^- is contained in $z \in [-1/2, 0]$. □

In the same way the first return maps $P_0^+ \colon \Sigma_0^+ \to \Sigma_0^+$, $P_1^- : \Sigma_1^- \to \Sigma_1^-$, and $P_1^+ \colon \Sigma_1^+ \to \Sigma_1^+$ are well defined diffeomorphism.

We now consider how the first return maps P_0^+ and P_0^- fit together.

Lemma 6.3. *If $p \in \Sigma_0^+ \cap \Sigma_0^-$ then $P_0^+(p) = P_0^-(p)$. If $q \in \Sigma_1^+ \cap \Sigma_1^-$ then $(P_1^+)^{-1}(q) = (P_1^-)^{-1}(q)$.*

Proof : A point $p = (x,y,z)$ belongs to $\Sigma_0^+ \cap \Sigma_0^-$ if and only if:
- either $y = 0$, $z = 0$ and $x \geq 2/3$; in this case $P_0^+(p) = P_0^-(p) = (2x, 0, 0)$
- or $y = 0$ $z = 1/2$ and $x \leq -2/3$; in this case $P_0^+(p) = P_0^-(p) = (2x, 0, 1/2)$.

□

The lemma allows us to define $P_0 \colon \Sigma_0 \to \Sigma_0$ as $P_0 = P_0^+$ on Σ_0^+ and $P_0 = P_0^-$ on Σ_0^-, and $(P_1)^{-1} \colon \Sigma_1 \to \Sigma_1$ as $(P_1)^{-1} = (P_1^+)^{-1}$ on Σ_1^+ and $(P_1)^{-1} = (P_1^-)^{-1}$ on Σ_1^-.

6.4. **In the quadrants.**

Recall that we denote by C^{++}, C^{+-},C^{-+} and C^{--} the quadrants $\{x \geq 0$ and $y \geq 0\}$,$\{x \geq 0$ and $y \leq 0\}$, $\{x \leq 0$ and $y \geq 0\}$, and $\{x \leq 0$ and $y \leq 0\}$. Each of the quadrants are X-invariant. As a direct consequence, the return maps defined above preserve the quadrants.

Notice that Σ_0^+ and Σ_1^+ are disjoint global sections of the vector field X restricted to C^{++}. Hence for any point $p \in \Sigma_0^+ \cap C^{++}$ the orbit segment joining p to $P_0^+(p)$ cuts Σ_1^+ in a unique point $P_{0,1}^{++}(p)$. The map $P_{0,1}^{++} : \Sigma_0^+ \cap C^{++} \to \Sigma_1^+ \cap C^{++}$ is a diffeomorphism. One defines analogously the diffeomorphism $P_{1,0}^{++} : \Sigma_1^+ \cap C^{++} \to \Sigma_0^+ \cap C^{++}$. Furthermore one has

$$P_{1,0}^{++} \circ P_{0,1}^{++} = P_0^+ \quad \text{and} \quad P_{0,1}^{++} \circ P_{1,0}^{++} = P_1^+$$

In the same way Σ_0^+ and Σ_1^- are disjoint global sections of the vector field X restricted to C^{-+}. This allows us to define the diffeomorphisms $P_{0,1}^{-+} : \Sigma_0^+ \cap C^{-+} \to \Sigma_1^- \cap C^{-+}$ and $P_{1,0}^{-+} : \Sigma_1^- \cap C^{-+} \to \Sigma_0^+ \cap C^{-+}$, and we have

$$P_{1,0}^{-+} \circ P_{0,1}^{-+} = P_0^+ \quad \text{and} \quad P_{0,1}^{-+} \circ P_{1,0}^{-+} = P_1^-$$

One defines in the same way the diffeomorphisms $P_{0,1}^{+-} : \Sigma_0^- \cap C^{+-} \to \Sigma_1^+ \cap C^{-+}$ and $P_{1,0}^{+-} : \Sigma_1^+ \cap C^{+-} \to \Sigma_0^- \cap C^{+-}$, $P_{0,1}^{--} : \Sigma_0^- \cap C^{--} \to \Sigma_1^- \cap C^{--}$ and $P_{1,0}^{--} : \Sigma_1^- \cap C^{--} \to \Sigma_0^- \cap C^{--}$,
and we have

$$P_{1,0}^{+-} \circ P_{0,1}^{+-} = P_0^- \quad \text{and} \quad P_{0,1}^{+-} \circ P_{1,0}^{+-} = P_1^+ \quad \text{on the quadrant } C^{+-}$$

$$P_{1,0}^{--} \circ P_{0,1}^{--} = P_0^- \quad \text{and} \quad P_{0,1}^{--} \circ P_{1,0}^{--} = P_1^- \quad \text{on the quadrant } C^{--}$$

Let $\delta > 0$ such that any segment of orbit of length 4δ meeting Σ_0 is disjoint from Σ_1. For all $\delta > 0$ small enough

- In the quadrant C^{++}, we define U^{++} (resp. V^{++}) as being the set of points $p \in C^{++}$ such that there is $t \in [-\delta, \delta]$ with $X_t(p) \in \Sigma_0^+$ (resp. $X_t(p) \in \Sigma_1^+$).
- In the quadrant C^{-+}, we define U^{-+} (resp. V^{-+}) as being the set of points $p \in C^{-+}$ such that there is $t \in [-\delta, \delta]$ with $X_t(p) \in \Sigma_0^+$ (resp. $X_t(p) \in \Sigma_1^-$).
- In the quadrant C^{--}, we define U^{--} (resp. V^{--}) as being the set of points $p \in C^{--}$ such that there is $t \in [-\delta, \delta]$ with $X_t(p) \in \Sigma_0^-$ (resp. $X_t(p) \in \Sigma_1^-$).
- In the quadrant C^{+-}, we define U^{+-} (resp. V^{+-}) as being the set of points $p \in C^{+-}$ such that there is $t \in [-\delta, \delta]$ with $X_t(p) \in \Sigma_0^-$ (resp. $X_t(p) \in \Sigma_1^+$).

For simplifying notation, we omit the dependence on δ of the sets U^{++}, V^{++}, U^{-+}, etc.

6.5. **A family of segment diffeomorphisms.**

Lemma 6.4. *There is a smooth function $\theta\colon [0,1] \times (0,+\infty)^2 \to [0,1]$ such that, for any $a, b > 0$ the map $\theta_{a,b}\colon [0,1] \to [0,1]$, defined as $\theta_{a,b}(x) = \theta(x,a,b)$, is an increasing diffeomorphism satisfying the following properties:*

- *$\theta_{a,b}(x) = ax$ for small x*
- *$\theta_{a,b}(x) = 1 - b(1-x)$ for x close to 1*
- *$\theta_{1,1}$ is the identity map.*

Given any oriented segments I, J and $a, b > 0$ we denote by $\theta_{a,b,I,J}\colon I \to J$ the diffeomorphism obtained by the composition of $\Phi_J^{-1} \circ \theta_{a\frac{\ell(I)}{\ell(J)}, b\frac{\ell(I)}{\ell(J)}} \circ \Phi_I$, where $\Phi_I : I \to [0,1]$ and $\Phi_J\colon J \to [0,1]$ are the unique affine increasing diffeomorphisms. Notice that $\theta_{a,b,I,J}\colon I \to J$ is a smooth orientation preserving diffeomorphism such that the derivative at the origin of I is a and the derivative at the end point of I is b.

6.6. **diffeomorphisms in the quadrants.** We fix $\alpha > 1$ such that $\log \alpha < \frac{1}{10} \log 2$.

Definition 6.1. *Let $I = [a, b]$, $J = [c, d]$ be two segments of $\mathbb{R}$ of size strictly larger than 2δ. We denote by $\Psi^+_{I,J} : I \to J$ the diffeomorphism defined as follows:*

- *for $t \in [a, a + \frac{\delta}{\alpha}]$ one defines $\Psi^+_{I,J}(t) = c + \alpha(t - a)$.*
- *for $t \in [b - \delta, b]$ one defines $\Psi^+_{I,J}(t) = d - (\alpha^{-1}(b - t))$.*
- *Let denote $I' = [a + \frac{\delta}{\alpha}, b - \delta]$, $J' = [c + \delta, d - \frac{\delta}{\alpha}]$. For $t \in I'$ one defines $\Psi^+_{I,J}(t) = \theta_{\alpha, \frac{1}{\alpha}, I', J'}(t)$.*

We denote by $\Psi^-_{I,J} : I \to J$ the diffeomorphism defined as follows:

- *for $t \in [a, a + \delta]$ one defines $\Psi^-_{I,J}(t) = c + \alpha^{-1}(t - a)$.*
- *for $t \in [b - \frac{\delta}{\alpha}, b]$ one defines $\Psi^-_{I,J}(t) = d - (\alpha(b - t))$.*
- *Let denote $I" = [a + \delta, b - \frac{\delta}{\alpha}]$, $J" = [c + \frac{\delta}{\alpha}, d - \delta]$. For $t \in I"$ one defines $\Psi^-_{I,J}(t) = \theta_{\frac{1}{\alpha}, \alpha, I", J"}(t)$.*

See Figure 7.

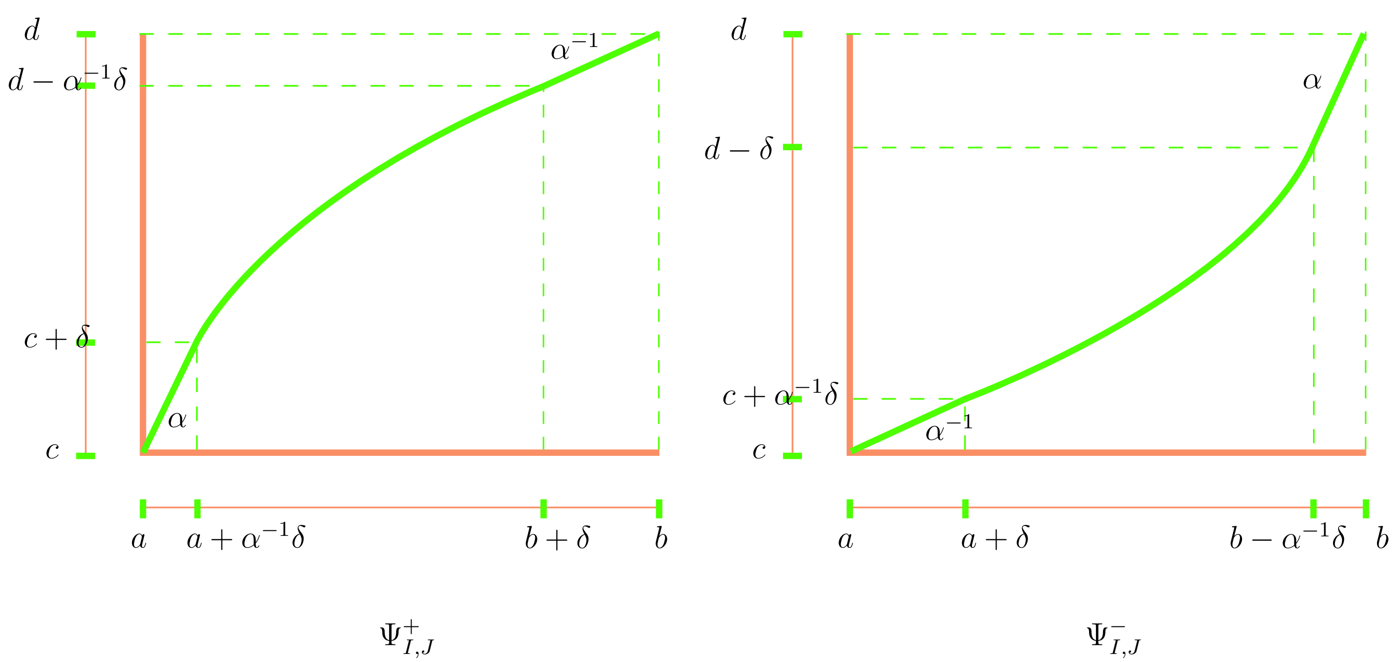

FIGURE 7. The maps $\Psi^+_{I,J}$ and $\Psi^-_{I,J}$, where $I = [a, b]$ and $J = [c, d]$.

Definition 6.2. *We define a diffeomorphism $f^{++} : C^{++} \to C^{++}$ as follows:*

- *for $p \in \Sigma^+_0 \cap C^{++}$ we state $f^{++}(p) = P^+_0(p)$*
- *for $q \in \Sigma^+_1 \cap C^{++}$ we state $f^{++}(p) = P^+_1(p)$*

- *consider* $p \in \Sigma_0^+ \cap C^{++}$ *and let* $q = P_{0,1}^{++}(p)$. *Let* $t_0 > 0$ *such that* $q = X_{t_0}(p)$, *and* $t_1 > 0$ *such that* $X_{t_1}(P_0^+(p)) = P_1^+(q)$. *One denote* $I = [0, t_0]$ *and* $J = [0, t_1]$. *One defines* f^{++} *on the orbit segment* $[p, q]^c$ *by* $f^{++}(X_t(p)) = X_s(P_0^+(p))$ *with* $s = \Psi_{I,J}^-(t)$.
- *consider* $p \in \Sigma_1^+ \cap C^{++}$ *and let* $q = P_{1,0}^{++}(p)$. *Let* $t_0 > 0$ *such that* $q = X_{t_0}(p)$, *and* $t_1 > 0$ *such that* $X_{t_1}(P_1^+(p)) = P_0^+(q)$. *One denote* $I = [0, t_0]$ *and* $J = [0, t_1]$. *One defines* f^{++} *on the orbit segment* $[p, q]^c$ *by* $f^{++}(X_t(p)) = X_s(P_1^+(p))$ *with* $s = \Psi_{I,J}^+(t)$.

Lemma 6.5. *The map* f^{++} *is well defined on* C^{++} *and it is a diffeomorphism of* C^{++}. *Furthermore* $f^{++}(U^{++})$ *is a subset of the interior of* U^{++} *(* U^{++} *is seen as a subset of* C^{++}*) and* $(f^{++})^{-1}(V^{++})$ *is a subset of the interior of* V^{++}.

Proof : The unique difficulty is on $\Sigma_0^+ \cap C^{++}$ and $\Sigma_1^+ \cap C^{++}$. However, by construction f^{++} sent the orbit segment of length 2δ centered at a point $p \in \Sigma_0^+ \cap C^{++}$ on the orbit segment for length $2\alpha^{-1}\delta$ centered at $P_0^{++}(p)$ as an homothety of ratio α^{-1}. As P_0^{++} is a diffeomorphism we get that f^{++} is a diffeomorphism in a neighborhood in C^{++} of $\Sigma_0^+ \cap C^{++}$. An analogous arguments holds in a neighborhood of $\Sigma_1^+ \cap C^{++}$. □

One defines analogously diffeomorphisms f^{+-},f^{-+},and f^{--} on C^{+-}, C^{-+} and C^{--}, respectively, such that the subset (of the quadrants) U^{+-}, U^{-+} and U^{--} are attracting regions and V^{+-}, V^{-+} and V^{--} are repelling regions.

Once again, we omit the dependence on δ of the definition of f^{++}, f^{+-}, f^{-+}, and f^{--}.

Notice that the diffeomorphisms f^{++}, f^{+-},f^{-+}, and f^{--} do not coincide on the intersection of the quadrants. All our construction consists in gluing this diffeomorphisms.

6.7. **In a neighborhood of the periodic orbit .** The aim of this section is to define a diffeomorphism of a neighborhood of the periodic orbit of X, and to glue it with the diffeomorphisms we have defined in the quadrants.

Notice that $\Sigma_0 \cap \{x = y = 0\} = \{(0, 0, 1/4), (0, 0, 3/4 = -1/4)\}$ and $\Sigma_1 \cap \{x = y = 0\} = \{(0, 0, 1/2), (0, 0, 0)\}$.

¿From now on, we asume that δ verifies that $\delta < \frac{1}{16}$.

One defines a Morse-Smale diffeomorphism $\Psi_0 \colon S^1 \to S^1$ in the following way:

- the restriction of Ψ_0 to the segment $I_0 = [0, \frac{1}{4}]$ is $\Psi^+_{[I_0,I_0]}$. By definition it is a linear expansion in $[0, \frac{\delta}{\alpha}]$ and it is a linear contraction in $[\frac{1}{4} - \delta, \frac{1}{4}]$. Furthermore, there are no fixed points in $(0, \frac{1}{4})$.
- the restriction of Ψ_0 to the segment $I_1 = [\frac{1}{4}, \frac{1}{2}]$ is $\Psi^-_{[I_1,I_1]}$. It is a linear contraction in $[\frac{1}{4}, \frac{1}{4} + \delta]$ and it is a linear expansion in $[\frac{1}{2} - \frac{\delta}{\alpha}, \frac{1}{2}]$. Furthermore, there are no fixed points in $(\frac{1}{4}, \frac{1}{2})$.
- the restriction of Ψ_0 to the segment $I_2 = [\frac{1}{2}, \frac{3}{4}]$ is $\Psi^+_{[I_2,I_2]}$. It is a linear expansion in $[\frac{1}{2}, \frac{1}{2} + \frac{\delta}{\alpha}]$ and it is a linear contraction in $[\frac{3}{4} - \delta, \frac{3}{4}]$. Furthermore, there are no fixed points in $(\frac{1}{2}, \frac{3}{4})$.
- the restriction of Ψ_0 to the segment $I_3 = [\frac{3}{4}, 1 = 0]$ is $\Psi^-_{[I_3,I_3]}$. It is a linear contraction in $[\frac{3}{4}, \frac{3}{4} + \delta]$ and a linear expansion in $[1 - \frac{\delta}{\alpha}, 1]$. Furthermore, there are no fixed points in $(\frac{3}{4}, 1)$.

The diffeomorphism Ψ_0 has precisely 4 fixed points: two sinks ($\frac{1}{4}$ and $\frac{3}{4}$) and two sources (0 and $\frac{1}{2}$). One denotes by f_0 the diffeomorphism of $\mathbb{R}^2 \times S^1$ defined as follows: for any $p = (x, y, z)$,

$$f_0(p) = X_{1+\Psi_0(z)-z}(p) = (2^{1+\Psi_0(z)-z}x, 2^{-(1+\Psi_0(z)-z)}y, \Psi_0(z)).$$

The diffeomorphism f_0 has precisely 4 fixed points: $(0, 0, 0), (0, 0, \frac{1}{4})(0, 0, \frac{1}{2})$ and $(0, 0, \frac{3}{4})$ which are saddle points. Notice that the closed $\{x = y = 0\}$ is a normally hyperbolic invariant curve for f_0.

6.8. **Gluing f_0 with f^{++},f^{+-}, f^{-+} and f^{--}.** On the quadrant C^{++} the surface Σ_0^+ contains the intersection of the horizontal disk $\{(x, y, z)|z = \frac{1}{4}, \sqrt{x^2 + y^2} < \frac{1}{4}\}$ with C^{++}, and Σ_1^+ contains the intersection of the horizontal disk $\{(x, y, z)|z = \frac{1}{2}, \sqrt{x^2 + y^2} < \frac{1}{4}\}$ with C^{++}(See Figure 5). As a consequence one gets that f^{++} coincides with f_0 in a neighborhood, in C^{++}, of the segment $\{(0, 0)\} \times [\frac{1}{4} - \delta, \frac{1}{2} + \frac{\delta}{\alpha}]$.

Definition 6.3. *We define $f_0^{++} : C^{++} \to C^{++}$ as follows:*
- *if $p \in \Sigma_0^+ \cup \Sigma_1^+$, then $f_0^{++}(p) = f^{++}(p)$*
- *if q belongs to a X-orbit segment $[p, P_{0,1}^{++}(p)]^c$, where $p \in \Sigma_0^+$, then $f_0^{++}(q) = f^{++}(q)$*
- *if q belongs to a X-orbit segment $[p, P_{1,0}^{++}(p)]^c$, where $p = (x, y, z) \in \Sigma_1^+$ with $\sqrt{x^2 + y^2} \geq \frac{1}{100}$ then $f_0^{++}(q) = f^{++}(q)$.*
- *if q belongs to a X-orbit segment $[p, P_{1,0}^{++}(p)]^c$, where $p = (x, y, z) \in \Sigma_1^+$ with $\sqrt{x^2 + y^2} \leq \frac{1}{100}$. Notice that $p = (x, y, \frac{1}{2}) \in \Sigma_1^+$ and $P_1^+(p) = (2x, \frac{1}{2}y, \frac{1}{2})$. Let $p' = P_{1,0}^{++}(p) \in \Sigma_0^+$, it holds that $p' = (x', y', \frac{1}{4})$ and $P_0^+(p') = (2x', \frac{1}{2}y', \frac{1}{4})$.*

  One deduces that the image of the segment $[p, p']^c$ by f_0 and by f^{++} is the segment $[P_1^+(p), P_0^+(p')]^c$. This allows us to consider the barycentral diffeomorphism, between f_0 and f^{++}:
  - *consider $t_0 > 0$ such that $f_0(q) = X_{t_0}(P_1^+(p))$*
  - *consider $t^{++} > 0$ such that $f^{++}(q) = X_{t^{++}}(P_1^+(p))$*
  - *let r denote $\sqrt{x^2 + y^2}$ (notice that r does not depend on $q \in [p, p']^c$)*
  - *let $t = \psi(100r)t_0 + (1 - \psi(100r))t^{++}$*

  Then we define $f_0^{++}(q) = X_t(P_1^+(p))$.

Lemma 6.6. *The map f_0^{++} is well defined and it is a diffeomorphism of the quadrant C^{++} which coincides with f_0 in the neighborhood of the circle $\{x = y = 0\}$ and coincides with f^{++} in $\sqrt{x^2 + y^2} \geq \frac{2}{100}$. Furthermore, f_0^{++} coincides with f^{++} on U^{++} (in particular, U^{++} is an attracting region for f_0^{++} i.e. $f_0^{++}(U^{++})$ is included in the interior of U^{++}). Finally f_0^{++} coincides with f^{++} on $(f^{++})^{-1}(V^{++})$ so that V^{++} is a repelling region.*

Proof : As we know that each point x in C^{++} belongs to a unique segment of the form $[p, p']^c$ (in the case $x \in \Sigma_0^+ \cup \Sigma_1^+$ then $x = p$) where p, p' are in $\Sigma_0^+ \cup \Sigma_1^+$ and $(p, p')^c \cap (\Sigma_0^+ \cup \Sigma_1^+) = \emptyset$, so the map f_0^{++} is well defined. Let

$$M^{++} = \bigcup_{\{p=(x,y,z)|z=\frac{1}{2},\sqrt{x^2+y^2}<\frac{1}{100}\}} [p, p']^c$$

where $p' = P^{++}_{1,0}(p) \in \Sigma^+_0$. Since $f^{++}_0 = f^{++}$ in $C^{++}\backslash M^{++}$ it follows that the restriction of f^{++}_0 to $C^{++}\backslash M^{++}$ is a diffeomorphism. Note that if $q = (x', y', z')$ verifies that $\sqrt{x'^2 + y'^2} \geq \frac{2}{100}$ then $q \in C^{++}\backslash M^{++}$, therefore $f^{++}_0(q) = f^{++}(q)$.

The projection of M^{++} onto $\{(x, y, z)|z = \frac{1}{2}, \sqrt{x^2 + y^2} < \frac{1}{100}\}$ along the orbits of the flow, the map $(x, y, z) \to \sqrt{x^2 + y^2}$, the maps ψ f_0, f^{++} and P^+_1 and the flow X are differentiable, therefore the map f^{++}_0 is a diffeomorphism restricted to M^{++}. Besides, in the case that $r < \frac{1}{100}$ but close enough to $\frac{1}{100}$ it holds that

$$t = \psi(100r)t_0 + (1 - \psi(100r))t^{++} = t^{++},$$

hence $f^{++}_0 = f^{++}$.

We have seen that f^{++} coincides with f_0 in a neighborhood of the segment $\{(0, 0)\} \times [\frac{1}{4} - \delta, \frac{1}{2} + \frac{\delta}{\alpha}]$ in C^{++}. Then in a neighborhood of $\{(0, 0)\} \times [\frac{1}{4} - \delta, \frac{1}{4} + \delta]$ in C^{++} $f^{++}_0 = f^{++} = f_0$ is a contraction of factor α^{-1}. It follows that in U^{++}, $f^{++}_0 = f^{++}$ is a contraction of factor α^{-1} and U^{++} is an attracting region of f^{++}_0.

Analogously, in a neighborhood of the segment $\{(0, 0)\} \times [\frac{1}{2} - \frac{\delta}{\alpha}, \frac{1}{2} + \frac{\delta}{\alpha}]$ in C^{++}, $f^{++}_0 = f^{++} = f_0$ is an expansion of factor α. Then V^{++} is a repelling region for f^{++}_0.

□

One defines in an analogous way the diffeomorphisms f^{+-}_0, f^{-+}_0, and f^{--}_0 verifying that they coincide with f_0 in a neighborhood of the circle $\{x = y = 0\}$, they induce linear contraction of ration $\frac{1}{\alpha}$ in the orbit segment crossing U^{+-},U^{-+} and U^{--}, respectively, and they induce linear dilations of ration α in the orbit segment crossing $(f^{+-}_0)^{-1}(V^{+-})$, $(f^{-+}_0)^{-1}(V^{-+})$, and $(f^{--}_0)^{-1}(V^{--})$ respectively.

7. On the sides of the quadrants

The diffeomorphisms f^{++}_0 and f^{+-}_0 are defined on C^{++} and C^{+-} whose intersection is the half plane $\{y = 0, x \geq 0\}$. Notice that, in $C^{++} \cap C^{+-}$, these diffeomorphisms satisfy the following properties:

- $f^{++}_0(x, y, z) = f^{+-}_0(x, y, z) = f_0(x, y, z)$ in the neighborhood of the circle $\{x = y = 0\}$. We fix a constant δ_2 such that all the diffeomorphisms $f^{\pm\pm}_0$ coincide with f_0 on $\{\sqrt{x^2 + y^2} \leq \delta_2\}$.
- $f^{++}_0(x, y, z) = f^{+-}_0(x, y, z)$ for every point (x, y, z) with $\sqrt{x^2 + y^2} = x \geq 2$. Furthermore f^{++}_0 and f^{+-}_0 define a diffeomorphism in the neighborhood of the half affine plane $\{y = 0, x \geq 2\}$
- $f^{++}_0 = f^{+-}_0$ on the union of the orbits segments of lengths $2\frac{\delta}{\alpha}$ centered at the points of $\Sigma^+_1 \cap \{y = 0, x \geq 0\}$; there, they map in a affine way the orbit segment of length $2\frac{\delta}{\alpha}$ centered at $p \in \Sigma^+_1$ onto the orbit segment of length 2δ centered at $P^+_1(p)$.

  Notice that $\Sigma^+_1 \cap \{y = 0, x \geq 0\}$ is the half horizontal straight line $\{y = 0, x \geq 0, z = \frac{1}{2}\}$.

In other words, f^{++}_0 and f^{+-}_0 coincide in the points of $C^{++} \cap C^{+-}$ which are out of the rectangle $\{x \in [\delta_2, 2], y = 0, z \in [\frac{-1}{2} + \frac{\delta}{\alpha}, \frac{1}{2} - \frac{\delta}{\alpha}]\}$.

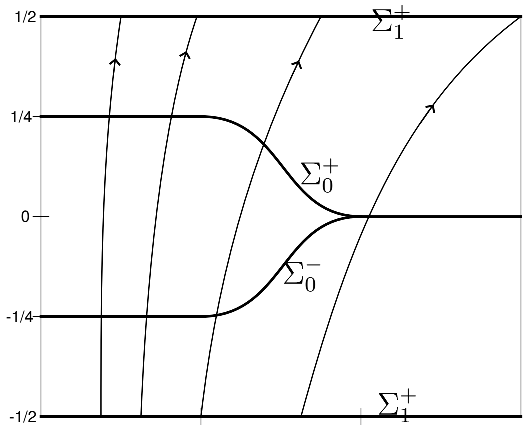

FIGURE 8. The surfaces Σ_0^-, Σ_0^+ and Σ_1 in $C^{++} \cap C^{+-}$.

The aim of the section is to define a diffeomorphism $f^{+\pm}$ in a neighborhood of this rectangle, and to glue it with f_0^{++} and f_0^{+-}, preserving the attracting regions U^{++} and U^{+-}.

The diffeomorphism $f^{+,\pm}$ needs to have some compatibility with f^{++} in C^{++} and with f^{+-} in C^{+-}. For that, we will build it by using sections of X which extend the cross sections Σ_0^+ and Σ_0^-. Analogously we will extend Σ_1^+ and Σ_1^- for building the diffeomorphisms in the other quadrants.

This extension of the sections will also be used for building the attracting and repelling regions U_ε and V_ε which are small extensions of the union of the attracting and repelling regions $U^{\pm,\pm}$ and $V^{\pm,\pm}$ in the quadrants.

7.1. **Extending the cross section $\Sigma_0^\pm$ and $\Sigma_1^\pm$.** On the cylinder $\{x \geq 0, y = 0\}$ of $\mathbb{R}^2 \times S^1$ (corresponding to an unstable separatrix of the periodic orbit of X), the surfaces Σ_0^+ and Σ_0^- induce each one a proper embedding of $[0, +\infty)$ and these two curves coincide for $x \geq \frac{2}{3}$. When $x \geq \frac{2}{3}$, they are the half straight line $[\frac{2}{3}, +\infty) \times \{0\} \times \{0\}$. These curves are contained in the respective boundaries of Σ_0^+ and Σ_0^-, which are contained the half spaces $y \geq 0$ and $y \leq 0$, respectively. On the other hand Σ_1^+ induces the half straight line $\{x \geq 0, y = 0, z = \frac{1}{2})\}$ (See Figure 8).

On this plane, the curves induced by Σ_0^+ and Σ_1^+ have been used for defining f_0^{++} and the curves induced by Σ_0^- and Σ_1^+ have been used for defining f_0^{-+}. We want now to define a map on a neighborhood of this separatrix that we can glue with f_0^{++} and f_0^{-+} in the corresponding quadrants. For that, we need to extend a little bit Σ_0^+ in the quadrant C^{-+} and Σ_0^- in the quadrant C^{++}.

The aim of this section is to build these extensions. In order to simplify the construction of the diffeomorphisms, we would like a good relative position of the extensions of Σ_0^+ and Σ_0^- with respect to Σ_0^- and Σ_0^+ respectively. Recall that Σ_0^+ and Σ_0^- are obtained from S_0^+ and S_0^- by pushing the points along the orbits of X in the direction of the planes $z = \frac{1}{4}$ and $z = -\frac{1}{4}$, respectively. For this reason, we will first extend the surface S_0^+ and S_0^-, and then we will push this surfaces along the orbit of X following the same rules as we have already done for S_0^+ and S_0^-.

7.1.1. *An extension of* Σ_0^-. A natural extension of S_0^- is the half helicoid S_0. However one cannot project S_0 continuously along the orbit of X on the plane $z = -\frac{1}{4}$. For this reason, we remove from S_0 its intersection with the cylinder $\{x = 0, y \geq 0, z \in \mathbb{R}/\mathbb{Z}\}$ (which is a stable separatrix of the periodic orbit).

We consider the surface $\tilde{S}_0^- = S_0 \setminus \{x = 0, y \geq 0\}$, obtained by removing from the half helicoid S_0 the union of the half straight line $\{x = 0, y \geq 0, z = \frac{1}{4} = \frac{-3}{4}\}$ with the circle $\{(0,0)\} \times \mathbb{R}/\mathbb{Z}$. In other words,

$$\tilde{S}_0^- = \{(x,y,z) \in S_0, (x,y) \neq (0,0), z \neq 1/4\}$$

.

It is a global cross section of X on the invariant open set $\mathbb{R}^2 \times S^1 \setminus \{x = 0, y \geq 0\}$.

Notice that, for $(x,y,z) \in \tilde{S}_0^- \subset \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ one can consider that $z \in (-\frac{3}{4}, \frac{1}{4})$ and this choice of the representant of z is continuous on $\tilde{S}_0^-$. Furthermore $X_{-z-\frac{1}{4}}(x,y,z)$ belongs to the plane $z = -\frac{1}{4}$ and the map $(x,y,z) \mapsto \pi_0^-(x,y,z) = X_{-z-\frac{1}{4}}(x,y,z)$ is a smooth projection of $\tilde{S}_0^-$ on the plane $\{z = -\frac{1}{4}\}$, which is a diffeomorphism on $\{z = -\frac{1}{4}\} \setminus \{x = 0, y \geq 0\}$.

We will now push every point $(x,y,z) \in \tilde{S}_0^-$ along the orbit of X in direction of $\pi_0^-(x,y,z)$, replacing (x,y,z) by $X_t(x,y,z)$ where $t = \psi(\sqrt{x^2+y^2})$ (ψ is the map built in section 6.5). The surface we get contains the complement, in the disk $\{z = -\frac{1}{4}, \sqrt{x^2+y^2} \leq \frac{1}{10}\}$ of the stable separatrix $\{x = 0, y \geq 0\}$. For this reason we add the whole disk to the projection.

We define

$$\begin{aligned} \tilde{\Sigma}_0^- &= \Big\{X_t(x,y,z), \text{ where } (x,y,z) \in \tilde{S}_0^-, \text{ and } t = \psi(\sqrt{x^2+y^2}) \cdot (-z - \tfrac{1}{4})\Big\} \\ &\cup \Big\{z = \tfrac{-1}{4} \text{ and } \sqrt{x^2+y^2} < \tfrac{1}{10}\Big\} \end{aligned}$$

We denote $A = \{X_t(x,y,z), \text{ where } (x,y,z) \in \tilde{S}_0^-, \text{ and } t = \psi(\sqrt{x^2+y^2}) \cdot (-z - \frac{1}{4})\}$ and $B = \{z = \frac{-1}{4} \text{ and } \sqrt{x^2+y^2} < \frac{1}{10}\}$.

Lemma 7.1. *The set* $\tilde{\Sigma}_0^-$ *is a smooth surface.*

Proof : Since S_0 with $(x,y) \neq (0,0)$ and $z \neq \frac{1}{4}$ is smooth, ψ, $\sqrt{x^2+y^2}$ and X are differentiable, it follows that A is a smooth surface. Moreover, it is diffeomorphic to $\{z = -\frac{1}{4}\} \setminus \{x = 0, y \geq 0\}$.

Notice that $\{z = \frac{-1}{4} \text{ and } \sqrt{x^2+y^2} < \frac{1}{10}\} \setminus \{x = 0, y \geq 0, z = \frac{-1}{4}\}$ is contained in A . Then $A \cup B = A \cup \{x = 0, 0 \leq y < \frac{1}{10}, z = \frac{-1}{4}\}$

Let $r \in \{x = 0, 0 \leq y < \frac{1}{10}, z = \frac{-1}{4}\}$. Any neighborhood of r in $A \cup B$ is a neighborhood of r in B, therefore one deduces that $\tilde{\Sigma}_0^-$ is an open surface. □

For every $\varepsilon \in (0, \frac{1}{10})$ one defines $\Sigma_{0,\varepsilon}^-$ as being the intersection of $\tilde{\Sigma}_0^-$ with the affine half space $y \leq \varepsilon$.

Lemma 7.2. *For* ε *small enough,* $\Sigma_{0,\varepsilon}^-$ *is a smooth surface with boundary, diffeomorphic to a half plane. Moreover,* $\Sigma_{0,\varepsilon}^- = \Sigma_0^- \cup C$ *where* C *is a strip.*

Proof : We have to show that $\Sigma^-_{0,\varepsilon}$ is transversal to the plane $y=\varepsilon$. Notice that for a big radius, for example $\sqrt{x^2+y^2}>2$ $\Sigma^-_{0,\varepsilon}=S_0$ in $\{\sqrt{x^2+y^2}>2\}\backslash\{x=0,y\geq 0\}$, and since $y=\varepsilon$ is transversal to the helicoid S_0 then $\Sigma^-_{0,\varepsilon}$ is transversal to $y=\varepsilon$ in $\{\sqrt{x^2+y^2}>2\}$.

Notice that when $\varepsilon=0$ it holds that $\Sigma^-_{0,\varepsilon}=\Sigma^-_0$, and it is transversal to $y=0$. Then in $\{\sqrt{x^2+y^2}\leq 2\}$, $\Sigma^-_{0,\varepsilon}$ is transversal to $y=\varepsilon$ if ε is small enough. So, we have proven that for ε small enough, $\Sigma^-_{0,\varepsilon}$ is a smooth surface with boundary.

We have that $\Sigma^-_{0,\varepsilon}=\Sigma^-_0\cup C$ where $C=\{(x',y',z')\in\tilde{\Sigma}^-_{0,\varepsilon}, 0\leq y'\leq\epsilon\}$. Recall that $\Sigma^-_{0,\varepsilon}=S_0$ in $\{\sqrt{x^2+y^2}\geq 2\}\backslash\{x=0,y\geq 0\}$, so in the case that $\sqrt{x^2+y^2}\geq 2$, the intersection between the helicoid surface S_0 and $\{(x,y,z)$ such that $0\leq y\leq\varepsilon\}$ is the union of two disjoint strips, C_1, C_2. Besides, we have proved that $\Sigma^-_{0,\varepsilon}$ is transversal to $y=\varepsilon$ if ε is small enough. Let C^u be the curve defined as $C^u=\Sigma^-_{0,\varepsilon}\cap\{\sqrt{x^2+y^2}\leq 2\}\cap\{y=u\}$. Then $\cup_{u\in[0,\varepsilon]}C^u$ is a strip that intersects C_1 and C_2. It follows that $C=\cup_{u\in[0,\varepsilon]}C^u\cup C_1\cup C_2$ is a strip.

Since Σ^-_0 is a half plane, it follows that $\Sigma^-_{0,\varepsilon}$ is diffeomorphic to a half plane. $\square$

7.1.2. *Extension of* Σ^+_0. One defines in a analogous way :

$$\tilde{S}^+_0=\{(x,y,z)\in S_0, (x,y)\neq(0,0), z\in(-1/4,3/4)\}$$

Then we define $\tilde{\Sigma}^+_0$ by pushing S^+_0 along the flow in the direction of the plane $z=\frac{1}{4}$, and by completing the resulting surface by adding the disc $\{z=\frac{1}{4}$ and $\sqrt{x^2+y^2}<\frac{1}{10}\}$. That is:

$$\begin{aligned}\tilde{\Sigma}^+_0 &= \{X_t(x,y,z),\ \text{where } (x,y,z)\in\tilde{S}^+_0, t=\psi(\sqrt{x^2+y^2})\cdot(-z+\tfrac{1}{4})\}\\ &\quad\cup\{z=\tfrac{1}{4} \text{ and } \sqrt{x^2+y^2}<\tfrac{1}{10}\}\end{aligned}$$

Notice that $\{z=\frac{1}{4}$ and $\sqrt{x^2+y^2}<\frac{1}{10}\}\setminus\{x=0,y\leq 0,z=\frac{1}{4}\}$ is contained in $\left\{X_t(x,y,z),\ \text{where } (x,y,z)\in\tilde{S}^+_0, t=\psi(\sqrt{x^2+y^2})\cdot(-z+\frac{1}{4})\right\}$. As for $\tilde{\Sigma}^-_0$, one deduces that $\tilde{\Sigma}^+_0$ is an open surface.

For every $\varepsilon\in(0,\frac{1}{10})$, one defines $\Sigma^+_{0,\varepsilon}$ as the intersection of $\tilde{\Sigma}^+_0$ with the affine half space $y\geq-\varepsilon$. As for $\Sigma^-_{0,\varepsilon}$, $\Sigma^+_{0,\varepsilon}$ is a surface with boundary diffeomorphic to the half plane, for small $\varepsilon>0$.

7.1.3. *Relative positions of* $\Sigma^+_{0,\varepsilon}$ *and* $\Sigma^+_{0,\varepsilon}$.

Lemma 7.3. *There is ε_0 such that for every $\varepsilon\in(0,\varepsilon_0)$ the intersection $\Sigma^+_{0,\varepsilon}\cap\Sigma^-_{0,\varepsilon}$ is equal to $\{(x,y,z)\in S_0, \sqrt{x^2+y^2}\geq\frac{2}{3}, |y|\leq\varepsilon\}$*

Proof :
Let (x,y,z) be a point in $\Sigma^+_{0,\varepsilon}\cap\Sigma^-_{0,\varepsilon}$, then $|y|\leq\varepsilon$.

Claim 1. *Any point $(x,y,z)\in\Sigma^+_{0,\varepsilon}\cap\Sigma^-_{0,\varepsilon}$ verifies that $z\neq\frac{1}{4}$ and $z\neq-\frac{1}{4}$.*

Proof : It is a consequence of the the fact that if $z=-\frac{1}{4}$ then $(x,y,z)\notin\Sigma^+_{0,\varepsilon}$ and if $z=\frac{1}{4}$ then $(x,y,z)\notin\Sigma^-_{0,\varepsilon}$. □

Hence any $(x,y,z)\in\Sigma^+_{0,\varepsilon}\cap\Sigma^-_{0,\varepsilon}$ is in the set A defined before lemma 7.1, that is, there exists $(x_1,y_1,z_1)\in\tilde{S}^-_0\subset S_0$ in the X orbit of (x,y,z). Analogously, there exists $(x_2,y_2,z_2)\in\tilde{S}^+_0\subset S_0$ in the X orbit of (x,y,z).

We know that $(x,y,z)=X_{t_1}(x_1,y_1,z_1)$ where $(x_1,y_1,z_1)\in\tilde{S}^-_0$, and $t_1=\psi(\sqrt{x_1^2+y_1^2})\cdot(-z_1-\frac{1}{4})$, and $(x,y,z)=X_{t_2}(x_2,y_2,z_2)$ where $(x_2,y_2,z_2)\in\tilde{S}^+_0$, and $t_2=\psi(\sqrt{x_2^2+y_2^2})\cdot(-z_2+\frac{1}{4})$.

Claim 2. $(x,y,z)\in\Sigma^+_{0,\varepsilon}\cap\Sigma^-_{0,\varepsilon}$ *verifies that* $\sqrt{x_1^2+y_1^2}>\frac{1}{10}$ *and* $\sqrt{x_2^2+y_2^2}>\frac{1}{10}$.

Proof : Let $(x_1,y_1,z_1)\in\tilde{S}^-_0$ such that $\sqrt{x_1^2+y_1^2}\leq\frac{1}{10}$ then the point of its orbit $(x,y,z)\in\Sigma^-_{0,\varepsilon}$ verifies that $z=-\frac{1}{4}$, hence $(x,y,z)\notin\Sigma^+_{0,\varepsilon}$. Analogously, the point $(x_2,y_2,z_2)\in\tilde{S}^+_0$ verifies that $\sqrt{x_2^2+y_2^2}>\frac{1}{10}$. □

Claim 3. $(x,y,z)\in\Sigma^+_{0,\varepsilon}\cap\Sigma^-_{0,\varepsilon}$ *verifies that* $\sqrt{x_1^2+y_1^2}\geq\frac{2}{3}$ *or* $\sqrt{x_2^2+y_2^2}\geq\frac{2}{3}$.

Proof : By lemma 4.1 we know that any segment of orbit with both extreme points in S_0 has length greater than $\frac{1}{2}$. Then by compactness there exists a small $l>0$ such that if $(x_1,y_1,z_1)\in S_0$ and $\frac{1}{10}\leq\sqrt{x_1^2+y_1^2}\leq\frac{2}{3}$ any segment of orbit by (x_1,y_1,z_1) that intersects S_0 in another point has length greater than $\frac{1}{2}+l$.

Let us assume that $(x_1,y_1,z_1)\in\tilde{S}^-_0$ verifies that $\frac{1}{10}\leq\sqrt{x_1^2+y_1^2}<\frac{2}{3}$ and $(x_2,y_2,z_2)\in\tilde{S}^+_0$ verifies that $\frac{1}{10}\leq\sqrt{x_2^2+y_2^2}<\frac{2}{3}$. It holds that $(x_1,y_1,z_1)\neq(x_2,y_2,z_2)$: suppose that $(x_1,y_1,z_1)=(x_2,y_2,z_2)$, then it can be proven that $t_1=\psi(\sqrt{x_1^2+y_1^2})\cdot(-z_1-\frac{1}{4})$, and $t_2=\psi(\sqrt{x_1^2+y_1^2})\cdot(-z_2+\frac{1}{4})$ have different sign then $X_{t_1}(x_1,y_1,z_1)\neq X_{t_2}(x_2,y_2,z_2)$.

Recall that $(x,y,z)\in\Sigma^+_{0,\varepsilon}\cap\Sigma^-_{0,\varepsilon}$.

- In the case $x\geq 0$ then $x_1\geq 0$ and since $(x_1,y_1,z_1)\in\tilde{S}^-_0$ then $z_1\in[-\frac{1}{4},\frac{1}{4})$. Since we push $\tilde{S}^-_0$ along the orbit of X in the direction of the plane $z=-\frac{1}{4}$, it follows that $(x,y,z)\in\Sigma^-_{0,\varepsilon}$ verifies that $z\in[-\frac{1}{4},\frac{1}{4})$. In the same way, $x_2\geq 0$ and since $(x_2,y_2,z_2)\in\tilde{S}^+_0$ then $z_2\in(-\frac{1}{4},\frac{1}{4}]$. Since we push $\tilde{S}^+_0$ along the orbit of X in the direction of the plane $z=\frac{1}{4}$, it follows that $(x,y,z)\in\Sigma^+_{0,\varepsilon}$ verifies that $z\in(-\frac{1}{4},\frac{1}{4}]$.

  Since in Σ^-_0 we have that $z\in[-\frac{1}{2},0]$ and in Σ^+_0 we have that $z\in[0,\frac{1}{2}]$ we can choose ε_0 such that for $\varepsilon<\varepsilon_0$ in $\Sigma^-_{0,\varepsilon}$ we have that $z\in[-\frac{1}{2}-\frac{l}{2},0+\frac{l}{2}]$ and in $\Sigma^+_{0,\varepsilon}$ we have that $z\in[0-\frac{l}{2},\frac{1}{2}+\frac{l}{2}]$, hence the point (x,y,z) verifies that $z\in[-\frac{1}{2}-\frac{l}{2},0+\frac{l}{2}]\cap[0-\frac{l}{2},\frac{1}{2}+\frac{l}{2}]$.

  It follows that $(x,y,z)\in\Sigma^+_{0,\varepsilon}\cap\Sigma^-_{0,\varepsilon}$ with $x\geq 0$ verifies that $z\in(-\frac{1}{4},\frac{1}{4})\cap[-\frac{1}{2}-\frac{l}{2},0+\frac{l}{2}]\cap[0-\frac{l}{2},\frac{1}{2}+\frac{l}{2}]=[-\frac{l}{2},\frac{l}{2}]$ if l is small enough. It follows that the segment of the orbit between (x,y,z) and (x_1,y_1,z_1) has length less or equal than $\frac{1}{4}+\frac{l}{2}$ and segment of the orbit between (x,y,z) and (x_2,y_2,z_2) has length less or equal than $\frac{1}{4}+\frac{l}{2}$. Then the segment of X orbit between (x_1,y_1,z_1) and (x_2,y_2,z_2) has length less or equal than $\frac{1}{2}+l$. This is a contradiction.

- In the case $x \leq 0$ then $x_1 \leq 0$ and since $(x_1, y_1, z_1) \in \tilde{S}_0^-$ then $z_1 \in (-\frac{3}{4}, -\frac{1}{4}]$, since we push $\tilde{S}_0^-$ along the orbit of X in the direction of the plane $z = -\frac{1}{4}$, it follows that $(x, y, z) \in \Sigma_{0,\varepsilon}^-$ verifies that $z \in (-\frac{3}{4}, -\frac{1}{4}]$. Since $x_2 \leq 0$, $(x_2, y_2, z_2) \in \tilde{S}_0^+$ then $z_2 \in [-\frac{3}{4}, -\frac{1}{4}) = [\frac{1}{4}, \frac{3}{4})$ and it follows that $(x, y, z) \in \Sigma_{0,\varepsilon}^+$ verifies that $z \in [-\frac{3}{4}, -\frac{1}{4})$. As before $(x, y, z) \in \Sigma_{0,\varepsilon}^+ \cap \Sigma_{0,\varepsilon}^-$ with $x \leq 0$ verifies that $z \in (-\frac{3}{4}, -\frac{1}{4}) \cap [-\frac{1}{2} - \frac{l}{2}, 0 + \frac{l}{2}] \cap [0 - \frac{l}{2}, \frac{1}{2} + \frac{l}{2}] = [-\frac{1}{2} - \frac{l}{2}, -\frac{1}{2} + \frac{l}{2}]$ if l is small enough.
  It follows that the segment of orbit between (x_1, y_1, z_1) and (x_2, y_2, z_2) has length less or equal than $\frac{1}{2} + l$, which is a contradiction.

So, we have proved that either the point (x_1, y_1, z_1) verifies that $\sqrt{x_1^2 + y_1^2} \geq \frac{2}{3}$ or the point (x_2, y_2, z_2) verifies that $\sqrt{x_2^2 + y_2^2} \geq \frac{2}{3}$. □

In the set of points $\{(x, y, z) | \sqrt{x^2 + y^2} \geq \frac{2}{3}\}$ we have that $\psi \equiv 0$ so either $t_1 = 0$ or $t_2 = 0$.

It follows that $\Sigma_{0,\varepsilon}^- \cap \Sigma_{0,\varepsilon}^+ = \{S_0, \sqrt{x^2 + y^2} \geq \frac{2}{3}, |y| < \epsilon\}$ and the lemma is proved. □

We denote $\Sigma_{0,\varepsilon} = \Sigma_{0,\varepsilon}^+ \cup \Sigma_{0,\varepsilon}^-$. As a consequence of lemma 7.3 one gets

Corollary 7.1. *For $\varepsilon > 0$-small enough it holds that $\Sigma_{0,\varepsilon}$ is a branched surface with boundary diffeomorphic to the quotient of two half planes $\{(x, y, -1), y \leq 1\}$ and $\{(x, y, 1), y \geq -1\}$ by the equivalence relation identifying the points $(x, y, -1)$ and $(x, y, 1)$ when $|x| \geq 1$.*

7.1.4. *Extension of Σ_1^+ and Σ_1^-.* We define analogously $\tilde{\Sigma}_1^-$ and $\tilde{\Sigma}_1^+$ by pushing the surfaces

$$\tilde{S}_1^- = \{(x, y, z) \in S_1, (x, y) \neq (0, 0), z \in (-\frac{1}{2}, \frac{1}{2})\}$$

and

$$\tilde{S}_1^- = \{(x, y, z) \in S_1, (x, y) \neq (0, 0), z \in (0, 1)\}$$

along the flow. Let us just give the formula for $\tilde{\Sigma}_1^-$ and $\tilde{\Sigma}_1^+$:

$$\begin{aligned} \tilde{\Sigma}_1^- &= \{X_t(x, y, z), \text{ where } (x, y, z) \in \tilde{S}_1^-, t = \psi(\sqrt{x^2 + y^2}) \cdot (-z)\} \\ & \quad \cup \{z = 0 \text{ and } \sqrt{x^2 + y^2} < \tfrac{1}{10}\} \\ \tilde{\Sigma}_1^+ &= \{X_t(x, y, z), \text{ where } (x, y, z) \in \tilde{S}_1^+, t = \psi(\sqrt{x^2 + y^2}) \cdot (-z + \tfrac{1}{2})\} \\ & \quad \cup \{z = \tfrac{1}{2} \text{ and } \sqrt{x^2 + y^2} < \tfrac{1}{10}\} \end{aligned}$$

Analogously we define

$$\Sigma_{1,\varepsilon}^+ = \tilde{\Sigma}_1^+ \cap \{x \geq -\varepsilon\}, \quad \Sigma_{1,\varepsilon}^- = \tilde{\Sigma}_1^- \cap \{x \leq \varepsilon\}, \text{ and } \Sigma_{1,\varepsilon} = \Sigma_{1,\varepsilon}^+ \cup \Sigma_{1,\varepsilon}^-.$$

7.1.5. *Return maps on $\Sigma_{0,\varepsilon}^+$ and $\Sigma_{0,\varepsilon}^-$.*

Lemma 7.4. *For every $\varepsilon > 0$ small enough, the first return map $P_{0,\varepsilon}^+$ of X is well defined on $\Sigma_{0,\varepsilon}^+$ and it is a diffeomorphism on its image. Moreover, there is $\varepsilon' \in (0, \varepsilon)$ such that $P_{0,\varepsilon}^+(\Sigma_{0,\varepsilon}^+) \subset \Sigma_{0,\varepsilon'}^+$.*

Furthermore, the return times on $\Sigma_{0,\varepsilon}^+$ are strictly larger than $\frac{4}{10}$.

Proof : Look at the universal cover. The half space $y \geq -\varepsilon$ is positively invariant by X, and the connected components of the lifts of $\Sigma^+_{0,\varepsilon}$ disconnect that half space. Furthermore, the coordinate z is bounded on each of the connected component of the lift of $\Sigma^+_{0,\varepsilon}$ but this coordinate tends to infinity on the positive orbit of X. One deduces that the positive orbit of every point $(x, y, z) \in \mathbb{R}^2 \times S^1$ with $y \geq -\varepsilon$ cuts infinitely many times $\Sigma^+_{0,\varepsilon}$ and the first return of the flow of X on $\Sigma^+_{0,\varepsilon}$ is well defined. Furthermore, the coordinates y is uniformly contracted by the (positive) flow of X. It follows that the image of the first return map is disjoint from the boundary of $\Sigma^+_{0,\varepsilon}$. As a consequence the return time is everywhere continuous (hence differentiable) and $P^+_{0,\varepsilon}$ is a diffeomorphism on its image. Furthermore, its image is contained is some $\Sigma^+_{0,\varepsilon'}$ for some $\varepsilon' \in (0, \varepsilon)$.

Since in Σ^+_0 we have that $z \in [0, \frac{1}{2}]$ we can choose ε_0 small enough in such a way that if $\varepsilon < \varepsilon_0$ we have that $z \in [0 - \frac{1}{20}, \frac{1}{2} + \frac{1}{20}]$ in $\Sigma^+_{0,\varepsilon}$, therefore the return times on $\Sigma^+_{0,\varepsilon}$ is strictly larger than $\frac{4}{10}$.

□

In the same way, one defines the first return map $P^-_{0,\varepsilon} \colon (\Sigma^-_{0,\varepsilon}) \to \Sigma^-_{0,\varepsilon'}$. The return times on $\Sigma^-_{0,\varepsilon}$ are strictly larger than $\frac{4}{10}$ and one verifies:

Lemma 7.5. *For $\varepsilon > 0$ small enough, if $(x, y, z) \in \Sigma^+_{0,\varepsilon} \cap \Sigma^-_{0,\varepsilon}$ then $P^+_{0,\varepsilon}(x, y, z) = P^-_{0,\varepsilon}(x, y, z)$*

Proof : Just notice that, for small ε if $x \in \Sigma^+_{0,\varepsilon} \cap \Sigma^-_{0,\varepsilon}$ then the absolute value of x is either strictly larger than $\frac{2}{3}$ or it is very close to $\frac{2}{3}$. As X expands uniformly the x coordinates and since the time returns on $\Sigma^+_{0,\varepsilon}$ and on $\Sigma^-_{0,\varepsilon}$ are lower bounded by $\frac{4}{10}$, one deduces that $P^+_{0,\varepsilon}(x, y, z)$ and $P^-_{0,\varepsilon}(x, y, z)$ have their x coordinates larger than $\frac{2}{3}$, hence these two points belong to the intersections $\Sigma^+_{0,\varepsilon} \cap \Sigma^-_{0,\varepsilon}$. As they are first return maps one deduces that these two points coincide. □

Previous lemma allows us to extend $P_0 \colon \Sigma_0 \to \Sigma_0$ to a smooth map $P_{0,\varepsilon} \colon \Sigma_{0,\varepsilon} \to \Sigma_{0,\varepsilon'}$, which coincides with $P^{\pm}_{0,\varepsilon}$ on $\Sigma^{\pm}_{0,\varepsilon}$.

Remark 8. *Be careful that $P_{0,\varepsilon}$ is not the first return map on $\Sigma_{0,\varepsilon}$: the orbit of a point x in $\Sigma^-_{0,\varepsilon}$ can intersect $\Sigma^+_{0,\varepsilon}$ before its first return on $\Sigma^-_{0,\varepsilon}$. For such a point $P_{0,\varepsilon}(x)$ is the second return on $\Sigma_{0,\varepsilon}$.*

Remark 9. *Notice that for every ε small, if x in $\Sigma_{0,\varepsilon}$, $P_{0,\varepsilon}(x) = X_t(x)$ with $t \geq \frac{4}{10}$.*

7.1.6. *Inverse return maps on $\Sigma^+_{1,\varepsilon}$ and $\Sigma^-_{1,\varepsilon}$.*

One would like to extend in the same way the first return maps defined on Σ^+_1 and Σ^-_1 to $\Sigma^+_{1,\varepsilon}$ and $\Sigma^-_{1,\varepsilon}$, respectively. However, the half space $x \geq -\varepsilon$ and $x \leq \varepsilon$ are not positively invariant by the flow of X. These two half space are invariant by the negative times of the flow of X. For this reason, one cannot extend the first return maps P^+_1 and P^-_1 on $\Sigma^+_{1,\varepsilon}$ and $\Sigma^-_{1,\varepsilon}$, respectively, but one can extend their inverses $(P^+_1)^{-1}$ and $(P^-_1)^{-1}$. Exactly as above, the extensions $(P^+_{1,\varepsilon})^{-1}$ and $(P^-_{1,\varepsilon})^{-1}$ coincide on the intersection $\Sigma^+_{1,\varepsilon} \cap \Sigma^-_{1,\varepsilon}$ allowing to define a smooth map $P^{-1}_{1,\varepsilon} \colon \Sigma_{1,\varepsilon} \to \Sigma_{1,\varepsilon'}$, for some $\varepsilon' \in (0, \varepsilon)$. As in the map $P_{0,\varepsilon}$ we have that if x in $\Sigma_{1,\varepsilon}$, $P^{-1}_{1,\varepsilon}(x) = X_t(x)$ with $t \leq -\frac{4}{10}$.

7.1.7. *Extension of the attracting and repelling regions $U^{\pm\pm}$ and $V^{\pm\pm}$.* Finally, for every small $\varepsilon > 0$ we define the compact sets U_ε and V_ε as being the union of the orbit segments of X of length 2δ through the points of $\Sigma_{0,\varepsilon}$ and $\Sigma_{1,\varepsilon}$, respectively.

In formula:

$$U_\varepsilon = \{p \in \mathbb{R}^2 \times S^1 | \exists t \in [-\delta, \delta], X_t(p) \in \Sigma_{0,\varepsilon}\}, \text{ and}$$
$$V_\varepsilon = \{p \in \mathbb{R}^2 \times S^1 | \exists t \in [-\delta, \delta], X_t(p) \in \Sigma_{1,\varepsilon}\}$$

Notice that U_ε contains the union $U^{++} \cup U^{+-} \cup U^{-+} \cup U^{--}$ and V_ε contains the union $V^{++} \cup V^{+-} \cup V^{-+} \cup V^{--}$

We fix now some ε_0 such that all the lemmas above apply.

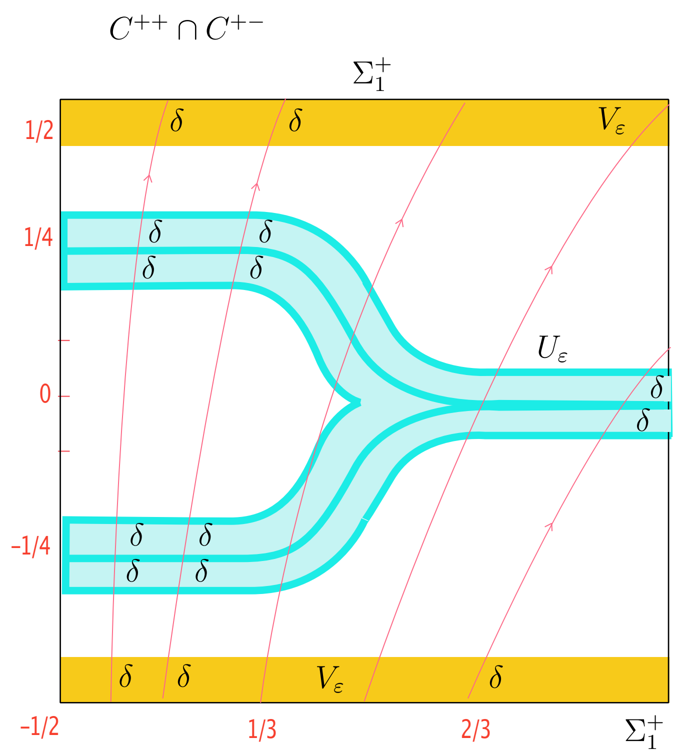

FIGURE 9. Neighborhood U_ε and V_ε $C^{++} \cap C^{+-}$.

7.2. **Diffeomorphism in the neighborhood of the side $C^{++} \cap C^{+-}$.** As we have done several times before, the strategy for building the diffeomorphism in a neighborhood of the unstable separatrix $C^{++} \cap C^{+-}$ (i.e. the half plane $x \geq 0, y = 0$) consists in cover this neighborhood by a 2-parameter family of orbit segments $I^u(x, y)$. For each of this segment, we will explain which segment will be its image. Finally we will explicit the diffeomorphism in restriction to each segment. However, the situation here is more complicate than in the previous cases because each segment I^u will contain a sub-segment $[A(x, y), B(x, y)] \subset I^u$ which need to be preserved. A difficulty is that the segment $[A(x, y), B(x, y)]$ would be degenerate (that is $A(x, y) = B(x, y)$) or not, depending on the value of (x, y).

Let us first build the family $I^u(x, y)$ of orbit segments.

7.2.1. *Cutting the orbits in compact segments.*

Remark 10. *For $\varepsilon > 0$ small enough, the intersections of $\Sigma_{1,\varepsilon}$ with $\{(x,y,z)|x \geq 0$ and $y \in [-\varepsilon,\varepsilon]\}$ is*

$$\Sigma_{1,\varepsilon} \cap \{(x,y,z)|x \geq 0 \text{ and } y \in [-\varepsilon,\varepsilon]\} =$$

$$= \Sigma_1^+ \cap \{(x,y,z)|x \geq 0 \text{ and } y \in [-\varepsilon,\varepsilon]\} \cup \{(x,y,0)|0 \leq x \leq \varepsilon \text{ and } y \in [-\varepsilon,\varepsilon]\}$$

Recall that Σ_1^+ is contained in the half space $\{x \geq 0\}$ and is a global section of X on $\{x \geq 0\}$. The first return map P_1^+ induces a diffeomorphisms of $\Sigma_1^+ \cap \{x \geq 0\}$ which contracts uniformly the coordinates y (by a uniformly bounded factor). Hence

$$P_1^+(\Sigma_1^+ \cap \{(x,y,z)|x \geq 0 \text{ and } y \in [-\varepsilon,\varepsilon]\}) \subset \Sigma_{1,\varepsilon} \cap \{(x,y,z)|x \geq 0 \text{ and } y \in [-\varepsilon,\varepsilon]\},$$

and the restriction of P_1^+ to $\Sigma_1^+ \cap \{(x,y,z)|x \geq 0$ and $y \in [-\varepsilon,\varepsilon]\}$ is a diffeomorphism onto its image.

Remark 11. *We have defined the inverse return map $P_{1,\varepsilon}^{-1}$ on $\Sigma_{1,\varepsilon}$. Clearly we have that the map $P_{1,\varepsilon}^{-1} \circ P_1^+$ induce the identity map on $\Sigma_1^+ \cap \{(x,y,z)|x \geq 0$ and $y \in [-\varepsilon,\varepsilon]\}$. For this reason, it is coherent to denote $P_{1,\varepsilon} = P_1^+$ on this set.*

The surface Σ_1^+ is a smooth half plane which coincides with the plane $z = \frac{1}{2}$ in a neighborhood of $(0,0,\frac{1}{2})$, which the helicoid S_1^+ for large radius, and which cuts transversally $\{y = 0\}$ exactly along the half straight line $\{x \geq 0, y = 0, z = \frac{1}{2}\}$.

One deduces that for small $\varepsilon > 0$, the intersection $\Sigma_1^+ \cap \{(x,y,z)|x \geq 0$ and $y \in [-\varepsilon,\varepsilon]\}$ is a graph over $\{(x,y)|x \geq 0$ and $y \in [-\varepsilon,\varepsilon]\}$. In other words we proved:

Lemma 7.6. *For every $\varepsilon > 0$ small enough, for every (x,y) with $x \geq 0$ and $y \in [-\varepsilon,\varepsilon]$, there is a unique $z = z(x,y) \in S^1$ such that $(x,y,z) \in \Sigma_1^+$.*

Furthermore, $z(x,y)$ is a smooth function which converges uniformly to $\frac{1}{2}$ when $|y|$ tends to 0.

We denote by $c(x,y)$ the time return of the point $(x,y,z(x,y))$ on Σ_1^+. In other words:

$$X_{c(x,y)}(x,y,z(x,y)) = P_1^+(x,y,z(x,y)).$$

Recall that the return time on Σ_1^+ belongs to $[\frac{3}{4},\frac{5}{4}]$ so that $c(x,y) \in [\frac{3}{4},\frac{5}{4}]$.

Remark 12. *The map $c(x,y)$ is a smooth function which tends uniformly to 1 when $|y|$ tends to 0*

We denote by $I^u(x,y)$ the orbit segment joining $(x,y,z(x,y))$ to $P_1^+(x,y,z(x,y))$.

We define

$$\mathcal{W}^{u+} = \bigcup_{x \geq 0, y \in [-\varepsilon,\varepsilon]} I^u(x,y).$$

Remark 13. *$\mathcal{W}^{u+}$ is a closed set containing $\{(x,y,z), x \geq 0, y \in [-\frac{\varepsilon}{4},\frac{\varepsilon}{4}]\}$. In particular $\mathcal{W}^{u+}$ is a closed neighborhood of the unstable separatrix $C^{++} \cap C^{+-} = \{x \geq 0, y = 0, z \in S^1\}$.*

7.2.2. *The intersection points of the segments $I^u(x,y)$ with $\Sigma_{0,\varepsilon}$.*

Each surface $\Sigma^+_{0,\varepsilon}$ and $\Sigma^-_{0,\varepsilon}$ cuts each segment $I^u(x,y)$ in exactly one point. We denote

$$A(x,y) = I^u(x,y) \cap \Sigma^-_{0,\varepsilon} \text{ and } B(x,y) = I^u(x,y) \cap \Sigma^+_{0,\varepsilon}.$$

Considering the time parametrization of the segments $I^u(x,y)$ the points $A(x,y)$ and $B(x,y)$ define two function $a(x,y)$ and $b(x,y)$ by

$$A(x,y) = X_{a(x,y)}(x,y,z(x,y)) \text{ and } B(x,y) = X_{b(x,y)}(x,y,z(x,y))$$

Lemma 7.7. *For small $\varepsilon > 0$, for all $x \geq 0$ and $y \in [-\varepsilon, \varepsilon]$ The maps $a(x,y)$ and $b(x,y)$ are differentiable. Furthermore*

- $a(x,y) \in (0,1)$ *and*
- $b(x,y) \in (0,1)$
- $b - a \in [0, \frac{1}{2}]$
- $\inf_{(x,y)} a(x,y) > 0$
- $\inf_{(x,y)} c(x,y) - b(x,y) > 0.$

(See figure 10)

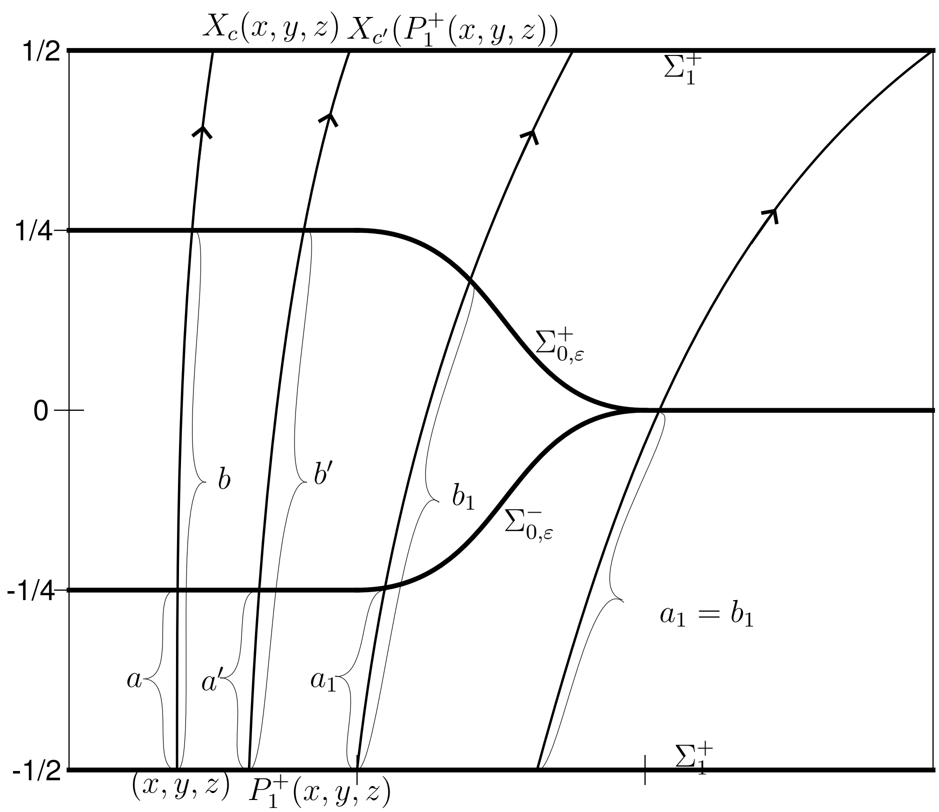

FIGURE 10. Maps a, b and c in $C^{++} \cap C^{+-}$

Proof : This is a consequence of the differentiability of the surfaces $\Sigma^-_{0,\varepsilon}$,$\Sigma^+_{0,\varepsilon}$, and Σ_1^+ when $x \geq 0$ and $y \in [-\varepsilon, \varepsilon]$ and the transversality of the flow to these surfaces. By construction the distance between $\Sigma^-_{0,\varepsilon}$ and $\Sigma^+_{0,\varepsilon}$ is less or equal than $\frac{1}{2}$ then $b-a\in[0,\frac{1}{2}]$, and the distances between Σ^+_1 and $\Sigma^-_{0,\varepsilon}$ and between $\Sigma^+_{0,\varepsilon}$ and Σ^+_1 are positive then $\inf_{(x,y)} a(x,y)>0$ and $\inf_{(x,y)} c(x,y)-b(x,y)>0$. Finally, Remark 12 implies $a(x,y)<1$ and $b(x,y)<1$. □

Remark 14. *Notice that our construction of the surfaces $\Sigma_{0,\varepsilon}$ and $\Sigma_{1,\varepsilon}$ are independent of the number $\delta>0$. Hence one may shrink $\delta>0$ so that $a\geq 3\delta$ and $c-b>3\delta$.*

7.2.3. *The return maps and the function a, b.*

We denote $p_1(x,y)$ the two first coordinates of $P^+_1(x,y,z(x,y))$, that is

$$P^+_1(x,y,z(x,y))=(p_1(x,y),z(p_1(x,y)).$$

Lemma 7.8. $A(p_1(x,y))=P_{0,\varepsilon}(A(x,y))$ *and* $B(p_1(x,y))=P_{0,\varepsilon}(B(x,y))$

Proof : Recall that $A(p_1(x,y))$ is the unique point in $I^u(p_1(x,y))$ in $\Sigma^-_{0,\varepsilon}$. The point $A(x,y)\in\Sigma^-_{0,\varepsilon}$ and $P_{0,\varepsilon}:\Sigma^-_{0,\varepsilon}\to\Sigma^-_{0,\varepsilon}$ then $P_{0,\varepsilon}(A(x,y))\in\Sigma^-_{0,\varepsilon}$. Since $(p_1(x,y),z(p_1(x,y)))$ is the unique point in Σ^+_1 in the segment of orbit between $A(x,y)$ and $P_{0,\varepsilon}(A(x,y))$ then $P_{0,\varepsilon}(A(x,y))\in I^u(p_1(x,y))$, and the first claim follows. The second claim of the lemma is analogous, using $P_{0,\varepsilon}:\Sigma^+_{0,\varepsilon}\to\Sigma^+_{0,\varepsilon}$ □

Next lemma is just a consequence of the fact that $\Sigma^+_{0,\varepsilon}\cap\Sigma^-_{0,\varepsilon}$ is equal to $\{(x,y,z)\in S_0,\sqrt{x^2+y^2}\geq\frac{2}{3},|y|\leq\varepsilon\}$.

Lemma 7.9. *There is $\varepsilon_0>0$ such that every $0<\varepsilon<\varepsilon_0$ one has:*

1. *For every $x\geq\frac{2}{3},y\in[-\varepsilon,\varepsilon]$ one has $a(x,y)=b(x,y)$, that is $A(x,y)=B(x,y)$;*
2. *for every $x\geq 0,y\in[-\varepsilon,\varepsilon]$ one has*
$$a(x,y)=b(x,y)\Longrightarrow a(p_1(x,y))=b(p_1(x,y))$$
3. *there is $\delta>0$ such that, for every $x\geq 0,y\in[-\varepsilon,\varepsilon]$ one has*
$$b(x,y)-a(x,y)\leq 5\delta\Longrightarrow a(p_1(x,y))=b(p_1(x,y))$$

As we have noted, the construction of the $\Sigma_{0,\varepsilon}$ and $\Sigma_{1\varepsilon}$ is independent on the choice of the number δ used in the construction of the diffeomorphisms. We fix from now on the number $\delta>0$ small enough for verifying the Lemma 7.9.

7.2.4. *Families of diffeomorphisms of the segments.* We will build a diffeomorphism such that it will map the orbit segments $I^u(x,y)$ on the segment $I^u(p_1(x,y))$; furthermore, it will have a special form in the neighborhood of the orbit segment joining $A(x,y)$ to $B(x,y)$. We will now explain the rule for building the diffeomorphism in the neighborhood of the orbit segment $[A(x,y),B(x,y)]^c$.

For that we will consider the time interval $[a(x,y)-\delta,b(x,y)+\delta]$ whose length is $r(x,y)=2\delta+b(x,y)-a(x,y)\in[2\delta,\frac{1}{2}+2\delta]$. The image will be contained in the time interval $[a(p_1(x,y))-\delta,b(p_1(x,y))+\delta]$ whose length is $s(x,y)=2\delta+b(p_1(x,y))-a(p_1(x,y))$. Furthermore, Lemma 7.9 implies that $r(x,y)\leq 7\delta\Longrightarrow s(x,y)=2\delta$.

This explains that we consider the region

$$R=\{(r,s)\in[2\delta,\frac{1}{2}+2\delta]^2\text{ such that }r\leq 7\delta\Longrightarrow s=2\delta\}$$

Recall that $\alpha>1$ is a number close to 1 already used for the definitions of f_{ext}, f^{++}.

Lemma 7.10. *There is a smooth family of smooth maps, $\xi_{r,s}\colon [0,r] \to [0,s]$, $(r,s) \in R$, verifying the following properties:*

(1) *$\xi_{r,s}$ induces an increasing diffeomorphism from $([0,r])$ to $[(1-\alpha^{-1})\delta, s-(1-\alpha^{-1})\delta]$;*
(2) *the derivatives of $\xi_{r,s}$ at 0 and r are equal to α^{-1}*

$$\xi'_{r,s}(0) = \xi'_{r,s}(r) = \alpha^{-1}$$

(3) *the derivative of $\xi_{r,s}$ is less than α^{-1} on the segment $[0,2\delta]$ and $[r-2\delta,r]$:*

$$0 \leq \xi'_{r,s}(t) \leq \alpha^{-1} \text{ for all } t \in [0,2\delta] \cup [r-2\delta,r]$$

(4) *if s is larger than 4δ, then the derivative of $\xi_{r,s}$ is equal to α^{-1} on the segment $[0,2\delta]$ and $[r-2\delta,r]$:*

$$s \geq 4\delta \Longrightarrow \xi'_{r,s}(t) = \alpha^{-1} \text{ for all } t \in [0,2\delta] \cup [r-2\delta,r]$$

(5) *the diffeomorphism $\xi_{r,s}$ is an affine map if $r = 2\delta$:*

$$\xi'_{2\delta,2\delta}(t) = \alpha^{-1}$$

(6) *if $r = s = \frac{1}{2} + 2\delta$ then ξ_{rs} is the map $t \mapsto \Psi_0(t - \frac{1}{4} - \delta) + \frac{1}{4} + \delta$, where Ψ_0 is the Morse-Smale diffeomorphism we defined on the periodic orbit (see section 6.7).*

Proof : Just notice that in the case that $r = s = \frac{1}{2} + 2\delta$ (where $s \geq 4\delta$) it holds that the restrictions of $\xi_{r,s}$ to $[0,2\delta]$ and $[r-2\delta,r]$ coincide with a traslation of the restrictions of Ψ_0 to $[-\frac{1}{4} - \delta, -\frac{1}{4} + \delta]$ and $[\frac{1}{4} - \delta, \frac{1}{4} + \delta]$, respectively; and we have that Ψ_0 is a linear contraction of factor α^{-1} in these intervals. So, there is no incompatibilities in the properties we ask for $\xi_{r,s}$. □

Remark 15. *By definition of the functions $\xi_{r,s}$, for every $(r,s) \in R$ one has:*

$$\xi_{r,s}([0,2\delta] \subset [(1-\alpha^{-1})\delta, (1+\alpha^{-1})\delta], \text{ and}$$

$$\xi_{r,s}([r-2\delta,r] \subset [s-(1+\alpha^{-1})\delta, s-(1-\alpha^{-1})\delta].$$

We are going to define a family of maps which help us to decide how we will map $I^u(x,y)$ on the segment $I^u(p_1(x,y))$. Recall that the times length of $I^u(x,y)$ is $c(x,y)$.

For $(a,b,c),(a',b',c')$ satisfying:

- $a,b,a',b' \in [0,1]$ and $c,c' \in [\frac{3}{4},\frac{5}{4}]$,
- $a \geq 3\delta$, $a' \geq 3\delta$, $c-b \geq 3\delta$, $c'-b' \geq 3\delta$, $a \leq b$, $a' \leq b'$
- and $b - a \leq 5\delta \Longrightarrow a' = b'$,

we define a diffeomorphism $\chi_{(a,b,c)(a',b',c')}\colon [0,c] \to [0,c']$ by its restrictions to the segments $[0,\alpha^{-1}\delta]$, $[\alpha^{-1}\delta, a-\delta]$, $[a-\delta, b+\delta]$, $[b+\delta, c-\alpha^{-1}\delta]$, and $[c-\alpha^{-1}\delta, c]$ as follows:

- the restriction $\chi_{(a,b,c)(a',b',c')}\colon [0,\alpha^{-1}\delta] \to [0,\delta]$, is the homothety of ratio α;
- the restriction $\chi_{(a,b,c)(a',b',c')}\colon [\alpha^{-1}\delta, a-\delta] \to [\delta, a'-\alpha^{-1}\delta]$ is the diffeomorphism $\theta_{\alpha,\alpha^{-1},[\alpha^{-1}\delta,a-\delta],[\delta,a'-\alpha^{-1}\delta]}$ defined in section 6.5 (it is a diffeomorphism whose derivative at the extremities are α and α^{-1} respectively);

- the restriction $\chi_{(a,b,c)(a',b',c')}\colon [a-\delta, b+\delta] \to [a'-\alpha^{-1}\delta, b'+\alpha^{-1}\delta]$ is obtained as follows:
$$\chi_{(a,b,c)(a',b',c')}(a-\delta+t) = a'-\delta+\xi_{rs}(t)$$
$$\text{with } r = 2\delta - a + b \quad \text{ and } s = 2\delta - a' + b'.$$
This is possible because our hypotheses on (a,b) and (a',b') imply that $(r,s) \in R$.
- the restriction $\chi_{(a,b,c)(a',b',c')}\colon [b+\delta, c-\alpha^{-1}\delta] \longrightarrow [b'+\alpha^{-1}\delta, c'-\delta]$ is the diffeomorphism $\theta_{\alpha^{-1},\alpha,[b+\delta,c-\alpha^{-1}\delta],[b'-\alpha^{-1}\delta,c'-\delta]}$; it is a diffeomorphism whose derivative at the extremities are α^{-1} and α respectively;
- the restriction $\chi_{(a,b,c)(a',b',c')}\colon [c-\alpha^{-1}\delta, c] \to [c'-\delta, c']$, is the affine homothety of ratio α;

See figures 11 and 12.

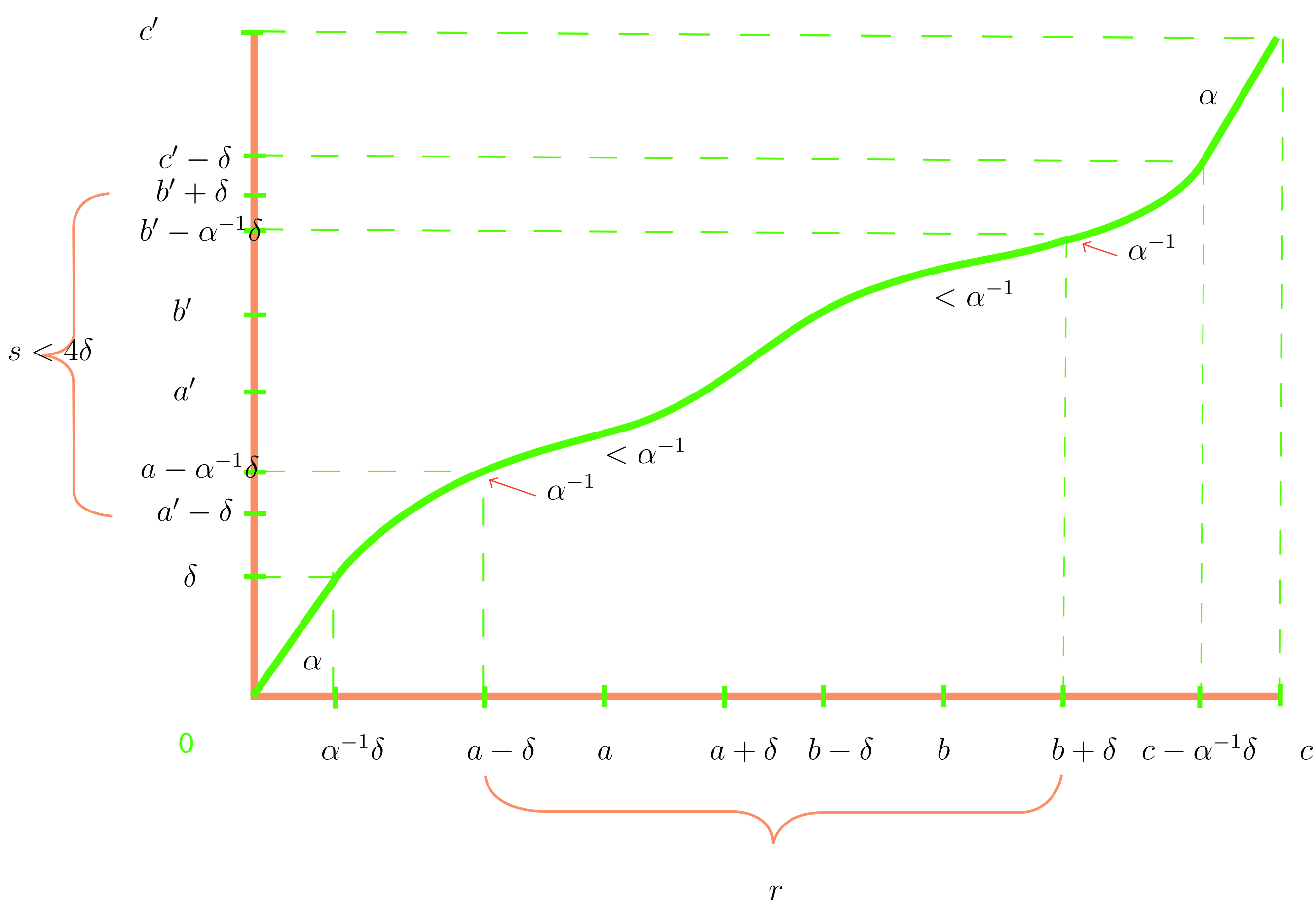

FIGURE 11. The map $\chi_{(a,b,c)(a',b',c')}$ in the case $s<4\delta$.

Lemma 7.11. *The family $\chi_{(a,b,c)(a',b',c')}$ is a smooth family of diffeomorphisms. Furthermore:*

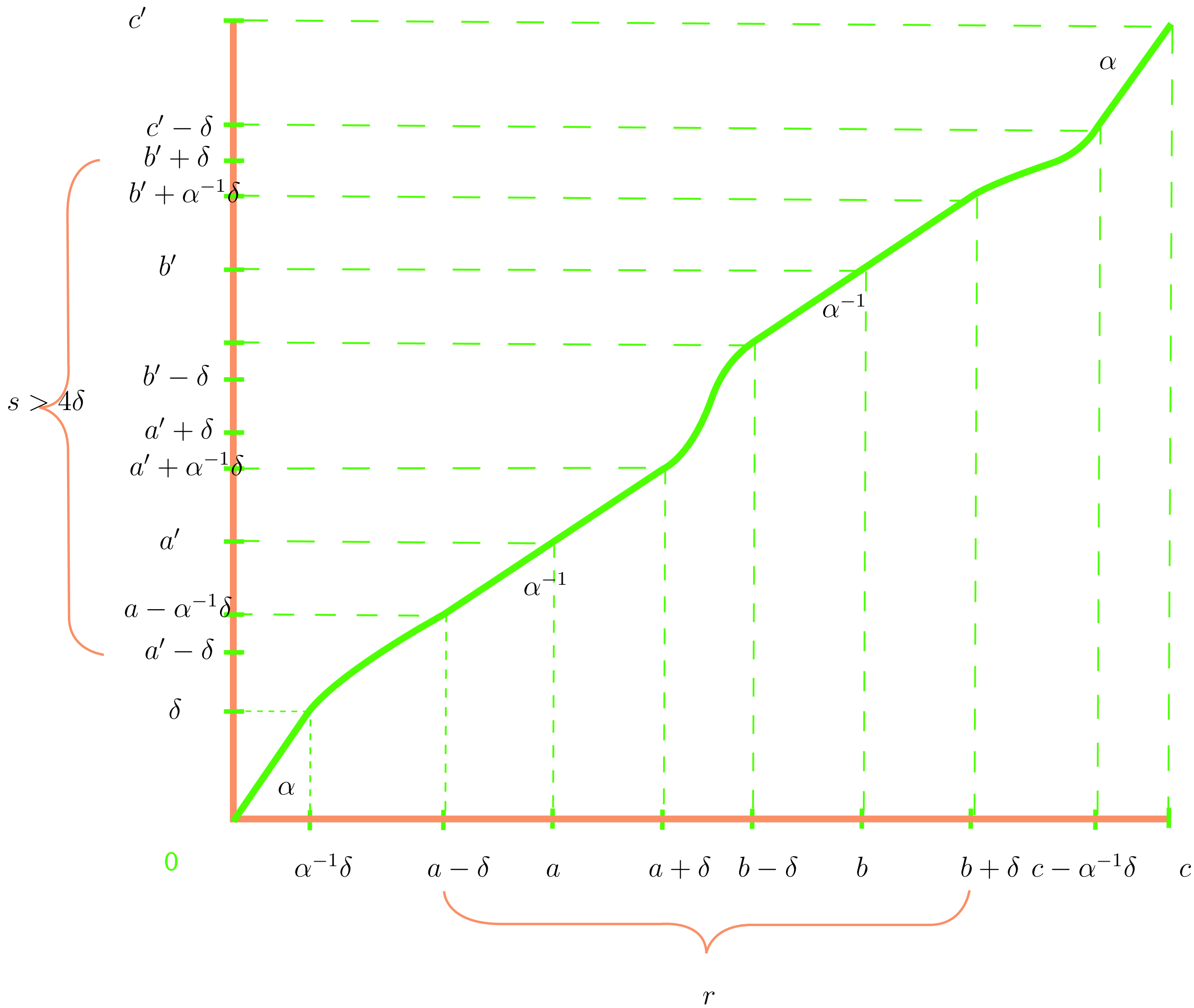

FIGURE 12. The map $\chi_{(a,b,c)(a',b',c')}$ in the case $s > 4\delta$.

(1) *If* $a = a' = \frac{1}{4}$, $b = b' = \frac{3}{4}$ *and* $c = c' = 1$ *then* $\chi_{(a,b,c)(a',b',c')}$ *coincides with the diffeomorphism* Ψ_0*;*

(2) *If* $a = b$ *and then* $a' = b'$ *then* $\chi_{(a,b,c)(a',b',c')}(a) = a'$*; furthermore,* $\chi_{(a,b,c)(a',b',c')}$ *coincides with* $\Psi^+_{[0,a],[0,a']}$ *on* $[0, a]$ *and with* $\Psi^-_{[a,c],[a',c']}$ *on* $[a, c]$*.*

Proof : In the case that $a = a' = \frac{1}{4}$, $b = b' = \frac{3}{4}$ and $c = c' = 1$, for verifying that $\chi_{(a,b,c)(a',b',c')}$ coincides with the diffeomorphism Ψ_0, recall that by last item of lemma 7.10 the restriction of $\chi_{(\frac{1}{4},\frac{3}{4},1)(\frac{1}{4},\frac{3}{4},1)}$ to $[\frac{1}{4}-\delta, \frac{3}{4}+\delta]$ coincides with Ψ_0.

By item 5 of lemma 7.10 the restriction of $\chi_{(a,b,c)(a',b',c')}$ to $[a-\delta, a+\delta]$ is a contraction of factor α^{-1} and the second claim follows. □

7.2.5. *The diffeomorphism $f^{+\pm}$.*

We are now ready to build the announced diffeomorphism $f^{+\pm}$ on a neighborhood of $C^{++} \cap C^{+-}$.

Recall that $\mathcal{W}^{u+}$ denotes the union of all the segment $I^u(x,y)$ and it is a neighborhood of $C^{++} \cap C^{+-}$.

Definition 7.1. *We denote by $f^{+\pm}\colon \mathcal{W}^{u+} \to \mathcal{W}^{u+}$ the map defined as follows. For every $(x,y,z) \in \Sigma_1^+ \cap \{|y| \le \varepsilon\}$ one has:*

- *$f^{+\pm}(I^u(x,y)) = I^u(p_1(x,y))$.*
- *Let us denote $a = a(x,y)$, $b = b(x,y)$, $c = c(x,y)$, $a' = a(p_1(x,y))$, $b' = b(p_1(x,y))$, and $c' = c(p_1(x,y))$.*
  For every $t \in [0, c(x,y)]$ we define
  $$f^{+\pm}(X_t(x,y,z)) = X_{t'}(P_1^+(x,y,z)) \text{ where } t' = \chi_{(a,b,c)(a',b',c')}(t).$$

Lemma 7.12. *The map $f^{+\pm}$ is well defined on $\mathcal{W}^{u+}$ and it is a diffeomorphism on its image.*

Proof : Two distinct segments $I^u(x,y)$ and $I^u(x',y')$ are either disjoint or the intersection is reduced to one extremity. This means that $(x',y') = p_1(x,y)$ or $(x,y) = p_1(x',y')$. In this case one verifies that the both values of $f^{+\pm}$ at the intersecting point are equal. So the map is well defined. The family of maps $\chi_{(a,b,c)(a',b',c')}$ is a smooth family of diffeomorphisms. Furthermore, the derivative at the extremities of each segment $I^u(x,y)$ is α. Hence, at the point $P_1^+(x,y,z(x,y)) = I^u(x,y) \cap I(p_1(x,y))$, the derivative of $f^{+\pm}$ restricted to $I^u(x,y)$ and to $I^u(p_1(x,y))$ are equal: the derivative of $f^{+\pm}$ at this point is well defined and equal to α. □

Notice that this new construction coincides with the previous maps f_0^{++} f_0^{+-} in a neighborhood of the periodic orbit and for large radius:

Lemma 7.13. *For every $(x,y,z) \in \Sigma_1^+ \cap \{|y| \le \varepsilon\}$, one has:*

- *if $x \le \frac{1}{100}$, then the restriction of $f^{+\pm}$ to $I^u(x,y)$ coincides with f_0 hence with f_0^{++} on C^{++} and with f_0^{+-} on C^{+-};*
- *if $x \ge \frac{2}{3}$, then the restriction of $f^{+\pm}$ to $I^u(x,y)$ coincides with f_0^{++} on C^{++} and with f_0^{+-} on C^{+-}.*

Proof : By the first item of lemma 7.11 we have that if $x \le \frac{1}{100}$ then $\chi_{(a,b,c)(a',b',c')}$ coincides with the diffeomorphism Ψ_0 and f_0 coincide with $f^{+\pm}$. The last claim is a consequence of item 1 of lemma 7.9 and item 2 of lemma 7.11. □

7.2.6. *Attracting and repelling regions.*

Next lemma asserts that the sets U_ε and V_ε are invariant by positive and negative iterates of $f^{+\pm}$, respectively:

Lemma 7.14. *For every small $\varepsilon' < \varepsilon$ one has*

- *for every $(x,y,z) \in \mathcal{W}^{u+} \cap U_{\varepsilon'}$, the image $f^{+\pm}(x,y,z)$ belongs to the interior of $U_{\varepsilon'}$; moreover, $Df^{+\pm}(x,y,z)$ contracts the vector field X by a factor less or equal than α^{-1}, and this factor is equal to α^{-1} if $x \notin [\frac{1}{6}, 2]$.*

- *for every $(x,y,z) \in \mathcal{W}^{u+}$ such that $f^{+\pm}(x,y,z)$ belongs to $V_{\varepsilon'}$, the point (x,y,z) belongs to the interior of $V_{\varepsilon'}$; moreover, $(Df^{+\pm})^{-1}(x,y,z)$ contracts the vector field X by a factor α^{-1}.*

Proof : Recall that there is $\varepsilon'' \in (0,\varepsilon')$ such that $P_{0,\varepsilon'}(\Sigma_{0,\varepsilon'}) \subset \Sigma_{0,\varepsilon''}$ and $P^{-1}_{1,\varepsilon'}(\Sigma_{1\varepsilon'}) \subset \Sigma_{1,\varepsilon''}$.

Let $(x',y',z') \in U_{\varepsilon'}$, then there exists $(x,y,z) \in \Sigma_1$ such that $X_t(x,y,z) = (x',y',z')$ with $t \in [0, c(x,y)]$. In fact, $(x',y',z') \in U_{\varepsilon'}$ if and only if $|t-a| < \delta$ or $|t-b| < \delta$. Since

$$f^{+\pm}(X_t(x,y,z)) = X_{t'}(P_1^+(x,y,z)) \text{ where } t' = \chi_{(a,b,c)(a',b',c')}(t),$$

the restriction $\chi_{(a,b,c)(a',b',c')} \colon [a-\delta, b+\delta] \to [a'-\alpha^{-1}\delta, b'+\alpha^{-1}\delta]$ is obtained by $\chi_{(a,b,c)(a',b',c')}(a-\delta+t) = a'-\delta+\xi_{rs}(t)$ with $r = 2\delta - a + b$ and $s = 2\delta - a' + b'$. Since $0 \le \xi'_{r,s}(t) \le \alpha^{-1}$ for all $t \in [0,2\delta] \cup [r-2\delta, r]$ it follows that the image $f^{+\pm}(x',y',z')$ belongs to the interior of $U_{\varepsilon'}$; moreover, $Df^{+\pm}(x',y',z')$ contracts the vector field X by a factor less or equal than α^{-1}.

Besides, the derivative of $\xi_{r,s}$ is equal to α^{-1} on the segment $[0,2\delta]$ and $[r-2\delta, r]$ if s is larger than 4δ (this is the case when x is smaller than $\frac{1}{6}$) and it is equal to α^{-1} if $r = 2\delta$ (this is the case when $|x| > 2$).

Analogously assume that $(x',y',z') \in \mathcal{W}^{u+}$ satisfies that $f^{+\pm}(x',y',z') \in V_{\varepsilon'}$. Consider $(x,y,z) \in \Sigma_1^+$, with $z = z(x,y)$ such that $(x',y',z') \in I^u(x,y)$, that is, there is $t \in [0, c(x,y)]$ with $X_t(x,y,z) = (x',y',z')$. Now $f^{+\pm}(x',y',z')$ belongs to $I^u(p_1(x,y))$ and let $t_1 \in [0, c(p_1(x,y))]$ such that $X_{t_1}(P_1^+(x,y,z)) = f^{+\pm}(x',y',z')$

The fact that $f^{+\pm}(x',y',z') \in V_{\varepsilon'}$ implies that one of the two following possibility is verified

- either $f^{+\pm}(x',y',z')$ belongs to a segment of orbit of size 2δ centered at a point of Σ_1^+. In that case either $t_1 \in [0,\delta]$ or $c(p_1(x,y)) - t_1 \in [0,\delta]$). Then either $t \in [0, \alpha^{-1}\delta]$ or $c(x,y) - t \in [0,\alpha^{-1}\delta]$, implying that (x',y',z') belongs to the interior of $V_{\varepsilon'}$
- or $f^{+\pm}(x',y',z')$ belongs to a segment of orbit of size 2δ centered at a point q of $\Sigma^-_{1,\varepsilon'} \cap \{(x,y,z), x \ge 0 \text{ and } y \in [-\varepsilon,\varepsilon]\} = \{(x,y,0), x \in [0,\varepsilon] \text{ and } y \in [-\varepsilon,\varepsilon]\}$. Hence $I^u(x,y)$ and $I^u(p_1(x,y))$ are in a very small neighborhood of the periodic orbit where $f^{+\pm}$ coincides with f_0 and whereΣ_1^+ coincides with the disc $z = -\frac{1}{2}$. Furthermore, $|\frac{1}{2} - t_1| < \delta$. One deduces that $|\frac{1}{2} - t| < \alpha^{-1}\delta$ proving that (x',y',z') belongs to the orbit segment of size $2\alpha^{-1}\delta$ centered at the point $P^{-1}_{1,\varepsilon'}(q) \in \Sigma^-_{1,\varepsilon''}$ for some $\varepsilon'' < \varepsilon'$. In particular (x',y',z') belongs to the interior of V'_{ε}.

Finally, in both cases, $(Df^{+\pm})^{-1}$ contracts the vector field X by a factor α^{-1}. □

Recall that the central derivative of f at a point (x,y,z) is $D^c f = \|Df_{(x,y,z)}(X(x,y,z)\|$. Note that in lemma 7.14 we have proved that for every small $\varepsilon' < \varepsilon$ one has

- for every $(x,y,z) \in \mathcal{W}^{u+} \cap U_{\varepsilon'}$ the central derivative $D^c f^{+\pm}(x,y,z)$ is less than α^{-1}, and it is equal to α^{-1} if $x \notin [\frac{1}{6}, 2]$.
- for every $(x,y,z) \in \mathcal{W}^{u+}$ such that $f^{+\pm}(x,y,z)$ belongs to $V_{\varepsilon'}$ the central derivative $(D^c f^{+\pm})^{-1}(x,y,z)$ is equal to α^{-1}.

7.3. **The diffeomorphisms in a neighborhood of the other sides.**

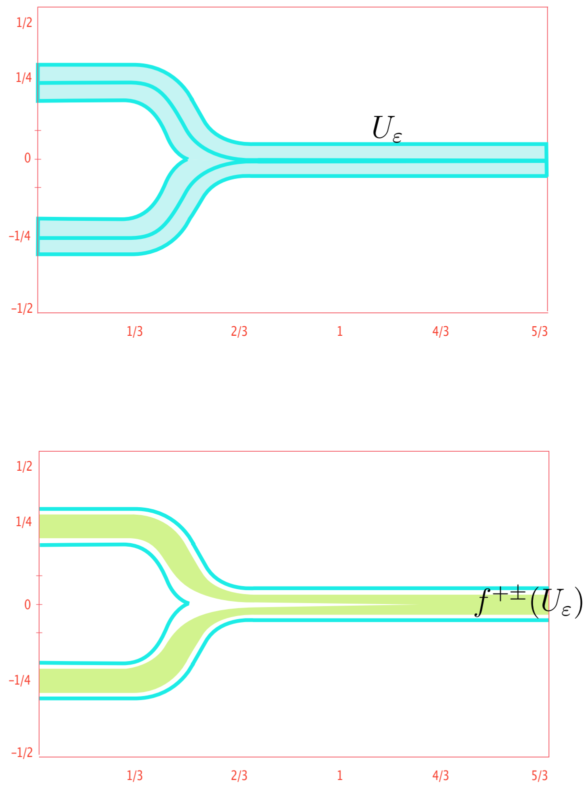

FIGURE 13. The map $f^{+\pm}(x,y,z)$ on $U_\varepsilon \cap \{y=0, x\geq 0\}$

7.3.1. *The diffeomorphism $f^{-\pm}$ in the neighborhood of $C^{--}\cap C^{-+}$.* Consider the union of orbit segments

$$\mathcal{W}^{u-} = \bigcup_{(x,y,z)\in\Sigma_1^-\cap\{|y|\leq\varepsilon\}} I^u(x,y)$$

where $I^u(x,y)$ is the segment of orbit joining a point $(x,y,z)\in\Sigma_1^-$ to its first return $P_1^-(x,y,z)$ on Σ_1^-. Then $\mathcal{W}^{u-}$ is a neighborhood of the unstable separatrix $C^{-+}\cap C^{--} = \{x\leq 0, y=0, z\in S^1\}$. We define $f^{-\pm}\colon \mathcal{W}^{u-}\to\mathcal{W}^{u-}$ exactly in the same way as $f^{+\pm}$.

We get

Lemma 7.15. *For every $(x,y,z)\in\Sigma_1^-\cap\{|y|\leq\varepsilon\}$, one has:*

- *if $x\geq -\frac{1}{100}$, then the restriction of $f^{-\pm}$ to $I^u(x,y)$ coincides with f_0 hence with f_0^{-+} on C^{-+} and with f_0^{--} on C^{--};*
- *if $x\leq -\frac{2}{3}$, then the restriction of $f^{-\pm}$ to $I^u(x,y)$ coincides with f_0^{-+} on C^{-+} and with f_0^{--} on C^{--}.*

Moreover for every small $\varepsilon'<\varepsilon$ one has

- *for every $(x,y,z)\in\mathcal{W}^{u-}\cap U_{\varepsilon'}$, the image $f^{-\pm}(x,y,z)$ belongs to the interior of $U_{\varepsilon'}$; moreover, the central derivative $D^c f^{-\pm}(x,y,z)$ is less or equal than α^{-1}, and it is equal to α^{-1} if $x\notin[-2,-\frac{1}{6}]$.*

- *for every* $(x,y,z) \in \mathcal{W}^{u-}$ *such that* $f^{-\pm}(x,y,z)$ *belongs to* $V_{\varepsilon'}$*, the point* (x,y,z) *belongs to the interior of* $V_{\varepsilon'}$*; moreover, the central derivative* $(D^c f^{-\pm})^{-1}(x,y,z)$ *is equal to* α^{-1}*.*

7.3.2. *The diffeomorphisms* $f^{\pm+}$ *and* $f^{\pm-}$ *in the neighborhood of* $C^{++} \cap C^{-+}$ *and* $C^{+-} \cap C^{--}$*. .*

As we already noticed, it is more convenient to consider the inverse of the return maps in the neighborhood of the stable manifold $\{x = 0\}$, because the flow expands the x coordinates: as a consequence the sets $\{|x| \leq \varepsilon\}$ is not invariant for the positive times of the flow but is invariant for negative times. For this reason we define:

$$\mathcal{W}^{s-} = \bigcup_{(x,y,z)\in\Sigma_0^- \cap\{|x|\leq\varepsilon\}} I^s(x,y)$$

where $I^s(x,y)$ is the segment of orbit joining a point $(x,y,z) \in \Sigma_0^-$ to its first negative return $(P_0^-)^{-1}(x,y,z)$ on Σ_0^-.

Analogously we define

$$\mathcal{W}^{s+} = \bigcup_{(x,y,z)\in\Sigma_0^+ \cap\{|x|\leq\varepsilon\}} I^s(x,y)$$

where $I^s(x,y)$ is the segment of orbit joining a point $(x,y,z) \in \Sigma_0^+$ to its first negative return $(P_0^+)^{-1}(x,y,z)$ on Σ_0^+.

Analogously to the construction of $f^{+\pm}$ and $f^{-\pm}$, we build diffeomorphisms

$(f^{\pm+})^{-1}\colon \mathcal{W}^{s+} \to \mathcal{W}^{s+}$ and $(f^{\pm-})^{-1}\colon \mathcal{W}^{s-} \to \mathcal{W}^{s-}$ which coincide with $(f_0)^{-1}$ for $|y| \leq \frac{1}{100}$, and with the inverse of f_0^{++}, f_0^{+-}, or f_0^{-+}, f_0^{--} (according to the corresponding quadrant), for $|y| \geq \frac{2}{3}$. The set $U_{\varepsilon'}$ is invariant by $f^{\pm+}$ and $f^{\pm-}$, for every $\varepsilon' < \varepsilon$; moreover $f^{\pm+}(U_{\varepsilon'}) \subset Int(U_{\varepsilon'})$, $f^{\pm-}(U_{\varepsilon'}) \subset Int(U_{\varepsilon'})$, and their central derivative is equal to α^{-1}. Finally $V_{\varepsilon'}$ is an attracting region for $(f^{\pm+})^{-1}$ and $(f^{\pm-})^{-1}$, for every $\varepsilon' \leq \varepsilon$; moreover the central derivative of $(f^{\pm+})^{-1}$ and of $(f^{\pm-})^{-1}$ are less or equal than α^{-1} and it is equal to α^{-1} if $|y| \notin [\frac{1}{6}, 2]$.

7.4. **Gluing all the pieces of the puzzle.** We will now glue the diffeomorphisms f_0^{++},f_0^{+-},f_0^{-+}, and f_0^{--} we have defined in the respective quadrants with the diffeomorphisms $f^{+\pm}$,$f^{-\pm}$,$f^{\pm+}$, and $f^{\pm-}$, in order to get a diffeomorphism of $\mathbb{R}^2 \times S^1$ which will be our local model of Axiom A diffeomorphism in a neighborhood of the boundary component of a Birkhoff section. Let us make now some easy observations which will help us in this construction.

(1) All these diffeomorphisms preserve every leaf of the 1-dimensional foliation generated by the vector field X. More precisely, each of them is of the form $(x,y,z) \mapsto X_t(x,y,z)$ where t is a strictly positive number depending smoothly on (x,y,z).
(2) Furthermore, for $f^{++}, \dots, f^{--}$, f_0, $f^{+\pm}, \dots, f^{\pm-}$, the orbit segment joining a point to its image is larger than the smallest orbit segment joining $\Sigma_{i,\varepsilon}$ to $\Sigma_{j,\varepsilon}$, $i \neq j \in \{0,1\}$, and smaller than two times the largest time return on $\Sigma_{i,\varepsilon}^+$ and on $\Sigma_{i,\varepsilon}^-$. For $\varepsilon = 0$ these time distances are $\frac{1}{4}$ and $2\frac{5}{4}$ respectively. For small ε, these time distances are larger than $\frac{1}{5}$ and smaller that 3, respectively. Hence

all these diffeomorphisms are on the form $p \mapsto X_t(p)$ with $t \in [\frac{1}{5}, 3]$. The diffeomorphisms $f_0^{++}, \dots, f_0^{--}$ are obtained as barycenters along the orbits of the diffeomorphism f_0 with $f^{++}, \dots, f^{--}$. One deduces that the same estimates hold for these diffeomorphisms.

(3) The neighborhood $\mathcal{W}^{u+}$ and $\mathcal{W}^{u-}$ of the unstable separatrices are invariant by these diffeomorphisms. More precisely $\mathcal{W}^{u+} \cap C^{++}$ is invariant by f_0^{++} and by $f^{+\pm}$. Analogously, the intersection of each of these neighborhood $\mathcal{W}^{u+}$ and $\mathcal{W}^{u-}$ with each quadrant are invariant by the corresponding diffeomorphisms.

(4) Recall that $\mathcal{W}^{u+}$ is union of orbits segment $I^u(x, y)$ with extremal points in Σ_1^+. The images $f_0^{++}(I^u(x, y))$ and $f^{+\pm}(I^u(x, y))$ are both the same orbit segment $I^u(p_1(x, y))$. Moreover, f_0^{++} and $f^{+\pm}$ coincide on the two extremal subsegments of $I^u(x, y)$ of length $\alpha^{-1}\delta$ with the affine dilation of ratio α. The same happens in the intersection of each of $\mathcal{W}^{u+}$ and $\mathcal{W}^{u-}$ with each quadrant.

(5) The neighborhood $\mathcal{W}^{s+}$ and $\mathcal{W}^{s-}$ of the stable separatrices are invariant by the inverse of these diffeomorphisms. More precisely $\mathcal{W}^{s+} \cap C^{++}$ is invariant by $(f_0^{++})^{-1}$ and by $(f^{+\pm})^{-1}$. Analogously, the intersection of each of these neighborhood $\mathcal{W}^{s+}$ and $\mathcal{W}^{s-}$ with each quadrant are invariant by the inverse of the corresponding diffeomorphisms.

(6) $\mathcal{W}^{s+}$ is union of orbits segment $I^s(x, y)$ with extremal points in Σ_0^+. The images $(f_0^{++})^{-1}(I^s(x, y))$ and $(f^{+\pm})^{-1}(I^s(x, y))$ are both the same orbit segment $I^s(p_0^{-1}(x, y))$. Moreover, $(f_0^{++})^{-1}$ and $(f^{+\pm})^{-1}$ coincide on the two extremal subsegments of $I^s(x, y)$ of length $\alpha^{-1}\delta$ with the affine dilation of ratio α. The same happens in the intersection of each of $\mathcal{W}^{s+}$ and $\mathcal{W}^{s-}$ with each quadrant.

(7) there is a neighborhood of the periodic orbit such that all the diffeomorphisms f_0^{++},f_0^{+-},f_0^{-+},f_0^{--} $f^{+\pm}$,$f^{-\pm}$, $f^{\pm+}$ and $f^{\pm-}$ coincide with f_0 (and their inverses coincide with f_0^{-1}.

Hence (by shrinking ε if necessary) we can assume that for every $(x, y, z) \in \mathcal{W}^{u+} \cap \mathcal{W}^{s+}$ then all these diffeomorphisms and their inverse coincide with f_0 and f_0^{-1}, respectively, on $I^u(x_1, y_1)$ and on $I^s(x_2, y_2)$ where $I^u(x_1, y_1)$ and $I^s(x_2, y_2)$ are the orbit segments containing (x, y, z). The same happens for the other intersections of $\mathcal{W}^{u+} \cup \mathcal{W}^{u-}$ with $\mathcal{W}^{s+} \cup \mathcal{W}^{s-}$ in the other quadrants.

We are now ready for gluing the pieces of the puzzle.

Definition 7.2. *We call* pre-model-diffeomorphism *and we denote by f_m the diffeomorphism defined as follows:*

- *if (x, y, z) belongs to the complement of $\mathcal{W}^{u+} \cup \mathcal{W}^{u-} \cup (f^{\pm+})^{-1}(\mathcal{W}^{s+}) \cup (f^{\pm-})^{-1}(\mathcal{W}^{s-})$ then $f_m(x, y, z)$ is $f_0^{++}(x, y, z)$,$f_0^{+-}(x, y, z)$,$f_0^{-+}(x, y, z)$ or $f_0^{--}(x, y, z)$ according to the quadrant containing (x, y, z).*
- *if $(x, y, z) \in \mathcal{W}^{u+} \cap C^{++}$; let $(x_1, y_1, z_1) \in \Sigma_1^+$ and t_1, t_2 such that:*
  - *$(x, y, z) \in I^u(x_1, y_1)$*
  - *$f^{+\pm}(x, y, z) = X_{t_1}(P_1^+(x_1, y_1, z_1))$.*
  - *$f_0^{++}(x, y, z) = X_{t_2}(P_1^+(x_1, y_1, z_1))$.*

  Then $f_m(x, y, z) = X_r(P_1^+(x_1, y_1, z_1)$ where $r = \psi(\frac{y_1}{\varepsilon})t_1 + (1 - \psi(\frac{y_1}{\varepsilon}))t_2$.

  We define f_m exactly in the same way on the intersection of $\mathcal{W}^{u+} \cup \mathcal{W}^{u-}$ with each of the quadrants;

- *so we have already defined f_m in the complement of $(f^{\pm+})^{-1}(\mathcal{W}^{s+})\cup(f^{\pm-})^{-1}(\mathcal{W}^{s-})$; we will now define $f_m^{-1}\colon \mathcal{W}^{s+}\cap C^{++}\to (f^{\pm+})^{-1}(\mathcal{W}^{s+})\cap C^{++}$.*
  if $(x,y,z)\in \mathcal{W}^{s+}\cap C^{++}$; let $(x_1,y_1,z_1)\in\Sigma_0^+$ and t_1,t_2 such that:
  – $(x,y,z)\in I^s(x_1,y_1)$
  – $(f^{\pm+})^{-1}(x,y,z)=X_{t_1}((P_0^+)^{-1}(x_1,y_1,z_1))$.
  – $(f_0^{++})^{-1}(x,y,z)=X_{t_2}((P_0^+)^{-1}(x_1,y_1,z_1))$.
  Then $(f_m)^{-1}(x,y,z)=X_r((P_0^+)^{-1}(x_1,y_1,z_1)$ where $r=\psi(\frac{x_1}{\varepsilon})t_1+(1-\psi(\frac{x_1}{\varepsilon}))t_2$. Considering the inverse map, this defines $f_m\colon(f^{\pm+})^{-1}(\mathcal{W}^{s+})\cap C^{++}\to\mathcal{W}^{s+}\cap C^{++}$.
  We define f_m exactly in the same way on the intersection of $(f^{\pm+})^{-1}(\mathcal{W}^{s+})\cup(f^{\pm-})^{-1}(\mathcal{W}^{s-})$ with each of the quadrants.

Lemma 7.16. *The map f_m defined above is well defined on $\mathbb{R}^2\times S^1$ and it is a diffeomorphism of $\mathbb{R}^2\times S^1$.*

Proof : We have that the maps $f^{+\pm},f^{-\pm},f^{\pm+}$, $f^{\pm-}$, $f_0^{++},f_0^{+-},f_0^{-+}$ and f_0^{--} are diffeomorphisms that map central arcs of the form $I^u(x_1,y_1,z_1)$ with $(x_1,y_1,z_1)\in\Sigma_1$ to $I^u(P_1(x_1,y_1,z_1))$ and their inverses map central arcs of the form $I^s(x_0,y_0,z_0)$ with $(x_0,y_0,z_0)\in\Sigma_0$ to $I^s(P_0^{-1}(x_0,y_0,z_0))$. Notice that the coefficient of the barycenter is constant on each segment $I^u(x_1,y_1,z_1)$ and $I^s(x_1,y_1,z_1)$. □

7.5. **Properties of the pre-model-diffeomorphism.**

Proposition 7.1. *The pre-model-diffeomorphism f_m verifies the following properties*

(1) *Every leaf of the 1-dimensional foliation defined by the flow of X is invariant by f_m. More precisely, for every (x,y,z) there is $t\in[\frac{1}{5},3]$, depending smoothly on (x,y,z), such that $f_m(x,y,z)=X_t(x,y,z)$.*
(2) *for every $\varepsilon'>0$ small enough the set $U_{\varepsilon'}$ is invariant by f_m, and the set $V_{\varepsilon'}$ is invariant by f_m^{-1}*
(3) *the central derivative D^cf_m is less or equal to α^{-1} on $U_{\varepsilon'}$ and the central derivative $D^c(f_m)^{-1}$ is less or equal to α^{-1} on $V_{\varepsilon'}$;*
(4) *there is a constant $\beta>0$ which is a lower bound for the central derivatives $D^c(f_m)$ on $U_{\varepsilon'}$ and $D^c(f_m)^{-1}$ on $V_{\varepsilon'}$;*
(5) *$D^cf_m(x,y,z)=\alpha^{-1}$ on $U_{\varepsilon'}\cap\{\sqrt{x^2+y^2}\in[0,\frac{1}{100}]\cup[3,+\infty)\}$ and $D^cf_m^{-1}(x,y,z)=\alpha^{-1}$ on $V_{\varepsilon'}\cap\{\sqrt{x^2+y^2}\in[0,\frac{1}{100}]\cup[3,+\infty)\}$.*
(6) *there is $N\in\mathbb{N}$ such that*
   - *for any point $(x,y,z)\in U_{\varepsilon'}$ there are at most N number $i\in\mathbb{N}$ such that $D^cf_m(f_m^i(x,y,z))\neq\alpha^{-1}$;*
   - *for any point $(x,y,z)\in V_{\varepsilon'}$ there are at most N number $i\in\mathbb{N}$ such that $D^cf_m^{-1}(f_m^{-i}(x,y,z))\neq\alpha^{-1}$;*
(7) *For all $\alpha'>\alpha$, there is $N>0$ such that for any $n\geq N$*
   - *for any point $(x,y,z)\in U_{\varepsilon'}$, one has*
   $$D^cf_m^n(x,y,z)\in(\alpha'^{-n},\alpha^{-n}]$$
   - *for any point $(x,y,z)\in V_{\varepsilon'}$, one has*
   $$D^cf_m^{-n}(x,y,z)\in(\alpha'^{-n},\alpha^{-n}]$$

Proof : Most of these properties had been proven before. The times estimates $t \in [\frac{1}{5}, 3]$ comes from the fact that f_m is obtained as a barycenter along the orbits of diffeomorphisms satisfying the same time estimates (see item 2 of the list of properties in Section 7.4).

Notice that the set of points of $U_{\varepsilon'}$ where the central derivative $D^c f_m$ is different of α^{-1} is contained in $U_{\varepsilon'} \cap \{|x| \in [\frac{1}{6}, 2]\} \cap \{|y| \leq \varepsilon'\}$, so in this compact set there is a constant $\beta > 0$ which is a lower bound for the central derivatives $D^c(f_m)$. It follows that there is $N \in \mathbb{N}$ such that for any point $(x, y, z) \in U_{\varepsilon'}$ there are at most N number $i \in \mathbb{N}$ such that $f_m^i(x, y, z)) \in U_{\varepsilon'} \cap \{|x| \in [\frac{1}{6}, 2] \cap \{|y| \leq \varepsilon'\}$, then there are at most N number $i \in \mathbb{N}$ such that $D^c f_m(f_m^i(x, y, z)) \neq \alpha^{-1}$. It follows that for all $\alpha' > \alpha$, there is $N > 0$ such that for any $n \geq N$ for any point $(x, y, z) \in U_{\varepsilon'}$ one has

$$D^c f_m^n(x, y, z) \in (\alpha'^{-n}, \alpha^{-n}].$$

The proofs for $V_{\varepsilon'}$ are analogous.

□

Corollary 7.2. *Choosing δ small enough, we have that for any $(x, y, z) \in U_\varepsilon$ if $f_m(x, y, z) = X_t(x, y, z)$ then $t \geq \frac{3}{10}$ and for any $(x, y, z) \in V_\varepsilon$ if $f_m^{-1}(x, y, z) = X_s(x, y, z)$ then $s \leq -\frac{3}{10}$.*

Proof : This is a consequence of item 2 of the previous proposition and the fact that the return times of $P_{0,\varepsilon}$ on $\Sigma_{0,\varepsilon}^+$ and $\Sigma_{0,\varepsilon}^-$, and those of $P_{1,\varepsilon}^{-1}$ on $\Sigma_{1,\varepsilon}^+$ and $\Sigma_{1,\varepsilon}^-$, are strictly larger than $\frac{4}{10}$. Then $t \geq \frac{4}{10} - 2\delta$. □

7.6. **The model- diffeomorphism.** We are going to define our final local diffeomorphism $f_{mod} : \mathbb{R}^2 \times S^1$ which will be our model of Axiom A diffeomorphism in a neighborhood of a connected component of the boundary of the Birkhoff section.

The aim is to glue the map f_m that we have already defined with a new map g, which will be the representant of f_{ext} in $\mathbb{R}^2 \times S^1$, when $\sqrt{x^2 + y^2} > 20$.

Recall that f_m restricted to $\sqrt{x^2 + y^2} > 20$ verifies that $f_m(x, y, z)$ is $f_0^{++}(x, y, z)$, $f_0^{+-}(x, y, z)$, $f_0^{-+}(x, y, z)$ or $f_0^{--}(x, y, z)$ according to the quadrant containing (x, y, z), and in this region $\Sigma_0 = S_0$ and $\Sigma_1 = S_1$.

Let $p = (x, y, z) \in \Sigma_0 \cap \{\sqrt{x^2 + y^2} > 20\}$ and let I_p be the segment of orbit joining p with $P_0(p) \in \Sigma_0$, then $f_m(I_p) = I_{P_0(p)}$. Moreover, f_m restricted to $I_p \cap U_{\varepsilon'}$ is contraction of factor α^{-1} and $(f_m)^{-1}$ restricted to $I_p \cap V_{\varepsilon'}$ is contraction of factor α^{-1}.

Now we are going to define the map $g : \bigcup_{\{p \in \Sigma_0 \cap \{\sqrt{x^2+y^2} > 20\}\}} I_p \to \mathbb{R}^2 \times S^1$ by $g(I_p) = I_{P_0(p)}$. Furthermore, the expression of g in restriction to the segment I_p is the same as the expression of f_{ext}: since I_p contains a unique point $q \in S_1$. Let $r, s, r', s' > 0$ such that $q = X_r(p)$, $P_0(p) = X_s(q)$ $P_1(q) = X_{r'}(P_0(p))$ and $P_0^2(p) = X_{s'}(P_1(q))$. Then

- for $t \in [0, r]$ $g(X_t(p)) = X_{t'}(P_0(p))$ with $t' = \Theta_{r,r'}(t)$
- for $t \in [0, s]$ $g(X_{-t}(P_0(p)) = X_{-t'}(P_0^2(p))$ with $t' = \Theta_{s,s'}(t)$

where $\Theta_{r,r'}$ had been defined in section 5.1.

Definition 7.3. *We call* model diffeomorphism *and we denote by f_{mod} the diffeomorphism defined as follows:*

Let $(x, y, z) \in I_p$; where $p = (x_1, y_1, z_1) \in \Sigma_0$ and t_1, t_2 such that:

- $f_m(x, y, z) = X_{t_1}(P_0(x_1, y_1, z_1))$.

- $g(x,y,z) = X_{t_2}(P_0(x_1,y_1,z_1))$.

Then $f_{mod}(x,y,z) = X_r(P_0(x_1,y_1,z_1)$ *where* $r = \psi(\frac{\sqrt{x_1^2+y_1^2}}{100})t_1 + (1-\psi(\frac{\sqrt{x_1^2+y_1^2}}{100}))t_2$.

Lemma 7.17. *The map f_{mod} defined above is well defined on $\mathbb{R}^2 \times S^1$ and it is a diffeomorphism of $\mathbb{R}^2 \times S^1$. Furthermore U_ε is an attracting region for f_{mod} and V_ε is an attracting region for f_{mod}^{-1}.*

Proof : Just notice that f_m and g are diffeomorphisms that send any central arc of the form I_p, where $p \in \Sigma_0$, to $I_{P_0(p)}$ and the coefficient of the barycenter is constant on each segment I_p. This property ensures the continuity and the differentiability out of the sections Σ_0. We get the differentiability on Σ_0 by noticing that f_m and g coincide on U_ε. This also implies that U_ε is an attracting region for f_{mod}.

The fact that f_m and g coincide on $f_m(V_\varepsilon)$ implies that V_ε is an attracting region for f_{mod}^{-1} as it is for f_m^{-1}.

□

Now we are in conditions to prove lemma 5.3

Proof of lemma 5.3 Let $U_{mod} = U_\varepsilon$ and $V_{mod} = V_\varepsilon$.

The items 1, 2, 6, 7 and 8 are verified by f_m and by g so they are verified by f_{mod}. Item 3 is just the definition of $U_\varepsilon = U_{mod}$ and $V_\varepsilon = V_{mod}$. Item 4 is because the restriction of f_{mod} to $(0,0) \times S^1$ coincide the restriction of f_m to $(0,0) \times S^1$ which is f_0. Item 5 is just by definition of g.

8. Dynamical properties of the diffeomorphisms f_r

8.1. The dynamics of f_r and the pseudo-Anosov map on the Birkhoff section.

The aim of this section is to describe dynamical properties of f_r (build in Section 5.3) which are not direct consequences of Lemma 5.3: these properties are more related with the precise construction of the model. More precisely:

Let N be the closed surface obtained from the Birkhoff section B_0 by identifying each boundary component with a point. We denote by $Sing \subset N$ the finite set whose points p_γ correspond to the boundary components γ of B_0 . Fried showed in [6] that the first return map P_{B_0} induces on N a pseudo-Anosov homeomorphisms $P_N \colon N \to N$, whose singular points are contained in $Sing$. We will show:

Proposition 8.1. (1) *For every large r, the Axiom A diffeomorphism f_r has exactly two basic pieces, one of them is an attractor and the other is a repeller; more precisely, in the notation of Section 5.3, the maximal invariant sets A_r and R_r of f_r in U_r and V_r, respectively, are connected transitive (indeed mixing) attractor and repeller, respectively;*

(2) *There is a continuous surjective projection $\pi_{A_r} \colon A_r \to N$ which induces a semi-conjugacy between $f_r|_{A_r}$ and P_N. Furthermore*
   - *the preimage $\pi_{A_r}^{-1}(p)$ is a single point if p does not belong to the unstable manifold of a point in Sing;*
   - *the preimage $\pi_{A_r}^{-1}(p)$ consists in exactly two points in the same central leaf if p belongs to any unstable separatrix of a point in Sing;*

- *and the preimage $\pi_{A_r}^{-1}(p)$ consists in $2n$ periodic points if $p = p_\gamma \in Sing$ where the linking number of γ is $\pm n$; in this case, the preimage of each unstable separatrices of p_γ consist of the union of two unstable separatrices corresponding to two successive periodic points in $\pi_{A_r}^{-1}(p_\gamma)$.*

Remark 16. *Item 2 of Proposition 8.1 means that A_r is obtained from the pseudo Anosov map P_N by* "opening" *the unstable separatrices of the periodic point in Sing. That is, the attractor A_r is obtained as a* "derived from pseudo-Anosov map" *which is the analogous of the derived from Anosov construction in* [13].

We start the proof by building a surface B_Σ, with boundary and corners, obtained by cutting the Birkhoff section B_0 along the local unstable manifolds of the orbits γ in ∂B.

8.2. The surface B_Σ.

Let us denote B_Σ the surface with boundary and corners which coincides with B_0 out of $\bigcup_\gamma O_\gamma$ and with $(h_r \circ \Gamma_\gamma)^{-1}(\Sigma_{0,n})$ on O_γ, where n is the linking number of γ. We denote by $P_\Sigma \colon B_\Sigma \to B_\Sigma$ the map which coincide with P_{B_0} outside O_γ and which is induced (via conjugacy by $h_r \circ \Gamma_\gamma$) by the map P_0 on each O_γ (recall that P_0 is the map built in Section 6.3, which coincides with P_0^+ on Σ_0^+ and with P_0^- on Σ_0^-).

Let U_Σ be the union of the orbit segments of length 2δ centered at the points of B_Σ.

Lemma 8.1. *$A_r \subset U_\Sigma$.*

Proof : Let denote $B_{\Sigma,\varepsilon}$ the branched surface with boundary and corners which coincides with B_0 out of $\bigcup_\gamma O_\gamma$ and with $(h_r \circ \Gamma_\gamma)^{-1}(\Sigma_{0,n,\varepsilon})$ on O_γ, where n is the linking number of γ and $\Sigma_{0,n,\varepsilon}$ is a lift of $\Sigma_{0,\varepsilon}$ to $\mathbb{R}^2 \times \mathbb{R}/|n|\mathbb{Z}$. Let $U_{\Sigma,\varepsilon}$ be the union of the orbit segments of length 2δ centered at the points of $B_{\Sigma,\varepsilon}$. Then for all ε there is $\varepsilon' < \varepsilon$ such that $P_{0,\varepsilon} \colon \Sigma_{0,n,\varepsilon} \to \Sigma_{0,n,\varepsilon'}$ and therefore one has that $f_r(U_{\Sigma,\varepsilon}) \subset U_{\Sigma,\varepsilon'}$. One deduces that the maximal invariant set of f_r in $U_{\Sigma,\varepsilon}$ is contained in $U_{\Sigma,0} = U_\Sigma$. □

Lemma 8.2. *Let $p, q \in B_\Sigma$. Assume that there is $t \in [0, 2\delta]$ such that $X_t(p) = q$. Then either $p = q$ or p and q belong to ∂B_Σ.*

Proof : The statement is true by construction if p or q are not in O_γ for some γ (δ is much smaller than the times return on B_0). Hence, it is enough to verify the same statement for the lift $\Sigma_{0,n}$ of the model, then it is enough to prove it for the model Σ_0 on $\mathbb{R}^2 \times S^1$. The statement is true if both p and q belong to Σ_0^+ or if both belong to Σ_0^-. So we can assume $p \in \Sigma_0^+$ and $q \in \Sigma_0^-$ (the case $p \in \Sigma_0^-$ and $q \in \Sigma_0^+$ is analogous). The fact that p and q are on the same orbit segment of X_{mod} implies that p and q belong to the plane $y = 0$ which is the unstable manifold of the periodic orbit of X_{mod}. If p or q belong to the intersection $\Sigma_0^+ \cap \Sigma_0^-$, then $p = q$ because, by construction , the first return time of X_{mod} of such a point in Σ_0 is much larger than 2δ. So $p \in \Sigma_0^+ \cap \{(x, y, z), y = 0, x \in [-\frac{2}{3}, \frac{2}{3}]\}$ and $q \in \Sigma_0^- \cap \{(x, y, z), y = 0, x \in [-\frac{2}{3}, \frac{2}{3}]\}$, that is p and q belong to the boundary of Σ_0. □

8.3. **Projections of the surface B_Σ on B_0 and on N.** Notice that the surfaces Σ_0^+ and Σ_0^- have been obtained by pushing the points of S_0^+ and S_0^- along the orbits of $Xmod$. The inverse of this construction induces diffeomorphisms $\pi_0^+\colon \Sigma_0^+\setminus\{(0,0)\}\times S^1 \to S_0^+\setminus\{(0,0)\}\times S^1$ and $\pi_0^-\colon \Sigma_0^-\setminus\{(0,0)\}\times S^1 \to S_0^-\setminus\{(0,0)\}\times S^1$. One easily verifies that $\pi_0^+ = \pi_0^-$ on $\Sigma_0^+\cap\Sigma_0^-$, inducing a diffeomorphism $\pi_0\colon \Sigma_0\setminus\{(0,0)\}\times S^1 \to S_0\setminus\{(0,0)\}\times S^1$. Furthermore, π_0 is the identity map on $\Sigma_0\cap S_0$ that is for $\sqrt{x^2+y^2}\geq \frac{2}{3}$.

Let $\tilde\pi_\Sigma\colon B_\Sigma\setminus\partial B_0 \to B_0\setminus\partial B_0$ be the map which is the identity map out of the O_γ, and which is induced by the map π_0 built above, in each of the O_γ. Let us state some properties of $\tilde\pi_\Sigma$:

- $\tilde\pi_\Sigma(p) = X_t(p)$ for some $t\in[-\frac{1}{4},\frac{1}{4}]$, for every $p\in B_\Sigma\setminus\partial B_0$.
- if $\tilde\pi_\Sigma(p)=\tilde\pi_\Sigma(q)$ for $p\neq q$ then p and q belong to the boundary of B_Σ.

Notice that $\tilde\pi_\Sigma$ may be seen as a continuous projection from $B_\Sigma\setminus\partial B_0$ on $N\setminus Sing = B_0\setminus\partial S_0$. One easily verifies that $\tilde\pi_\Sigma$ extends in a unique way in a projection $\pi_\Sigma\colon B_\Sigma\to N$.

Remark 17.
- *π_Σ is injective on $B_\Sigma\setminus\partial B_\Sigma$.*
- *$P_N\circ\pi_\Sigma = \pi_\Sigma\circ P_\Sigma$, that is π_Σ induces a semi conjugacy between P_Σ and the pseudo-Anosov map P_N*

Corollary 8.1. *Consider a segment of X-orbit I included in U_Σ. Then $I\cap B_\Sigma$ consists in at most 2 point p, q ($p\neq q \Rightarrow p,q\in\partial B_\Sigma$) whose images by π_Σ are equal $\pi_\Sigma(p)=\pi_\Sigma(q)$*

Let $\pi_{U_\Sigma}\colon U_\Sigma\to N$ be the map defined as follows: for every $q\in U_\Sigma$ there is a point $p\in B_\Sigma$ such that q belongs to the orbit segment of length 2δ centered at p. One defines $\pi_{U_\Sigma}(q)=\pi_\Sigma(p)$. The corollary above implies that this point is well defined. Furthermore

Corollary 8.2. *The map $\pi_{U_\Sigma}\colon U_\Sigma\to N$ is continuous and induces a semi conjugacy between the restriction of f_r to U_Σ and the pseudo Anosov homeomorphisms P_N*

Proof : The semi conjugacy property is easily deduce from the semi conjugacy property of π_Σ and the fact that, if q belongs that the orbit segment of length 2δ centered at $p\in B_\Sigma$ then $f_r(q)$ is contained in the orbit segment of length $2\alpha^{-1}\delta$ centered at $P_\Sigma(p)$. □

8.4. Proof of Proposition 8.1.

8.4.1. *Transitivity and semi-conjugacy with the pseudo-Anosov map.* The diffeomorphism f_r satisfies the Axiom A and strong transversality. Furthermore the compact set $U_{\Sigma,\varepsilon}$ is an attracting region (its image is included in its interior). So it contains a basic piece of f_r which is a transitive attractor Λ; it is contained in the maximal invariant set A_r and hence contained in U_Σ.

The attractor Λ cannot be a periodic point because f_r is partially hyperbolic and the strong unstable bundle has dimension equal to 1. Recall that, by Claim 2 of the proof of Proposition 5.1 the dimension of the stable direction of every point in A_r is 2. So the unstable manifold of every point in Λ has dimension 1. Consider any point $p\in\Lambda$. As Λ is an attractor, the unstable manifold $W^u(p)$ is contained in Λ hence in U_Σ.

Consider a point $p\in\Lambda$ which does not belong to the unstable manifold of a periodic point $q\in\partial B_\Sigma$, and let us denote by $\Lambda_0\subset\Lambda$ the closure of $W^u(p)$. Recall that every basic piece of an Axiom A diffeomorphism splits in a disjoint union of compact sets which are

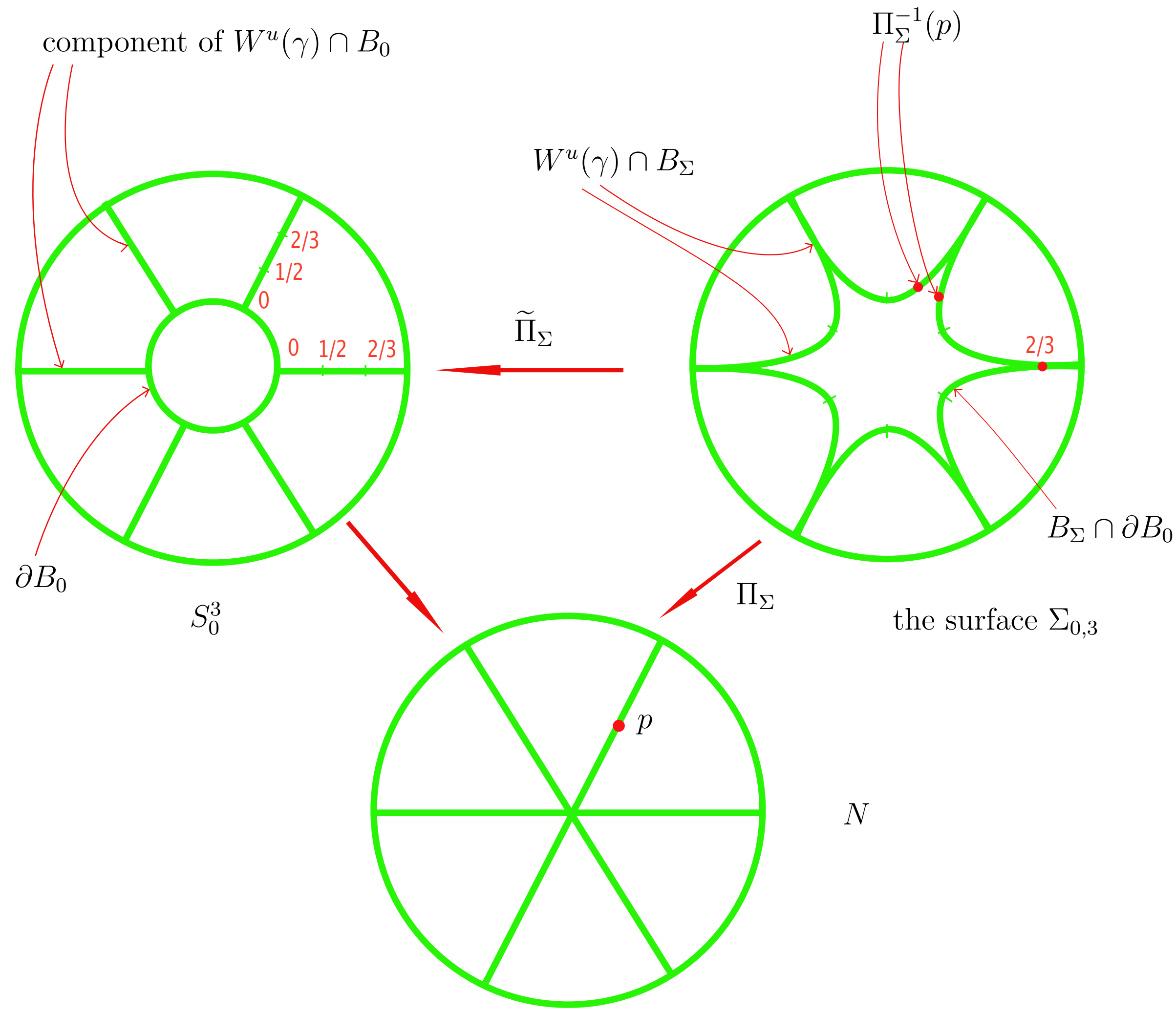

FIGURE 14. The surface N in a neighborhood of a 6-prongs singularity, the surface $S_{0,3}$(B_0 in a neighborhood of γ with $n(\gamma, B) = 3$) and the surface $\Sigma_{0,3}$.

cyclically permuted by the diffeomorphism, and such that the return map is mixing (see [15]); the compact sets of this decomposition will be called the mixing components of the basic piece. Furthermore, these mixing components of a basic piece are built as follows: consider a point in the basic piece, the corresponding mixing component is the closure of the transverse intersection between its invariant (stable and unstable) manifolds. So Λ_0 is the mixing component of Λ which contains p; let i denote the period of Λ_0.

Using Corollary 8.2, one deduces that the projection $\pi_{U_\Sigma}(W^u(p))$ is a (regular) leaf of the unstable foliation of P_N. As P_N is a pseudo-Anosov map, every unstable leaf is dense in N. As a consequence, one gets that $\pi_{U_\Sigma}(\Lambda_0) = N$.

This implies that, for every $p \in B_\Sigma \setminus \partial B_\Sigma$, the orbit segment of length 2δ centered at p contains at least a point in Λ_0. By compactness of Λ_0 it follows that for every $p \in B_\Sigma$, the orbit segment of length 2δ centered at p contains at least a point in Λ_0.

Assume now that $K \subset A_r$ is a compact set which is invariant by f_r^j for some $j > 0$ with $K \neq \Lambda_0$ and let q be a point of K. As $K \subset A_r \subset U_\Sigma$ there is $q_0 \in B_\Sigma$ such that q belongs to the orbit segment I of length 2δ centered at q_0, and this segment contains a point $p \in \Lambda_0$. Now the iterates $f_r^{nj}(I)$, $n \in \mathbb{N}$ contain the points $f_r^{nj}(q) \in K$ and $f_r^{nj}(p) \in \Lambda_0$; furthermore the length $\ell(f_r^{nj})(I)$ is bounded by $2\delta\alpha^{-nj}$, hence tends to 0. This implies that $K \cap \Lambda_0 \neq \emptyset$. This implies that U_Σ does not contain any other mixing component of a basic piece than Λ_0, furthermore implies that $\Lambda = \Lambda_0$, that is a connected mixing attractor.

This ends the proof of item 1) of Proposition 8.1: the compact set A_r is the unique (connected and mixing) attractor of f_r.

8.4.2. *Injectivity defect of the semi-conjugacy.* Let us now prove item 2). We have already seen that the restriction of π_{U_Σ} to A_r is a continuous surjection on N inducing a semi conjugacy between f_r and P_N. Let us prove that the restriction of π_{U_Σ} to A_r is injective on the complement of the unstable manifolds of the periodic orbits in $A_r \cap \partial B_0$.

Claim 1. *Let $p, q \in B_\Sigma$ be two different points, and let $I \subset U_\Sigma$ and $J \subset U_\Sigma$ be the maximal orbit segment containing p and q respectively (recall that these segments have length bounded by 4δ) Assume that $f_r(I)$ and $f_r(J)$ are contained in a same orbit segment $L \subset U_\Sigma$. Then $p \in \partial B_\Sigma$ and $q \in \partial B_\Sigma$.*

Proof : By construction, $f_r(I)$ and $f_r(J)$ are contained in the orbit segment in U_Σ through the points $P_\Sigma(p)$ and $P_\Sigma(q)$, respectively. Now Lemma 8.2 implies that either $P_\Sigma(p) = P_\Sigma(q)$ or these points belong to the boundary of B_Σ. In both cases, p and q belong to ∂B_Σ: the map P_Σ is injective out of ∂B_Σ and the pre-image of ∂B_Σ is contained in ∂B_Σ. □

Claim 2. *For every connected component $\gamma \subset \partial B_0$ the intersection $W^u(\gamma) \cap A_r$ is the union of the (1-dimensional) unstable manifolds $W^u(p)$ for $p \in Per(f) \cap A_r \cap \gamma$.*

Proof : If $q \in W^u(\gamma) \cap A_r$ then the α-limit of q is equal to one of the periodic orbit contained in γ (because γ is a normally hyperbolic invariant circle and the restriction of f_r to γ is Morse-Smale). As the α-limit is contained in A_r we proved the claim. □

Consider now a point $p \in A_r$ which does not belong to the unstable manifold of a point in $Per(f_r) \cap \partial B_0 \cap A_r$. Let $q \in B_\Sigma$ such that p belongs to the maximal orbit segment $I \subset U_\Sigma$ containing q. We will show

Claim 3. *$I \cap A_r = \{p\}$.*

Proof : For that we consider a point $p_1 \in A_r \cap I$

According to Claim 2, the segment I is disjoint from $W^u(\gamma)$ for all $\gamma \subset \partial B_0$. So Claim 1 implies that $f_r^{-1}(I)$ contains a unique point q_1 in B_Σ. One deduces by Claim 1 that $f_r^{-1}(p)$ and $f_r^{-1}(p_1)$ belongs to the the maximal orbit segment in U_Σ containing q_1. Hence we can iterate the process, building a sequence of points $q_n \in B_\Sigma$ such that the maximal orbit segment I_n in U_Σ containing q_n contains $f_r^{-n}(p)$ and $f_r^{-n}(p_1)$. Let $J_n \subset I_n$ be the orbit segment joining $f_r^{-n}(p)$ to $f_r^{-n}(p_1)$. As J_n is contained in U_Σ we get that the length of these segments satisfies: $\ell(J_{n-1}) \leq \alpha^{-1}\ell(J_n) \leq \alpha^{-1}4\delta$. So $\ell(J_0) \leq 4\delta\alpha^{-n}$ for every n, that is $p = p_1$. This ends the proof of the claim. □

This proves that the projection π_{U_Σ} is injective on the complement of the unstable manifolds of the periodic orbits in $A_r \cap \partial B_0$. For ending the proof of the proposition it remains to consider the restriction of π_{U_Σ} to the intersection of U_Σ with the unstable manifold $W^u(\gamma)$ where γ is a component of ∂B_0.

8.4.3. *Dynamics of the repeller.* Note that in an analogous way it can be proved that there is a continuous surjective projection $\pi_{R_r}\colon R_r \to N$ which induces a semi-conjugacy between $f_r|_{R_r}$ and P_N. Furthermore

- the preimage $\pi_{R_r}^{-1}(p)$ is a single point if p does not belong to the stable manifold of a point in $Sing$;
- the preimage $\pi_{R_r}^{-1}(p)$ consists in exactly two points in the same central leaf if p belongs to one stable separatrix of a point in $Sing$;
- and the preimage $\pi_{R_r}^{-1}(p)$ consists in $2n$ periodic points if $p = p_\gamma \in Sing$ where the linking number of γ is $\pm n$; in that case, the preimage of each stable separatrices of p_γ consist of the union of two stable separatrices corresponding to two successive periodic points in $\pi_{R_r}^{-1}(p_\gamma)$.

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