GRAPHS OF FREE GROUPS AND THEIR MEASURE EQUIVALENCE

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This work concerns the Geometric Group Theory of an interesting class of groups that can be obtained as graphs of free groups. These groups are called *Quadratic Baumslag–Solitar groups*, and are defined by graphs of groups that have infinite cyclic edge groups, and whose vertex groups are either infinite cyclic, or surface groups $\pi_1(S)$ that are "attached by their boundary", meaning that the edge groups of the adjacent edges correspond to the subgroups generated by the boundary classes of S. More generally, we may also take S to be a 2-orbifold.

The first part of the thesis studies JSJ decompositions for groups. We prove that, in most cases, the defining graph of groups of a Quadratic Baumslag–Solitar group is a JSJ decomposition in the sense of Rips and Sela [36]. This generalizes a result by Forester [11].

The second part studies measure equivalence between groups. It involves the concept of measure free factors of a group, which is a generalization of that of free factors, in a measure theoretic context. We find new families of cyclic measure free factors of free groups and some virtually free groups, following a question by D. Gaboriau [16].

Then we characterize the Quadratic Baumslag–Solitar groups that are measure equivalent to a free group, according to their defining graphs of groups.

BIOGRAPHICAL SKETCH

Juan Francisco Alonso was born in Montevideo, Uruguay on November 4, 1981. He attended Universidad de la República, Montevideo, Uruguay. He enrolled in 2000 and graduated in 2005 with the title of Licentiate in Mathematics.

In 2006 he entered Cornell University as a graduate student. He obtained his PhD from Cornell in 2012.

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CHAPTER 1
INTRODUCTION

The present work features some advances in two main areas of Geometric Group Theory. One of them is the JSJ decomposition for groups. The other is within the theory of measure equivalence, specifically studying the groups that are measure equivalent to a free group and their measure free factors.

The JSJ decomposition depends on Bass–Serre theory, which deals with *graphs of groups* (Definition 2.1.2). These are graphs whose edges and vertices are assigned groups in such a way that each edge group injects into the groups of the adjacent vertices. Such a structure naturally gives the data for an associative iteration of amalgamated products and HNN extensions, whose result is called the *fundamental group* of the graph of groups in question.

To understand a group G, it is often useful to decompose it as an amalgamated product or an HNN extension over a subgroup that belongs to a well-understood class of groups, such as trivial groups, finite groups or cyclic groups. More generally, consider all possible factorizations of G as a graph of groups with edge stabilizers in some single class of groups.

For various specified families of edge groups, it is possible to show the existence of a single graph of groups decomposition, from which all of these factorizations can be obtained. This is called a *JSJ decomposition* of G (over subgroups in the given class), although the notion is imprecise on how the other factorizations of G are to be obtained from the JSJ decomposition. A classical example is the case over trivial edge groups. All the maximal decompositions of a finitely generated group G over the class of trivial groups can be obtained from the Grushko decomposition of G (i.e. the complete factorization with respect to the free product).

The letters JSJ stand for Jaco, Shalen and Johannson. Their results in [23] and [24] can be interpreted as proving the existence of a JSJ decomposition for 3-manifold groups over subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$. It was these works that motivated the study of JSJ decompositions over non-trivial subgroups (i.e. aside form the Grushko decomposition). Various existence theorems were obtained by Kropholler [30], Rips and Sela [38], [36], Bowditch [3], Dunwoody and Sageev [6], Fujiwara and Papasoglu [12], Dunwoody and Swenson [7] and Scott and Swarup [37]. In [18] and [19] Guirardel and Levitt propose a precise definition of JSJ decomposition, which is verified by the graphs of groups constructed in most of the works mentioned here. Notions of JSJ decomposition that do not fit their definition include the constructions in [7] and [37], as well as the compatibility JSJ decomposition in [20].

In this work we focus on the JSJ decomposition as defined by Rips and Sela [36], for finitely presented one-ended groups, with infinite cyclic edge stabilizers (stated here as Theorem 3.2.4).

It is not always clear how to recognize whether a given graph of groups is a JSJ decomposition of its fundamental group. This is the problem that will be treated in this work. In [11], Forester studied the *Generalized Baumslag–Solitar (GBS) groups*, which are defined by graphs of groups whose vertex and edge stabilizers are infinite cyclic. He showed that the defining graph of a GBS group is a JSJ decomposition, under mild hypotheses, see 5.1.3 and 5.1.4.

In the present work we introduce the *Quadratic Baumslag–Solitar (QBS) groups*. They are defined by graphs of groups whose edge groups are infinite cyclic, and whose vertex groups can be either infinite cylic or *quadratically hanging* surface groups. (For the meaning of quadratically hanging see Definition 3.1.1). It is clear that the GBS groups defined by Forester [11] are a subclass of the QBS groups. This concrete class of groups will be used as the main example and application of our techniques in the mentioned areas.

We show that, under some conditions, the defining graph of a QBS group is a JSJ decomposition, generalizing the result of Forester for GBS groups [11]. Specifically, we prove the following. (The definitions for the new terms are referred below).

Theorem. 5.2.2 *Let Γ be a QBS graph, $G = \pi_1(\Gamma)$. Suppose that Γ is reduced, has no leaves, and satisfies the following conditions:*

1. *Each edge e of Γ has integer labels $m_e^+, m_e^- > 1$.*
2. *Each GBS component Γ_i of Γ is reduced, and T_{Γ_i} is not a point or a line.*

Then Γ is a Rips-Sela JSJ decomposition for G

A *QBS graph* is the defining graph of a QBS group. See section 2.2 for the definition of a reduced graph of groups. For the edge labels, see Definition 4.2.1. The GBS components and the leaves of a QBS graph are defined at the begining of section 5.2.

To obtain this theorem, we start from Forester's result about GBS graphs (5.1.3, 5.1.4), and apply a combination of Theorems 4.2.5 and 4.1.2. These theorems deal with *universality* (condition 4 in Theorem 3.2.4) and *unfoldedness* (Definition 2.2.1), that are the two main conditions for a graph of groups (with cyclic edge groups) to be a JSJ decomposition (in the sense of Rips and Sela). Theorem 4.1.2

gives a general criterion for unfoldedness, while Theorem 4.2.5 is used to show the universality of a graph of groups, from the universality of certain subgraphs. This same combination of Theorems 4.2.5 and 4.1.2 admits other applications than the one treated in this thesis. It may be used to recognize a graph of groups as a JSJ decomposition of its fundamental group, provided it arises from known JSJ decompositions by adding more quadratically hanging surface vertices (see Definition 3.1.1).

On the other hand, the notion of *measure equivalence* between groups was introduced by M. Gromov [17], as an analog of quasi-isometry in the context of actions on measure spaces. It has been widely studied since then, for example the works of Dye [8], [9], Ornstein and Weiss [33], and Furman [13] characterized the measure class of $\mathbb{Z}$, found to be exactly the infinite amenable groups. This motivates the study of which groups are measure equivalent to the non-abelian free groups, F_n with $n \geq 2$ (which are all virtually isomorphic, thus measure equivalent).

We will be working with the notion of *treeability*, in the sense of Pemantle and Peres [35], which is equivalent to being measure equivalent to a free group, as shown by G. Hjorth [22].

The concept of *measure free factor* (Definition 7.1.3) was introduced by D. Gaboriau in [16], as a tool for the study of measure equivalence. He was able to find many groups that are measure equivalent to the free group F_2, by showing that this class is closed under certain amalgamated products. Namely, those amalgamations $A *_C B$ where A and B are measure equivalent to F_2 and C is a measure free factor of either A or B (see 7.1.6 for the precise statement). In this work, Gaboriau [16] poses the question of which elements of F_2 generate a cyclic measure free factor.

He shows that such an element cannot be a proper power, and also finds the first non-trivial example (see 7.1.4).

Gaboriau's results were then used by M. Bridson, M. Tweedale and H. Wilton [4] to prove that hyperbolic *limit groups* (defined by Z. Sela [39]) are measure equivalent to F_2. This provides a wide set of examples of groups in the measure equivelence class of F_2, and it naturally raises the question of whether all non-abelian limit groups are in this class. The importance of measure free factors in the study of this problem is explained in that same work [4], and it arises from the structure of limit groups as iterated amalgamations and HNN extensions, found by Z. Sela [39] and O. Kharlampovich and A. Myasnikov [27], [28]. Thus, new measure free factors of free or limit groups give rise to more limit groups in the measure class of F_2.

This work advances the study of measure free factors of free groups, by finding some new infinite families of elements of F_n that generate cyclic measure free factors of F_n. Namely, we prove the following.

Theorem. 8.3.1 *Let $F = \langle x, y_1, \ldots, y_k \rangle$ be a free group of rank $k+1$. Then an element of the form*

$$w = xy_1x^{m_1}y_1^{-1}y_2x^{m_2}y_2^{-1}\cdots y_kx^{m_k}y_k^{-1}$$

generates a measure free factor of F, where $m_1, \ldots, m_k$ are arbitrary integers.

Theorem. 8.3.2 *Let $G = F_2 = \langle a, b \rangle$. Then any element of the form $w = a^kb^n$ for $k \neq 0$ and $n \neq 0$ generates a measure free factor of G.*

Since conjugates of measure free factors are also measure free factors, Theorem 8.3.2 gives that all the words of the form $a^kb^na^p$ with $n \neq 0, k \neq -p$ generate measure free factors of F_2. These are exactly all the *three-letter words* of F_2 that are not proper powers. In the special case of F_2, Theorem 8.3.1 says that the words of the form $aba^m b^{-1}$ generate measure free factors of F_2.

This produces new examples of groups that are measure equivalent to a free group, for instance the limit groups $F_n *_w F_n$ and $F_n *_w (\langle w \rangle \times \mathbb{Z}^m)$, where w is one of the elements mentioned in Theorems 8.3.1 or 8.3.2.

We also find measure free factors of some virtually free groups.

Theorem. 8.4.1 *Let $G = \langle a_1, \ldots, a_n, s_1, \ldots, s_k | s_1^{n_1} = 1, \ldots, s_k^{n_k} = 1 \rangle \cong F_n * \mathbb{Z}_{n_1} * \cdots * \mathbb{Z}_{n_k}$. If $v \in F_n$ generates a measure free factor of F_n, then $w = vs_1^{p_1} \cdots s_k^{p_k}$ generates a measure free factor of G for any $p_1, \ldots, p_k$.*

The measure free factors obtained by Gaboriau [16] are the boundary subgroups of certain surface groups (orientable with positive genus, see 7.1.4). Theorem 8.4.1 allows us to generalize this to boundary subgroups of some 2-orbifold groups (with positive genus, see Corollary 8.4.2, or with many cone-points, see 9.3.4).

The main tool in the proof of these results is Theorem 8.1.2. It allows us to pass to finite index subgroups in the problem of checking if a cyclic subgroup is a measure free factor. Also, it provides a partial converse to the Kurosh theorem for Borel equivalence relations of A. Alvarez [1].

Finally, we apply our results about measure equivalence to the case of the QBS groups. We determine which QBS graphs have a treeable fundamental group (i.e. that is measure equivalent to a free group), as follows.

Theorem. 9.4.3 *Let Γ be a QBS graph, and $G = \pi_1(\Gamma)$. Then G is a treeable group if and only if all of the following conditions hold.*

1. *The EGBS components of* Γ *are either rooted trees, double or triple trees, or flowers. I.e. they define treeable GBS groups.*

2. *No EGBS component of* Γ *is adjacent to more than two QH vertices of type* $S_{2,2}$.

3. *If* Λ *is an EGBS component of* Γ *adjacent to two QH vertices* v_1, v_2 *of type* $S_{2,2}$*, then* Λ *is a rooted tree, the path* β *in* Λ *from* v_1 *to* v_2 *is made of simple edges, and the root of* Λ *can be taken in* β*. (I.e. all directed edges in* Λ *point towards* β*).*

4. *If* Λ *is an EGBS component of* Γ *adjacent to one QH vertex* v_1 *of type* $S_{2,2}$*, then* Λ *is a rooted tree, and the path* β *in* Λ *from* v_1 *to the root of* Λ *contains at most one directed edge, whose degree is* 2.

The definition of EGBS components can be found at the begining of section 9.4, and the one for vertices of type $S_{2,2}$ is right before Theorem 9.4.3. The graph types named *rooted trees*, *double/triple trees*, and *flowers* are introduced in section 9.2, as well as the types of edges involved in them. In particular, a GBS group is treeable if and only if it is amenable, and in that case it is isomorphic to either $\mathbb{Z}$, $\mathbb{Z}^2$ or the fundamental group of a Klein bottle.

We remark that Y. Kida [29] studied the general problem of the measure equivalence for the classical Baumslag–Solitar groups $BS(n,m)$. That is, the problem of whether $BS(n,m)$ is measure equivalent to $BS(n',m')$. He finds that, in most cases, measure equivelence between Baumslag–Solitar groups implies isomorphism.

The chapters are organized as follows.

Chapters 2 and 3 cover the background material on the JSJ decomposition: chapter 2 introduces the basics of Bass–Serre theory and chapter 3 gives the specifics of the Rips–Sela JSJ decomposition. In chapter 4 we discuss the conditions of universality and unfoldedness, proving Theorems 4.1.2 and 4.2.5. These results are applied in chapter 5 to the QBS groups. There we introduce the GBS and QBS groups, and recall Forester's results [11] about GBS groups. Then we prove that QBS groups are one-ended with the exeption of $\mathbb{Z}$ (Proposition 5.2.1), and give the proof of Theorem 5.2.2 mentioned above.

Chapters 6 and 7 provide background on the theory of measure equivalence that is relevant to this work. Specifically, chapter 6 gives an introduction to Borel equivalence relations, treeability and cost, as well as to the induced and coinduced actions. Chapter 7 recalls the definitions and previous results concerning measure free factors. Also in chapter 7, we define *common measure free factors* (Definition 7.2.1) and prove a version for HNN extensions (Proposition 7.2.3) of Gaboriau's theorem about amalgamations (Theorem 7.1.6).

Chapter 8 contains the proofs of the Theorems 8.3.1, 8.3.2 and 8.4.1 just mentioned, as well as that of Theorem 8.1.2 on which they depend. We also generalize the theorems of Gaboriau about the boundary subgroup of surface groups (7.1.4 and 7.1.5). This is done first for non-orientable surfaces (Lemma 8.2.1). Then we give a version for slightly more general systems of disjoint simple closed curves instead of the boundary subgroup (Proposition 8.2.2), and generalize it to some 2-orbifolds (Corollary 8.4.2). The final version of this series of generalizations is Corollary 9.3.4, which is obtained in chapter 9. Chapter 9 is devoted to Theorem 9.4.3 stated above. It also provides the necessary background on amenability and on 2-orbifold covers, including the proof of 9.3.4.

Chapters 6, 7 and 8 can be read independently from the previous chapters. Also, chapter 9 is mostly independent from chapters 2 through 5, except for the definitions of GBS and QBS graphs, and of edge labels (Definition 4.2.1).

CHAPTER 2

BASS–SERRE THEORY

The JSJ decomposition for groups is a topic in Bass–Serre theory. This theory studies groups with respect to their decompositions as graphs of groups (Definition 2.1.2), which can be seen as associative iterations of amalgamated products and HNN extensions. Equivalently, as the fundamental Theorem 2.1.6 will establish, it studies group actions on simplicial trees. This chapter provides a quick review of this theory, a comprehensive treatment can be found in the book by Serre [40].

2.1 Graphs of groups and the Bass–Serre theorem

First we recall the notions of *amalgamated products* and *HNN extensions*. These classical constructions motivated the theory of graphs of groups.

Definition 2.1.1. *Let A, B and C be groups, and $\alpha : C \to A$, $\beta : C \to B$ be injective homomorphisms. Assume also that A and B are given by the presentations $A = \langle S_1 | R_1 \rangle$, $B = \langle S_2 | R_2 \rangle$.*

1. *The* amalgamated product *of A and B over C is the group defined by*

$$A *_C B = \langle S_1, S_2 | R_1, R_2, \alpha(c) = \beta(c) \text{ for } c \in C \rangle$$

2. *When $A = B$, we can define the* HNN extension *of A over C by the presentation*

$$A *_C = \langle S_1, t | R_1, t\alpha(c)t^{-1} = \beta(c) \text{ for } c \in C \rangle$$

It is a well known result (see [42]) that the natural maps from the groups A, B and C to the amalgamated product or HNN extension are injective. Thus we will think of A, B and C as subgroups of their amalgamation or HNN extension. These constructions arise in algebraic topology, specifically in the computation of fundamental groups via Van Kampen's theorem. The amalgamated product is the group extension that appears in the statement of the Van Kampen theorem (see [21]). The HNN extension can be seen as the fundamental group of a mapping torus, corresponding to a partial homeomorphism on a space.

In the special case when C is trivial, the amalgamated product is just the free product $A * B$, and the HNN extension becomes $A * \mathbb{Z}$. For a non-trivial example, consider the Baumslag–Solitar groups, that are defined by the presentation

$$BS(n,m) = \langle a, b | ba^n b^{-1} = a^m \rangle$$

for $n \neq 0$ and $m \neq 0$. It is clear that $BS(n,m)$ is an HNN extension of the form $\mathbb{Z} *_{\mathbb{Z}}$ for the homomorphisms $\alpha(c) = c^n$, $\beta(c) = c^m$. Another example is the surface group corresponding to the closed orientable surface of genus 2, which has a presentation of the form

$$\langle a, b, c, d | [a, b][c, d] = 1 \rangle$$

This group decomposes as an amalgamated product of the form $F_2 *_{\mathbb{Z}} F_2$, where the first factor is generated by a, b and the second one by c, d.

For a *graph* we understand a pair of sets $\Gamma = (V, E)$, the vertex and edge set of Γ respectively, together with two maps $s, t : E \to V$, which give the *source* and *target* of an edge. If $e \in E$ is such an edge, the vertices $s(e)$, $t(e)$ will be called the *endpoints* of e. Thus our graphs have oriented edges, and admit loops (edges e with $s(e) = t(e)$) and multiple edges (different edges having the same endpoints). We will usually drop the maps s and t from the notation, thus we will say that an edge e has endpoints v^+, v^- to mean that $s(e) = v^-$ and $t(e) = v^+$.

Definition 2.1.2. *A* graph of groups *consists on the following:*

1. *A connected graph* Γ*.*
2. *A group* G_v *for each vertex* v *of* Γ*.*
3. *A group* G_e *for each edge* e *of* Γ*, and two injective homomorphisms*

$$\partial_e^+ : G_e \to G_{v^+}$$

$$\partial_e^- : G_e \to G_{v^-}$$

where v^+*,* v^- *are the endpoints of* e*.*

This is denoted by $(\Gamma, G, \partial^+, \partial^-)$, or simply by Γ. Note that even if the endpoints of an edge e agree, i.e. $v^+ = v^-$, there are still two different maps ∂_e^+ and ∂_e^-, one for the source and one for the target of e.

If T is a spanning tree for Γ, let $\pi_1(\Gamma, T)$ be defined by the following presentation.

- Generators: the elements of G_v for the vertices $v \in V(\Gamma)$, and an element t_e for each edge $e \in E(\Gamma)$, $e \notin T$.
- Relations: the relations in G_v for each vertex v, and

$$\partial_e^+(g) = \partial_e^-(g) \qquad \text{for } e \in T,\ g \in G_e$$

$$t_e \partial_e^+(g) t_e^{-1} = \partial_e^-(g) \qquad \text{for } e \in E(\Gamma),\ e \notin T,\ g \in G_e$$

This group is called the *fundamental group* of Γ. This notation is motivated by the special case with trivial vertex and edge groups, in which it agrees with the usual fundamental group of the graph. Notice that the fundamental group does not depend on the orientation of the edges. It is useful to have some orientation on the edges for Definition 2.1.2, but it does not matter which particular orientation we pick. The fundamental group does not depend on the spanning tree T either, see [40] for a proof.

Proposition 2.1.3. *If T, S are two spanning trees for Γ, then $\pi_1(\Gamma, T) \cong \pi_1(\Gamma, S)$.*

Thus we will often drop T from the notation. When G is a group and $G \cong \pi_1(\Gamma)$ for a graph of groups Γ, we say that Γ is a *splitting* of G. Note that one-edge splittings correspond to decompositions of G as an amalgamated product, in case the endpoints are different, or an HNN extension if the edge is a loop. In fact, this may be taken as the definition of amalgamated product or HNN extension.

A *subgraph* A of a graph Γ is a collection of vertices and edges of Γ such that every edge in A has its endpoints also in A. It is clear that A has a natural graph structure. Notice that any subset of edges of Γ defines a subgraph, consisting on those edges and their endpoints.

Definition 2.1.4. *Let Γ be a graph of groups and $A \subset \Gamma$ be a connected subgraph. We define the* collapse Γ/A *to be the graph of groups resulting from the following construction.*

1. *Remove from Γ all edges and vertices in A and replace them by a single vertex w. For the remaining edges, endpoints in A are replaced by w.*
2. *Put $G_w = \pi_1(A)$. The remaining vertices and edges of Γ keep the same vertex/edge groups.*
3. *Edge inclusion maps are the same as in Γ, possibly composed by an inclusion into $\pi_1(A)$.*

Topologically, the construction of Γ/A can be seen as collapsing the subgraph A to a point w. Collapsing isolated vertices has no effect. Thus we usually abuse notation and say that we collapse the edges in A. In particular, the collapse of a single edge e means the collapse of the subgraph formed by e and its endpoints.

Proposition 2.1.5. *If $A \subset \Gamma$ is a connected subgraph, then $\pi_1(\Gamma) \cong \pi_1(\Gamma/A)$.*

This gives some sort of associativity when computing the fundamental group of Γ. We say that Γ is a *refinement* of the splitting Γ/A. Through this proposition, we can see general splittings as iteration of amalgamated products and HNN extensions.

The other central concept in Bass–Serre theory is that of actions on simplicial trees. A *simplicial tree* is a CW complex of dimension 1 which is contractible. We call *vertices* and *edges* to its 0-cells and 1-cells respectively. When we say that a group G acts on a simplicial tree X, we assume that G acts by homeomorphisms, preserving vertices and edges. In this situation, we also say that X is a *simplicial G-tree*. For our uses, we will also require that if an element $g \in G$ stabilizes the endpoints of an edge e of X, then g fixes every point in e. Notice that we do not allow edge inversions.

The main point of Bass–Serre theory is to relate the splittings of a group with its actions on simplicial trees. Given an action $G \curvearrowright X$ on a simplicial tree, we can construct a graph of groups as follows.

- The underlying graph is $\Gamma = X/G$.
- If $\tilde{x} \in X$ is a vertex or edge, and $x \in \Gamma$ is its projection, then G_x is isomorphic to $\mathrm{Stab}_G(\tilde{x})$.

Since the action on X is without edge inversions, then the edges of $\Gamma = X/G$ can be given an orientation. We did not mention the edge maps, but they are straightforward. Namely, if we identifiy G_x with $\mathrm{Stab}_G(\tilde{x})$ for some specific $\tilde{x}$, these maps are subgroup inclusions possibly composed with conjugations. We will abuse notation and refer to this graph of groups as X/G.

The following is the main result in Bass–Serre theory. It establishes a correspondence between the splittings of a group and its actions on simplicial trees. It is crucial for the theory of JSJ decompositions, and will be used repeatedly in this work.

Theorem 2.1.6. *(Bass–Serre) Let G be a group. Then,*

1. *For every action of G on a simplicial tree X, we have that $\pi_1(X/G) \cong G$. (i.e. X/G is a splitting of G).*
2. *Given a splitting Γ of G, there exists an action of G on a simplicial tree T_Γ such that T_Γ/G is isomorphic to Γ.*

The tree T_Γ in the theorem is called the *Bass–Serre tree* of Γ.

We will give some examples, in order to illustrate the theorem. First, let Γ be a graph with only one edge e and two different endpoints v^-, v^+. Let $G_{v^+} = \langle a \rangle$, $G_{v^-} = \langle b \rangle$ and $G_e = \langle c \rangle$, all of them infinite cyclic. Put $\partial_e^+(c) = a^n$, $\partial_e^-(c) = b^m$ for $m, n \in \mathbb{Z}$ non-zero. Then $G = \pi_1(\Gamma)$ is an amalgamated product. In this case, the Bass–Serre tree T_Γ can be described as follows. Vertices projecting to $v^\pm$ correspond to left cosets of $G_{v^\pm}$ in G, and the edges correspond to the cosets of G_e. Adjacence between an edge and a vertex corresponds to inclusion of the respective cosets. For this particular example, vertices projecting to v^+ have valence n and those projecting to v^- have valence m. A vertex of the form gG_{v^+} is stabilized by the subgroup $gG_{v^+}g^{-1}$, and the element gag^{-1} permutes the neighboring edges cyclically. An analog is true for vertices projecting to v^-.

Bass–Serre trees are not locally finite in general, as the next example can show. Consider a closed orientable surface S, containing a system of disjoint simple closed curves $\alpha_1, \ldots, \alpha_n$ that are *essential*, i.e. not homotopic to a point. Also assume they are not homotopic to each other. Observe that we can *cut S along* $\alpha_1, \ldots, \alpha_n$, obtaining a disjoint union of surfaces with boundary, whose interiors agree with the components of $S \setminus (\alpha_1 \cup \ldots \cup \alpha_n)$. This gives rise to a graph of groups Γ, whose underlying graph is dual to the partition of S by the curves α_i. The edge groups are cyclic, each one generated by its corresponding class $[\alpha_i]$. The vertex groups are the fundamental groups of the components of S cut along $\alpha_1, \ldots, \alpha_n$. Van Kampen's theorem gives that Γ is a splitting of $G = \pi_1(S)$. The Bass–Serre tree can be viewed geometrically. Consider the universal cover of S. First assume that S is hyperbolic, so the universal cover can be identified with the disk $\mathbb{D}$. The preimage in $\mathbb{D}$ of each curve α_i consists on disjoint simple infinite curves that are made up of the different lifts of α_i. Together, these preimages cut $\mathbb{D}$ into regions that project to the components of S cut along $\alpha_1, \ldots, \alpha_n$. This division of $\mathbb{D}$ into regions has a dual graph, which can be identified with the Bass–Serre tree T_Γ. The action of G on $\mathbb{D}$ by deck transformations leaves the mentioned division invariant, and it induces the action on T_Γ.

In case S is the torus, it must be $n = 1$. The graph Γ corresponds to the HNN extension $\mathbb{Z}*_\mathbb{Z}$ where both edge maps are the identity. The Bass–Serre tree is a line, where the vertex group acts trivially and the stable letter acts as a translation.

Given a simplicial G-tree X, a subgroup $H \leq G$ *acts elliptically* on X, or is an *elliptic subgroup* with respect to X, if there is a point in X that is fixed by every element of H. Notice that if $X = T_\Gamma$ is the Bass–Serre tree of a graph of groups Γ, then a subgroup of $G = \pi_1(\Gamma)$ acts elliptically on X if and only if it is conjugate into one of the vertex groups of Γ. If an element $g \in G$ does not fix any point in X, it is said to act *hyperbolically*, or to be *hyperbolic* with respect to X.

Let X, Y be simplicial G-trees. A *morphism* $f : X \to Y$ is a G-equivariant map, which becomes simplicial after taking some subdivision of the edges of X. Recall that a map between trees is simplicial if it is continuous, and takes each simplex of the domain onto a simplex of the range by a linear map. Notice that edges may be collapsed to vertices. Morphisms are the natural notion of maps between G-trees. The following fact relates morphisms with elliptic subgroups.

Proposition 2.1.7. *There is a morphism $X \to Y$ if and only if every elliptic subgroup of X is also elliptic in Y.*

If Γ is a graph of groups and $A \subset \Gamma$ is a connected subgraph, then there is a morphism $T_\Gamma \to T_{\Gamma/A}$ from the Bass–Serre tree of Γ to the one of Γ/A, according to the proposition. Thus, the existence of a morphism between Bass–Serre trees $T_\Gamma \to T_{\Gamma^*}$ can be thought to be a generalization of Γ being a refinement of Γ^*.

2.2 Elementary deformations and foldings

Here we introduce some important transformations on graphs of groups.

Let Γ be a graph of groups. Let e be an edge of Γ and v^+, v^- its endpoints. First suppose that $v^+ \neq v^-$ and ∂_e^- is an isomorphism. That is, $G_e = G_{v^-} = C$ and $G_{v^+} = A$ with $C \subset A$. In this situation, the collapse of the edge e is called an *elementary collapse*. Note that v^+ and v^- are identified to a single vertex $\bar{v}$, and

$G_{\bar{v}} = A$ (through the isomorphism $A *_C C \cong A$).

The inverse of an elementary collapse is called an *elementary expansion*, and these transformations are the *elementary deformations*, which were introduced by Forester [10].

Now we consider another kind of transformations, called foldings. There are two types of folding associated to an edge e, depending on whether e has different endpoints or is a loop. Suppose that e has different endpoints and that $G_e = C \subset C_1 \subset A = G_{v^+}$ and $B = G_{v^-}$. Get Γ_1 from Γ by redefining $G_e = C_1$ and $G_{v^-} = C_1 *_C B$. We have $\pi_1(\Gamma) = \pi_1(\Gamma_1)$ by the isomorphism $A *_C B \cong A *_{C_1} (C_1 *_C B)$. In this case we say that Γ_1 is a *folding* of Γ, and that the folding occurs at the vertex v^+.

The other case of folding is when e is a loop, with vertex $v^+ = v^- = v$. Let $G_e = C$, $G_v = A$, and suppose that $\partial_e^+(C) \subset C_1 \subset A$. This time make Γ_1 with $G_v = A *_C t_e C_1 t_e^{-1}$ and $G_e = C_1$. The fundamental group is again preserved, and this transformation is also called *folding*. Making some abuse of notation, we say that the folding occurs at v^+ in the case just described, and at v^- if we use ∂_e^- instead.

Looking at the Bass–Serre trees, when there is a folding we have a map $T_\Gamma \to T_{\Gamma_1}$, simplicial and equivariant (in particular, it is a morphism). If $x \in T_\Gamma$ is a lift of v^+ with stabilizer gAg^{-1}, then this map identifies the edges coming from x and projecting to e, by the action of gC_1g^{-1}. Locally at x, it looks like "folding". In [2], Bestvina and Feighn explain this from the viewpoint of graphs of groups.

Notice that both elementary collapses and foldings are examples of morphisms between Bass–Serre trees.

If e is an edge of Γ, let Γ_e be the graph of groups obtained by collapsing the components of $\Gamma - e$. The next definition will be one of the defining conditions for the Rips–Sela JSJ decomposition.

Definition 2.2.1. *A splitting* Γ *is* unfolded *when either:*

1. Γ *has only one edge, and there is no folding onto it. That is, there is no* Γ_0 *such that* Γ *is obtained as a folding of* Γ_0*.*
2. Γ *has several edges, and* Γ_e *is unfolded for all of them.*

In general, it is hard to check whether a splitting is unfolded or not. Section 4.1 is devoted to this problem. Moreover, given a group, it is not obvious that it always admits an unfolded splitting. Theorem 3.2.4 establishes such existence for a wide class of groups.

CHAPTER 3

THE RIPS–SELA JSJ DECOMPOSITION

Now we discuss the version of JSJ decomposition that concerns us. This is the Rips–Sela JSJ decomposition, named after Theorem 3.2.4, obtained by E. Rips and Z. Sela [36]. Given a group G, we are interested in all its splittings over infinite cyclic subgroups (what we call $\mathbb{Z}$-splittings). A JSJ decomposition is one such $\mathbb{Z}$-splitting that encodes them all, in the sense that any other $\mathbb{Z}$-splitting can be obtained from it, in a specific manner. Theorem 3.2.4 proves the existence of such JSJ decompositions, under general assumptions. The aim of this chapter is to explain this with some detail, as well as to obtain the minimal conditions for a $\mathbb{Z}$-splitting to be a JSJ decomposition (Corollary 3.2.6).

3.1 $\mathbb{Z}$-Splittings and quadratically hanging subgroups

A $\mathbb{Z}$-*splitting* of the group G is a splitting whose edge groups are infinite cyclic. That is, a graph of groups Γ, with $\pi_1(\Gamma) \cong G$ and $G_e \cong \mathbb{Z}$ for all edges of Γ.

There is a special type of vertex groups that may occur in a $\mathbb{Z}$-splitting, and they are necessary for defining the JSJ decomposition.

Definition 3.1.1. *Let Γ be a graph of groups. A vertex group G_v is* quadratically hanging *(QH) if*

1. *$G_v \cong \pi_1(S)$ where S is a 2-orbifold. That is to say, it has one of the following presentations*

$$\langle a_1, \ldots, a_g, b_1, \ldots, b_g, p_1, \ldots, p_m, s_1, \ldots, s_n | s_i^{k_i} = 1, \Pi_k p_k \Pi_i s_i \Pi_j [a_j, b_j] = 1\rangle$$

$$\langle a_1, \ldots, a_g, p_1, \ldots, p_m, s_1, \ldots, s_n | s_i^{k_i} = 1, \Pi_k p_k \Pi_i s_i \Pi_j a_j^2 = 1\rangle$$

We require S to be different from the disk ($\langle p_1 | p_1 = 1\rangle = 1$), the cylinder ($\langle p_1, p_2 | p_1 p_2 = 1\rangle \cong \mathbb{Z}$), and a disk with one cone-point ($\rangle p_1, s_1 | s_1^k = 1, p_1 s_1 = 1\langle \cong \mathbb{Z}_k$).

2. *The edges from v are in correspondence with the components of ∂S. Moreover, if these edges are $e_1, \ldots, e_m$, then we have $\partial_{e_i} : G_{e_i} \to \langle p_i \rangle$ (note that p_i is the boundary class corresponding to e_i), and G_{e_i} is non trivial.*

We will see that a QH vertex that arises in any $\mathbb{Z}$-splitting of a group G affects the JSJ decompositions of G. Thus it will be useful to make the following definition.

Definition 3.1.2. *Let G be a group. Then $P \subset G$ is a* QH subgroup *if there is a $\mathbb{Z}$-splitting Γ_P of G with P occuring as a QH vertex group.*

Our definition of QH vertex differs slightly from the one originally used by Rips and Sela in [36], in which they require the maps $\partial_{e_i} : G_{e_i} \to \langle p_i \rangle$ to be onto. This does not change the QH subgroups, since the additional condition on the QH vertex can be met by performing elementary expansions on Γ_P.

Definition 3.1.3. *We say that a $\mathbb{Z}$-splitting is* reduced *if it does not admit elementary collapses, except possibly on the edges connecting to QH vertices.*

Let Γ_1, Γ_2 be one-edged $\mathbb{Z}$-splittings of G, with edge groups C_1, C_2 respectively. That is, G is written as an amalgamation or HNN extension over C_i. We say that Γ_1 is *elliptic* in Γ_2 if the subgroup C_1 acts elliptically in T_{Γ_2}, the Bass–Serre tree of Γ_2. Otherwise, we say that Γ_1 is *hyperbolic* in Γ_2.

Proposition 3.1.4. [36, Theorem 2.1] *Let G be freely indecomposable, and Γ_1, Γ_2 be one-edged $\mathbb{Z}$-splittings of G. Then Γ_1 is elliptic in Γ_2 if and only if Γ_2 is elliptic in Γ_1.*

Consider a surface group $G = \pi_1(S)$. For simplicity, assume that S is closed, orientable, and with negative Euler characteristic. Let α_1, α_2 be two essential simple closed curves in S, and C_1, C_2 be the subgroups of G generated by their respective homotopy classes. For each $i = 1, 2$, G admits a one-edged splitting Γ_i that has C_i as edge group. The vertex groups of Γ_i correspond to the components of S cut along α_i.

Notice that Γ_1 is elliptic in Γ_2 exactly when α_1 and α_2 are disjoint (or can be made disjoint by homotopy). In that case, cutting S along the multi-curve $\alpha_1 \cup \alpha_2$ gives a new $\mathbb{Z}$-splitting of G that refines both Γ_1 and Γ_2.

On the other hand, consider a general group G and one-edged $\mathbb{Z}$-splittings Γ_1, Γ_2. If Γ_1 is hyperbolic in Γ_2, then there cannot be a common refinement of Γ_1 and Γ_2. In some cases, such as the conditions of Theorem 3.2.4, it is possible to show that this situation must arise from intersecting curves in a QH subgroup of G.

These examples show that in general we cannot have a *maximal* $\mathbb{Z}$-splitting, i.e. so that any other $\mathbb{Z}$-splitting is the image of it by a morphism.

3.2 The Rips–Sela theorem

We will now state the fundamental theorem of Rips and Sela, which proves the existence of certain $\mathbb{Z}$-splittings that will be called JSJ decompositions. It applies to *one-ended* groups, that are defined as follows.

Definition 3.2.1. *A space X is* one-ended *if there is an increasing sequence of compact sets K_n, such that $X = \cup_n K_n$ and $X - K_n$ is connected for all n.*

Definition 3.2.2. *A group G is* one-ended *if one/all of its Cayley graphs is/are* *one-ended.*

Equivalently, for G finitely generated, G is one-ended if it acts freely and cocompactly on a one-ended space.

Consider a class of groups $\mathcal{A}$, such as trivial, finite or cyclic groups. We say that a group G *splits over* $\mathcal{A}$ if it admits a non trivial graph of groups decomposition with edge groups in $\mathcal{A}$. For example, G splits over infinite cyclic groups if it admits a non trivial $\mathbb{Z}$-splitting.

According to a theorem of Stallings [41], a finitely generated infinite group is one-ended if and only if it does not split over finite groups. Thus it makes sense to study the splittings over infinite cyclic groups, $\mathbb{Z}$-splittings, of such a group as a next step.

Definition 3.2.3. *A simple closed curve in a 2-orbifold S is* weakly essential *if it is not nullhomotopic, nor boundary parallel, nor the core of a Moebius band embedded in S, and does not circle around a cone-point.*

Theorem 3.2.4. *(Rips–Sela) Let G be a finitely presented one-ended group. Then there is a reduced, unfolded $\mathbb{Z}$-splitting Γ of G satisfying the following conditions:*

1. *(a) A vertex group of Γ can either be a QH vertex group, or be elliptic in every $\mathbb{Z}$-splitting of G.*

   (b) Edge groups are elliptic in every $\mathbb{Z}$-splitting of G.

   (c) Every maximal QH subgroup of G is conjugate to a QH vertex group of Γ.

2. *Let Γ_1 be a one-edged $\mathbb{Z}$-splitting of G, with edge group C. Suppose that Γ_1 is hyperbolic in some other one-edged $\mathbb{Z}$-splitting. Then there is a QH vertex*

group $G_v = \pi_1(S)$ of Γ, and a weakly essential simple closed curve $\gamma \subset S$ such that C is conjugate to the group generated by $[\gamma] \in G_v \subset G$.

3. *If Γ_1 is a one-edged $\mathbb{Z}$-splitting of G that is elliptic in every other one-edged $\mathbb{Z}$-splitting, then there is a morphism $T_\Gamma \to T_{\Gamma_1}$.*

4. *Let Γ_1 be any $\mathbb{Z}$-splitting of G. Then there is a $\mathbb{Z}$-splitting $\hat{\Gamma}$, which is a refinement of Γ obtained by splitting some QH vertex groups along weakly essential simple closed curves, and a morphism $T_{\hat{\Gamma}} \to T_{\Gamma_1}$.*

A splitting Γ as in the theorem is called a *cyclic JSJ decomposition*, or *Rips–Sela JSJ decomposition* of G. In this work we will only consider this version of JSJ decomposition. Due to our definition of QH vertices (Definition 3.1.1), our JSJ decompositions may differ a bit from the ones in [36], but they agree after elementary expansions at the QH vertices.

Condition 4 in the theorem is called *universality*, and graphs of groups that satisfy it will be called *universal*. It says how every $\mathbb{Z}$-splitting of a group G can be obtained from a JSJ decomposition. Also, it is because of universality that the splitting in the theorem verifies the general definition of a JSJ decomposition (over infinite cylic groups), given by Guirardel and Levitt in [18] and [19]. We will not need that definition, however.

There is some redundancy in the conditions for a Rips–Sela JSJ decomposition, as the following proposition shows.

Proposition 3.2.5. *Let G be a one-ended group. Suppose Γ is a reduced $\mathbb{Z}$-splitting of G satisfying universality (condition 4 of Theorem 3.2.4). Then it also satisfies conditions 1, 2 and 3 of 3.2.4.*

Proof. For 1(a) and 1(b), let Γ_1 be any $\mathbb{Z}$-splitting of G. Let $\hat{\Gamma}$ and $f : T_{\hat{\Gamma}} \to T_{\Gamma_1}$ be the refinement and the morphism given by universality. If G_v is a vertex group of Γ that is not QH, then it is still elliptic in $\hat{\Gamma}$, and so it is elliptic in Γ_1. This proves 1(a). The edge groups of Γ are also elliptic in $\hat{\Gamma}$, and so they are elliptic in Γ_1. This gives 1(b).

Now we prove condition 2. Let Γ_1 be a one-edged $\mathbb{Z}$-splitting of G that is hyperbolic in some other $\mathbb{Z}$-splitting. Let $\hat{\Gamma}$ be the refinement of Γ given by condition 4, and $f : T_{\hat{\Gamma}} \to T_{\Gamma_1}$ the corresponding morphism. Take e an edge in T_{Γ_1}, let $C = \mathrm{Stab}_G(e)$ be its stabilizer subgroup and $K = f^{-1}(e)$ be its pre-image under f. There are two kinds of edges in $\hat{\Gamma}$: those that were already present in Γ, and those that were obtained by cutting the surfaces of QH vertices along simple closed curves. Since $f(K) = e$, K is not a single point and it meets the interior of an edge e_1. Then $\mathrm{Stab}_G(e_1) \subset C$. Moreover, since C is cyclic, the generator of $\mathrm{Stab}_G(e_1)$ is a power of the one of C. If e_1 was of the first kind, then C would be elliptic in every $\mathbb{Z}$-splitting of G, which is a contradiction against our assumption on Γ_1. Thus e_1 is of the second kind, and K does not meet the interior of any edges of the first kind. Let K^+ be the union of the edges e' of $T_{\hat{\Gamma}}$ so that $\mathrm{Stab}_G(e')$ intersects C in a non-trivial subgroup. Then K^+ is connected and contains K. (If $C = \langle c \rangle$, then $K^+ = \bigcup_{n\geq 1} \mathrm{Fix}(c^n)$ which is an increasing union of connected sets). The same reasoning used for e_1 shows that K^+ does not contain edges of the first kind. (Recall that an element g is elliptic if and only if g^n is elliptic for any $n \neq 0$).

Now let v be the QH vertex of Γ that corresponds to e_1. Let Γ_0 be the splitting of $G_v = \pi_1(S)$ obtained by cutting S along the same simple closed curves as in $\hat{\Gamma}$. Then there is a copy of T_{Γ_0} embedded in $T_{\hat{\Gamma}}$ that contains e_1. Notice that if $g : T_{\hat{\Gamma}} \to T_\Gamma$ is the map that collapses all edges of the second kind, then g collapses

T_{Γ_0} to a vertex w in the orbit of v. So $\mathrm{Stab}_G(T_{\Gamma_0}) = \mathrm{Stab}_G(w)$ and it is conjugate to $G_v = \pi_1(S)$. Observe that K^+ must be contained in T_{Γ_0}, since it can't cross edges of the first kind. In particular, any fixed point of C lies in T_{Γ_0}, and so it is mapped to w by g. Thus $C \subset \mathrm{Stab}_G(w)$ that is conjugate to G_v. And $C = \mathrm{Stab}_G(e_1)$, since a simple closed curve represents a primitive element of $\pi_1(S)$. This proves condition 2.

Now lets prove condition 3. The setup is the same as in the previous case, but this time Γ_1 is elliptic in every $\mathbb{Z}$-splitting of G. This time, K cannot meet the interior of any edge of the second kind. To see that, suppose that K intersects the interior of an edge e_1 of the second kind. Let $G_v = \pi_1(S)$ be the QH vertex group of Γ corresponding to e_1, and let α be the simple closed curve in S such that $\mathrm{Stab}_G(e_1)$ conjugates to $\langle[\alpha]\rangle$. Since K meets the interior of e_1, we have $\mathrm{Stab}_G(e_1) \subset C$. Thus, if we write $C = \langle c \rangle$, we get that c^n is conjugate to $[\alpha]$ for some n. Let β be a simple closed curve in S that intersects α non-trivially and minimally. Then consider the one-edged splitting Γ_2 of G obtained from $[\beta]$. Since $[\alpha]$ acts hyperbolically on T_{Γ_2}, so does c. Thus Γ_1 is hyperbolic in Γ_2 (and viceversa, by 3.1.4), which goes against our assumption. So K does not intersect any edges of the second kind. This was shown for $K = f^{-1}(e)$ where e was any edge in T_{Γ_1}, so all the edges of the second kind are collapsed to points under f. Let $g : T_{\hat{\Gamma}} \to T_\Gamma$ be the map obtained by collapsing the edges of the second kind. Then f factors through g, and so we obtain the morphism in condition 3.

Finally, for condition 1(c), let H be a QH subgroup of G. Let Γ_1 be a $\mathbb{Z}$-splitting realizing it as a QH vertex. Write $H = \pi_1(S)$ as given by Γ_1. Again, condition 4 gives a morphism $f : T_{\hat{\Gamma}} \to T_{\Gamma_1}$ for some refinement $\hat{\Gamma}$ of Γ as before.

If c is the class of a boundary component of S, then c acts elliptically on $T_{\hat{\Gamma}}$.

To see that, note that some power of c fixes an edge e of T_{Γ_1} (the incident edge at v corresponding to this boundary curve), and $f^{-1}(e)$ meets the interior of some edge e_1. We obtain that $\mathrm{Stab}_G(e_1) \subset \mathrm{Stab}_G(e) \subset \langle c \rangle$, thus e_1 is fixed by a power of c.

Consider the action of H on $T_{\hat{\Gamma}}$ by restriction, and let $\hat{T}$ be a minimal subtree for this action. Then the boundary classes of S are elliptic in $\hat{T}$, since they are elliptic in $T_{\hat{\Gamma}}$.

Consider the decomposition Γ_H of H induced by $\hat{T}$. If e is an edge in $\hat{T}$, then $\mathrm{Stab}_H(e) \subset \mathrm{Stab}_G(e)$, so the edge groups of Γ_H are either trivial or infinite cyclic. Since the boundary classes of S are elliptic in $\hat{T}$, then Γ_H can be extended to Γ_2, a splitting of G obtained by refining Γ_1. And since G is one-ended, all edge groups of Γ_2 are infinite cyclic. Hence all edge groups of Γ_H are infinite cyclic.

Using Corollary 4.2.3 (below), Γ_H is obtained by splitting S along some disjoint, weakly essential simple closed curves. Now, if e is an edge in $\hat{T}$, then $\mathrm{Stab}_H(e)$ is generated by a conjugate of one of these curves. So $\mathrm{Stab}_H(e) = \mathrm{Stab}_G(e)$ since the generator of $\mathrm{Stab}_H(e)$ is primitive. And it is also hyperbolic in some $\mathbb{Z}$-splitting of G, so e is of the second kind.

We conclude as in the proof of condition 2, obtaining that H is conjugate into G_v, for v a QH vertex of Γ. □

Thus we obtain the minimal conditions needed to determine when a $\mathbb{Z}$-splitting is a JSJ decomposition.

Corollary 3.2.6. *Let G be a one-ended group. If Γ is a reduced, unfolded $\mathbb{Z}$-splitting of G that verifies universality (condition 4 from Theorem 3.2.4), then it is a Rips–Sela JSJ decomposition for G.*

Universality is the main feature of a JSJ decomposition, as we have explained before. We ask for it to be reduced because a graph of groups can always be trivially refined by elementary expansions. Unfoldedness is a somewhat technical condition, it expresses that the edge groups of a JSJ decomposition are as small as they can be.

CHAPTER 4

EXTENSIONS OF JSJ GRAPHS BY SURFACE VERTICES

Now we will develop general tools that are useful for proving that a graph of groups Γ is a JSJ decomposition of $\pi_1(\Gamma)$. These tools will be used in chapter 5 to construct new examples of the JSJ decomposition. First we discuss the unfoldedness condition, giving a general criterion that broadens a similar result by Forester [11]. Then we turn to the universality condition, where the main result is Theorem 4.2.5. The idea of this theorem is to start with a universal graph, and to extend it by adding new QH vertices while preserving universality.

4.1 Criterion for unfoldedness

We give a criterion for the unfoldedness of a general $\mathbb{Z}$-splitting. It is a generalization of Proposition 5.1.4, due to Forester, and the proof follows the same lines.

Lemma 4.1.1. *Let G be a freely indecomposable group. Suppose that Γ is a $\mathbb{Z}$-splitting of G, e is an edge of T_Γ with endpoints v_0, v_1 and $H \leq Stab_G(v_1)$ contains $Stab_G(e)$ properly. If Γ_1 is a non trivial unfolding of Γ_e at the endpoint v_0 of e, then H cannot be elliptic in Γ_1.*

In the statement of the lemma, we abused notation and still called e, v_0 and v_1 their respective projections in Γ and Γ_e. Recall that Γ_e is the graph obtained from Γ by collapsing all edges but the projection of e.

Proof. Let X be the Bass–Serre tree corresponding to Γ_e and Y the one coresponding to Γ_1. Notice that X can be obtained from T_Γ by collapsing the components of $T_\Gamma - Ge$.

Let $q : T_\Gamma \to X$ be the quotient map, and $f : Y \to X$ be the folding map. Let e' be an edge of Y, with endpoints v_0', v_1', such that $f(e') = q(e)$ and the fold occurs at v_0'.

Let g be the generator of $\mathrm{Stab}_G(e)$ and g^m the one of $\mathrm{Stab}_G(e')$. We know $m \neq 0$ since G is freely indecomposable, and so $|m| > 1$ since the fold is non trivial ($\mathrm{Stab}_G(e')$ is strictly contained in $\mathrm{Stab}_G(e) = \mathrm{Stab}_G(q(e))$). We may assume $m > 1$, the case for $m < -1$ being analogous.

Define Y_0, $g^k Y_1$ for $k = 0, \ldots, m-1$ to be the components of Y minus the edges $g^k e'$, containing v_0', $g^k v_1'$ respectively. Also let X_0, X_1 be the components of $X - q(e)$ containing $q(v_0)$, $q(v_1)$, and T_0, T_1 the ones of $T_\Gamma - e$ containing v_0, v_1. Observe that $f(Y_0) = X_0$, $f(g^k Y_1) = X_1$, $q(T_0) = X_0$ and $q(T_1) = X_1$.

Seeking a proof by contradiction, suppose that H is elliptic in Γ_1. Thus H fixes a point x' in Y. Since $g \in H$, and g fixes no point of $g^k Y_1$ for any k, we get that x' must belong to Y_0. Then H fixes the point $x = f(x')$ in X_0, and stabilizes the subtree $q^{-1}(x)$ in T_0.

Now, e separates $q^{-1}(x)$ from v_1, and H stabilizes both. So H must also stabilize e, which is a contradiction, since H contained $\mathrm{Stab}_G(e)$ strictly. □

Now we are ready to prove the mentioned result, which gives an unfoldedness criterion for universal $\mathbb{Z}$-splittings.

Theorem 4.1.2. *Let G be a one-ended group. Suppose that Γ is a reduced $\mathbb{Z}$-splitting of G satisfying universality. If every edge group is a proper subgroup of its neighboring vertex groups, then it is unfolded, and is therefore a cyclic JSJ decomposition for G.*

Proof. Again, suppose that Γ is not unfolded. Let e be an edge of Γ and Γ_1 a non trivial unfolding of Γ_e. Let v_0 and v_1 be the endpoints of e, when considered in T_Γ, and assume the unfolding occurs at v_0.

By the universality of Γ, it has a refinement $\hat{\Gamma}$, obtained as in condition 4 of 3.2.4, that admits a morphism $T_{\hat{\Gamma}} \to T_{\Gamma_1}$. Let w_0, w_1 be the vertices of e as an edge of $T_{\hat{\Gamma}}$, that correspond to v_0, v_1 respectively. Put $H = \mathrm{Stab}_G(w_1)$.

Since H is elliptic in $\hat{\Gamma}$ and there is a morphism $T_{\hat{\Gamma}} \to T_{\Gamma_1}$, then H must also be elliptic in Γ_1.

On the other hand, $H \leq \mathrm{Stab}_G(v_1)$ and it contains $\mathrm{Stab}_G(e)$. If v_1 is not a QH vertex, then it doesn't get split in the refinement $\hat{\Gamma}$. So $H = \mathrm{Stab}_G(v_1)$, which contains $\mathrm{Stab}_G(e)$ strictly by hypothesis. And if v_1 is a QH vertex, with $G_{v_1} = \pi_1(S)$, then H is conjugate to $\pi_1(S_0)$ where S_0 is a component of S cut by some weakly essential simple closed curves. Thus H is not cyclic, and therefore must contain $\mathrm{Stab}_G(e)$ strictly.

By the Lemma 4.1.1, H cannot be elliptic in Γ_1, which is a contradiction. $\square$

4.2 Universal graphs and extensions by QH vertices

In this section we deduce the universality of a $\mathbb{Z}$-splitting, given the universality of certain subgraphs of it. We start with some preliminaries.

Definition 4.2.1. *Let Γ be a $\mathbb{Z}$-splitting of a finitely generated group, and e an edge in Γ. Let v^+ and v^- be the endpoints of e, and a be a generator of G_e. Define m_e^+ as the supremum of the m such that $\partial_e^+(a) = b^m$ for some $b \in G_{v^+}$. Define m_e^- in the same manner.*

The number m_e^+ will be called the *label* of e at the endpoint v^+. (With some abuse of notation, for when e is a loop, it gets two labels, one for each boundary map). We remark that it is possible to have $m_e^+ = +\infty$, although this will not happen in the cases that concern us. If v^+ is a QH vertex with $G_{v^+} = \pi_1(S)$, then the element b in the definition is the class of the boundary component of S corresponding to ∂_e^+. In particular m_e^+ is finite. Also, in the case when G_{v^+} is cyclic, the element b is a generator of G_{v^+} and the label is also finite.

The following theorem, due to Zieschang, will be crucial in the proof of 4.2.5. The proof is referred, and the corollary results from iterated use of the theorem.

Theorem 4.2.2. [45, Theorem 4.12.1, pag 140] *Let S be a 2-orbifold with boundary components $\gamma_1, \ldots, \gamma_n$. Let Δ be a one-edged $\mathbb{Z}$-splitting of $\pi_1(S)$ in which $[\gamma_1], \ldots, [\gamma_n]$ are elliptic. Then there is a weakly essential simple closed curve c in S such that Δ is obtained by cutting S along c (via Van-Kampen's theorem).*

Corollary 4.2.3. *Let S be a 2-orbifold with boundary components $\gamma_1, \ldots, \gamma_n$. If Δ is a general $\mathbb{Z}$-splitting in which $[\gamma_1], \ldots, [\gamma_n]$ are elliptic, then Δ is obtained by cutting S along $c_1, \ldots, c_m$, disjoint weakly essential simple closed curves.*

We will also need the following simple lemma about coverings of surfaces and 2-orbifolds.

Lemma 4.2.4. *Let S be a connected 2-orbifold with boundary that is neither a disk nor a cylinder with cone-points. Then there is a 4-sheeted cover $\hat{S}$ of S, such that every boundary component γ of S is covered by two boundary components $\hat{\gamma}_0$, $\hat{\gamma}_1$ of $\hat{S}$, and each one is a double cover of γ.*

Proof. Assume S is an orientable surface, the general case is analogous. Write

$$\pi_1(S) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g, p_1, \ldots, p_m | \Pi_k p_k \Pi_j [a_j, b_j] = 1\rangle$$

Observe that if the genus is positive, then the kernel of the map $\pi_1(S) \to \mathbb{Z}_2$ sending a_1 to 1 and all other generators to 0 defines a double cover S_0 of S where each boundary component of S is covered by two homeomorphic copies of itself. On the other hand, when m is even, the map $\pi_1(S) \to \mathbb{Z}_2$ sending all p_i to 1 and a_j, b_j to 0 is a well defined homomorphism, and its kernel gives a double cover S_1 of S in which each boundary component of S is covered twice by a single boundary curve of S_1. Notice that S_1 always has positive genus, by the Euler characteristic computation for a finite cover.

Combining these covers produces the desired 4-sheeted cover in the cases when m is even, or m is odd but S has positive genus. There remains the case of a sphere with an odd number of punctures. In this case, we have

$$\pi_1(S) = \langle p_1, \dots, p_m | p_1 \cdots p_m = 1 \rangle$$

Consider the map $\pi_1(S) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ that sends $p_1, \dots, p_{m-2}$ to $(1, 0)$, p_{m-1} to $(0, 1)$ and p_m to $(1, 1)$. This is a well defined homomorphism, and gives the desired covering.

□

The following result is the main point of this section. Under some conditions, it allows us to recognize the universality of a $\mathbb{Z}$-splitting built from the union of smaller universal graphs and some extra QH vertices.

Theorem 4.2.5. *Let Γ be a $\mathbb{Z}$-splitting of the one-ended group G. Let $V = \{v_1, \dots, v_m\}$ be a subset of the QH vertices of Γ, such that their corresponding 2-orbifolds are not disks nor cylinders with cone-points. Let $\Gamma_1, \dots, \Gamma_k$ be the components of the subgraph spanned by the vertices not in V, and put $G_i = \pi_1(\Gamma_i)$. Assume that*

1. *If e is an edge with endpoints in V, then $m_e^+, m_e^- > 1$.*

2. *If the vertices $v_j \in V$ and $w \in \Gamma_i$ are connected by an edge, then w is not a QH vertex of Γ_i.*

3. *Each G_i is one-ended, and each Γ_i satisfies universality as a $\mathbb{Z}$-splitting of G_i.*

Then Γ satisfies universality.

Proof. First we observe that if w is a vertex of Γ_i, then it is QH in Γ if and only if it is QH in Γ_i. If w is QH in Γ_i, then it has no more incident edges in Γ by condition 2, and so it is also QH in Γ. And if w is QH in Γ, then it cannot be connected by an edge to $v_j \in V$, for that would cause G_i to be freely decomposable. (To see that, let $p_1, \ldots, p_n$ be the boundary classes in G_w, and suppose that the edge assigned to p_1 is not in Γ_i. Observe that $p_2, \ldots, p_n$ are part of a free basis for G_w. This induces a free splitting of G_w that allows us to refine Γ_i to a graph with some trivial edge groups).

Let Γ' be a $\mathbb{Z}$-splitting of G, and $T' = T_{\Gamma'}$ its Bass–Serre tree. Consider the action of G_i on T' by restriction of the action of G. Passing to a minimal invariant subtree, G_i acts cocompactly and with cyclic edge stabilizers (since G_i is one-ended). So this action gives rise to a $\mathbb{Z}$-splitting of G_i. By universality of Γ_i, there is a refinement $\hat{\Gamma}_i$, and a morphism $T_{\hat{\Gamma}_i} \to T'$, so that $\hat{\Gamma}_i$ is obtained from Γ_i by splitting QH vertex groups along weakly essential simple closed curves. Then all the non-QH vertex groups, and all the edge groups of Γ_i are elliptic in Γ'.

This proves that all the non-QH vertex groups of Γ are elliptic in Γ', since V consists only of QH vertices.

It also implies that if an edge e has an endpoint in some Γ_i, then G_e is elliptic in Γ': If e is contained in Γ_i we have already shown it. If e has endpoints $v_j \in V$ and $w \in \Gamma_i$, then w is non-QH by condition 2, and so G_w is elliptic in Γ'. Since $G_e \subset G_w$, then G_e must also be elliptic in Γ'.

Claim: All edge groups of Γ are elliptic in Γ'.

Proof of the claim: If e has an endpoint in some Γ_i, we have already proved it.

Now let e be an edge with endpoints $v^{\pm} \in V$ (which can be the same vertex).

Let $\gamma^{\pm}$ be the boundary components of the orbifolds $S^{\pm}$ corresponding to $G_{v^{\pm}}$, so that $\partial_e^{\pm} : G_e \to \langle[\gamma^{\pm}]\rangle$. Let $H_e = \langle[\gamma^+], [\gamma^-]\rangle \subset G$ be the subgroup generated by the classes of $\gamma^{\pm}$. Note that H_e is a GBS group.

If either $m_e^+ > 2$ or $m_e^- > 2$, then the splitting of H_e with edge e satisfies the conditions in Forester's theorem (5.1.3), that are direct consequences of those over $m_e^{\pm}$. So it is a JSJ decomposition of H_e, and so G_e is elliptic in T' (as we have done for the Γ_i).

If $m_e^+ = m_e^- = 2$, we proceed by contradiction. Suppose G_e is hyperbolic in Γ', and let c be the generator of G_e. Take an edge e' of Γ' that has a lift to T' lying on the axis of c. Then $\Gamma'' = \Gamma'_{e'}$ is a one-edged $\mathbb{Z}$-splitting of G in which c is hyperbolic. Let $T'' = T_{\Gamma''}$ be its Bass–Serre tree, and let a be the generator of the edge group of Γ''.

On one hand, we consider the subgroup H_e. Note that $H_e = \pi_1(K)$ where K is a Klein bottle. (K is obtained by gluing two Möbius bands by their boundaries. In this case γ^+ and γ^- are the core circles of the Möbius bands, and c is their common boundary circle). The action of H_e on T'' by restriction gives rise to a $\mathbb{Z}$-splitting of H_e (for H_e is freely indecomposable). Note that c is hyperbolic in it, since it is so in T''. So this $\mathbb{Z}$-splitting is non trivial, and we can take $b \in H_e$ a generator of an edge group. Now observe that the edge groups of this decomposition of H_e are all conjugate in G into $\langle a \rangle$. This is so because the only elements that fix an edge of T'' are the conjugates of a power of a. So we obtain an element $b \in H_e$, $b \neq 1$, which is conjugate to a power of a.

On the other hand, we consider the subgroup M constructed as follows.

Take the graph formed by the vertices in V and the edges of Γ with endpoints in V and both labels equal to 2. Let Δ be the component of this graph that contains e. For each vertex $v_j \in \Delta$ write $G_{v_j} = \pi_1(S_j)$, where S_j is the orbifold that corresponds to v_j as a QH vertex of Γ. Let $\hat{S}_j$ be the 4-sheeted cover of S_j given by Lemma 4.2.4. These covers can be extended to a 4-sheeted cover of the whole graph Δ, that can be constructed as follows. Define the graph $\hat{\Delta}$ to have the same vertices as Δ, with $\pi_1(\hat{S}_j) < \pi_1(S_j)$ as vertex group at v_j. And for each edge f of Δ, we put in four edges f_0, f_1, f_2 and f_3 in $\hat{\Delta}$, with infinite cyclic edge groups. The boundary maps are described as follows: Suppose v_j is an endpoint of f and δ is the boundary component of S_j corresponding to f. Then let δ_0 and δ_1 be the boundary components of $\hat{S}_j$ that cover δ and assign f_0, f_2 to δ_0 and f_1, f_3 to δ_1. So the generaror of G_{f_0} maps to $[\delta_0]$ and similarly for the others. This is a 4-sheeted cover, in the sense that $\pi_1(\hat{\Delta}) < \pi_1(\Delta)$ with index 4. (This is best seen by building a presentation 2-complex of $\pi_1(\Delta)$, using S_j for the vertex v_j, and tubes for the edges. Then extend the covers $\hat{S}_j$ of S_j to covers of the tubes.) Note that the labels of the edges of $\hat{\Delta}$ are all 1. The local picture at each edge is as in the example on figure 4.2.

Now let M be the subgroup of $\pi_1(\hat{\Delta})$ generated only by the vertex groups and

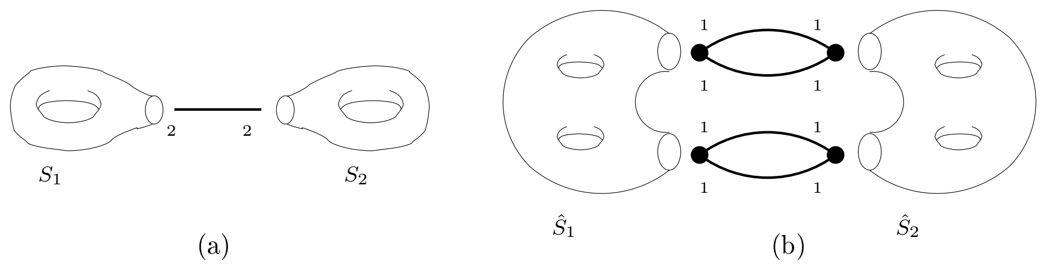

Figure 4.1: Example of the cover $\hat{\Delta}$ **(a)** Original graph Δ, with two QH vertices and an edge with $m_e^{\pm} = 2$. **(b)** Its cover $\hat{\Delta}$

the stable letters of the f_0 and f_1 edges. This is equivalent to saying that M is the fundamental group of the graph resulting from $\hat{\Delta}$ by erasing all f_2 and f_3 edges (and keeping the f_0 and f_1 edges). So $M = \pi_1(S)$, where S is the orbifold that results from gluing the $\hat{S}_j$ along their boundary curves, so that two boundary curves are identified if they are connected by an edge of $\hat{\Delta}$. Note that in this subgroup, c is the class of one of the common boundaries of $\hat{S}^+$ and $\hat{S}^-$ that corresponds to a lift of the edge e. (Say, to e_0). Lets call this curve β, so that $[\beta] = c$.

If p is a boundary curve of S, then some power of $[p]$ is in an edge group G_f of Γ, so that f is not in Δ. (All boundaries corresponding to edges in Δ were glued). Since f is not in Δ, but connects to a vertex in Δ, we know that f is one of the edges for which we have already proved that G_f is elliptic in Γ'. Thus, the classes of the boundary curves of S are elliptic in Γ'. (And so in Γ'').

Again, restrict to M the action on T''. This gives a $\mathbb{Z}$-splitting of M, in which c is hyperbolic and all the boundary classes of S are elliptic. By Corollary 4.2.3, this decomposition of M is obtained by cutting S along disjoint, weakly essential simple closed curves. Let α be one of these curves, so that it intersects β essentially (i.e. the intersection cannot be removed by homotopy). There must be such α, since $c = [\beta]$ is hyperbolic in this decomposition.

Now, since $[\alpha]$ is a generator of an edge group in the $\mathbb{Z}$-splitting of M induced by T'', then $[\alpha]$ must be conjugate in G to a power of a. This is by the same argument we used for the element b.

Since both $[\alpha]$ and b are conjugate to a power of a, then they have the same dynamics in every action of G on a tree. That is to say, in a given G-tree, they are either both elliptic or both hyperbolic, depending on the behaviour of a.

For the contradiction, consider Γ^*, the one-edged splitting of G over $[\gamma^-]$. This splitting is obtained from Γ_e by folding at v^+. In the case of an amalgamation, Γ_e corresponds to $A *_{\langle c \rangle} B$ and Γ^* to $A *_{\langle [\gamma^-] \rangle} (H_e *_{\langle c \rangle} B)$. The case of an HNN extension is similar.

In both cases H_e is contained in a vertex group, so b must be elliptic in Γ^*. We will show that $[\alpha]$ is hyperbolic in Γ^*. This will give the contradiction, thus proving the claim.

Consider the action of M on T_{Γ^*} by restriction. It gives a splitting of $M = \pi_1(S)$ in which the boundary classes are elliptic, so we may use the Corollary 4.2.3 again. This time $c = [\beta]$ stabilizes an edge on T_{Γ^*}, thus β is one of the curves that cut S to form this decomposition. Since α intersects β essentially, then $[\alpha]$ must be hyperbolic in this splitting of M, and therefore in Γ^*. $\Diamond$

Thus far we know that all non-QH vertex groups and all edge groups of Γ are elliptic in Γ'. For each QH vertex v of Γ, write $G_v = \pi_1(S_v)$ where S_v is the corresponding orbifold. Then G_v acts on T' by restriction. Since edge groups of Γ are elliptic in Γ', it follows that the boundary classes of S_v act elliptically on T'. Applying Corollary 4.2.3, the $\mathbb{Z}$-splitting of G_v induced by its action on T' is obtained by cutting S_v along some disjoint, weakly essential simple closed curves. The vertex groups of this decomposition correspond to the pieces of S_v after the cutting, and are elliptic in Γ'. Also note that each boundary curve of S_v lies in exactly one of these pieces. So the splitting of G_v is compatible with Γ, giving rise to a refinement of Γ.

Let $\hat{\Gamma}$ be the refinement of Γ that results from splitting all the QH vertex groups G_v as above. Then all vertex and edge groups of $\hat{\Gamma}$ are elliptic on Γ'. Equivalently, there is a morphism $T_{\hat{\Gamma}} \to T'$. Since Γ' was an arbitrary $\mathbb{Z}$-splitting of G, this concludes the proof. $\square$

CHAPTER 5

EXAMPLES OF THE JSJ DECOMPOSITION

We will provide some explicit classes of graphs of groups that are a JSJ decomposition of their fundamental group. The first of these examples was obtained by M. Forester [11]. He defined the Generalized Baumslag–Solitar graphs, and in most cases, recognized them as JSJ decompositions. In some sense, they are the simplest example of $\mathbb{Z}$-splittings since their vetex groups are also cyclic. Then we will apply the techniques developed in chapter 4 to the Generalized Baumslag–Solitar graphs, producing examples of JSJ decompositions that have QH vertices. These will give what we have called Quadratic Baumslag–Solitar groups.

5.1 Generalized Baumslag–Solitar groups

Here we recall the relevant definitions and results form the work of Forester [11].

Definition 5.1.1. *A* Generalized Baumslag–Solitar (GBS) graph *is a graph of groups in which all vertex and edge groups are infinite cyclic.*

Note this is a special case of $\mathbb{Z}$-splitting. A *GBS group* is a group obtained as a fundamental group of a GBS graph, and a *GBS tree* is the associated Bass–Serre tree.

The one-edged GBS graphs yield two well known classes of groups. Suppose Γ is a GBS graph with one edge e, and let $n, m \in \mathbb{Z}$ represent the maps $\partial_e^+, \partial_e^-$ under the standard homomorphism $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$. In the case of an HNN extension, $\pi_1(\Gamma)$ is the Baumslag–Solitar group $BS(n, m)$, by definition. In the case of an amalgamation, when n and m are relatively prime then $\pi_1(\Gamma)$ is the fundamental group of the complement of a torus knot in $\mathbb{R}^3$, winding n times in the horizontal direction and m times in the vertical one. (See [21] for a proof).

Lemma 5.1.2. [11, Lemma 2.6] *Let* Γ *be a GBS graph,* $G = \pi_1(\Gamma)$*. Assume* $G \ncong \mathbb{Z}$*, and let* $T = T_\Gamma$ *be the Bass–Serre tree of* Γ*. Then:*

1. *G is not free.*
2. *G acts freely on $T \times \mathbb{R}$.*
3. *G is torsion-free, one-ended and has cohomological dimension 2.*
4. *T contains an invariant line if and only if $G \cong \mathbb{Z} \rtimes \mathbb{Z}$ (i.e. either $\mathbb{Z}^2$ or the Klein bottle group $\langle a, b | a^2 b^2 = 1\rangle$).*

The following is the most general statement about JSJ decompositions of GBS groups.

Theorem 5.1.3. [11, Theorem 2.15] *Let* Γ *be a GBS graph,* $G = \pi_1(\Gamma)$*. Suppose* Γ *is reduced, unfolded, and* T_Γ *is not a point or a line* ($G \ncong \mathbb{Z}, \mathbb{Z} \rtimes \mathbb{Z}$)*. Then* Γ *is a JSJ decomposition of* G*.*

To give explicit examples, one must check unfoldedness for GBS graphs. As it turns out, most of them are unfolded.

Proposition 5.1.4. [11, Proposition 2.17] *Let* Γ *be a GBS graph. If every edge group is a proper subgroup of its neighboring vertex groups, then* Γ *is unfolded.*

This result may be obtained from Lemma 4.1.1, so we give this proof below.

Proof. Suppose Γ is a GBS graph in the conditions of 5.1.4. Notice that if Γ is not a single vertex, then $G = \pi_1(\Gamma) \ncong \mathbb{Z}$ and so it is one-ended. If Γ is not unfolded, then there is an edge e of Γ and a non trivial unfolding Γ_1 of Γ_e. In the Bass–Serre tree T_Γ, let v_0 be the endpoint of e at which the unfolding occurs, and v_1 be the other endpoint. Let e' be the edge of T_{Γ_1} with stabilizer contained in $\mathrm{Stab}_G(e)$. Put $H = \mathrm{Stab}_G(v_1)$. Then $\mathrm{Stab}_G(e') \leq \mathrm{Stab}_G(e) \leq H$, where both inclusions are strict (the first one because the unfolding is non trivial, the second one by the hypothesis of 5.1.4). These three subgroups are infinite cyclic, and $\mathrm{Stab}_G(e')$ is elliptic in Γ_1, so H must also be elliptic in Γ_1 (if g^n acts elliptically on a tree, so acts g). This contradicts Lemma 4.1.1. □

The combination of the last two statements (5.1.3 and 5.1.4) allows us to recognize most GBS graphs as JSJ decompositions of their fundamental groups.

5.2 Quadratic Baumslag–Solitar groups

Now we consider graphs of groups Γ with edge groups infinite cyclic, and vertex groups either QH surface groups or infinite cyclic. We will call these graphs *Quadratic Baumslag–Solitar (QBS) graphs.* For simplicity, we restrict the QH vertex groups to be surface groups instead of general 2-orbifold groups. Notice that in a GBS graph all labels are finite, and easily computed from the boundary maps as indicated in the remarks after definition 4.2.1.

A group G will be called a *QBS group* if it can be written as $\pi_1(\Gamma)$, where Γ is a QBS graph.

If Γ is a QBS graph, let $\Gamma_1, \ldots, \Gamma_k$ be the components of the subgraph spanned by the non-QH vertices. That is, the components that are left after removing all QH vertices and the edges connecting to them. Note that each Γ_i is then a GBS graph. The Γ_i will be called the *GBS components* of Γ.

A GBS component of Γ will be called a *leaf* if it is reduced to a single vertex w, and is attached to only one edge e with $G_w = G_e$.

Proposition 5.2.1. *Let Γ be a reduced QBS graph with no leaves, and $G = \pi_1(\Gamma)$. Assume $G \ncong \mathbb{Z}$. Then G is one ended.*

This is a corollary of [43, Theorem 18]. We also give a proof here.

Proof. Let X be the complex constructed as follows.

- For each QH vertex v of Γ, let $X_v = S$ be its corresponding surface, i.e. $G_v = \pi_1(S)$.

- For each cyclic vertex v of Γ, put in a circle $X_v \cong S^1$.

- For each edge e, glue in a cylinder $X_e \cong [-1, 1] \times S^1$ along its boundary. The gluing maps are such that they induce $\partial_e^{\pm}$ in the fundamental groups. More explicitly, if $v^{\pm}$ are the endpoints of e, then the gluing maps are of the form $g_e^{\pm} : \{\pm 1\} \times S^1 \to X_{v^{\pm}}$, so that $(g_e^{\pm})_* = \partial_e^{\pm}$ in the fundamental groups.

By Van-Kampen's theorem, $G = \pi_1(X)$. Let $\tilde{X}$ be its universal cover. The goal will be to show that $\tilde{X}$ is one-ended.

Let $\Gamma_1, \ldots, \Gamma_k$ be the GBS components of Γ.

Let $X_i \subset X$ be the union of X_v, X_e with $v, e \in \Gamma_i$, i.e. the restriction of this complex to the subgraph Γ_i.

Notice that the complete lift of X_i consists of disjoint copies of $T_{\Gamma_i} \times \mathbb{R}$ as in Lemma 5.1.2. The complete lift of $X_v \cong S$ for v a QH vertex consists of disjoint copies of $\tilde{S}$, the universal cover of S.

We call the components of the mentioned lifts *fundamental pieces*. The fundamental pieces are connected by bands $[-1,1]\times\mathbb{R}$ glued along their boundary lines. In the case of a QH piece $\tilde{S}$, they are glued to the lifts of the boundaries of S. In the case of a GBS piece Γ_i, they are glued to vertical lines $\{x\}\times\mathbb{R}$, for x a vertex of T_{Γ_i}.

In order to show that $\tilde{X}$ is one-ended, we need to exhibit an increasing sequence of compact sets $\{K_n\}_{n>0}$ that covers X such that $X - K_n$ is connected for all n. Consider the $T_{\Gamma_i}\times\mathbb{R}$ metric or the hyperbolic $\tilde{S}$ metric on each fundamental piece, and the euclidean metric on each band. (These merics are not invariant, but that won't be important). A subset of $\tilde{X}$ will be called *convex* if its intersection with each fundamental piece, band and copy of $\tilde{S}$ is convex with respect to the corresponding (geodesic) metrics. It is fairly clear that a convex set is simply connected, and that there are arbirarily large convex compact sets. Thus $\tilde{X}$ can be covered by an increasing sequence of convex compact sets. There remains to show that a convex compact set has connected complement in $\tilde{X}$.

Let $K\subset\tilde{X}$ be compact and convex. Let Y be a fundamental piece.

If Y is a lift of X_i, then $Y\cong T_{\Gamma_i}\times\mathbb{R}$. In case T_{Γ_i} is not a point, then Y is one-ended, and $Y-K$ has a single component (since $Y\cap K$ is convex). Thus all points of $Y-K$ are in the same component of $\tilde{X}-K$. If T_{Γ_i} is a point, then there must be a surface (QH) piece connected to Y by a band (if not, $G\cong\mathbb{Z}$). We will deal with this case later on.

Suppose Y is the universal cover $\tilde{S}$ of the surface $S=X_v$ for a QH vertex v.

Let $\gamma \subset \partial\tilde{S}$ be a boundary line. It is connected by a band B_0 to another fundamental piece Y_γ. If Y_γ corresponds to a QH vertex, then $Y_\gamma = \tilde{S}_\gamma$, the universal cover of a surface S_γ. Put $B_\gamma = B_0$.

If Y_γ is a lift of some X_i, there must be a QH piece connected to Y_γ other than Y: if not, then $\Gamma_i = \{w\}$ and $G_e = G_w$ where e is the edge between v and w (so that there are no more lifts of $S = X_v$ connected to X_i), and there are no other vertices adjacent to w (no QH pieces projecting to other surfaces). This cannot happen, since then Γ_i would be a leaf of Γ.

Pick one of these QH pieces and call it $\tilde{S}_\gamma$. It is connected by a band B_1 to Y_γ. Recall that $Y_\gamma \cong T_{\Gamma_i} \times \mathbb{R}$ where B_0 is attached to a vertical $\{x_0\} \times \mathbb{R}$, and so is B_1 to $\{x_1\} \times \mathbb{R}$. Let α be the geodesic of T_{Γ_i} between x_0 and x_1. Then the union of B_0, B_1 and $\alpha \times \mathbb{R} \subset Y_\gamma$ is a band. We call it B_γ.

Thus, for each boundary line γ of $Y = \tilde{S}$, we have another QH piece $\tilde{S}_\gamma$, and a band B_γ connecting $\tilde{S}$ to it. Let D_1 be the union of all of these pieces and bands. Then D_1 is homeomorphic to a disk with some boundary lines (those of the $\tilde{S}_\gamma$ not attached to B_γ).

We can use the same procedure with the boundary lines of D_1 and so on, obtaining D_k for $k \geq 1$. Then D_k is included in the interior of D_{k+1}, and each D_k is homeomorphic to a disk with boundary lines.

By compactness, there is some n such that $K \cap D_n$ is in the interior of D_n. Since this interior is one-ended, $D_n - K$ has a single component.

So, again, all points of $Y - K$ are in the same component of $\tilde{X} - K$. The same is true for the points in a band attached to Y, thus for the case that was left.

That covers all the points in $\tilde{X} - K$, so it has only one component. □

The following is the main theorem of this chapter. It allows us to recognize most QBS graphs as the JSJ decomposition of their fundamental groups.

Theorem 5.2.2. *Let Γ be a QBS graph, $G = \pi_1(\Gamma)$. Suppose that Γ is reduced and satisfies the following conditions:*

1. *Each edge e of Γ has integer labels $m_e^+, m_e^- > 1$.*
2. *Each GBS component Γ_i of Γ is reduced, and T_{Γ_i} is not a point or a line.*

Then Γ is a Rips–Sela JSJ decomposition for G

Proof. Let V be the set of QH vertices of Γ. The components of Γ minus V are the GBS components Γ_i of Γ. By condition 1 and 5.1.4, each Γ_i is unfolded. This, together with condition 2, allows us to apply 5.1.3 (Forester's result). We conclude that each Γ_i is a JSJ decomposition of $G_i = \pi_1(\Gamma_i)$. Notice that by 5.1.2, the G_i are one-ended. By these facts and condition 1, we have verified the hypotheses of Theorem 4.2.5 for Γ and V. So Γ satisfies universality. Now we can use Theorem 4.1.2 to conclude that Γ is unfolded. Therefore Γ is a JSJ decomposition of G, by 3.2.6. □

When some edge label equals 1, then Γ may fail to be a JSJ decomposition. This was already true for GBS graphs. In figure 5.1 there is an example, in which the edge e with a label equal to 1 is not in a GBS component. However, if in the same figure we change the label 1 for some $m_e^- > 1$, and make $k = 1$ instead, we do get a JSJ decomposition (by 4.2.5 and then 4.1.2), which is not covered by Theorem 5.2.2.

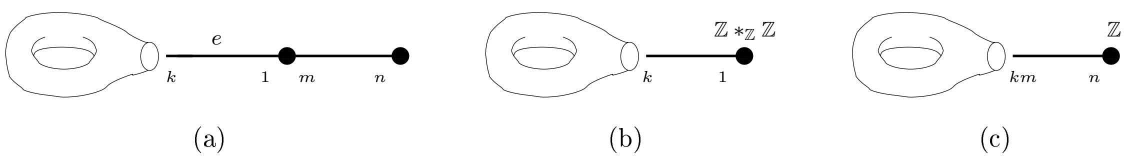

Figure 5.1: **(a)** A QBS graph that satisfies universality (for $m, n > 1$), but with $m_e^- = 1$. It admits an unfolding at the surface vertex, as shown in (b) and (c). **(b)** The one-edged splitting corresponding to the edge e of the graph in (a). **(c)** An unfolding of the splitting in (b).

CHAPTER 6

MEASURE PRESERVING GROUP ACTIONS

Here we review some of the theory of Borel equivalence relations on standard Borel spaces, and the notion of cost, an invariant of an equivalence relation preserving a probability measure. This material is covered with detail in the book by Kechris and Miller [26]. We also review the induced and coinduced actions. These topics form the background for our study on measure free factors.

6.1 Graphings, treeings and cost

A *standard Borel space* is a measurable space X (a set X, together with a σ-algebra of subsets of X), which is isomorphic to a Borel subset of a compact metric space with the σ-algebra of Borel sets. It is a well known result that standard Borel spaces of the same cardinality are isomorphic (see for instance [25]), which is the reason for the term *standard*. When X is a standard Borel space, we will refer to the subsets in its σ-algebra as the Borel subsets of X. We will also call an isomorphism of measurable spaces between standard Borel spaces a *Borel isomorphism*.

Since countable products of compact metric spaces are compact and metrizable, we get that a countable product of standard Borel spaces has a natural structure of standard Borel space.

A *Borel equivalence relation* on the standard Borel space X is an equivalence relation $E \subset X \times X$ which is a Borel subset of $X \times X$.

Borel equivalence relations are one of the basic objects of our work. The main example we will consider is when a group G acts on X by Borel automorphisms.

Then the equivalence relation E_G^X, whose classes are the G-orbits, is Borel. It is a fact that every Borel equivalence relation with countable classes comes from the action of some group (Feldman–Moore theorem, [26, Theorem 15.1]), but we will not need to make use of this.

On the other hand, every group admits such an action: G acts on the compact metric space $\{0,1\}^G$ by shifting of the coordinates. The action is by homeomorphisms, thus it is Borel.

Now we turn to *graphings*, which play the role of generators for these equivalence relations.

Let X be a standard Borel space. A *partially defined isomorphism* on X is a map $\varphi : A \to B$, where A and B are Borel subsets of X and φ is a Borel isomorphism between them. For example, if $E = E_G^X$ is an orbit equivalence relation, then the action of an element $g \in G$ restricted to any Borel subset of X defines a partial isomorphism.

If $\Phi = \{\varphi_i : A_i \to B_i\}_{i \in I}$ is a family of partial isomorphisms, the equivalence relation *generated* by Φ is the minimal Borel equivalence relation $E \subset X \times X$ containing the set

$$\{(x, \varphi_i(x)) : x \in A_i, i \in I\}$$

we will denote it by E_Φ.

Definition 6.1.1. *A* graphing *for E is a family $\Phi = \{\varphi_i\}_{i \in I}$ of partially defined Borel isomorphisms on X that generates E*

As the name suggests, a graphing defines a graph structure on each equivalence class. Consider the (directed) graph with vertex set X and edges $(x, \varphi_i(x))$ for

$x \in A_i$, $i \in I$. Then its connected components are the equivalence classes of E, and this gives the graph structure of each class.

As an example, let $E = E_G^X$ be the orbit relation induced by a free action $G \curvearrowright X$. If $S = \{s_i\}_{i \in I}$ is a generating set for G, then the maps $\varphi_i : X \to X$ s.t. $\varphi_i(x) = s_i \cdot x$ form a graphing for E_G^X. In this case $A_i = B_i = X$ for all i. The graph structure is the same for every orbit, and agrees with the Cayley graph of (G, S).

Now let μ be a finite measure on the standard Borel space X. We say that the equivalence relation E *preserves* μ, or is *measure preserving*, if it admits a graphing $\Phi = \{\varphi_i : A_i \to B_i\}$ in which every transformation φ_i preserves μ, i.e. $(\varphi_i)_* \mu|_{A_i} = \mu|_{B_i}$. It is easy to check that *every* graphing of a measure preserving equivalence relation satisfies that property.

If $G \curvearrowright X$ is a Borel action, the orbit relation E_G^X preserves μ if and only if G acts by measure preserving Borel automorphisms of X. In this case we say that the action is measure preserving. When μ is a probability measure, we call it *probability measure preserving*.

As an example, consider the shift action $G \curvearrowright \{0,1\}^G$. It preserves a probability measure, given by the product of the counting measure on $\{0, 1\}$. Namely, it is the unique measure that assigns value $1/2$ to each basic set of the form $A_{g,i} = \{(a_h)_{h \in G} : a_g = i\}$ (for $g \in G, i = 0, 1$).

In the context of measure preserving Borel actions, we want to relax the definition of a free action, to mean free almost everywhere.

Definition 6.1.2. *A measure preserving Borel action $G \curvearrowright X$ is called* free *if the set of $x \in X$ such that $g \cdot x = x$ for a non-trivial $g \in G$ has measure zero.*

The only equivalence relations we will need to consider will be the ones induced by free probability measure preserving Borel actions, and their sub-relations. The shift action $G \curvearrowright \{0,1\}^G$ just considered is an example of a free action, under this definition.

For a measure preserving equivalence relation, we can define its *cost*, an invariant introduced by G. Levitt [31], and studied extensively by D. Gaboriau [14], [15]. In the analogy between graphings and group generators, the cost would correspond to the rank.

Definition 6.1.3. *Let X be a standard Borel space with a finite measure μ.*

1. *Let $\Phi = \{\varphi_i : A_i \to B_i\}$ be a measure preserving graphing. The* cost *of Φ is*

$$C_\mu(\Phi) = \sum_i \mu(A_i)$$

2. *Let E be a Borel measure preserving equivalence relation on X. The cost of E is the infimum of the costs of its graphings, i.e.*

$$C_\mu(E) = \inf\{C(\Phi) : \Phi \textit{ is a graphing for } E\}$$

The analogy with the rank is justified by the following result, which is a consequence of Theorem 6.1.10 below.

Theorem 6.1.4. *(Gaboriau) The cost of any free Borel probability measure preserving action of the free group F_n is n.*

It is clear that $C_{\lambda\mu} = \lambda C_\mu$ both for graphings and equivalence relations. When μ is a probability, we drop it from the notation, writing C for C_μ. This may also happen when the measure is clear from the context.

The cost may also be defined for groups, where it turns out to be an interesting invariant.

Definition 6.1.5. *The* cost *of a group G is the infimum of the costs $C(E_G^X)$ over all free Borel probability measure preserving actions $G \curvearrowright X$ on standard Borel probability spaces.*

The cost of G will be denoted by $C(G)$.

Remark 6.1.6. *A group G has* fixed price *if every free Borel probability measure preserving action of G has cost $C(G)$. It is an open problem to determine if every countable group has fixed price.*

The most important kind of graphings are *treeings*, which we define now. Treeings are the analog of free bases for a free group, in the context of measure preserving Borel equivalence relations.

Definition 6.1.7. *Let E be a measure preserving Borel equivalence, and Φ be a graphing for E. We say that Φ is a* treeing *if the graph induced by Φ on each class of E is a tree for* almost every class *(the property holds on the complement of an E-saturated set of measure zero).*

Not every measure preserving Borel equivalence relation admits a treeing. For instance, free actions of non-amenable groups of cost 1 (like $F_2 \times \mathbb{Z}$) do not. (See Theorem 9.1.2). If it does, it is called *treeable*. As an example, consider the relation E_G^X induced by a free measure preserving Borel action $G \curvearrowright X$, and the graphing Φ given by a generating set S of G. In this case, Φ is a treeing if and only if G is a free group and S is a free basis. If G is not free, it is still possible that E_G^X may admit a treeing for some free measure preserving Borel action. However, to determine which groups do so is an open problem.

Definition 6.1.8. *A group G is* treeable *if there exists a free probability measure preserving Borel action $G \curvearrowright X$ such that E_G^X is treeable.*

It should be pointed out that there is another notion, that sometimes is also called treeability. A group G is *strongly treeable* if E_G^X is treeable for *every* free probability measure preserving Borel action $G \curvearrowright X$. This clearly implies that G is treeable. It is an open question whether the converse holds, i.e. whether every treeable group is strongly treeable.

The following theorems, due to Gaboriau, are the fundamental results in this theory. We will employ them repeatedly in the sequel.

Theorem 6.1.9. *[15, Theorem 5] Let E be a measure preserving Borel equivalence relation with countable classes. If E is treeable and $F \subset E$ is a sub-equivalence relation, then F is also treeable.*

Thus a subgroup of a treeable group is also treeable. There is a close relationship between treeings and cost, given by the following theorems.

Theorem 6.1.10. *[15, Theorem 1] Let E be a measure preserving Borel equivalence relation on the standard Borel space X. If Φ is a treeing for E then $C_\mu(\Phi) = C_\mu(E)$.*

If the cost is finite, the converse holds.

Theorem 6.1.11. *[15, Theorem 1, Proposition I.11] Let E be a measure preserving Borel equivalence relation on the standard Borel space X, with $C_\mu(E) < \infty$. A graphing Φ for E is a treeing if and only if $C_\mu(\Phi) = C_\mu(E)$.*

Theorem 6.1.12. *[26, Proposition 30.5] Let $G \curvearrowright X$ a free probability measure preserving Borel action. If E_G^X is treeable then $C(G) = C(E_G^X)$.*

Finally, we recall *complete sections*, which are often useful to compute cost.

Definition 6.1.13. *Let E be a Borel equivalence relation on the standard Borel space X. A* complete section *for E is a Borel subset $A \subset X$ meeting every class of E.*

If E has countable classes and is measure preserving, then a complete section always has positive measure. By disregarding sets of measure zero, we need only ask that a complete section meets almost every class. If E is an equivalence relation on X, and $A \subset X$, one defines the *restriction* of E to A as follows.

$$E|_A = \{(x, y) \in A \times A : (x, y) \in E\}$$

The relationship between cost and complete sections is given by the formula below.

Theorem 6.1.14. *[15, Proposition II.6] Let E be a measure preserving Borel equivalence relation with countable classes, on the standard Borel measure space X. Let $A \subset X$ be a complete section for E. Then*

$$C_\mu(E) = C_{\mu|_A}(E|_A) + \mu(X \setminus A)$$

6.2 Induced and coinduced actions

In this section we present tools for extending an action of a subgroup of a group G to an action of G. Specifically, if we have groups $H \leq G$, and an action $H \curvearrowright X$, we wish to construct an action $G \curvearrowright Y$ that contains, in some sense, the action of H on X. These tools are the *induced* and *coinduced* actions.

Definition 6.2.1. *Let $H \leq G$ be a subgroup and $H \curvearrowright X$ be an action. Define*

$$\mathrm{CoInd}_H^G X = \{\psi : G \to X / \psi(gh^{-1}) = h \cdot \psi(g) \text{ for } h \in H, g \in G\}$$

with the action of G given by $g \cdot \psi(k) = \psi(g^{-1}k)$ for $g, k \in G$.

The coinduced action satisfies the following general properties, which are easy to prove:

1. The map $p : \mathrm{CoInd}_H^G X \to X$ taking ψ to $\psi(1)$ is H-equivariant and surjective.
2. If $\{g_i\}_{i \in I}$ is a set of representatives of G/H, then the map

   $$\mathrm{CoInd}_H^G X \to X^{G/H} \text{ s.t. } \psi \mapsto \{\psi(g_i)\}_{g_i H \in G/H}$$

   is a bijection. (Note: we write an element of A^B as $\{a_b\}_{b \in B}$, that is, a collection of elements of A indexed by the elements of B).
3. Let G act on $X^{G/H}$ as follows: if $g \in G$, $f \in X^{G/H}$ then put

   $$(g \cdot f)_{g_i H} = h f_{g_j H} \text{ where } g^{-1} g_i = g_j h^{-1} \text{ for } h \in H$$

   With respect to this action, the map defined in (2) is an equivariant isomorphism.
4. If $H \curvearrowright X$ is free, then so is $G \curvearrowright \mathrm{CoInd}_H^G X$.
5. If $H \curvearrowright (X, \mu)$ is a probability measure preserving Borel action, then $\mathrm{CoInd}_H^G X$ can be given the product Borel structure and the product measure of $X^{G/H}$. The action $G \curvearrowright \mathrm{CoInd}_H^G X$ is Borel and probability measure preserving.

Some of the interesting properties of an action are preserved under coinduction, as the next lemma shows. It is the key for our applications of this construction.

Lemma 6.2.2. *Let $H \leq G$, and $H \curvearrowright X$ be a free probability measure preserving Borel action with a graphing Φ that generates E_H^X. Let $Y = \mathrm{CoInd}_H^G X$. There exists a graphing $\hat{\Phi}$ that generates E_H^Y and has $C(\hat{\Phi}) = C(\Phi)$. Moreover, if Φ is a treeing so is $\hat{\Phi}$.*

Proof. Let $\Phi = \{\varphi_i : A_i \to B_i\}_{i \in I}$. By subdividing the sets A_i, we can assume that each φ_i is of the form $\varphi_i(x) = h_i \cdot x$ for $h_i \in H$ and all $x \in A_i$. Now define $\hat{A}_i = p^{-1}(A_i)$, $\hat{B}_i = p^{-1}(B_i)$ and $\hat{\varphi}_i(y) = h_i \cdot y$ for all $y \in \hat{A}_i$. Put $\hat{\Phi} = \{\hat{\varphi}_i\}$. Then $\hat{\Phi}$ generates E_H^Y by equivariance of the map p and freeness of the action $H \curvearrowright X$. We also have that $C(\hat{\Phi}) = C(\Phi)$, by definition of the product measure: if we identify Y with $X^{G/H}$ then p is the projection onto the coordinate of the coset $1H$ and $\hat{A}_i = p^{-1}(A_i)$ is a basic set. Thus the measure of $\hat{A}_i$ with respect to the product measure is $\mu(A_i)$. Finally, if Φ is a treeing, then $\hat{\Phi}$ must also be a treeing, since a non-trivial cycle in the graphing $\hat{\Phi}$ would project under p to a non-trivial cycle of Φ in X. □

Now we turn to the induced action. In our context, it is only useful for extensions of finite index. However, it will be the main tool for subsequent chapters.

Definition 6.2.3. *Let $H \leq G$ be a subgroup and $H \curvearrowright X$ be an action. Define*

$$\mathrm{Ind}_H^G X = (X \times G)/H$$

where the quotient is by the right diagonal action of H ($(x, g) \cdot h = (h^{-1} \cdot x, gh)$). Let G act on $X \times G$ by left multiplication on the second coordinate. This induces the action of G on $\mathrm{Ind}_H^G X$.

These are the main properties of the induced action, whose proofs are straightforward:

1. If $\{g_i\}_{i\in I}$ is a set of representatives of G/H, then the map

$$X \times G \to X \times (G/H) \text{ s.t. } (x,g) \mapsto (g_i^{-1}g\cdot x, g_iH) \text{ where } g \in g_iH$$

   is invariant by the right diagonal action of H. This map induces a bijection between $\mathrm{Ind}_H^G X$ and $X \times (G/H)$.

2. Let G act on $X \times (G/H)$ as follows:

$$g\cdot(x, g_iH) = (h\cdot x, g_jH) \text{ where } gg_i = g_jh \text{ for } h \in H$$

   This action makes the bijection in (1) into an equivariant isomorphism.

3. If $\{g_i\}$ is a set of representatives of G/H, then we can write $\mathrm{Ind}_H^G X = X \times G/H = \bigcup_i (X \times \{g_iH\})$, which is a union of disjoint copies of X.

4. The inclusion $X \to X \times \{H\} \subset \mathrm{Ind}_H^G X$ is H-equivariant. We call $X_0 = X \times \{H\}$.

5. X_0 is a complete section for the orbits of G on $\mathrm{Ind}_H^G X$. Its translates are of the form $X \times \{g_iH\} = g_iX_0$.

6. The restriction to X_0 of the orbit equivalence of G on $\mathrm{Ind}_H^G X$ is the orbit equivalence of H on $X_0 = X$. In symbols

$$E_G^{\mathrm{Ind}_H^G X}|_{X_0} = E_H^{X_0} \cong E_H^X$$

7. If $H \curvearrowright X$ is probability measure preserving and the index of H in G is finite, we consider the product measure on $X \times (G/H)$ where G/H has the counting measure, and we rescale it by the index of H in G. This gives an invariant probability measure on $\mathrm{Ind}_H^G X$.

CHAPTER 7
MEASURE FREE FACTORS

Here we discuss the notion of measure free factors of groups. We will recall the results of D. Gaboriau [16] on this subject. One of them concerns the problem of finding such measure free factors, giving the first non-trivial examples. The other gives an application of measure free factors to prove the treeability of an amalgamation. We will extend this application to the case of HNN extensions, introducing the concept of common measure free factors.

7.1 Definitions and known results

First we need to give a definition for the free product of equivalence relations.

Definition 7.1.1. *Let E_1, E_2 be Borel equivalence relations on the standard Borel space X. We say that E_1 and E_2 are* orthogonal*, and write $E_1 \perp E_2$, if for every cycle (x_i), $i \in \mathbb{Z}_{2n}$, of elements of X such that*

1. *$(x_i, x_{i+1}) \in E_1$ for all i odd.*
2. *$(x_i, x_{i+1}) \in E_2$ for all i even.*

we have that $x_i = x_{i+1}$ for some i.

Definition 7.1.2. *Let E be a measure preserving Borel equivalence relation. We say that E is the* free product *of the Borel sub-equivalence relations E_1, E_2, and write $E = E_1 * E_2$, if*

1. *E is generated by their union $E_1 \cup E_2$.*

2. *There is a full measure set B, saturated by E, so that $E_1|_B \perp E_2|_B$. (We say that $E_1 \perp E_2$ for almost every class of E).*

The definition of the measure free product of several equivalence relation is a straightforward generalization.

These definitions reflect the notion of free product for groups. Specifically, if $G \curvearrowright X$ is a free measure preserving Borel action, and G splits as $G = H * K$, then it is easy to check that $E_G^X = E_H^X * E_K^X$.

Definition 7.1.3. *[16, Definition 3.1] A subgroup $H \leq G$ is a* measure free factor *of G if there exists a free probability measure preserving Borel action of G on a standard Borel space X, and a Borel equivalence relation E' on X such that*

$$E_G^X = E_H^X * E'$$

From the remark above, it is clear that if H is a free factor of G then it is also a measure free factor. It is also an easy fact that the image of a measure free factor by an automorphism of G is again a measure free factor of G. Free factors are not the only examples of measure free factors, as shown by the following theorems of Gaboriau.

Theorem 7.1.4. *[16, Theorem 3.2] Let $F = \langle a_1, b_1, \ldots, a_g, b_g \rangle$ be a free group of rank $2g$. Then the element $w = [a_1, b_1] \cdots [a_g, b_g]$ generates a measure free factor of F.*

Corollary 7.1.5. *[16, Corollary 3.5] Let S be an orientable surface with boundary, of genus at least 1. Let $\gamma_1, \ldots, \gamma_k$ be the boundary curves of S. Then the boundary subgroup $\langle [\gamma_1], \ldots, [\gamma_k] \rangle \leq \pi_1(S)$ is a measure free factor of $\pi_1(S)$.*

In general, finding measure free factors is hard. Gaboriau [16] posed the problem of finding which elements of a free group generate a cyclic measure free factor. It is shown in [16] that such an element cannot be a proper power.

Measure free factors can be used to construct new treeable groups, via amalgamated products. The following is obtained by combining the statements of Theorems 3.13 and 3.17 of [16].

Theorem 7.1.6. *Let $G = G_1 *_\Lambda G_2$, where G_i are treeable groups, and Λ is a measure free factor of G_1. Let also $H \leq G_2$ be a measure free factor of G_2. Then G is treeable, and $H \leq G$ is a measure free factor of G.*

In particular, if Λ is also a measure free factor of both G_1 and G_2, then Λ is a measure free factor of the amalgamation $G = G_1 *_\Lambda G_2$.

7.2 Common measure free factors

We would like to have a result similar to 7.1.6 for HNN extensions. Let G be a group, $H \leq G$ a subgroup and $\alpha : H \to G$ an injective homomorphism. We define the HNN extension $G*_H$ by the following presentation:

$$G*_H = \langle G, t | tht^{-1} = \alpha(h) \text{ for } h \in H \rangle$$

To conclude that $G*_H$ is treeable we need stronger hypotheses: We still assume that H is a measure free factor of G, and also that $\alpha(H)$ is contained in a subgroup $H' \leq G$ so that H and H' are *common measure free factors* of G, as defined next.

Definition 7.2.1. *Let G be a group, and $H, K \leq G$ subgroups. Then H and K are* common measure free factors *of G if there exists a free probability measure* *preserving Borel action $G \curvearrowright X$ on a standard Borel probability space, and a Borel equivalence relation E' on X such that*

$$E_G^X = E_H^X * E_K^X * E'$$

For more than two subgroups the definition is similar. Notice that if H and K are common measure free factors, then $H \cap K = \{1\}$ unless $H = K$ and the subgroup generated by H and K in G is isomorphic to $H * K$.

Lemma 7.2.2. *Let $G = G_1 * G_2$, and let H_1, H_2 be a measure free factors of G_1, G_2 respectively. Then H_1 and H_2 are common measure free factors of G.*

Proof. For $i = 1, 2$, let $G_i \curvearrowright X_i$ be free probability measure preserving Borel actions such that $E_{G_i}^{X_i} = E_{H_i}^{X_i} * E_i'$. Fix graphings Ψ_i of E_i'. Take $Y_i = \mathrm{CoInd}_{G_i}^G X_i$, and $Y = Y_1 \times Y_2$ with the diagonal action of G. Then we get $E_{G_i}^Y = E_{H_i}^Y * E_i''$, where E_i'' is generated by $\hat{\Psi}_i$, the graphing obtained as in Lemma 6.2.2 from a graphing Ψ_i of E_i'. Since $G = G_1 * G_2$ we have

$$E_G^Y = E_{G_1}^Y * E_{G_2}^Y = E_{H_1}^Y * E_1'' * E_{H_2}^Y * E_2''$$

which gives the lemma.

$\square$

The following is the version of Theorem 7.1.6 for HNN extensions. The proof uses the same techniques as those used by Gaboriau in his proof of 7.1.6.

Proposition 7.2.3. *Let G be a treeable group, H, H', K be subgroups of G, and $\alpha : H \to H'$ be an injective homomorphism. If H, H' and K are common measure free factors of G, then the HNN extension $\Gamma = G *_H$ is treeable and $K \leq \Gamma$ is a measure free factor of Γ.*

Proof. Let $G \curvearrowright X$ be a free probability measure preserving Borel action such that E_G^X is treeable and $E_G^X = E_H^X * E_{H'}^X * E_K^X * E'$. This can be done by taking a diagonal action $G \curvearrowright X = X_0 \times X_1$ where $E_G^{X_0}$ is treeable and $E_G^{X_1}$ splits as the desired free product.

Let $Y = \mathrm{CoInd}_G^\Gamma X$. Take treeings Φ_H, Φ_K and Ψ for E_H^X, E_K^X and E' respectively, which exist by 6.1.9, since E_G^X is treeable. Let $\hat{\Phi}_H$, $\hat{\Phi}_K$ and $\hat{\Psi}$ be the treeings obtained from them by Lemma 6.2.2, applied to $Y = \mathrm{CoInd}_G^\Gamma X$. By the same argument as in 6.2.2 (projecting by p), we can obtain that $E_G^Y = E_H^Y * E_{H'}^Y * E_K^Y * E''$ where E'' is the equivalence relation generated by $\hat{\Psi}$. Finally, consider the Borel automorphism of Y given by the action of the stable letter t, which we still denote by t, i.e. $t : Y \to Y$ s.t. $t(y) = t \cdot y$.

Put $\Omega = \{t\} \cup \hat{\Phi}_{H'} \cup \hat{\Phi}_K \cup \hat{\Psi}$, which is a graphing on Y. We claim that Ω is a treeing for E_Γ^Y.

To see that Ω generates E_Γ^Y, first observe that it clearly generates $E_{H'}^Y$, E_K^Y and E''. Also, since $E_H^Y = t^{-1} E_{H'}^Y t$, we see that Ω generates E_G^Y. Since Γ is generated by G and t, it is now clear that Ω generates E_Γ^Y.

In order to see that there are no non-trivial cycles a.e., suppose $\phi_{i_1} \cdots \phi_{i_n}(y) = y$ for y in a set of positive measure, where $\phi_j \in \Omega$ and $\phi_{i_1} \cdots \phi_{i_n}$ is a reduced word. This gives rise to a reduced word $\gamma_{i_1} \cdots \gamma_{i_n}$ with letters in $G \cup \{t, t^{-1}\}$ representing 1 in Γ. If all of its letters are in G, we get a contradiction, since $\hat{\Phi}_{H'} \cup \hat{\Phi}_K \cup \hat{\Psi}$ is a treeing of a subrelation of E_G^Y (each one is a treeing, and they are mutually orthogonal since $E_G^Y = E_H^Y * E_{H'}^Y * E_K^Y * E''$). If some γ_i equals t or t^{-1}, that contradicts the normal form for an HNN extension, since the treeing $\hat{\Phi}_{H'} \cup \hat{\Phi}_K \cup \hat{\Psi}$ generates an equivalence relation which is orthogonal to E_H^Y, thus no element of

H appears in the normal form of $\gamma_{i_1} \cdots \gamma_{i_n}$.

$\square$

Considering common measure free factors gives also the following refinement of 7.1.6, which is obtained by the same argument we used for 7.2.3.

Proposition 7.2.4. *Let $G = G_1 *_\Lambda G_2$, where G_i are treeable groups. Let $K \leq G_1$ and assume that Λ and K are common measure free factors of G_1. Also let $H \leq G_2$ be a measure free factor of G_2. Then G is treeable, and H and K are common measure free factors of G.*

CHAPTER 8

FINITE COVERS AND FINDING NEW MEASURE FREE FACTORS

We will now address the problem of finding measure free factors of free groups, and sligthly more general treeable groups. This problem was posed by D. Gaboriau [16]. First we develop the main technical tool, Theorem 8.1.2, which gives a way of passing to finite index subgroups in the problem of showing that some cyclic subgroup is a measure free factor. This result is closely related to the Kurosh theorem for Borel equivalence relations obtained by A. Alvarez [1], but going in the converse direction. This theorem will then be applied to find new families of measure free factors, as well as to extend Gaboriau's theorem about the boundary subgroup of surface groups to the case of non-orientable surfaces and 2-orbifolds.

8.1 Lifting to finite covers

In this section we prove Theorem 8.1.2, which is the main tool in the proofs of our results about measure free factors. Our general goal is to show that some element $w \in G$ of a treeable group G generates a measure free factor of G. This result will allow us to pass to a finite index subgroup H of G, replacing $\langle w \rangle$ by a suitable subgroup of H. First we explain what this suitable subgroup is, starting from the geometric viewpoint.

Let G be a group and $H \leq G$ a subgroup of finite index $n = [G : H]$. Consider a complex X_G with $\pi_1(X_G) = G$, and the n-sheeted covering space $X_H \to X_G$ corresponding to $H \leq G$. Recall that the conjugacy classes in G correspond to the homotopy classes of closed curves in X_G.

Let $w \in G$ and take a closed curve γ in X_G that represents the conjugacy class of w. The pre-image of γ in X_H is a union of closed curves $\gamma_1, \ldots, \gamma_k$, where γ_i is the union of m_i lifts of γ. In that sense, γ_i "covers" γ with index m_i. Each of these curves γ_i defines a conjugacy class $[w_i]$ in H, where we can write $w_i = g_i^{-1} w^{m_i} g_i$ for some $g_i \in G$.

More explicitly, let $p_0 \in X_G$, $p \in X_H$ be basepoints (with p projecting to p_0), and let p_i be a pre-image of p_0 so that γ_i can be obtained as the lift of γ^{m_i} starting at p_i. Take a curve α_i in X_H going from p to p_i, and let $g_i^{-1} \in G = \pi_1(X_G, p_0)$ be the homotopy class of the projection of α_i. Then we obtain $[\alpha_i \gamma_i \alpha_i^{-1}] = g_i^{-1} w^{m_i} g_i$.

Notice that the choice of α_i corresponds to the choice of the representative g_i in the coset $g_i H$, which also corresponds to the choice of w_i as representative of its conjugacy class in H. On the other hand, the choice of p_i corresponds to the choice of g_i in the right coset $\langle w \rangle g_i$. Thus the g_i form a set of representatives of the double cosets in $\langle w \rangle \backslash G / H$.

This motivates the following definitions.

Definition 8.1.1. *Suppose $H \leq G$ is a subgroup of finite index, and $w \in G$.*

(a) If $g \in G$, let $m(g)$ be the minimum t such that $g^{-1} w^t g \in H$. We will say that the element $g^{-1} w^{m(g)} g$ is the lift *of w to H with respect to g.*

(b) Let $\{g_1, \ldots, g_k\}$ be a set of representatives of the double cosets in $\langle w \rangle \backslash G / H$. Put $m_i = m(g_i)$, and $w_i = g_i^{-1} w^{m_i} g_i$. Then we say that the set $\{w_1, \ldots, w_k\}$ is a complete lift *of w to H.*

This definition of lifts follows the one used by J. Manning (Definition 1.4 in [32]), with the difference that here we have different complete lifts for the various choices of double coset representatives, instead of defining it as a set of conjugacy classes. It bears this same relationship with the more general notion of *elevations*, introduced by D. Wise [44].

We are now ready to state the main result of this section.

Theorem 8.1.2. *Let G be a treeable group of finite cost. Let $w \in G$, and $H \leq G$ be a subgroup of finite index $n = [G : H]$. Take $\{g_1, \ldots, g_k\}$, a set of representatives of the double cosets in $\langle w \rangle \backslash G/H$, and let $K = \langle w_1, \ldots, w_k \rangle \leq H$ be the subgroup generated by the corresponding complete lift of w to H. Assume that*

1. *K is free of rank k, i.e. $w_1, \ldots, w_k$ is a free basis of K.*
2. *K is a measure free factor of H.*

Then $\langle w \rangle$ is a measure free factor of G

Proof. Let $H \curvearrowright X$ be a free probability measure preserving Borel action that realizes K as a measure free factor of H (so we have $E_H^X = E_K^X * E'$), and such that E_H^X is treeable. There is an action that satisfies both properties simultaneously: Take an action $H \curvearrowright Z_1$ that realizes K as a measure free factor of H, and an action $H \curvearrowright Z_2$ such that $E_H^{Z_2}$ is treeable. Then put $X = Z_1 \times Z_2$ and let H act on X by the diagonal action.

Consider the induced action of G on $Y = \mathrm{Ind}_H^G X = X \times G/H$, and put $X_i = X \times \{g_i H\}$. By reindexing if necessary, we may assume $g_1 \in H$, so X_1 is the standard embedding of X into $\mathrm{Ind}_H^G X$ (that we called X_0 in the properties following Definition 6.2.3). We can also assume that $g_1 = 1$: replacing g_i by $g_i g_1^{-1}$ changes K into $g_1 K g_1^{-1}$, which satisfies the same hypotheses.

Define $\phi_i : X_1 \to X_i$ by $\phi_i(x) = g_i \cdot x$. Also take Φ' a treeing for E', which exists since E_H^X is treeable (here we identify X_1 with X via the standard embedding). Consider the graphing $\Phi = \{w, \phi_2, \ldots, \phi_k\} \cup \Phi'$. We will show that Φ is a treeing of E_G^Y. This would give $E_G^Y = E_{\langle w \rangle}^Y * E''$ (E'' generated by $\{\phi_2, \ldots, \phi_k\} \cup \Phi'$), thus proving the result.

To see that Φ generates E_G^Y, recall that X_1 is a complete section and notice that every translate of X_1 is of the form $w^t g_i X_1$ (since the g_i are a set of representatives of the double cosets in $\langle w \rangle \backslash G/H$, the $w^t g_i H$ cover all the cosets of G/H). We get that $w^t g_i X_1 = w^t X_i = w^t \phi_i(X_1)$, so we only need to show that Φ generates the restriction $E_G^Y|_{X_1} = E_H^{X_1}$. Take $x \in X_1$. Notice that

$$w_i \cdot x = g_i^{-1} w^{m_i} g_i \cdot x = \phi_i^{-1} w^{m_i} \phi_i(x)$$

which is a word on Φ that is defined at x, since $x \in X_1$ and $w^{m_i} g_i \cdot x \in X_i$ (noticing that $X_i = g_i X_1 = g_i g_i^{-1} w^{m_i} g_i X_1 = w^{m_i} g_i X_1$). Together with Φ', the elements $w_1, \ldots, w_k$ generate E_H^X, so we obtain that Φ generates $E_H^{X_1}$. Thus Φ generates E_G^Y.

Now, on one hand

$$C(E_G^Y) = (n-1)\mu(X) + C(E_G^Y|_{X_1}) = (n-1)\mu(X) + C(E_H^X)$$

by the formula for a complete section (Theorem 6.1.14), so

$$C(E_G^Y) = (n-1)\mu(X) + C(E') + k\mu(X) = (n+k-1)\mu(X) + C(E')$$

since K is free of rank k. On the other hand

$$C(\Phi) = \mu(Y) + (k-1)\mu(X) + C(\Phi') = (n+k-1)\mu(X) + C(E')$$

Thus, by Theorem 6.1.11, Φ is a treeing of E_G^Y.

$\square$

As we pointed out, Theorem 8.1.2 can be seen as a partial converse to the main result of [1] by A. Alvarez, that we state below. Notice that Alvarez's result is stated purely in terms of equivalence relations and does not involve a group action.

Theorem 8.1.3. [1, Théorème 1] *Let E be a Borel equivalence relation on the standard Borel space X. Assume that $E = *_i E_i$ is the countable free product of the subrelations E_i ($i \in I$). Let S be a subrelation of E. Then*

$$S = *_i(*_{k_i \in K(i)} S_{k_i}) * T$$

where T is a treeable subrelation of E, and for each k_i in a countable set $K(i)$ there is a partial Borel isomorphism $\phi_{k_i} \in [[E]]$ defined in a Borel subset $A_{k_i} \subset X$, such that

$$S_{k_i} = (\phi_{k_i}^{-1} E_i|_{\phi_{k_i}(A_{k_i})}\phi_{k_i}) \cap S$$

Moreover, for each i there is k_i s.t. $A_{k_i} = X$ and $S_{k_i} = E_i \cap S$.

The set $[[E]]$ in the statement is the set of the partial Borel isomorphisms $\phi : A \to B$ of X such that $(x, \phi(x)) \in E$ for all $x \in A$.

8.2 Curves on surfaces

Here we extend the results of D. Gaboriau on measure free factors of surface groups.

Lemma 8.2.1. *Let S be a surface with boundary, and $\gamma_1, \ldots, \gamma_k$ be its boundary curves. If the genus of S is at least 1, then the boundary subgroup $\langle[\gamma_1], \ldots, [\gamma_k]\rangle \leq \pi_1(S)$ is a measure free factor of $\pi_1(S)$.*

Proof. When S is orientable, this is Gaboriau's Theorem 7.1.5. If S is non-orientable with one boundary component, take the orientable double cover $\hat{S}$. By the former case, the boundary components of $\hat{S}$ generate a measure free factor of $\pi_1(\hat{S})$. On the other hand, the boundary curves of $\hat{S}$ are the lifts of the boundary curve of S to this two-sheeted cover. So Theorem 8.1.2 applies.

In case there is more than one boundary component, write

$$\pi_1(S) = \langle a_1, \dots, a_g, c_1 \dots, c_{k-1} \rangle$$

where the boundary classes are $[\gamma_1] = a_1^2 \cdots a_g^2 c_1 \cdots c_{k-1}$, and $[\gamma_j] = c_{j-1}$ for $j > 1$. We just showed that $a_1^2 \cdots a_g^2$ generates a measure free factor of the subgroup $\langle a_1, \dots, a_g \rangle$. Since

$$\pi_1(S) = \langle a_1, \dots, a_g \rangle * \langle c_1 \dots, c_{k-1} \rangle$$

we get that the boundary subgroup $\langle a_1^2 \cdots a_g^2, c_1, \dots, c_{k-1} \rangle$ is a measure free factor. □

Proposition 8.2.2. *Let S be a surface with boundary, and $\gamma_1, \dots, \gamma_k$ be its boundary curves. Suppose that $\alpha = \{\alpha_1, \dots, \alpha_n\}$ is a family of disjoint essential simple closed curves on S, and $S_1, \dots, S_t$ are the components of S cut along α. If S_j has genus at least 1 for every j, then the subgroup*

$$\langle [\gamma_1], \dots, [\gamma_k], [\alpha_1], \dots, [\alpha_n] \rangle \leq \pi_1(S)$$

is a measure free factor of $\pi_1(S)$.

Proof. Consider the graph of groups Γ induced by cutting S along α. First consider the case where all the curves in α are two sided. Then there is one vertex for each component S_j, and one edge for each curve α_i. The vertex groups are $\pi_1(S_j)$, and the edge groups are $\langle [\alpha_i] \rangle$. Let $\alpha_i^{\pm}$ be the sides of a tubular neighborhood of

α_i. Then $[\alpha_i^{\pm}]$ are the images of $[\alpha_i]$ in the adjacent vertex groups. Since each S_j has genus at least one, Lemma 8.2.1 gives that the boundary subgroup of $\pi_1(S_j)$ is a measure free factor of $\pi_1(S_j)$. This boundary subgroup is freely generated by the $[\alpha_i^{\pm}]$ and the $[\gamma_p]$ that lie in S_j. Now $\pi_1(S)$ is obtained as an iteration of amalgamated products and HNN extensions of the vertex groups $\pi_1(S)$. Each such extension gives the identification

$$[\alpha_i] = [\alpha_i^-] \quad [\alpha_i] = t_i[\alpha_i^+]t_i^{-1}$$

where $t_i = 1$ in the case of an amalgamation, and is a new generator in the case of an HNN extension. Then Proposition 7.2.3 gives the proposition.

If the α_i are not necessarily two sided, let $\beta = \{\alpha_1, \ldots, \alpha_m\}$ be the curves of α that are cores of Möbius bands in S. Cut S along β, forming $S|_\beta$, and apply the former case to it. Notice that $[\alpha_i^2]$ for $i \leq m$ are boundary curves of $S|_\beta$. Since $\pi_1(S)$ is obtained from $\pi_1(S|_\beta)$ by amalgamating $\langle[\alpha_i]\rangle$ along $[\alpha_i^2]$ for $i \leq m$, then the result is obtained by 7.2.4.

□

8.3 Cyclic measure free factors of the free groups

In this section we apply the results of the last section to find some cyclic measure free factors of free groups.

Theorem 8.3.1. *Let $F = \langle x, y_1, \ldots, y_k \rangle$ be a free group of rank $k+1$. Then an element of the form*

$$w = xy_1x^{m_1}y_1^{-1}y_2x^{m_2}y_2^{-1}\cdots y_kx^{m_k}y_k^{-1}$$

generates a measure free factor of F, where $m_1, \ldots, m_k$ are arbitrary integers.

Proof. We can assume all m_i are different from 0, by reordering and applying Lemma 7.2.2.

Let $H = \langle x, c_1, \ldots, c_k \rangle$ be another free group on $k+1$ generators, and put $v = xc_1 \cdots c_k$. Then we can obtain F as an HNN extension:

$$F = \langle H, y_1, \ldots, y_k | c_j = y_j x^{m_j} y_j^{-1}, \text{ for } j = 1, \ldots, k \rangle$$

The standard inclusion $H \hookrightarrow F$ maps v to w, so we may regard w as an element of H. A natural complex with fundamental group H is a $k+2$-punctured sphere S, whose boundary components represent $x, c_1, \ldots, c_k$ and $w = v = xc_1 \cdots c_k$. We will identify these boundary curves with their representatives in $H = \pi_1(S)$. Starting from S, we can build a complex with fundamental group F by attaching cylinders C_i, glued to S by their boundary curves. More explicitly, one of the boundary components of C_i is identified with c_i, and the other is glued to x by an attaching map of degree m_i, so it represents x^{m_i} in the fundamental group (see Figure 8.3). We call this complex X. Notice that w is the only boundary component of S that was not attached to a cylinder in X.

Next we construct a finite cover $\hat{S} \to S$, proceeding as follows: Let $p > 0$ be an integer with $(p, m_j) = 1$ for all $j = 1, \ldots, k$ and $(p, k+1) = 1$ (e.g. p prime, large enough). Let $\hat{H}$ be the kernel of the morphism $H \to \mathbb{Z}_p$ sending the generators $x, c_1, \ldots, c_k$ to 1, and let $\hat{S}$ be the cover corresponding to $\hat{H}$. Notice that it has index p, and its boundary components are represented by $\hat{x} = x^p$, $\hat{c}_j = c_j^p$ and $\hat{w} = w^p$. This is because the images of x, c_j and w in $\mathbb{Z}_p$ have order p. By these reasons we obtain that $\hat{w} = w^p$ is a complete lift of w to $\hat{H}$. Now, by computing the Euler characteristic, we can see that $\hat{S}$ is a surface of positive genus. Thus we may apply Lemma 8.2.1 to conclude that its boundary subgroup (which is freely generated by $\hat{x}, \hat{c}_1, \ldots, \hat{c}_k, \hat{w}$), is a measure free factor of $\hat{H}$.

Now we extend $\hat{S}$ to a p-sheeted cover $\hat{X} \to X$. Consider the standard p-sheeted cover $\hat{C}_i \to C_i$ of each cylinder C_i, and glue each of its boundary components to $\hat{S}$ along the curves $\hat{c}_i$ and $\hat{x}^{m_i}$ respectively. Since $(p, m_i) = 1$, the covering maps $\hat{S} \to S$ and $\hat{C}_i \to C_i$ agree on the glued boundaries and they give rise to a covering map $\hat{X} \to X$.

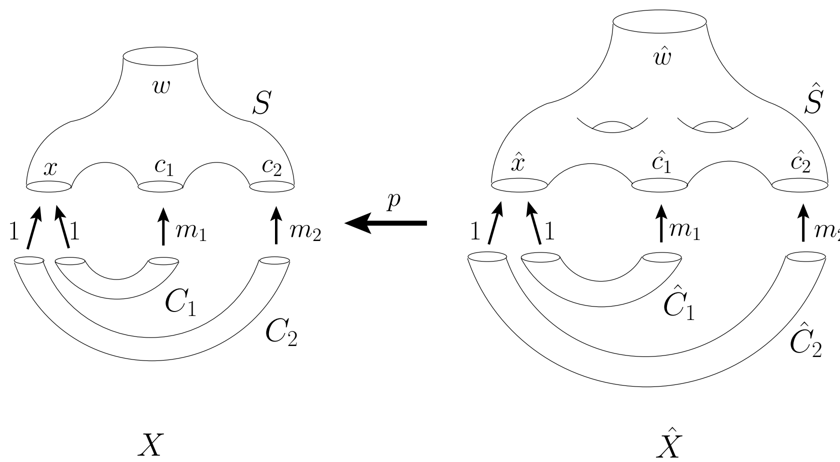

Figure 8.1: **Left:** Sketch of the complex X for $k = 2$. The gluing maps are labeled by their degrees. **Right:** The p-sheeted cover $\hat{X}$, sketched in the same fashion.

Let $\hat{F}$ be the index p subgroup of F corresponding to the cover $\hat{X}$. Then $\hat{F} \cap H = \hat{H}$, and $\hat{w} = w^p$ is also a complete lift of w to $\hat{F}$. By our construction of $\hat{X}$, we can write $\hat{F}$ as an HNN extension:

$$\hat{F} = \langle \hat{H}, l_1, \ldots, l_k | \hat{c}_j = l_j \hat{x}^{m_j} l_j^{-1}, \text{ for } j = 1, \ldots, k \rangle$$

Now by Proposition 7.2.3, $\hat{w}$ generates a measure free factor in $\hat{F}$, so by Theorem 8.1.2, we get that w generates a measure free factor of F. $\square$

Theorem 8.3.2. *Let $G = F_2 = \langle a, b \rangle$. Then any element of the form $w = a^k b^n$ for $k \neq 0$ and $n \neq 0$ generates a measure free factor of G.*

Proof. It is enough to consider the case when $k, n > 0$, for one can apply the automorphism taking a, b to $a^{\text{sign}(k)}$, $b^{\text{sign}(n)}$. Also, we can assume $k, n > 1$, for if either one equals 1, then w is already a free factor.

Let Γ be the rose on two petals, labeled by a and b respectively. So $\pi_1(\Gamma) = G$. Consider the graph $\hat{\Gamma}$ constructed as follows: It has kn vertices, v_j for $j \in \mathbb{Z}_{kn}$. As for the edges, for each $j \in \mathbb{Z}_{kn}$ there is an oriented edge labeled by b going from v_j to v_{j+1}, and an oriented edge labeled by a going from v_j to v_{j+n}. The orientations and labeling of the edges give a projection map $\hat{\Gamma} \to \Gamma$, which is a covering of index kn. Let $H = \pi_1(\hat{\Gamma}, v_0)$ be the corresponding subgroup of G.

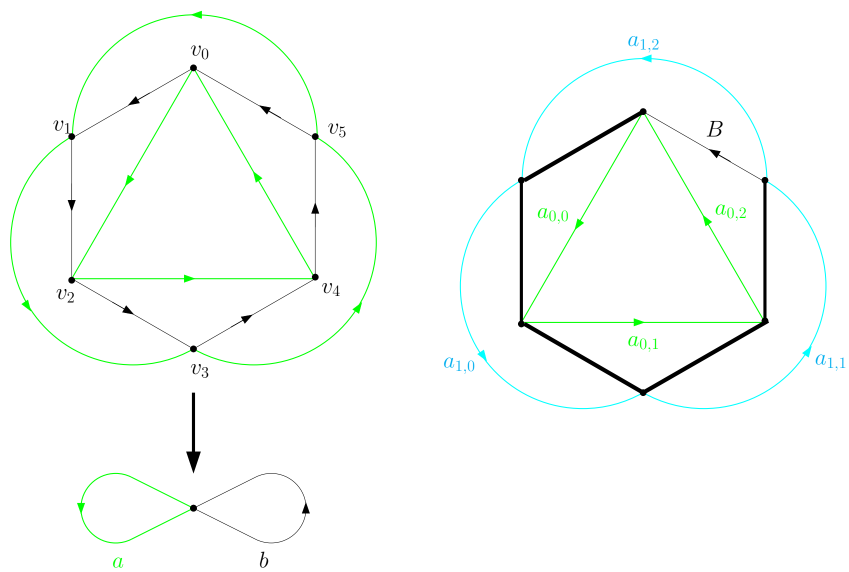

Figure 8.2: **Left:** An example of the cover $\hat{\Gamma}$ for $k = 3, n = 2$. **Right:** The generators for H, where the bold edges form the spanning tree.

Consider the lift to $\hat{\Gamma}$ of a curve representing w in Γ. If it starts at $v_j \in \hat{\Gamma}$, then it ends at v_{j+n}. This is because the lift of a^k is closed, ending in v_j, and the lift of b^n that starts at v_j ends at v_{j+n}. Thus w^k is the minimal power of w whose lift from v_j is a closed curve in $\hat{\Gamma}$. This holds true for any j, as the cover is normal.

Let $t = 0, \ldots, n-1$ be a set of representatives of the cosets of $n\mathbb{Z}_{kn} \cong \mathbb{Z}_k$ in

$\mathbb{Z}_{kn}$. Let γ_j be the lift of w^k starting from v_t, and let $w_t = [\beta_t \gamma_t \beta_t^{-1}]$ where β_t is the lift of b^t starting at v_0. Then $w_0, \ldots, w_{n-1}$ are a complete lift of w to H.

Now we will write the w_t in a suitable basis for H. Let T be the spanning tree of $\hat{\Gamma}$ consisting of all the edges labeled by b except (v_{kn-1}, v_{kn}). Then the edges not in T correspond to a free basis of H. Name them as follows: B will stand for the generator corresponding to (v_{kn-1}, v_{kn}), and $a_{t,i}$, for $t = 0, \ldots, n-1$ and $i \in \mathbb{Z}_k$, will stand for the edge going from the vertex v_{t+ni} to the vertex $v_{t+n(i+1)}$. This covers all edges not in T, observing that the edges labeled by a are arranged in n cycles of length k, each spanning a coset of $n\mathbb{Z}_{kn}$ in $\mathbb{Z}_{kn}$.

Writing w_t in this basis, we get

$$w_t = (a_{t,0} \cdots a_{t,k-2} a_{t,k-1})(a_{t,1} \cdots a_{t,k-1} a_{t,0}) \cdots (a_{t,k-1} a_{t,0} \cdots a_{t,k-2})B$$

Let $A_t = a_{t,0} \cdots a_{t,k-1}$. Then w_t is the product of all cyclic conjugates of A_t and B.

Let $y_{t,i} = (a_{t,0} \cdot a_{i-1})^{-1}$. Then B, A_t, $y_{t,i}$, for $t = 0, \ldots, n-1$ and $i = 0, \ldots, k-2$ also forms a free basis of H. In this new basis, w_t reads as

$$w_t = A_t x_{t,0} A_t x_{t,0}^{-1} \cdots x_{t,k-2} A_t x_{t,k-2}^{-1} B$$

Using the previous result 8.3.1, we know that $v_t = A_t x_{t,0} A_t x_{t,0}^{-1} \cdots x_{t,k-2} A_t x_{t,k-2}^{-1}$ is a measure free factor of $H_t = \langle A_t, x_{t,i} \text{ for } i = 0, \ldots, k-2 \rangle = \langle a_{t,i} \text{ for } i = 0, \ldots k-1 \rangle$. Now

$$H = H_0 * \cdots * H_{n-1} * \langle B \rangle$$

So by Lemma 7.2.2 the subgroup $M = \langle v_0, \ldots, v_{k-1}, B \rangle$ is a measure free factor of H, and the given basis generates it freely. Since $w_t = v_t B$, then $w_0, \ldots w_{k-1}, B$ is also a free basis of M. So $K = \langle w_0, \ldots, w_{n-1} \rangle$ is a measure free factor of H, freely generated by a complete lift of w. Theorem 8.1.2 finishes the proof. □

Considering the conjugates of the words of the form $a^k b^n$ gives the following.

Corollary 8.3.3. *An element of the form $w = a^k b^n a^p$ with $k \neq -p$, $n \neq 0$, generates a measure free factor of $F_2 = \langle a, b\rangle$.*

Observe that these are exactly all the 3–letter words (i.e. of the form $a^k b^n a^p$) that are not a proper power.

8.4 Measure free factors of virtually free groups

Finally, we consider measure free factors of the virtually free groups that are free products of free groups and finite cyclic groups.

Theorem 8.4.1. *Let $G = \langle a_1, \ldots, a_n, s_1, \ldots, s_k | s_1^{n_1} = 1, \ldots, s_k^{n_k} = 1\rangle \cong F_n * \mathbb{Z}_{n_1} * \cdots * \mathbb{Z}_{n_k}$. If $v \in F_n$ generates a measure free factor of F_n, then $w = v s_1^{t_1} \cdots s_k^{t_k}$ generates a measure free factor of G for any $t_1, \ldots, t_k$.*

Proof. We use induction on k. The base case $k = 0$ is trivial. For the inductive step, consider the subgroup K generated by $a_1, \ldots, a_n, s_1, \ldots, s_{k-1}$, and the subgroup H generated by the conjugates $K_j = s_k^j K s_k^{-j}$ for $j = 0, \ldots, n_k - 1$. The natural presentation of K is analogous to the one of G, only with $k-1$ generators of torsion. On the other hand, we shall see that $H \cong K_0 * \cdots * K_{n_k - 1}$ and has index n_k in G.

To prove this we consider the complex X corresponding to the given presentation of G. We use orbifold notation: X consists on a wedge of circles in correspondence with the generators $a_1, \ldots, a_n, s_1, \ldots, s_k$, where the circle for s_j is capped by a disk D_j with a cone-point of degree n_j. Then H corresponds to the n_k-sheeted branched cover $\hat{X} \to X$ constructed as follows: Let $\hat{D}_k \to D_k$ be the branched covering given by the map $\mathbb{D} \to \mathbb{D}/z \to z^{n_k}$ (identifying D_k and $\hat{D}_k$ with the unit disk $\mathbb{D} \subset \mathbb{C}$, with the cone-point at 0). Let $x_0, \ldots, x_{n_k-1}$ be the preimages of the basepoint of X in $\hat{D}_k$, and $X_0, \ldots, X_{n_k-1}$ be copies of the presentation complex of K. We get $\hat{X}$ by wedging each X_j to $\hat{D}_k$ at x_j, and the covering map $\hat{X} \to X$ is the natural extension of the branched cover $\hat{D}_k \to D_k$ (see Figure 8.4).

Notice that the single preimage in $\hat{X}$ of the cone-point of D_k has degree 1, i.e. is no longer a cone-point. So $\pi_1(\hat{X}) \cong \pi_1(X_0) * \cdots * \pi_1(X_{n_k-1})$, giving that $H = \pi_(\hat{X}) \cong K_0 * \cdots * K_{n_k-1}$ and has index n_k in G.

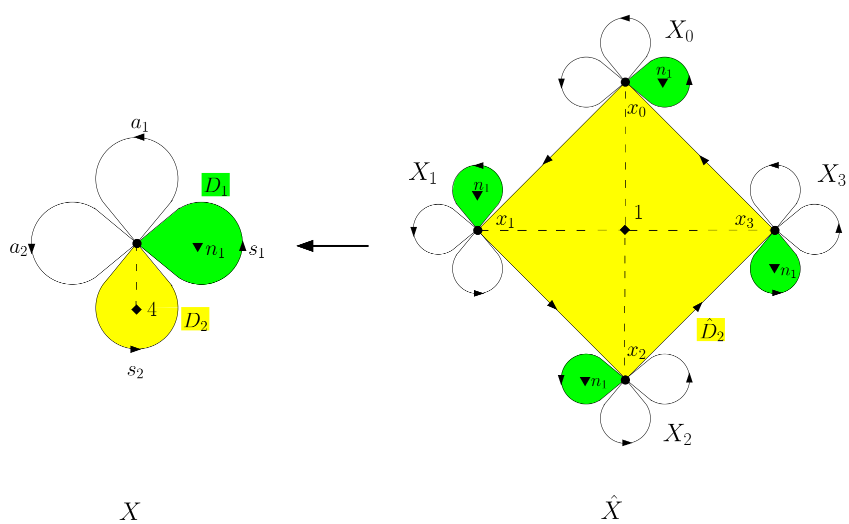

Figure 8.3: **Left:** The complex X for $n = 2$, $k = 2$ and $n_2 = 4$. **Right:** Sketch of the 4-sheeted branched cover $\hat{X}$, and the subcomplexes X_i used in the proof.

Now we will describe a complete lift for w. Let $u = vs_1^{t_1} \cdots s_{k-1}^{t_{k-1}} \in K$, and $u_j = s_k^j u s_k^{-j} \in K_j$ for $j = 0, \ldots, n_k - 1$. Consider also $d = (t_k, n_k)$ and $m = n_k/d$. The lift of w^m to $\hat{X}$ starting at x_i is

$$w_i = u_i \cdots u_{i+(m-1)t_k}$$

the product of the u_{i+lt_k} for $l = 0, \ldots, m-1$, in that order. Then $w_0, \ldots, w_{d-1}$

form a complete lift of w to H.

For each $i = 0, \ldots, d-1$ write $L_i = *_{l=0}^{m-1} K_{i+lt_k}$. By induction hypothesis, u generates a measure free factor of K, and so does u_j in K_j by conjugation. Then the subgroup $M_i \leq L_i$, generated by u_{i+lt_k} for $l = 0, \ldots, m-1$, is a measure free factor of L_i, using Lemma 7.2.2. Observe that w_i is a free factor of M_i, and so it is a measure free factor of L_i (this is an easy consequence of the definition of measure free factor). Finally, since $H = *_{i=0}^{d-1} L_i$, the subgroup generated by the complete lift $w_0, \ldots, w_{d-1}$ is free of rank d, and a measure free factor of H (again by 7.2.2). We finish by applying Theorem 8.1.2.

□

This allows us to extend the results in section 8.2 to the exact analogues for 2-orbifold groups.

Corollary 8.4.2. *Let S be a 2-orbifold with boundary, and $\gamma_1, \ldots, \gamma_k$ be its boundary curves. Suppose that $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a family of disjoint essential simple closed curves on S, and $S_1, \ldots, S_t$ are the components of S cut along α. If S_j has genus at least 1 for every j, then the subgroup*

$$\langle [\gamma_1], \ldots, [\gamma_k], [\alpha_1], \ldots, [\alpha_n] \rangle \leq \pi_1(S)$$

is a measure free factor of $\pi_1(S)$.

Proof. The arguments used in the proof of 8.2.1 and 8.2.2 allows us to reduce the corollary to the case with only one boundary component ($k = 1$) and no curves α_i ($n = 0$). Then we can write $\pi_1(S)$ as either

$$\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g, s_1, \ldots, s_m | s_1^{n_1} = \cdots = s_m^{n_m} = 1 \rangle$$

where $[\gamma_1] = [a_1, b_1] \cdots [a_g, b_g] s_1 \cdots s_m$, or

$$\pi_1(S) = \langle a_1, \ldots, a_g, s_1, \ldots, s_m | s_1^{n_1} = \cdots = s_m^{n_m} = 1 \rangle$$

where $[\gamma_1] = a_1^2 \cdots a_g^2 s_1 \cdots s_m$. In both cases we conclude using Proposition 8.4.1, together with Lemma 8.2.1.

$\square$

CHAPTER 9

TREEABILITY OF QUADRATIC BAUMSLAG–SOLITAR GROUPS

Here we consider the problem of finding which QBS groups are treeable, or equivalently, measure equivalent to a free group. This is a particular case of the clasification of QBS groups with respect to measure equivalence. For the classical Baumslag–Solitar groups the general problem was studied by Y. Kida [29]. We find that a finitely generated GBS group is treeable if and only if it is isomorphic to $\mathbb{Z}$, $\mathbb{Z}^2$ or the fundamental group of a Klein bottle. Then we determine the finite QBS graphs whose fundamental groups are treeable.

9.1 Amenability and cost

First we recall the background material on amenability that is relevant for the work at hand. For proofs we refer to the book by Kechris and Miller [26]. See also the book by Paterson [34], for a general treatment of amenability.

Definition 9.1.1. *A countable group G is* amenable *if it admits a left-invariant, finitely additive probability measure defined on all subsets of G.*

Recall that finite additivity means that $\mu(A \cup B) = \mu(A) + \mu(B)$ if A and B are disjoint. We remark that asking for countable additivity, the usual property for a measure, would not be possible if G is infinite. That μ is left-invariant means that $\mu(gA) = \mu(A)$ for every $g \in G$ and $A \subset G$.

The following are the standard properties of amenable groups that we will need.

1. [26, Example 5.2] $\mathbb{Z}$ is amenable.

2. [26, Proposition 5.6] A subgroup of an amenable group is still amenable.

3. [26, Corollary 5.7] An abelian group is amenable. More generally, solvable groups are amenable.

4. [26, Proposition 5.6] If $N \trianglelefteq G$, then G is amenable if and only if both N and G/N are amenable.

5. [26, Propositions 5.8, 5.10] F_n is not amenable for $n > 1$.

Our interest in amenable groups comes from the next result. It combines the results of Ornsten and Weiss [33], Furman [13] and Theorems 6.1.11 and 6.1.12 due to Gaboriau. Recall that a group G is *strongly treeable* if E_G^X is treeable for *every* free probability measure preserving action $G \curvearrowright X$ on a standard Borel space X. It has *fixed price* if $C(G) = C(E_G^X)$ for *every* free probability measure preserving action $G \curvearrowright X$ as before. Notice that strongly treeable groups have fixed price by 6.1.12.

Theorem 9.1.2. [26, Corollary 31.2] *Let G be an infinite countable group. The following are equivalent:*

1. *G is amenable.*
2. *G is strongly treeable and $C(G) = 1$*
3. *G is treeable and $C(G) = 1$*

So amenable groups have fixed price. We remark that, by this theorem, a group of cost 1 is treeable if and only if it is amenable. The other result on amenability we want to consider computes the cost of an amalgamation or an HNN extension over an amenable group. It will be used in combination with Theorem 9.1.2 in order to decide the treeability of GBS groups.

Theorem 9.1.3. [15, Théorème VI.7] *Let G_1, G_2 be groups of fixed price, and Λ an amenable group. Then:*

1. *An amalgamation of the form $G_1 *_\Lambda G_2$ has fixed price and*

$$C(G_1 *_\Lambda G_2) = C(G_1) + C(G_2) - C(\Lambda)$$

2. *An HNN extension of the form $G_1 *_\Lambda$ has fixed price and*

$$C(G_1 *_\Lambda) = C(G_1) + 1 - C(\Lambda)$$

This result can also be found in [26] as Theorems 36.1 and 37.1.

9.2 Treeability for Generalized Baumslag–Solitar groups

We will restrict our work to finite GBS graphs (recall section 5.1). In this context, we will prove that a GBS group is treeable if and only if it is amenable, and that happens when it is isomorphic to $\mathbb{Z}$, $\mathbb{Z}^2$ or the fundamental group of the Klein bottle.

Lemma 9.2.1. *A finitely generated GBS group has fixed price and its cost equals 1.*

Proof. This is a direct application of Theorem 9.1.3. Recall that a GBS graph has vertex and edge groups infinite cyclic, thus amenable, with fixed price and cost 1. □

This shows that finitely generated GBS groups are treeable exactly when they are amenable, recalling Theorem 9.1.2.

Lemma 9.2.2. *A GBS group is amenable if and only if it is isomorphic to* $\mathbb{Z}$, $\mathbb{Z}^2$ *or the Klein bottle group.*

Proof. Let Γ be a GBS graph and $G = \pi_1(\Gamma)$. Assume that G is not isomorphic to $\mathbb{Z}$, $\mathbb{Z}^2$ or the Klein bottle group. We will show that G contains a non-abelian free subgroup, so it cannot be amenable. Let T be the Bass–Serre tree associated to Γ. By Lemma 5.1.2, T does not have an invariant line. Then there must be two elements $g, h \in G$ that act hyperbolically on T, with different translation axes. Let L be the length of the intersection between these axes, and let $n > 1$ be large enough so that $nl(g), nl(h) > L$, where l stands for the translation length of an element of G (defined by $l(g) = \inf\{d(x, gx) : x \in T\}$). Then the elements g^n, h^n have translation lengths $l(g^n) = nl(g)$ and $l(h^n) = nl(h)$, but their axes intersect in a segment of lesser length L. It is a well known fact that in this case they generate a free group of rank 2 in G. (This was proved by Culler and Morgan in [5, Lemma 2.6]). □

Now we turn to the problem of finding which GBS graphs have amenable fundamental groups. We start with the one-edged graphs.

Lemma 9.2.3. *Let* Γ *be a GBS graph with a single edge* e*, and* $G = \pi_1(\Gamma)$*. Assume that* G *is amenable. Then:*

1. *If* Γ *has two vertices, then either* $m_e^+ = 1$ *or* $m_e^- = 1$*, and* $G \cong \mathbb{Z}$*, or* $m_e^+ = m_e^- = 2$ *and* G *is isomorphic to the Klein bottle group.*
2. *If* Γ *has one vertex, then* $m_e^+ = m_e^- = 1$ *and* G *is isomorphic to either* $\mathbb{Z}^2$ *or to the Klein bottle group.*

Proof. If Γ has either one vertex, or two vertices with $m_e^\pm > 1$, then its Bass–Serre tree T_Γ is minimal (i.e. they do not have invariant subtrees). In this case, it is easy to observe that T_Γ is a line exactly when $m_e^+ = m_e^- = 1$ for an HNN extension, or $m_e^+ = m_e^- = 2$ for an amalgamation. □

In order to find which GBS groups are treeable, we will need the fact that a cyclic measure free factor cannot be generated by a proper power.

Proposition 9.2.4. *Let G be a treeable group, $w \in G$ non-trivial, and assume that $w = v^n$ for $v \in G$, $n > 1$. Then the cyclic subgroup $\langle w \rangle$ is not a measure free factor of G.*

Proof. We will assume that $\langle w \rangle$ is a measure free factor of G, and reach a contradiction. Let $C = \langle c \rangle \cong \mathbb{Z}$ be an infinite cyclic group generated by c. Consider the amalgamation $H = G *_{w=c^3} C$. From Theorem 7.1.6 and the assumption that $\langle w \rangle$ is a measure free factor, we get that H is treeable. Then by Theorem 6.1.9, every subgroup of H is treeable. On the other hand, consider $K = \langle v \rangle *_{w=c^3} C$. It is clearly a subgroup of H, and it is also one of the GBS groups found to be non-amenable in Lemma 9.2.3. Thus $K \leq H$ is not treeable, a contradiction. □

In order to describe the GBS graphs that define amenable groups, we need to establish some notation. Let Γ be any GBS graph and e be an edge of Γ, with endpoints v^-, v^+. We say that e is *allowed* if the subgraph spanned by e (which is e together with its endpoints) satisfies the conditions of Lemma 9.2.3, and it is *forbidden* otherwise. A *double edge* is an edge with disctinct endpoints that satisfies $m_e^- = m_e^+ = 2$. Notice that a double edge is allowed. An allowed edge is *closed* if it is either a loop ($v^+ = v^-$ with $m_e^+ = m_e^- = 1$) or a double edge. An edge is *open* if it is allowed and not closed, and we further distinguish two types.

An open edge is called *simple* if $m_e^+ = m_e^- = 1$, and is called *directed* otherwise. A directed edge with $m_e^- = 1$ and $m_e^+ = m > 1$ is said to have *degree* m, and to *point towards* v^+ (and similarly with the roles of v^+ and v^- interchanged).

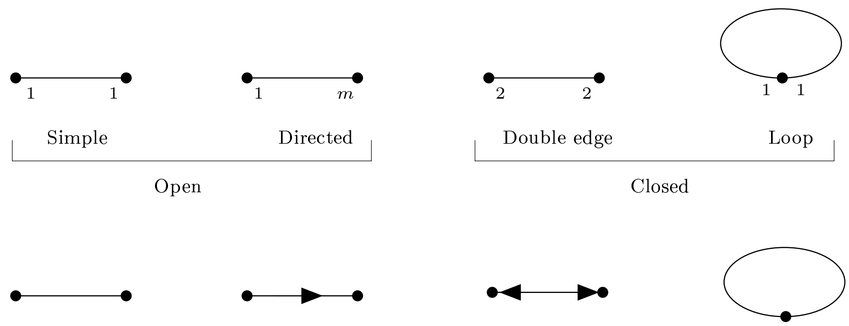

Figure 9.1: **Upper row:** Classification of the allowed edges, according to their labels. We assume $m > 1$. **Lower row:** The graphical notation we will be using for these edges.

If we consider the subgraph Δ_e spanned by e in the previous discussion, we observe that $\pi_1(\Delta_e)$ is amenable exactly when e is an allowed edge. If e is closed, then $\pi_1(\Delta_e)$ is isomorphic to $\mathbb{Z}^2$ or the Klein bottle group. On the other hand, if e is open then $\pi_1(\Delta_e)$ is cyclic. When e is directed towards v^+ then $\pi_1(\Delta_e) = G_{v^+}$. Finally, if e is simple we have $\pi_1(\Delta_e) = G_e = G_{v^\pm}$.

Now we describe the finite GBS graphs that turn out to define amenable groups. They must contain only allowed edges, since subgroups of amenable groups are amenable. We define the following types of GBS graphs:

1. *Rooted trees.* These graphs are trees in which all edges are open, and there exists a vertex r, called a root, so that all directed edges point towards r. (Or more precisely, point towards the endpoint that is closest to r). The root is not necessarily unique, we can observe that the set of possible roots spans a subtree whose edges are simple. The fundamental group is cyclic and equals the vertex group at the root.

2. *Double trees.* These graphs have a core subgraph which is a double edge e with endpoints v_1, v_2. Removing the edges in the core subgraph leaves two rooted trees with roots at v_1 and v_2 respectively. Clearly, the fundamental group of a double tree is that of its core subgraph, which is a Klein bottle group.

3. *Triple trees.* There is a core subgraph Γ_0, which consists of a chain of open edges $e_1, \ldots, e_n$ where e_i has endpoints v_i, v_{i+1}. The edges in the middle of the chain $e_2, \ldots, e_{n-1}$ are simple, while e_1 and e_n are directed of degree 2, pointing towards v_1 and v_{n+1} respectively (i.e. pointing away from the chain). Removing the edges of Γ_0 leaves rooted trees, whose roots can be taken in Γ_0. The fundamental group is again the group of the core, $\pi_1(\Gamma_0)$, and it is a Klein bottle group.

4. *Flowers.* These have a core subgraph Γ_0 which is either a loop (with labels equal to 1) or a cycle of simple edges. Removing the edges of this core subgraph leaves rooted trees, whose roots can be taken in Γ_0. Again, the fundamental group of a flower is that of its core, which is either $\mathbb{Z}^2$ or a Klein bottle group.

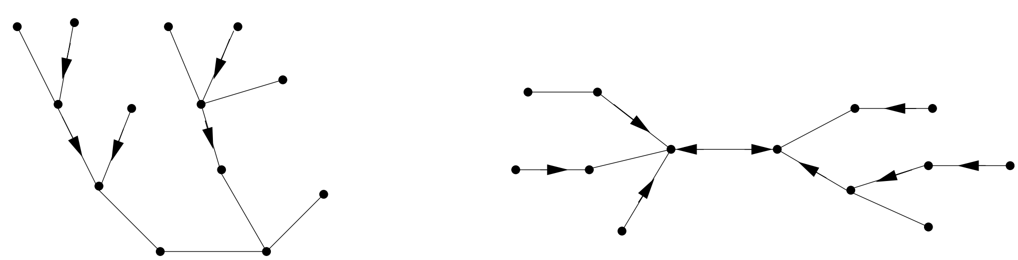

Figure 9.2: Examples of the GBS graphs defining amenable groups. **Left:** A rooted tree. **Right:** A double tree.

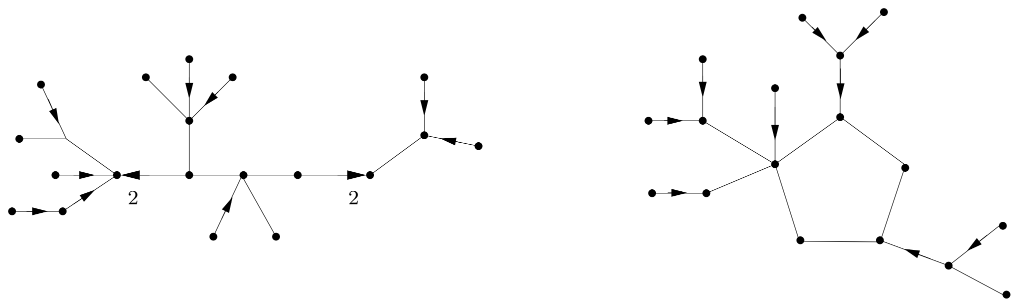

Figure 9.3: Examples of the GBS graphs defining amenable groups. **Left:** A triple tree. **Right:** A flower.

Proposition 9.2.5. *Let* Γ *be a finite GBS graph and* $G = \pi_1(\Gamma)$. *Then* G *is treeable if and only if* G *is isomorphic to either* $\mathbb{Z}$, $\mathbb{Z}^2$ *or the fundamental group of a Klein bottle. If this is the case, then* Γ *is either a rooted tree, a double or triple tree, or a flower, as defined above.*

Proof. We have already obtained the first statement, combining Lemma 9.2.1 with Theorem 9.1.2 and then applying Lemma 9.2.2.

For the second statement, we can assume that G is amenable. First notice that the fundamental group of the underlying graph of Γ injects into $G = \pi_1(\Gamma)$, as the subgroup generated by the stable letters of the standard presentation. Since G cannot contain a subgroup isomorphic to F_2, we see that Γ must have the underlying graph of a tree or a flower.

Claim 1: If the underlying graph of Γ is that of a flower, then the edges in the core cycle must have labels equal to 1.

Proof of Claim 1: Let e be an edge in the mentioned cycle. If e is a loop, we apply Lemma 9.2.3. Otherwise, let $e, e_1, \ldots, e_n$ be the edges forming the cycle. Let Δ be the subgraph spanned by $e_1, \ldots, e_n$, and $H = \pi_1(\Delta)$.

If H is cyclic, let $h \in H$ be its generator. If not, then $H = \pi_1(S)$ where S is either a torus or a Klein bottle, and there is a weakly essential simple closed curve γ such that $[\gamma]$ is elliptic in Δ. This curve is unique, since any two disctinct weakly essential simple closed curve in S intersect. In this case put $h = [\gamma]$. (Alternatively, we can observe that all elliptic elements of a GBS graph are commensurable, and pick h as a common root for the generators of G_{v^-} and G_{v^+}, where v^-, v^+ are the endpoints of e).

Let c be the generator of G_e, and t be the stable letter corresponding to the cycle. So we see that $c = h^k = th^pt^{-1}$ for some k, p, where m_e^- divides k and m_e^+ divides p. On the other hand, the subgroup generated by h and t is an HNN extension of the form

$$\langle h, t | h^k = th^pt^{-1} \rangle$$

which is a one-edged GBS graph of labels $|k|, |p|$. By Lemma 9.2.3, and the fact that subgroups of G must be amenable, we have that $|k| = |p| = 1$. So $m_e^- = m_e^+ = 1$, and e must be simple. Note that e was chosen arbitrarily as an edge of the cycle, thus all edges on the cycle must be simple. $\lozenge$

Now notice that if Γ is a GBS graph and e is a simple edge in Γ, then e can be collapsed, yielding a graph $\Gamma_1 = \Gamma/e$ that is also a GBS graph. Moreover, this collapse preserves the labels of the rest of the edges in Γ. We can repeat this procedure until there are no more simple edges. This only involves the collapse of subtrees made of simple edges, so the classes of graphs in the statement are preserved. Notice also that a cycle of simple edges becomes a loop, which is a closed edge. Thus it is enough to prove the proposition when there are no simple edges.

In the rest of the proof, we will assume that Γ has no simple edges.

Claim 2: Γ has at most one closed edge.

Proof of Claim 2: Consider the Bass–Serre tree T_Γ. Each closed edge e gives rise to an element $g_e \in G$ that acts hyperbolically on T_Γ. If e is a loop, then g_e is the corresponding stable letter, and if e is a double edge, then $g_e = h^- h^+$ where $h^\pm$ are the generators of $G_{v^\pm}$, the vertex groups of the endpoints of e. Notice that, in both cases, the axis of g_e in T_Γ projects to e under the quotient $\Gamma = T_\Gamma / G$.

Now, suppose that e_1, e_2 are different closed edges in Γ, and let g_1, g_2 be their corresponding hyperbolic elements. Then g_1 and g_2 do not commute, and moreover, their axes in T_Γ intersect in at most one point. But then there are some powers of g_1 and g_2 that generate a non-abelian free subgroup of G. Thus G cannot be amenable, and we reach a contradiction, proving the claim. $\Diamond$

Collapsing a directed edge yields another GBS graph defining G. Now we provide the explicit description of this collapse, that will be needed for the rest of the proof. Let e_1 be a directed edge and v_0, v_1 be its endpoints. Assume e_1 points towards v_1 and has degree m. When we collapse e, we identify v_0 and v_1 to a single vertex $\bar{v}_1$ with vertex group equal to G_{v_1}. The labels of the edges adjacent to v_1 do not change, and those of the edges adjacent to v_0 get multiplied by m. Observe that a directed edge pointing towards v_0 transforms into a directed edge pointing towards $\bar{v}_1$, and the degree multiplies by m. On the other hand, a directed edge of degree k pointing away from v_0 becomes an edge with labels m and k, which may be forbidden or be a double edge.

The last part of the proof is split in two cases.

Claim 3: If there is one closed edge in Γ, then Γ must be a double tree or a flower.

Proof of Claim 3: Let e be the closed edge, and v any of its endpoints. If e is the only edge we are done, and Γ is a double tree (if e is a double edge), or a flower (if e is a loop). Otherwise, let e_1 be another edge adjacent to v. We assumed there are no simple edges, and we showed there can be no more closed edges (Claim 2), so e_1 must be a directed edge. We shall prove that e_1 points towards v. Let v_1 be the other endpoint of e_1, and let h_1 be the generator of G_{v_1}. If e_1 points towards v_1, then g_e and $h_1 g_e$ both act hyperbolically in T_Γ and they do not commute. As before, this is a contradiction since G is amenable. Alternatively, we can see that the Bass–Serre tree of the subgraph spanned by e and e_1 injects into T_Γ and has no invariant line, reaching the same contradiction.

Thus any edge e_1 that is adjacent to e must be directed and point towards e. We want to prove that this is the case for all the other edges of Γ. (Precisely, that they point towards the endpoint that is closest to e). This can be done by induction. Assume that all the edges at distance less than k from e point towards e. Let e_2 be an edge at distance $k+1$ from e. Then e_2 must be adjacent to an edge e_1 at distance k from e. Suppose e_2 points away from e. Then e_2 and e_1 are pointing away from each other, and collapsing e_2 turns e_1 into either a forbidden edge or a double edge. But G is amenable, so it does not admit a GBS graph decomposition with forbidden edges, or two closed edges. So e_2 must point towards e.

This just proved that in the case there is a closed edge, Γ must be a double tree or a flower. $\Diamond$

Claim 4: If Γ contains no closed edges, then Γ is either a rooted tree or a triple tree.

Proof of Claim 4: If Γ is not a rooted tree, then there must be a vertex v with two or more adjacent edges pointing away from v. This uses the condition that Γ is finite. Assume this is the case. Let $e_1, \ldots, e_n$ be the edges pointing away from v, $m_1, \ldots, m_n$ their respective degrees and $v_1, \ldots, v_n$ their other endpoints. (So $n \geq 2$). Let Γ' be the graph of groups obtained from collapsing e_1. From our description of the collapse of a directed edge, we see that Γ' is a GBS graph with fundamental group G, where v and v_1 identify to a single vertex $\bar{v}_1$. Also, for each $j \geq 2$, the edge e_j turns into an edge e'_j with endpoints at v_j and $\bar{v}_1$, and the labels of e'_j at $v_j, \bar{v}_1$ are m_j, m_1 respectively. Since Γ' cannot have forbidden edges, we get that $m_1 = \cdots = m_n = 2$. On the other hand, Γ' cannot have more than one double edge, so $n = 2$. Moreover, we get from Claim 3 that Γ' has to be a double tree. Notice that the edges of Γ other than $e_1, \ldots, e_n$ are still directed when passing to Γ', and the direction in which they point is preserved. So if Γ' is a double tree, then Γ must be a triple tree. $\Diamond$

The combination of Claims 3 and 4 finishes the proof, showing that if G is amenable then Γ is in one of the four classes of graphs mentioned in the statement.

□

9.3 Orbifold covers

Here we prove some results about 2-orbifolds and their fundamental groups that we will need for the classification of the treeable QBS groups. The main tool for this purpose are some specific covering spaces.

Lemma 9.3.1. *Let S be a 2-orbifold with more than two boundary curves $\gamma_1, \ldots, \gamma_n$ ($n > 2$), and let $k > 0$ be an odd integer. There exists a k-sheeted cover $\hat{S} \to S$ such that $\hat{S}$ has positive genus and boundary curves $\hat{\gamma}_1, \ldots, \hat{\gamma}_n$, where for each i, $\hat{\gamma}_i$*

covers Γ_i *with degree* k*.*

Proof. This is similar to the cover constructed in the proof of 8.3.1. Decompose S into pairs of pants, possibly with cone-points in their interiors, and notice that this reduces the lemma to the case of a pair of pants. This is because the covers obtained by applying the lemma to these pairs of pants glue back to form a cover of S. Let S be a pair of pants with cone points. Then

$$\pi_1(S) = \langle p_1, p_2, p_3, s_1, \ldots, s_m | s_i^{n_i} = 1, p_1 p_2 p_3 s_1 \cdots s_m = 1 \rangle$$

Let K be the kernel of the homomorphism $\phi : \pi_1(S) \to \mathbb{Z}_k$ onto the cyclic group of order k, such that $\phi(s_j) = 0$ and $\phi(p_1) = \phi(p_2) = 1$. Let $\hat{S} \to S$ be the cover corresponding to $K \leq \pi_1(S)$. This is a k-sheeted cover, since $[\pi_1(S) : K] = k$. The boundary curves of $\hat{S}$ correspond to p_1^k, p_2^k and p_3^k, since $\phi(p_i)$ has order k in $\mathbb{Z}_k$. (Noting that $\phi(p_3) = -2$, and k is odd). Finally, $\hat{S}$ has genus 1, which can be found by an Euler characteristic computation. □

Lemma 9.3.2. *Let* S *be a* 2*-orbifold with* n *boundary components. Suppose that one of the following conditions holds.*

1. S *has positive genus and* $k > 0$*.*
2. S *contains two cone-points of degree* m *and* $k > 0$ *divides* m*.*

Then there is a k*-sheeted cover* $\hat{S}$ *with* kn *boundary components that project homeomorphically to those of* S*.*

Proof. Embed S into $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. We write points in $\mathbb{C} \times \mathbb{R}$ in coordinates of the form (z, t). In case condition 1 holds, assume that S does not meet the axis $\overrightarrow{Ot} = \{0\} \times \mathbb{R}$, and that one handle of S goes around this axis. If condition 2 holds, then assume that S meets $\overrightarrow{Ot}$ exactly at the two cone-points x_1, x_2 of degree m. Also, in both cases, assume that all the boundary curves of S are null-homotopic in $\mathbb{R}^3 - \overrightarrow{Ot}$. It is easy to see that such an embedding exists.

Consider the map $f : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ defined by $f(z,t) = (z^k, t)$. Put $\hat{S} = f^{-1}(S)$. Then $f : \hat{S} \to S$ is a covering map, which in the case of condition 2 is branched at x_1 and x_2. This cover verifies the lemma. $\square$

The next result is also of importance for the theory of measure free factors.

Proposition 9.3.3. *Let S be a 2-orbifold with boundary curves $\gamma_1, \ldots, \gamma_n$. Assume that either of the following conditions holds.*

1. *S has at least two cone-points, one of them with degree bigger than 2.*
2. *S has at least three cone-points.*

Then the boundary subgroup $\langle [\gamma_1], \ldots, [\gamma_n] \rangle$ is a measure free factor of $\pi_1(S)$.

Proof. Using the arguments in the proof of 8.2.1, we reduce this to the case of only one boundary component. The case for positive genus is already known, so we can assume that S is a disk with cone-points. Let $y \in S$ be the cone-point of least degree p. Let $\hat{S}_1 \to S$ be the p-sheeted branched cover obtained by embedding S into $\mathbb{D} \subset \mathbb{C}$ so that y maps to 0, and taking the pre-image under the map $\mathbb{D} \to \mathbb{D}/z \to z^p$.

Let $x \in S$ be another cone-point, of degree k. Its preimage in $\hat{S}_1$ consists on p cone-points $x_1, \ldots, x_p$ of degree k. Since x_1 and x_2 have the same degree, we can apply Lemma 9.3.2 in its case 2 (with $m = k$). We get a cover $\hat{S}_2 \to \hat{S}_1$ with k boundary components.

If $k > 2$, set $\hat{S}_3 = \hat{S}_2$. Otherwise, $k = p = 2$ and condition 2 in the statement holds, so there is another cone-point $w \in S$. Its preimage under $\hat{S}_2 \to \hat{S}_1 \to S$ consists on kp different cone points of the same degree m as w. Take w_1, w_2 two of them and apply Lemma 9.3.2 to $\hat{S}_2$, to get a cover $\hat{S}_3 \to \hat{S}_2$. Then $\hat{S}_3$ has mk boundary components, and $mk > 2$ since the degree of a cone-point is by definition at least 2.

Now we apply Lemma 9.3.1 to $\hat{S}_3$. We get some cover $\hat{S} \to \hat{S}_3$ where $\hat{S}$ has positive genus. Consider then the cover of S given by the composition

$$\hat{S} \to \hat{S}_3 \to \hat{S}_2 \to \hat{S}_1 \to S$$

Since $\hat{S}$ has positive genus, its boundary subgroup is a measure free factor of $\pi_1(\hat{S})$. On the other hand, the subgroup $\pi_1(\hat{S}) \leq \pi_1(S)$ is of finite index, and the boundary classes of $\hat{S}$ give a complete lift of the boundary class of S to $\pi_1(\hat{S})$. Thus we can apply Theorem 8.1.2, and this finishes the proof.

□

Let S be a 2-orbifold. We say that S *has enough cone-points* if any of the conditions in Proposition 9.3.3 holds. Proposition 9.3.3 also allows for a further generalization of 8.4.2, that is our most general result on measure free factors of 2-orbifold groups.

Corollary 9.3.4. *Let S be a 2-orbifold with boundary, and $\gamma_1, \ldots, \gamma_k$ be its boundary curves. Suppose that $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a family of disjoint essential simple closed curves on S, and $S_1, \ldots, S_t$ are the components of S cut along α. Suppose also that for every j, S_j has either genus at least 1 or enough cone-points. Then the subgroup*

$$\langle [\gamma_1], \ldots, [\gamma_k], [\alpha_1], \ldots, [\alpha_n] \rangle \leq \pi_1(S)$$

is a measure free factor of $\pi_1(S)$.

Proof. Propositions 7.2.3 and 7.2.4 allow us to reduce this to the already known cases of Proposition 9.3.3 and Corollary 8.4.2. □

The following is an easy consequence of 9.3.1 and 9.3.3. Together with Proposition 9.3.3, it will be used to classify the treeable QBS groups (Theorem 9.4.3).

Corollary 9.3.5. *Let* S *be a* 2*-orbifold that is not a cylinder or a disk with cone-points. Let* $k \geq 3$ *be an odd integer. Then there exist a* k*-sheeted cover* $\hat{S} \to S$ *such that:*

1. *Each boundary component of* S *is covered by a single boundary component of* $\hat{S}$*, by a map of degree* k*.*
2. $\hat{S}$ *has either positive genus or enough cone-points.*
3. *The boundary classes of* $\hat{S}$ *generate common measure free factors of* $\pi_1(\hat{S})$

Proof. If S has 3 or more boundary components, this is direct from Lemma 9.3.1 and Corollary 8.4.2. The remaining case is when S is a cylinder with cone-points. The standard k-sheeted cover of the cylinder gives a cover $\hat{S} \to S$ that satsifies condition 1. On the other hand, each cone-point of S has exactly k preimages in $\hat{S}$, that are cone-points of the same degree. Since $k \geq 3$, we get that $\hat{S}$ has enough cone-points, and by Proposition 9.3.3, it satisfies condition 3.

□

9.4 Treeable Quadratic Baumslag–Solitar groups

Now we study the treeability of the QBS groups, aiming towards a classifiaction similar to Proposition 9.2.5 for GBS groups. Although in section 5.2 we restricted the QBS graphs to have surface QH vertex groups, for our present purpose we will allow QH vertices associated with general 2-orbifold groups.

If a QBS graph defines a treeable group, then by Theorem 6.1.9 we get that its GBS components also define treeable groups, so they are as in Proposition 9.2.5. This condition is usually not sufficient however. In the presence of QH vertices, there may be larger GBS subgroups, that we define next.

Let Γ be a QBS graph. Let v be a QH vertex of Γ, where $G_v = \pi_1(S)$ for a 2-orbifold S with boundary curves $\gamma_1, \ldots, \gamma_n$. Let $e_1, \ldots, e_n$ be the edges that are adjacent to v, ordered so that G_{e_j} injects into $\pi_1(\gamma_j) = \langle [\gamma_j] \rangle \subset G_v$ through the corresponding boundary map. Some of these edges may be counted twice, if they are loops with base vertex v. We will modify Γ as follows, obtaining a possibly disconnected graph of groups. Remove the QH vertex v but not its adjacent edges e_j, and give each edge e_j a new endpoint v_j for the one that was removed. Then set $G_{v_j} = \pi_1(\gamma_i)$, and the boundary maps are defined naturally. Notice that if e_j is a loop based at v, it gets two new endpoints, one for each boundary component.

Repeating this process, we remove all the QH vertices of Γ, and we are left with a possibly disconnected graph of groups Γ^*. The connected components of Γ^* will be called *Extended GBS (EGBS) components* of Γ. Notice these EGBS components are indeed GBS graphs. Although they are not subgraphs of Γ in general, their fundamental groups are subgroups of $G = \pi_1(\Gamma)$ in a natural way.

Moreover, each GBS component is contained in a unique EGBS component.

The rest of the EGBS components are one-edged, corresponding to the edges of Γ whose endpoints are at QH vertices.

We can build another QBS graph Γ', starting from Γ^* and putting back the QH vertices of Γ. If v is such a QH vertex, we connect it with new edges $e'_1, \ldots, e'_n$ to the vertices $v_1, \ldots, v_n$ of Γ^* that were created when removing v. We put $G_{e'_j} = G_{v_j} = \pi_1(\gamma_j)$, in the notation used above.

With this construction, Γ' is also a QBS graph with fundamental group G. The GBS components of Γ' are exactly the EGBS components of Γ. Also notice that collapsing the new edges e'_j we obtain Γ. We will say that Γ' is the *extended graph* of Γ.

It is clear that for $G = \pi_1(\Gamma)$ to be treeable, then the EGBS components of Γ must be as in Proposition 9.2.5.

Proposition 9.4.1. *Let Γ be a finite QBS graph and $G = \pi_1(\Gamma)$. Assume that Γ satisfies the following:*

1. *All QH vertex groups of Γ are fundamental groups of 2-orbifolds with either positive genus or enough cone-points.*
2. *All EGBS components of Γ define treeable GBS groups.*

Then G is treeable.

Proof. Consider the extended graph Γ' constructed above. By the first condition and Corollary 9.3.4, the boundary subgroup of every QH vertex group G_v of Γ is a measure free factor of G_v. Moreover, if $G_v = \pi_1(S)$, then the boundary classes of S generate common measure free factors of G_v (since they freely generate the boundary subgroup).

On the other hand, let Γ'' be the graph of groups obtained from Γ' by collapsing the EGBS components of Γ. So Γ'' is a graph of groups with the same QH vertices as Γ, and the other vertex groups are the GBS groups defined by the EGBS components of Γ. By the second assumption, all of these GBS vertex groups are treeable. Notice also that each edge of Γ'' connects a QH vertex with a GBS vertex, and its edge group is generated by a boundary class of the QH vertex group. So we observe that Γ' defines an iteration of amalgamations and HNN extensions, each one satisfying the conditions of 7.2.4 or 7.2.3. Since $\pi_1(\Gamma'') = G$, we get the result. □

In order to classify the treeable QBS groups, we will look at their finite index subgroups. These finite index subgroups will be obtained via finite covers of presentation complexes, which will extend suitable covers of the 2-orbifolds coming from their QH vertices. To find such extensions, we will need the following result about GBS groups.

Lemma 9.4.2. *Let Γ be a GBS graph and $G = \pi_1(\Gamma)$. Let $k > 0$ be relatively prime to all the labels of Γ. Then there exists $H \leq G$ a subgroup of index k, and a GBS graph $\hat{\Gamma}$ isomorphic to Γ such that:*

1. *The underlying graphs of Γ and $\hat{\Gamma}$ are identified.*
2. *The vertex and edge groups $\hat{G}_v, \hat{G}_e$ of $\hat{\Gamma}$ are subgroups of index k in the corresponding vertex and edge groups G_v, G_e of Γ.*
3. *$H = \pi_1(\hat{\Gamma})$, in particular, $H \cong G$.*

4. If $g \in G$ generates a vertex or edge group of Γ, then $\{g^k\}$ is a complete lift of g to H.

Proof. There is a standard construction to build a complex X with fundamental group G, starting from the data of Γ. For each vertex $v \in \Gamma$, we put in a circle S^1_v, and for each edge e we put in a cylinder $C_e \cong S^1 \times [0,1]$. Clearly, $\pi_1(S^1_v) \cong G_v$ and $\pi_1(C_e) \cong G_e$, and we choose specific isomorphisms. Then we glue the cylinders C_e to the circles S^1_v along their boundaries, according to the data of Γ. Namely, if e has endpoints v^-, v^+, we glue the components of ∂C_e to the circles $S^1_{v^-}, S^1_{v^+}$ through maps f^-_e, f^+_e that induce $\partial^-_e, \partial^+_e$ at the level of fundamental groups. Note that the degrees of these gluing maps are m^-_e, m^+_e respectively.

Now we construct a k-sheeted cover $p : \hat{X} \to X$ in a similar fashion as we did in the proof of 8.3.1. Let $p_v : \hat{S}^1_v \to S^1_v$ and $p_e : \hat{C}_e \to C_e$ be the standard k-sheeted covers of the circles S^1_v and the cylinders C_e. Observe that the maps $f^\pm_e \circ p_e : \partial^\pm \hat{C}_e \to S^1_v$ lift to the covers $p_v : \hat{S}^1_v \to S^1_v$. These lifts will be called $\hat{f}^\pm_e$, and will be used to glue the boundary components of $\hat{C}_e$ to the circles $S^1_{v^\pm}$. This gives rise to the complex $\hat{X}$. By the construction of the gluing maps of $\hat{X}$, we see that the covering projections p_v, p_e extend to a map $p : \hat{X} \to X$. Since k and $m^\pm_e$ are relatively prime, the gluing maps $\hat{f}^\pm_e$ have degrees $m^\pm_e$. So p is a local homeomorphism, and hence a covering map.

It is clear from the construction that the complex $\hat{X}$ comes from a graph of groups $\hat{\Gamma}$ which is isomorphic to Γ, and satisfies conditions 1 and 2 of the statement.

Put $H = \pi_1(\hat{X})$. Then H is an index k subgroup of G, and also $H = \pi_1(\hat{\Gamma})$.

If $g \in G$ generates either G_v or G_e, for a vertex v or an edge e of Γ, then g is represented by a simple closed curve α in either S^1_v or C_e. Its complete lift is a simple closed curve $\hat{\alpha}$ in $\hat{S}_v^1$ or $\hat{C}_e$ that covers α with degree k. Then the homotopy class of $\hat{\alpha}$ corresponds to g^k, and this gives a complete lift of g to H. $\square$

Let $S_{2,2}$ be the disk with two cone-points of degree 2. By Lemma 9.3.2, $S_{2,2}$ has a double cover which is a cylinder. Recall, on the other hand, that the 2-orbifolds that do not arise in QH vertices are the cylinder and the disks with one cone-point. Out of the 2-orbifolds that do arise in QH vertices, $S_{2,2}$ is the only one not covered in 9.3.3 or 9.3.5. In fact, the boundary subgroup of $\pi_1(S_{2,2}) \cong \mathbb{Z}_2 * \mathbb{Z}_2$ is not a measure free factor. (This is a consequence of $\mathbb{Z}_2 * \mathbb{Z}_2$ being amenable).

We say that a QH vertex in a graph of groups is of *type* $S_{2,2}$ if its corresponding 2-orbifold is $S_{2,2}$. Note that such a vertex is adjacent to exactly one edge.

We present now the main result of this chapter, classifying the QBS graphs that define treeable QBS groups. In the statement, we say that a QH vertex v of Γ is adjacent to a EGBS component Λ of Γ when v and Λ are joined by an edge in the extended graph Γ'.

Theorem 9.4.3. *Let Γ be a QBS graph, and $G = \pi_1(\Gamma)$. Then G is a treeable group if and only if all of the following conditions hold.*

1. *The EGBS components of Γ are either rooted trees, double or triple trees, or flowers. I.e. they define treeable GBS groups.*
2. *No EGBS component of Γ is adjacent to more than two QH vertices of type $S_{2,2}$.*
3. *If Λ is an EGBS component of Γ adjacent to two QH vertices v_1, v_2 of type $S_{2,2}$, then Λ is a rooted tree, the path β in Λ from v_1 to v_2 is made of simple edges, and the root of Λ can be taken in β. (I.e. all directed edges in Λ point towards β).*

4. *If Λ is an EGBS component of Γ adjacent to one QH vertex v_1 of type $S_{2,2}$, then Λ is a rooted tree, and the path β in Λ from v_1 to the root of Λ contains at most one directed edge, whose degree is 2.*

Proof. We will work with the extended graph Γ'.

First we show that the conditions are necessary. Condition 1 is clear from Proposition 9.2.5. In order to prove the other conditions, we will consider an index 2 subgroup of G. It will be obtained from the QBS graph Γ'' that we construct next.

Let $V_{2,2}$ be the set of QH vertices of type $S_{2,2}$ in Γ'. First we define a graph of groups Γ'_0 with the same underlying graph as Γ'. The edge groups are the same as in Γ', as well as the vertex groups for the vertices not in $V_{2,2}$. If $v \in V_{2,2}$, then its vertex group in Γ'_0 is the boundary subgroup of $G_v \cong \pi_1(S_{2,2})$. Note that this group is cyclic, and is equal to G_e where e is the edge adjacent to v in Γ'.

Now we construct the graph Γ''. Put in two copies Γ'_1, Γ'_2 of Γ_0. Then, for each $v \in V_{2,2}$, connect their corresponding copies $v_1 \in \Gamma'_1$, $v_2 \in \Gamma'_2$ by a simple edge e_v. That is, the edge group of e_v is identified with the vertex groups of v_1 and v_2 (which are equal to the boundary subgroup of G_v). It is clear that Γ'' is a QBS graph.

On the other hand, consider the presentation complex $X' = X_{\Gamma'}$. For the EGBS components, this agrees with the complex in the proof of Lemma 9.4.2. For the QH vertices v with $G_v = \pi_1(S)$, we put in the 2-orbifolds S, glued by their boundaries as indicated by the graph Γ'. We define X_Δ in the same way for a general QBS graph Δ.

It is not hard to see that the complex $X'' = X_{\Gamma''}$ is a double cover of X': On the subcomplexes $X_{\Gamma'_i} \subset X''$ the covering map $X'' \to X'$ restricts to the obvious identification $X_{\Gamma'_i} \to X_{\Gamma'_0} \subset X'$. On the other hand, for each $v \in V_{2,2}$ the cylinder corresponding to e_v forms the double cover of the copy of $S_{2,2}$ that corresponds to v.

That shows that the subgroup $G'' = \pi_1(\Gamma'')$ has index 2 in G. So G is treeable if and only if G'' is treeable. In particular, the GBS components of Γ'' must be as in Proposition 9.2.5 (yielding amenable groups).

Now we describe the GBS components of Γ''. Each EGBS component Λ of Γ gives rise to a GBS component Λ_0 of Γ'_0. More precisely, Λ_0 is just Λ with one extra vertex and simple edge for each $v \in V_{2,2}$ that is adjacent to Λ. Note that adding simple edges does not change the graph types of Proposition 9.2.5. Let Λ_1, Λ_2 be the copies of Λ_0 embedded in Γ'_1 and Γ'_2. If Λ is not adjacent to any QH vertex of type $S_{2,2}$, then Λ_1 and Λ_2 are two different GBS components of Γ''. Otherwise, let Λ'' be the subgraph of Γ'' given by the union of Λ_1, Λ_2 and all edges e_v for the $v \in V_{2,2}$ that are adjacent to Λ. Notice that Λ'' is a GBS component of Γ''. It is also clear that all GBS components of Γ'' are of this form.

Suppose $v \in V_{2,2}$ is adjacent to Λ. Then Λ must be a rooted tree, since otherwise Λ'' would have two different core subgraphs (as described in the definition of double and triple trees, and flowers), one in Λ_1 and the other in Λ_2.

Suppose that $v_1, v_2 \in V_{2,2}$ are both adjacent to the EGBS component Λ of Γ. Let α be the path in Λ'_0 going from v_1 to v_2, and call α_1, α_2 to the respective copies in Γ'_1 and Γ'_2. Then we get a cycle in Λ'', obtained from the concatenation $\gamma = \alpha_1 e_{v_2} \bar{\alpha}_2 \bar{e}_{v_1}$. Now, by Proposition 9.2.5, Λ'' must be a flower. In particular, it cannot have more non-trivial cycles. Thus Λ cannot be adjacent to more than two QH vertices of type $S_{2,2}$, proving condition 2. We also get that the cycle γ is the core of the flower Λ''. This happens exactly when all edges in α are simple and all directed edges in Λ point towards α. This proves condition 3, where the curve β is the part of α contained in Λ. (That is, α minus the two simple edges that are at the beginning and the end).

Now assume that only one $v \in V_{2,2}$ is adjacent to Λ. Take the path α going from v to the root r of Λ_0' (which agrees with the root of Λ). Again, let α_1, α_2 be the copies of α in Γ_1', Γ_2'. Consider the path $\gamma = \bar{\alpha}_1 e_v \alpha_2$. Since r is the root of Λ_0', we get that the directed edges of Λ'' that are not in γ point towards γ. On the other hand, a directed edge in α gives rise to two directed edges in γ, pointing away from each other. By Proposition 9.2.5, there are only two possibilities: Either all edges of α are simple, and Λ'' is a rooted tree, or α contains exactly one directed edge of degree 2, and Λ'' is a triple tree. This proves condition 4 (where β is the curve α without its initial simple edge).

It is important to notice that conditions 2, 3 and 4 are equivalent to say that the EGBS components of Γ'' are either rooted, double or triple trees, or flowers.

Now we prove that the conditions are sufficient for G to be treeable. Recall from the above arguments that G is treeable if and only if G'' is treeable, and that Γ'' contains no QH vertices of type $S_{2,2}$. This reduces the proof to the case where there are no QH vertices of type $S_{2,2}$. On the other hand, by Proposition 9.3.3 we know that the boundary subgroups of the disks with enough cone-points are measure free factors. Thus, by applying Theorem 7.1.6, we may reduce the proof to the case where there is no QH vertices corresponding to disks with cone-points.

So assume that Γ is a QBS graph and that its QH vertices do not correspond to disks with cone-points (they have valence at least 2). Let $k \geq 3$ be an odd integer that is relatively prime to all the labels of the edges of Γ'. If v is a QH vertex of Γ', let S_v be its corresponding 2-orbifold (so $G_v = \pi_1(S_v)$). Let $\hat{S}_v$ be the cover of S_v obtained from Corollary 9.3.5. On the other hand, if Λ is a EGBS component of Γ, let $\hat{\Lambda}$ be the graph obtained from Lemma 9.4.2.

We construct a graph of groups $\hat{\Gamma}'$ as follows: The underlying graph of $\hat{\Gamma}'$ is the same as that of Γ. On the underlying subgraph of Λ, we put the vertex and edge groups of $\hat{\Lambda}$. If v was a QH vertex of Γ', we assign it the vertex group $\hat{G}_v = \pi_1(\hat{S}_v)$. Finally, if e was an edge connecting a QH vertex $v \in \Gamma'$ to another vertex $w \in \Gamma'$ (that must be in an EGBS component by construction of Γ'), set $\hat{G}_e$ to be the index k subgroup of G_e.

Notice that the vertex v is Quadratically Hanging in $\hat{\Gamma}'$ and that the edges e adjacent to v are simple. To see that, let g_e, g_w be the generators of G_e and G_w, with $\partial_e^+(g_e) = g_w$. Also let γ be the boundary curve of S_v with $\partial_e^-(g_e) = [\gamma]$. Then g_e^k generates G_e by our definition, g_w^k generates $\hat{G}_w$ by Lemma 9.4.2, and $[\gamma]^k$ is a boundary class of $\hat{S}_v$ by Corollary 9.3.5. We get, in particular, that $\hat{\Gamma}'$ is a QBS graph.

Look at the complex $\hat{X}' = X_{\hat{\Gamma}'}$. We observe that the covering maps $\hat{S}_v \to S_v$ for the QH vertices $v \in \Gamma'$, glue well with the covering maps $X_{\hat{\Lambda}} \to X_\Lambda$ for the EGBS components Λ of Γ. This gives rise to a k-sheeted covering map $\hat{X}' \to X'$. Then the subgroup $H = \pi_1(\hat{\Gamma}')$ has index k in G, and it is enough to show that H is treeable.

Lemma 9.4.2 implies that the EGBS components of $\hat{\Gamma}'$ are isomorphic to that of Γ', so they also have the graph types in Proposition 9.2.5. On the other hand, Corollary 9.3.5 says that the QH vertices of $\hat{\Gamma}$ are the fundamental groups of 2-orbifolds with either positive genus or enough cone-points. So we can apply Proposition 9.4.1, and we get that H is treeable. This finishes the proof.

□

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