# HYPERBOLIC SETS AND ENTROPY AT HOMOLOGICAL LEVEL

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TESE SUBMETIDA EM CUMPRIMENTO PARCIAL
PARA A OBTENÇÃO DE GRAU DE
DOUTOR EM CIÊNCIAS

INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA–IMPA
RIO DE JANEIRO, BRASIL
FEVEREIRO 2014

*To José and Aciencia.*

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# Abstract

In the context of partially hyperbolic diffeomorphisms, with one-dimensional center direction, of $d$-dimensional torus isotopic to an Anosov diffeomorphism where the isotopy is contained in the set of partially hyperbolic diffeomorphism, we show that it is possible to approximate the homological entropy by the entropy of hyperbolic sets. Such an approximation is performed with hyperbolic sets so that the unstable index coincides with the index at the level of homology. Some examples are given which might shed light on which are the good questions in the higher dimensional center case. Concerning the *upper bound* problem, as to whether the homological entropy is an upper bound for topological entropy, the examples show us that, when the central direction is no longer one dimensional, the homological entropy is not an upper bound for the topological entropy.

# Acknowledgements

Agradezco a Enrique Pujals, orientador, por las observaciones y valiosas sugerencias a este trabajo; por su amistad y paciencia al, repetidamente, explicarme de un mismo asunto; por su oportuno apoyo y estímulo para atentar a trabajos posdoctorales. Es una honra tenerlo como orientador.

Agradezco a Rafael Potrie, co-orientador, tanto por su amistad como por el interés que manisfestó em mi trabajo. Durante mi visita al IMERL en Montevideo, desde un início, de Rafael recibí buena recepción, soy grato por el conocimiento matemático y por el trato extremamente amable que de él recibí. De lo que se habla en este trabajo, se reflejan varias charlas que mantuve con él.

Agradezco a los miembros de la banca examinadora, Marcelo Viana, Carlos Moreira, José Maria Pacífico y Pablo Carrasco, por comentarios a este trabajo.

Por su amor (difícil de medir), simplicidad, total apoyo y paciencia, expreso mi gratitud a mi querida familia: Mis padres, José Roldán y Aciencia Daquilema. Mis hermanos, Laura, Marcelino, Lucía, José. Mi cuñada Rosy. Mis sobrinas, Keila, Noemi y Raquel esencias de mi vida.

Explicito mi agradecimiento: a Cristina Lizana y Samuel Barbosa por su amistad y apoyo en la preparación del examen de calificación. A Ana Tercia, Artem Raibekas y Pablo Barrientos por su compañerismo y amistad. A Pablo Carrasco por tomar tiempo para escuchar y esclarecer dudas. En ellos, mi gratitud a todos los colegas "dinamistas". Y a las "fieras" de Uruguay.

Por el ambiente internacional, por el propicio ambiente de amistad, agradezco a mis compañeros, profesores y funcionarios del Impa, con quienes tuve el privilégio de aprender, tanto de matemática como de "futilidades" de la vida. Destaco la amabilidad y eficiencia de Andrea Nascimento. Agradezco al personal agradable de manutención y portería cuya labor hacen del Impa un lugar con orden.

Agradezco a Caitlin Nelligan por su amistad y aporte en correcciones de inglés, a todos los 'tios' – Javier, Luis, Laura, Miguel, Alejo, Pablo – que compartieron su amistad durante mis últimos meses en el Impa. Esta gente la conocí por medio de Misha y María Campaña, a quienes también agradezco por su amistad.

Mi agradecimiento profundo a la gran familia de la IASD, que, dondequiera que he ido, siempre me ha abrazado y prestado soporte familiar. De manera especial agradezco a Genildo Medeiros, Eliana Takamoto, Michelle Alves, Marta Monteiro, Savio Souza e Isabel Bezerra, por todos aquellos momentos que compartieron y por la paciencia que tuvieron conmigo en mis momentos estresantes. Agradezco también a Alejandro Moyano y Rosa Reinoso, instrumentos valiosos, a quienes yo y mi familia les debemos el conocimiento de la *verdad presente*.

Juan Carlos García fué mi tutor de tesis de pregrado y fué quien me animó a aplicar al Impa. Digo de él que es un gran amigo y apasionado por las matemáticas, se ha dedicado a incentivar el espíritu de un buen número de alumnos al gusto por las matemáticas.

El trabajo tuvo soporte financiero de CNPq. Finalmente, agradezco a Rio de Janeiro por "suas belas paisagens".

*"Junto a los números, la humildad define la fortaleza de un matemático"*

Rio de Janeiro, Brasil
Febrero 20, 2014

Mario Roldán.
*Ecuatoriano*

# Introduction

Let us consider $f\colon M \to M$ a diffeomorphism of a compact manifold. Concerning continuous maps and their action at an algebraic level, it is interesting to ask what features of a dynamical system are constrained by their actions on the homology level. A property which may hold for all dynamical systems is the so called *Entropy Conjecture* (formulated in [19] by M. Shub), which relates the topological entropy of a dynamical system with the logarithm of the modulus of the largest eigenvalue of the induced map on the homology level. This interesting problem leads to obtaining lower bounds for the topological entropy. The precise statement goes as follows:

**Entropy Conjecture**: For any $C^1$-map $f\colon M \to M$ of a compact manifold $M$, $h_{top}(f)$ is bounded from below by $h_H(f)$, the logarithm of the spectral radius of the linear map $f_*$ induced of $f$ on the total homology of $M$ with real coefficients.

While only the regularity of a system ($C^\infty$-diffeomorphism, see [23]), or just the structure of the manifold itself ($d$-Torus, see [16]) makes the conjecture valid, in the best case scenario, the conjecture remains open in general. If one desires to have an initial, although not profound idea about the sets where the conjecture holds, see [11].

At this point we consider the following question which is analyzed in this work:

*Under what condition is it possible to approximate the entropy (seen in the level of homology) of a diffeomorphism by the entropy of hyperbolic sets? Is it possible to do so with hyperbolic sets such that the index coincides with the index at the level of homology?*

To make our question clearer, we may formulate it in another way.

**BASIC QUESTION:** For a $C^1$ diffeomorphism $f\colon M \to M$, consider $u_0 \in \mathbb{N}$ such that $h_H(f) = \log sp(f_{*,u_0})$ and assume that $f_{*,1}$ is hyperbolic. When, or under which condition does there exist uniformly hyperbolic sets $\Lambda_\ell \subset M$ with hyperbolic splitting of the tangent bundle, $T_{\Lambda_\ell}M = E^u_\ell \oplus E^s_\ell$, such that $\dim E^u_\ell = u_0$ and

$$\limsup_{\ell} \; h_{top}(f\mid_{\Lambda_\ell}) \geq h_H(f)?$$

The number $u_0 \in \mathbb{N}$ such that $h_H(f) = \log sp(f_{*,u_0})$ is called *algebraic index* or *homological index.*

Note that it is now more clear that the question posed above is related to the entropy conjecture. It follows immediately, then, that if a diffeomorphism satisfies the inequality above, it also satisfies the entropy conjecture.

We are not trying to claim another conjecture or something of the sort. What we want to do is to study a possible refinement of the conjecture for some cases where it is already known that the entropy conjecture is valid. When the manifold is the $d$-torus, Misiurewicz and Przytycki (see [16]) have proved that the entropy conjecture turns out to be true for arbitrary continuous maps. Our refinement will have the $d$-torus as an ambient manifold, and a partially hyperbolic system with a one dimensional center bundle will be considered over it.

The importance of this refinement lies in the fact that a relationship can be established between the index in the ambient manifold (good index of hyperbolic sets), and the index at homological level. We try, with this refinement, to understand the structure of invariant sets that contributes to the growth of topological entropy. The prevailing idea for many systems is that hyperbolic sets are the heart of dynamic complexity.

The lower bound proposed by the entropy conjecture is clearly not optimal since it is easy to construct diffeomorphisms in every isotopy class with an arbitrarily large entropy. On the other hand, some families of maps may verify that the entropy in homology is also an upper bound: to determine if a family of maps satisfies such a bound is another interesting question which will be addressed in this work.

In Chapter 1, terminology and tools are recalled.

In Chapter 2, we begin by giving an overview of examples where our principal question has a positive answer (keeping in mind the upper bound problem): (1) Hyperbolic *linear automorphisms* of the $d$-torus. This is the first example to be considered, being that the basic question arises from it by observing its properties. As a prototype, it is worthwhile to keep in mind and consider the Thom-Anosov diffeomorphism (cat map). (2) *Anosov systems* defined on a compact manifold, satisfying orientability of the unstable bundle $E^u$ and the manifold $M$ itself. Concerning the upper bound problem, the previously mentioned examples are, so far, all that is known in the general case of Anosov diffeomorphisms. (3) *Smooth examples* on surfaces. These examples, due to Katok, are no longer so "trivial"; the importance is that the approximation problem by hyperbolic sets is tackled entirely assuming existence of

hyperbolic measures. (4) *Absolutely* partially hyperbolic diffeomorphism of $\mathbb{T}^3$ isotopic to Anosov. This is the first example in a non-hyperbolic setting to be analyzed, which is important for our context.
In addition, this chapter also contains a short summary of the results obtained.

In Chapter 3, we prove that partially hyperbolic diffeomorphisms not necessarily absolute, which are isotopic to the linear Anosov automorphism along a path of partially hyperbolic diffeomorphisms with one-dimensional center direction, satisfy the inequality formulated in the basic question. In this setting, it is true that the homological entropy is an upper bound for the topological entropy (this type of result was proved in [4]). In contrast, two examples of partially hyperbolic diffeomorphisms constructed over 4-torus and presented herein show that one should not expect the same result when the central direction is no longer one-dimensional. The examples will help us draw conclusions supposing the central direction fails to be one-dimensional.

In Chapter 4, we discuss some key questions for future work.

# Chapter 1

# Preliminaries

In this chapter, definitions and terminology related to the complexity of a system, normal hyperbolicity and foliation tools, are introduced.

## 1.1 Exponential Growth Rate

In this section some non-negative numbers which measure the complexity of the system are introduced. Given a sequence of numbers $(a_n) \subset (0, \infty]$, the following limit defined by

$$\tau(a_n) = \limsup_{n \to \infty} \frac{1}{n} \log a_n$$

is used to denote the *exponential growth rate*. By this we mean that, for $n$ big, $a_n$ is of the order $e^{n\tau}$. The numbers (given by the topological entropy, metric entropy, entropy at homology level, volume growth of the foliation) that will appear in the definitions below will indeed be the exponential growth rate of a particular given sequence.

**Topological Entropy**: *"The most glorious number in dynamic"* (see [12]) is a non-negative number which measures the exponential complexity of the orbits of a system. To be more precise, consider a continuous map, $f : M \to M$, of a compact metric space $M$, with distance function $d$. A set $E \subset M$ is said to be $(n, \varepsilon)$-*separated*, if for every $x \neq y \in E$ there exists $i \in \{0, \ldots, n-1\}$, such that $d(f^i x, f^i y) \geq \varepsilon$. Let $s(n, \varepsilon)$ be the maximal cardinality of an $(n, \varepsilon)$-separated set of $M$, notice that by compactness, this number is finite.

One defines

$$h(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon)$$

which represents the exponential growth rate of a sequence $a_n = s(n, \varepsilon)$. The *topological entropy*, as defined by Bowen, is the "greatest" of all those growth rates, by

$$h_{top}(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon) = \sup_{\varepsilon > 0} h(f, \varepsilon).$$

Therefore, with the expression above, in saying "exponential complexity of the orbits of a system", we want to give it the following suggestive meaning that appears in the literature: suppose we have a device to observe a dynamical system ('observe' meaning to locate all points of a system), which has resolution $\varepsilon > 0$ ('resolution' meaning that our device only considers points whose distances between them is at least $\varepsilon$). In this sense:

- $s(n, \varepsilon)$ is the number, at least, of different orbit segment $I_x^n = \{x, \ldots, f^{n-1}x\}$ that the observer is able to identify.
- $h(f, \varepsilon)$ is the exponential growth rate obtained with precision $\varepsilon > 0$, which is improved when $\varepsilon$ goes to zero.

The same value of $h_{top}(f)$ is obtained by considering the exponential growth rate of a sequence $a_n = r(n, \varepsilon)$, where $r(n, \varepsilon)$ denotes the minimal cardinality of a $(n, \varepsilon)$-spanning set. A set $E \subset M$ is said to be a $(n, \varepsilon)$-*spanning* set, if for every $x \in M$

there exists $y \in E$, such that $d(f^i x, f^i y) < \varepsilon$ for all $i \in \{0, \ldots, n-1\}$. Bowen's definition works for metric spaces, compact or not.

For a better understanding as well as for a source of background reading and a list of standard elementary properties of topological entropy, we recommend the books [13, 22].

**Metric Entropy**: We begin with a notion of ergodic measure. A Borel measure $\mu$ is said to be *$f$-invariant*, if $\mu(f^{-1}A) = \mu(A)$ for all measurable sets $A$. An $f$-invariant measure $\mu$ is said to be *ergodic*, if the only measurable sets $A$ with $f^{-1}(A) = A$ satisfy $\mu(A) = 0$ or $\mu(A) = 1$.

In a manner similar to the way the topological entropy via $(n, \varepsilon)$-spanning sets was defined, we are going to define the *metric entropy* as defined by Katok (which is equivalent to the classical definition in the case of diffeomorphism), with respect to the Borel probability $f$-invariant ergodic measure, $\mu$. In the literature this number is said to be one which measures the complexity of the orbits of a system that are relevant for a measure, $\mu$.

For $0 < \delta < 1$, $n \in \mathbb{N}$ and $\varepsilon > 0$, a finite set of $E \subset M$ is called an $(n, \varepsilon, \delta)$-*covering* set if the union of the all $\varepsilon$-balls, $B_n(x, \varepsilon) = \{y \in M | \ d(f^i x, f^i y) < \varepsilon \text{ for all } i\}$, centered at points $x \in E$ has $\mu$-measure greater than $\delta$. Subsequently, we consider the set $N_\delta(n, \varepsilon, \delta)$ as being the smallest possible cardinality of a $(n, \varepsilon, \delta)$-covering set (i.e. $N_\delta(n, \varepsilon, \delta) = \min\{\sharp E \mid \ E \text{ is a } (n, \varepsilon, \delta) - \text{covering set}\}$). Once again, the metric entropy of $f$ with respect to measure $\mu$ is an exponential growth rate of a sequence $a_n = N_\delta(n, \varepsilon, \delta)$, i.e.,

$$h_\mu(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_\delta(n, \varepsilon, \delta).$$

It was observed by Katok that $h_\mu(f)$ is independent of $\delta$.

A very significant result in the theory relating the metric entropy and the topological entropy is the so called *entropy variational principle*, which states that the topological entropy is achieved by taking the supremum of the metric entropies of all invariant measures of the systems:

$$h_{top}(f) = \sup_\mu h_\mu(f).$$

If there is $\mu$ such that $h_{top}(f) = h_\mu(f)$, then $\mu$ is said to be a *maximal* entropy measure.

**Entropy at Homology Level**: Homology theory associates with every continuous map $f\colon M \to M$ a sequence of homomorphisms $f_{*,k}$ on the homology groups with real coefficients $f_{*,k}\colon H_k(M,\mathbb{R}) \to H_k(M,\mathbb{R}),\quad k = 0,\dots,d = \dim M$.

Certain topological properties of the continuous map $f$ are reflected in algebraic properties of the homomorphisms $f_{*,k}$. The *homological entropy*, denoted by $h_H(f)$, is defined as

$$h_H(f) = \max_k \log sp(f_{*,k}).$$

Notice that this real number maximizes the above over all $k \in \{0,\cdots,d\}$, and it is the exponential growth rate of the sequence $a_n = \|f^n_{*,k}\|$.

*Remark* 1.1.1. The homological entropy measures how much the dynamic disfigures the manifold at algebraic level. The $d$-torus has a $CW$-complex structure, and therefore $h_H(f)$ measures how much and how many times the dynamic mixes up the cells.

*Remark* 1.1.2. Recall that, for a bounded linear operator $A\colon X \to X$ on a real or complex vector space $X$, if $\|.\|$ is a norm on $X$, the *spectral radius* of $A$, $sp(A)$, is defined as $sp(A) = \lim \|A^n\|^{1/n}$. Whenever all norms on $X$ are equivalent the above limit does not depend on the choice of the norm. Using the Jordan normal form theorem we see that the spectral radius exists and is equal to the supremum among the absolute values of the eigenvalues of $J$, where $J$ is a Jordan matrix associated to $A$. The well known property (we shall use here) in the theory is that the action induced in homology is invariant under homotopy.

**Volume Growth of the Foliation**: Here we shall clarify what we mean by "exponential growth rate of unstable disks". For this, consider $f\colon M \to M$ a diffeomorphism on a $d$-dimensional compact Riemannian manifold, $M$. Let us consider $\mathcal{W} = \{W(x)\}_{M\ni x}$ a $u$-dimensional foliation (for a brief review of foliation see section 1.2) on $M$, which is invariant under $f$, i.e. $fW(x) = W(fx)$. For any $x \in M$, let $W_r(x)$ be the $u$-dimensional disk on $W(x)$ centered at $x$, with radius $r > 0$.

Letting $vo\ell(W)$ denote the *volume* of a sub-manifold $W$ computed with respect to the induced metric on $W$, one can consider for each disk $W_r(x)$ the exponential volume growth rate of its iterated under the application $f$, as

$$\chi_W(x, r) = \limsup_{n\to\infty} \frac{1}{n} \log vo\ell(f^n W_r(x)),$$

where we then consider the maximum volume growth rate of $\mathcal{W}$

$$\chi_{\mathcal{W}}(f) = \sup_{M\ni x} \chi_{\mathcal{W}}(x, r),$$

a quantity that is independent of the radius $r > 0$ (it is easy to see). Therefore, when we refer to "exponential growth rate of unstable disks", it means that we are dealing with a partially hyperbolic diffeomorphism and it is considered the corresponding unstable foliation (see section 2.2 for definitions involved).

## 1.2 Some Tools

In this section we start with a few definitions and terminology that will be used throughout the remaining sections.

**Non-degeneracy**: Remember that a $u$-form $\omega$ on a manifold $M$ (is denoted by $\Lambda^u(M)$ the set of all them) is a choice for each $M \ni x$ of an alternating $u$-multilinear map, $\omega_x \colon T_xM \times \cdots \times T_xM \to \mathbb{R}$, which depends smoothly on $x$. A *closed* (i.e. the exterior derivative is zero) $u$-form $\omega \in \Lambda^u(M)$ is said to be *non-degenerate* on a tangent subbundle $E^u \subset TM$ (with fiber dimension equal to $u$) if for every $M \ni x$ and any set of linearly independent vector $\{\upsilon_1, \dots, \upsilon_u\} \subset E^u_x$ the following holds, $\omega_x(\upsilon_1, \dots, \upsilon_u) \neq 0$. We can say that no-degeneracy means to ask for a closed $u$-form definite positive.

**Cone Structure**: Consider a $d$-dimensional vector space $V$ with a inner product $\langle \cdot, \cdot \rangle$ and a $u$-dimensional subspace $E \subset V$. Let $E \oplus E^\perp = V$ be a splitting of $V$. Given $0 < \alpha \in \mathbb{R}$, we define the *$\alpha$-cone* with *core* $E$, denoted by $\mathcal{C}^u_\alpha(E)$, as the set

$$\mathcal{C}^u_\alpha(E) = \left\{ \upsilon + \upsilon^\perp \mid \left\| \upsilon^\perp \right\| \leq \alpha \left\| \upsilon \right\| \right\}.$$

A continuous *cone field* on a subset $K$ of a manifold $M$ is a continuous association of cones $\{\mathcal{C}_x\}$ in the vector spaces $T_xM$ of tangent vectors on $M$, $x \in K$.

Let $N$ be an embedded $C^1$-submanifold of $M$. $N$ is said to be *tangent to the cone field* $\mathcal{C}^u_\alpha(E)$ if, $T_xN$, the tangent subspace to $N$ at each point $N \ni x$ is contained in the corresponding cone $\mathcal{C}^u_\alpha(E_x)$.

A useful condition for partial hyperbolicity involving cone fields is the so called *cone criterium*, which establishes that partially hyperbolicity is equivalent to having cone fields on $M$ with invariance and increase property for forward iteration and backward iteration.

**Proposition 1.2.1** (Cone criterium). *For $f\colon M \to M$ a $C^1$-diffeomorphism the following are equivalent:*

1. *$f$ is a partially hyperbolic diffeomorphism.*

2. *There exist, $C^{cu}, C^{cs}$ cone fields and values $N \in \mathbb{N}$, $1 < \lambda \in \mathbb{R}$ such that:*

   – $D_x f^N(\overline{C}^{cu}(x)) \subset C^{cu}(f^N x)$,

   – $D_x f^{-N}(\overline{C}^{cs}(x)) \subset C^{cs}(f^{-N} x)$,

   – $\left\|D_x f^{-N} v\right\| > \lambda \left\|v\right\|$ *for every* $v \notin C^{cu}(x)$,

   – $\left\|D_x f^{N} v\right\| > \lambda \left\|v\right\|$ *for every* $v \notin C^{cs}(x)$.

**Lyapunov Exponents**: For a $C^1$-diffeomorphism $f$ of a $d$-dimensional manifold $M$ and an ergodic measure $\mu$, from Oseledet's theorem (see [15]) there exist real numbers $\lambda_1 < \lambda_2 \cdots < \lambda_m$ ($m \leq d$), called *Lyapunov exponents*, and a decomposition $T_xM = E_1(x) \oplus \cdots \oplus E_m(x)$ such that for every $1 \leq j \leq m$ and every $0 \neq v \in E_j(x)$ we have

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \left\|D_x f^n v\right\| = \lambda_j.$$

Now we recall the notion of hyperbolic measure: An $f$-invariant ergodic measure $\mu$ is said to be *a hyperbolic measure* if

- None of the Lyapunov exponents for $\mu$ are zero.

- There exist Lyapunov exponents with different signs.

Thus, provided we have a hyperbolic measure $\mu$, there exists $s \in \mathbb{N}$ such that

$$\lambda_1 < \lambda_2 \cdots < \lambda_s < 0 < \lambda_{s+1} < \cdots < \lambda_m.$$

To simplify terms we consider the following definition,

**Definition 1.2.1.** Let $\mu$ be a hyperbolic ergodic measure and a number $u \in \mathbb{N}$. It is said that $\mu$ has *index* equal to $u$ if there exist $u$ positive Lyapunov exponents. Here the exponents are counted with multiplicity.

**Normally Hyperbolic Foliation and Lamination**: Here we recall some results about the persistence of normally hyperbolic compact laminations, a work due to Hirsch-Pugh-Shub, (see [8] for a comprehensive approach). We review the concepts involved in the definition.

A *continuous foliation*, $\mathcal{F} = \{\mathcal{F}_x\}_{M \ni x}$, on a manifold $M$ is a division of $M$ into disjoints submanifold, $\mathcal{F}_x$, called *leaves* of the foliations (the leaves all have the same dimension, say $u$) where each leaf is connected (it need not be a closed subset of the manifold) such that each point $M \ni p$ admits a neighborhood $\mathcal{U} \ni p$ and there exists a homeomorphism $\varphi\colon D^u \times D^{m-u} \longrightarrow \mathcal{U}$ (called *foliation box*) sending each $D^u \times \{y\}$ into the leaf through $\varphi(0, y)$. If the foliations boxes are $C^1$, $\mathcal{F}$ is said to be $C^1$*-foliation.* A $C^1$-foliation is said to be a $C^1$*-lamination* if the tangent planes of the leaves give a continuous $u$-plane subbundle of $TM$.

Consider a lamination $\mathcal{L}$ and let $f\colon M \to M$ be a $C^1$-diffeomorphism *preserving* lamination $\mathcal{L}$ (the dynamic sent each leaf into a leaf). We will say that $\mathcal{L}$ is *normally hyperbolic* if there exists a $Df$-invariant splitting $TM = E^u \oplus T\mathcal{L} \oplus E^s$ of the tangent bundle, where the decomposition is partially hyperbolic.

Two laminations $\mathcal{L}_f$ and $\mathcal{L}_g$ invariant under $f$ and $g$ respectively are said to be *leaf conjugate* if there exists a homeomorphism $h$ such that for every $M \ni x$, $h$ carries laminae to laminae, i.e. $h(\mathcal{L}_f(x)) = \mathcal{L}_g(hx)$, and at level of leaves $h$ behaves as a usual conjugation, i.e. $h(\mathcal{L}_f(fx)) = \mathcal{L}_g(g \circ hx)$.

A lamination $\mathcal{L}$, which is preserved by $f$, is *structurally stable* if there exists a neighborhood $\mathcal{U} \ni f$ such that each $g \in \mathcal{U}$ admits some $g$-invariant lamination, $\mathcal{L}_g$, which is leaf conjugated to $\mathcal{L}$.

We highlight two results in the theory concerning the conditions that are required to keep intact a lamination under dynamic perturbations.

**Theorem.** *(1). If $f$ is a $C^1$ diffeomorphism of $M$ which is normally hyperbolic at the $C^1$-foliation $\mathcal{F}$, then $f$ is plaque expansive. (2). Let $f$ be normally hyperbolic to the $C^1$-lamination $\mathcal{L}$. If $f$ is plaque expansive, then $(f, \mathcal{L})$ is structurally stable.*

For a formal definition of *plaque expansivity* we refer the reader to [8].

**Potpourri of Notations**: We now consider some notations used hereafter. (1) The notion of homotopic first comes to mind. $f, g \in \mathrm{Diff}(M)$ are *homotopic* if there exists a continuous map $h\colon M \times [0,1] \to M$ (the homotopy) with $h(\cdot, 0) = f$, $h(\cdot, 1) = g$. (2) If $\lambda$ is an eigenvalue of a linear map $A\colon V \to V$, its *eigenspace* $E_\lambda$ is the set $\big\{v \in V \mid (A - \lambda I)^k(v) = 0\big\}$. (3) Two submanifolds $V, W$ of $M$ verify the *transversal intersection* condition at $x \in V \cap W$ if we have that $T_xV + T_xW = T_xM$. (4) The angle $\angle(E, F)$ between two vector subspaces $E$ and $F$ is the minimum angle between vectors $v \in V, w \in W$. (5) For $x_n, x \in M$ and vector subspaces $E_n \in T_{x_n}M$, $E_0 \in T_xM$, is denoted by $E_n \to E_0$ to mean the convergence in the Grassmannian sense.

# Chapter 2

# Examples and Statement of Results

In this chapter, we begin by giving a review of examples that the literature provide us for which the basic question mentioned in the introduction is positively answered. Notice that, the question has a negative answer if the action induced on the first homology group is not hyperbolic, for example it is possible to have a diffeomorphism with positive entropy, which simultaneously lacks hyperbolic sets. This can be done even in the partially hyperbolic cases, look for example at, $Anosov \times Id$ in $\mathbb{T}^3$. Then, the precise statements of results are given in subsequent section focusing on partially hyperbolic diffeomorphisms.

## 2.1 Overview of Some Known Examples

In this section we review some examples with the required property of the basic question, among which we want to highlight the linear Anosov diffeomorphisms and absolutely partially hyperbolic diffeomorphisms on 3-torus, which are discussed later. Before starting with the examples, we summarized with a lemma some properties related to a continuous map of the torus and its lift to the universal covering.

**Lemma 2.1.1.** *Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ a continuous map on the d-torus. Then,*

1. *The lift of $f$ to the universal covering map, $\widetilde{f}\colon \mathbb{R}^d \to \mathbb{R}^d$, can be written as $\widetilde{f} = A + p$ where $A$ is a linear map with $A|_{\mathbb{Z}^d}$ being a homomorphism of $\mathbb{Z}^d$, and $p$ is a continuous map which is $\mathbb{Z}^d$-periodic.*
2. *$f$ is homotopic to $f_A$. Here $f_A$ is the induced map of $A$ on the torus.*
3. *The matrix of $A\colon \mathbb{R}^d \to \mathbb{R}^d$ and $f_{*,1}\colon H_1(\mathbb{T}^d, \mathbb{R}) \to H_1(\mathbb{T}^d, \mathbb{R})$ are the same.*

*Remark* 2.1.1. Recall that a diffeomorphism $f\colon M \to M$ is called Anosov if there exist values $\lambda \in (0,1)$ and $C > 0$ with a $Df$-invariant splitting $TM = E^u \oplus E^s$ of the tangent bundle of $M$ , such that for all $n > 0$ and $x \in M$

$$\left\|D_x f^n|_{E^s_x}\right\| < C\lambda^n \quad \text{and} \quad \left\|D_x f^{-n}|_{E^u_x}\right\| < C\lambda^n.$$

The linear map $A\colon \mathbb{R}^d \to \mathbb{R}^d$ is called the *linear part* of $f$. If $f\colon \mathbb{T}^d \to \mathbb{T}^d$ is Anosov, then the linear part is, as well (see [13]).

**Hyperbolic Toral Automorphism**:

An example that sheds some light on the basic question is the linear one. Let $f_A\colon \mathbb{T}^d \to \mathbb{T}^d$ be a hyperbolic linear automorphism induced by $A \in SL(d, \mathbb{Z})$ with decomposition $T\mathbb{T}^d = E^u_A \oplus E^s_A$ of the tangent bundle of index $u = \dim E^u_A$. Then,

$$h_{top}(f_A) = \log sp(A_{*,u}).$$

This is achieved because if $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ are the eigenvalues (generalized and counted with multiplicity) of $A$, we know that (see [22])

$$h_{top}(f_A) = \sum_{|\lambda_i|>1} \log|\lambda_i|.$$

On the other hand, (to simplify calculations, we consider coefficients in $\mathbb{C}$) for coefficients in field $\mathbb{C}$ for both homology and cohomology, the universal coefficient theorem (classical theorem in algebraic topology) implies that cohomology is the exact dual of homology. Then, by induction on $d$ it follows from the Künneth formula that $H_k(\mathbb{T}^d, \mathbb{C})$ is isomorphic to a direct sum of $\binom{d}{k}$ copies of $\mathbb{C}$.

Because $\mathbb{C}$ is algebraically closed, we know that there exists a basis $\alpha_1 \dots \alpha_d$ such that $f_A^{*,1}$ is represented by an upper triangular matrix whose diagonal entries are exactly the eigenvalues of $A$, we call this matrix $B$. Let us denote by $\tau_1 \dots \tau_d$ the dual basis. With respect to this basis, $f_{A*,1}\colon H_1(\mathbb{T}^d, \mathbb{C}) \longrightarrow H_1(\mathbb{T}^d, \mathbb{C})$ is given by $B^\top$ and thus $sp(f_{A*,1}) = sp(A)$. Now, the cohomology group $H^k(\mathbb{T}^d, \mathbb{C})$ is a module with basis the "cup" products, $\{\alpha_{i_1}, \cup \cdots \cup \alpha_{i_k} \mid i_1 < \cdots < i_k\}$. Therefore, if we denote by $\{\tau_{i_1}, \cup \cdots \cup \tau_{i_k} \mid i_1 < \cdots < i_k\} \subset H_k(\mathbb{T}^d, \mathbb{C})$ the corresponding dual basis, then by using the cup product and basic properties relating it (see [7] for instance), we have:

$$\begin{aligned}
\langle \alpha_i \cup \alpha_j, f_*(\tau_i \cup \tau_j) \rangle &= \langle f^*(\alpha_i \cup \alpha_j), \tau_i \cup \tau_j \rangle \\
&= \langle f^*\alpha_i \cup f^*\alpha_j), \tau_i \cup \tau_j \rangle \\
&= \langle \sum_\ell b_{i\ell}\alpha_\ell \cup \sum_m b_{jm}\alpha_m, \tau_i \cup \tau_j \rangle \\
&= b_{ii}b_{jj} - b_{ij}b_{ji} = b_{ii}b_{jj} \\
&= \lambda_i \lambda_j.
\end{aligned}$$

Therefore, if the eigenvalues are sorted in order of decreasing absolute value, we have that $sp(f_{A*,2}) = |\lambda_1\lambda_2|$. By similar accounts we have that $sp(f_{A*,j}) = |\lambda_1 \cdots \lambda_j|$. Then, by hypothesis, $h_H(f) = \log sp(A_{*,u})$.

Notice that, in this case, the hyperbolic set $\Lambda_\ell$ is the whole $d$-torus which in fact has unstable index equal to $u$.

**Anosov System with Orientable Bundle**:

The following nice result is due to Ruelle and Sullivan, (see [17]). Here it is considered an Anosov diffeomorphism with orientable stable and unstable bundles. As such, one has a similar result to the Anosov lineal case. The idea of its proof is as follows: On the one hand, an equality is established between the two entropies, topological and homological, by observing the classical Lefschetz formula (see [13]),

$$L(f) = \sum_{Fix(f)\ni x} i(f,x) = \sum_{i=0}^{d} (-1)^i tr(f_{*,i}).$$

Since $E^u$ is orientable one can get that $|L(f^n)| = \sharp Fix(f^n)$, and thus

$$\sharp Fix(f^n) \leq \dim H_*(M,\mathbb{R})(sp(f_*))^n.$$

Therefore, $h_{top}(f) \leq h_H(f)$ because the exponential growth rate of hyperbolic periodic points coincides with the topological entropy. On the other hand, the entropy conjecture holds for Anosov diffeomorphisms. Thus, we have $h_{top}(f) = h_H(f)$. Now we need to make explicit the homological entropy in terms of the dimension of the unstable bundle. However, the result is a particular case of a study relating geometric currents. Therefore, we have,

**Theorem 2.1.2.** *If $f\colon M \to M$ is an Anosov diffeomorphism and the unstable and stable sub-bundle $E^u_f, E^s_f$ are orientable, then $h_{top}(f) = h_H(f) = \log sp(f_{*,u})$ where $u = \dim E^u_f$.*

**Smooth Examples on Surfaces**:

Let us begin by stating one of Katok's results (see [10]). It is assumed that $f\colon M \to M$ is $C^{1+\alpha}$-diffeomorphism, $\alpha > 0$, on a manifold $M$.

**Theorem 2.1.3.** *Let $\mu$ be an $f$-invariant ergodic and hyperbolic measure such that $h_\mu(f) > 0$. Then for every $\varepsilon > 0$, we can find $\Lambda_\varepsilon \subset M$ a hyperbolic set such that $h_{top}(f \mid_{\Lambda_\varepsilon}) > h_\mu(f) - \varepsilon$.*

By considering the case where $M$ is a surface, the result can be refined. In fact, by the variational principle of entropy, we know that

$$h_{top}(f) = \sup_{\mu} h_\mu(f).$$

Furthermore, by Ruelle's inequality for entropy we have that one of the Lyapunov exponents of $\mu$ (assuming $h_\mu(f) > 0$) is positive and the other is negative.

In other words, for surfaces, the topological entropy of a $C^{1+\alpha}$-system can be approximated by topological entropy of hyperbolic sets of index one.

Moreover, what we need to observe is that if the induced action on homology level $f_{*,1} \colon H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$ is hyperbolic, we also have a similar approximation of $h_H(f)$ by entropy of hyperbolic sets of index one. This is due to Manning's result (cited in [11]), which establishes that $h_{top}(f) \geq \log sp(f_{*,1})$.

**Absolutely P.H. on $\mathbb{T}^3$ :**

For the next result, we consider $f \colon \mathbb{T}^3 \to \mathbb{T}^3$ an absolutely partially hyperbolic diffeomorphism (see next section for definition). Let $A \colon \mathbb{T}^3 \to \mathbb{T}^3$ be a hyperbolic linear automorphism, such that $f$ is homotopic to $A$. Denote with $\lambda^s(\mu) \leq \lambda^c(\mu) \leq \lambda^u(\mu)$ the Lyapunov exponents associated to an ergodic $f$-invariant measure, $\mu$. Then, Ures in [20] has proved,

**Theorem 2.1.4.** *There exists, $\mu$, a unique ergodic $f$-invariant entropy maximizing measure of $f$. This measure is hyperbolic and the central Lyapunov exponent, $\lambda^c(\mu)$, has the same sign as the linear part, $\lambda^c(A)$.*

Our interest in this result lies not just in the fact that the measure is hyperbolic, but also in that the sign of the central exponent is preserved.

Then, if $f\colon \mathbb{T}^3 \to \mathbb{T}^3$ is $C^1$, there exist $\Lambda_\ell \subset \mathbb{T}^3$ hyperbolic sets with $\dim E^u_{\Lambda_\ell} = 2$ (we can assume $\dim E^u_A = 2$) such that

$$\limsup_{\ell\to\infty} h_{top}(f|_{\Lambda_\ell}) \geq h_H(f) = \log sp(f_{*,2}).$$

## 2.2 Statements of Results

In this section, before stating the precise results it is essential to mention a result due to Fisher, Potrie, and Sambarino (see [4]). In the study of dynamical coherence for partially hyperbolic diffeomorphism of torus, they have established the existence of a unique maximal entropy measure in the case where a partially hyperbolic diffeomorphism can be connected to a hyperbolic linear automorphism via a path which remains within a set of partially hyperbolic diffeomorphisms. For the existence and uniqueness of the measure, the central bundle is required to be one-dimensional.

We start by briefly introducing a simplified partial hyperbolicity terminology.

**Definition 2.2.1.** To say that $f\colon M \to M$ is *partially hyperbolic* diffeomorphism we need three conditions:

**(a)** *Splitting condition*: There exists a continuous splitting, $TM = E^u \oplus E^c \oplus E^s$ of the tangent bundle, which is $Df$-invariant i.e., $DfE^\sigma_x = E^\sigma_{fx}$ for $\sigma = u, c, s$.

**(b)** *Domination condition*: There exists $N > 0$ such that for every $x \in M$ and unit vectors $v^\sigma \in E^\sigma$ $(\sigma = u, c, s)$ we have

$$\left\|D_x f^N v^s\right\| < \left\|D_x f^N v^c\right\| < \left\|D_x f^N v^u\right\|.$$

**(c)** *Contraction property*: $\left\|Df^N|_{E^s}\right\| < 1$ and $\left\|Df^{-N}|_{E^u}\right\| < 1$.

If the inequality above holds for unit vectors belonging to the bundles of different points, then we say that $f$ is *absolutely partially hyperbolic*, i.e.,

$$\left\|D_x f^N v^s\right\| < \left\|D_y f^N v^c\right\| < \left\|D_z f^N v^u\right\|$$

for all $x, y, z \in M$ and $v^\sigma \in E_p^\sigma$ unit vectors ($\sigma = u, c, s$ and $p = x, y, z$), respectively.

*Remark* 2.2.1. For partially hyperbolic diffeomorphisms, it is a well-known fact that there are foliations $\mathcal{W}^u$ (unstable one) and $\mathcal{W}^s$ (stable one) tangent to the distributions $E^u$, $E^s$ respectively. Moreover, the bundles $E^u$ and $E^s$ are uniquely integrable (see [8]).

Let us start by setting up the set where the results will be held. The set of all partially hyperbolic diffeomorphisms on the $d$-torus are denoted by $\mathsf{PH}(\mathbb{T}^d)$.
Consider $A \in SL(d, \mathbb{Z})$, a $d \times d$ matrix with integer entries and determinant 1. Let $T\mathbb{T}^d = E_A^u \oplus E_A^c \oplus E_A^s$ be the dominated splitting associated with the induced diffeomorphism by $A$ (which is also denoted by $A$). We consider the subset of all partially hyperbolic diffeomorphisms, all of them having the same dimension of stable and unstable bundle, i.e,

$$\mathsf{PH}_{A,u,s} = \left\{ f \in \mathsf{PH} \mid f \sim A,\, \dim E_f^u = u,\ \dim E_f^s = s \right\}.$$

To simplify notation, $\mathsf{PH}_A(\mathbb{T}^d)$ will be written instead of $\mathsf{PH}_{A,u,s}(\mathbb{T}^d)$, where the dimension of the bundles is implicitly understood.

Now, we consider $\mathsf{PH}_A^0(\mathbb{T}^d)$ to be the connected component of $\mathsf{PH}_A(\mathbb{T}^d)$ containing the lineal map $A$.

Then, in [4] it is proved that:

**Theorem 2.2.1** (Fisher, Potrie, Sambarino)**.** *For every* $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ *such that* $\dim E^c_f = 1$*, there exists a unique maximal entropy measure which has entropy equal to the linear part.*

In the context presented above, the first refinement of entropy conjecture is performed over $\mathsf{PH}^0_A(\mathbb{T}^d)$, considering those who have one-dimensional central direction. However, the result is valid on any connected component containing some diffeomorphism which admits a special closed form, which we will discuss later.

**Theorem A.** *Let* $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ *with* $\dim E^c_f = 1$*. Then the following holds:*

1. *Let* $\mu$ *be an ergodic* $f$*-invariant measure,* $0 < \varepsilon_0 < |\lambda^c_A|$*. If* $h_\mu(f) > h(A) - \varepsilon_0$ *then* $\mu$ *is a hyperbolic measure with index* $u = \dim E^u_A$*.*

2. *For every* $\varepsilon > 0$ *there exists a hyperbolic set* $\Lambda_\varepsilon \subset \mathbb{T}^d$ *such that*

$$h_{top}(f \mid \Lambda_\varepsilon) \geq \log sp(f_{*,u}) - \varepsilon$$

*where* $u = \dim E^u_{\Lambda_\varepsilon}$*, and* $h_H(f) = \log sp(f_{*,u})$*.*

*Remark* 2.2.2. In particular, the unique maximal entropy measure given by Theorem 2.2.1 is actually a hyperbolic measure.

We wonder what happens if the central direction is no longer one-dimensional. In this direction, we present two examples of partially hyperbolic diffeomorphisms on $\mathbb{T}^4$, both of them with two-dimensional center direction. Among other things, both examples show us that homological entropy is not an *upper bound* for the topological entropy. The first one ensures hyperbolic measure with "good index", which is sufficient to approximate the homological entropy, i.e.,

**Theorem B.** *There exists an open subset* $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^4)$ *such that any* $f \in \mathcal{U}$ *is an* absolutely *partially hyperbolic diffeomorphism with* $\dim E^c_f = 2$. *Furthermore,*

1. $f$ *is homotopic to a hyperbolic linear map* $A$ *that has unstable index* 2.

2. $f$ *satisfies* $h_{top}(f) \geq h_{top}(f \mid W^c) > h_H(f) = \log sp(A_{*,2})$.

3. *Let* $\mu$ *be an ergodic measure such that* $h_\mu(f) > \log sp(A_{*,1})$ *then* $\mu$ *is a hyperbolic measure with index* 2.

4. $f$ *is topologically transitive.*

In the second theorem, although hyperbolic measure is guaranteed with entropy as close as desired to the topological entropy, its index is no longer correct. We believe that there exists a hyperbolic measure with "good index" in order to approximate the homological entropy.

**Theorem C.** *There exists an open subset* $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^4)$ *such that any* $f \in \mathcal{U}$ *is a* pointwise *partially hyperbolic diffeomorphism with* $\dim E^c_f = 2$. *Furthermore,*

1. $f$ *is homotopic to a hyperbolic linear map* $A$ *that has unstable index* 1.

2. $f$ *satisfies* $h_{top}(f) \geq h_{top}(f \mid W^c) > h_H(f) = \log sp(A_{*,1})$.

3. *Any ergodic measure* $\mu$, *such that* $h_\mu(f) > \log sp(A_{*,1})$ *is a hyperbolic measure with index* 2. *If* $\Lambda \subset \mathbb{T}^4$ *is a hyperbolic set with index* 1 *then* $h_{top}(f|\Lambda) \leq h_{top}(A)$.

4. $f$ *is topologically transitive.*

# Chapter 3

# Proofs of the Theorems

In this chapter, to prove Theorem A we combine the existence of a closed differential $u$-form $\omega \in \Lambda^u(\mathbb{T}^d)$, which is non trivial on $\mathcal{W}_A^u$, the unstable bundle of hyperbolic linear automorphism $A$ and a result, due to Saghin, relating the exponential growth rate of unstable disks. As was noted in the previous chapter, for an Anosov system it is possible to assume some extra conditions to get equality of both entropies. As such, we also consider here the *upper bound* problem for the one-dimensional center bundle cases, whose result is naturally expected.

In order to prove Theorem B we appeal to a source of many examples, namely the skew-products. The example constructed in Theorem C will be derived from Anosov. On the one hand, to get the first example we start from a product of two linear Anosov system and locally modify the dynamic of one of the coordinates. The modification will have torus leafs as center fibers, and on one of them the dynamic remains hyperbolic which is sufficient to achieve the required properties. On the other hand, the second example starts with a indecomposable linear Anosov diffeomorphism which is not a product, we locally modify the central fiber to get enough entropy while maintaining control over the domination of bundles.

## 3.1 Theorem A

In a certain sense, Theorem A is a generalization of Theorem 2.1.4 when the dimension of the unstable bundle is larger than one. What we need to do is to study the corresponding central Lyapunov exponent, and to do this we need some relation between the exponential growth rate of unstable discs and the eigenvalues of the linear map. The relation shall allow us to conclude that the central exponent is nonzero. What is more, in order to establish a relationship between the index of a hyperbolic set and its homology index, it will be necessary that the central exponent has the same sign as the linear part. We recall that this family of maps satisfies that the homological entropy is indeed an upper bound for topological entropy, (see [4]).

**Theorem 3.1.1.** *Let* $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$ *with* $\dim E^c_f = 1$*. Then the following holds:*

1. *Let* $\mu$ *be an ergodic* $f$*-invariant measure,* $0 < \varepsilon_0 < |\lambda^c_A|$*. If* $h_\mu(f) > h(A) - \varepsilon_0$ *then* $\mu$ *is a hyperbolic measure with index* $u = \dim E^u_A$*.*

2. *For every* $\varepsilon > 0$ *there exists a hyperbolic set* $\Lambda_\varepsilon \subset \mathbb{T}^d$*, such that*

$$h_{top}(f \mid \Lambda_\varepsilon) \geq \log sp(f_{*,u}) - \varepsilon,$$

*where* $u = \dim E^u_{\Lambda_\varepsilon}$ *and* $h_H(f) = \log sp(f_{*,u})$*.*

While on $\mathbb{T}^3$, the result follows directly from the quasi-isometry property of unstable leaves; however, in a larger dimension it follows from the existence of a closed $u$-form which is required to be non-degenerate on the unstable direction. So, to start dealing with the problem, the next 'key' proposition shows that, to have a positive definite differential form is a non-isolated property, as well as it being a property that extends to the closure.

**Proposition 3.1.2.** *The existence of a closed u-form, $\omega$, which is non-degenerate on the unstable bundle $E^u$, is an open and closed condition in $\mathsf{PH}_A(\mathbb{T}^d)$.*

*Proof.* The condition of being open is trivial, indeed, take $f \in \mathsf{PH}_A(\mathbb{T}^d)$ and let $\omega \in \Lambda^u(\mathbb{T}^d)$ be a closed $u$-form which we assume to be non-degenerate on $E^u_f$ bundle.

We need to show that there exists a neighborhood $\mathcal{U}$ of $f$ in $\mathsf{PH}_A(\mathbb{T}^d)$, such that for every $g \in \mathcal{U}$ there exists a closed $u$-form, $\omega_g$, which is non-degenerate on $E^u_g$. We have that there exists $0 < \alpha \in \mathbb{R}$ such that $\omega_x$ is positive definite over a cone $\mathcal{C}(E^u_f(x), \alpha)$. Notice that by the continuity of bundles, $\alpha$ can be taken locally constant. Furthermore, by using a compactness argument, $\alpha$ does not depend on $x \in \mathbb{T}^d$. On the other hand, for $g \in \mathsf{PH}(\mathbb{T}^d)$ $C^1$-close enough to $f$, also we have that $E^u_g(x) \subset \mathcal{C}(E^u_f(x), \alpha)$. So the same $u$-form, $\omega$, works for $g$.

In order to prove the closed property, consider the $C^1$-convergence $f_n \to f$ where $f_n, f \in \mathsf{PH}_A(\mathbb{T}^d)$ and a closed $u$-form, $\omega_n$, non-degenerate on $E^u_n$ the unstable bundle corresponding to $f_n$. To construct a non-degenerated closed $u$-form associated with $f$, we attempt to approach $n$ sufficiently so that the 'cone axis', where the form $\omega_n$ is positive, becomes $C^0$-close to the unstable bundle of $f$. Although this can be done in a uniform way and as closely as possible, the problem is that the cone angle may be small, and as a result not large enough to contain the unstable bundle of $f$. However this is corrected by pushing forward the unstable bundle of $f$ so that it fits within the cone.

To do this, take $N > 0$ big enough to ensure that the $E^{cs}_N$ and $E^u_f$ bundles become transversal to each other. We can do this because:

**(a)** $E^{cs}_f$ is transversal to $E^u_f$.

**(b)** $E^{cs}_n \longrightarrow E^{cs}_f$ $C^0$ in a uniform way.

So, there exists $0 < \alpha_0 \in \mathbb{R}$ such that $\angle(E^{sc}_N(x), E^u_f(x)) \geq \alpha_0$ for all $x \in \mathbb{T}^d$. We can also modify $N$ in order to have $\angle(E^u_N(x), E^u_f(x)) < \delta$ for every $x \in \mathbb{T}^d$, where $\delta > 0$ is small.

Associated to this $\delta > 0$, there exists $m \in \mathbb{N}$ so that for every $x \in \mathbb{T}^d$ and $\upsilon \in E^u_f(x)$ we have

$$D_x f^m_N(\upsilon) \in \mathcal{C}(E^u_N(f^m_N x), \beta).$$

To finish the proof of proposition we define $\omega_f \colon \mathbb{T}^d \to \Lambda^u(\mathbb{T}^d)$ a $u$-form given by

$$\omega_f = (f^m_N)_* \omega_N.$$

Notice that $\omega_f$ is indeed a closed form, and by construction it is also non-degenerate on unstable bundle $E^u_f$. □

*Remark* 3.1.1. Notice that if a diffeomorphism $f$ has a closed $u$-form which is non-degenerate on unstable bundle, then any element of the connected component that contains $f$ also has the same property.

The next two theorems help us conclude the result. The first theorem considers $f$ a $C^1$-partially hyperbolic diffeomorphism with one dimensional center direction, where Hua, Saghin and Xia proved a refined version of the Pesin-Ruelle inequality. See [9]. And the second one, due to Saghin, establishes the relation between the exponential growth rate of unstable discs and its corresponding at homology. See [18].

**Theorem 3.1.3** (Hua, Saghin, Xia). *Let $\nu$ an ergodic $f$-invariant measure and $\lambda^c(\nu)$ its Lyapunov exponent corresponding to the center distribution. Then the inequality holds* $h_\nu(f) \leq \lambda^c(\nu) + \chi_u(f)$.

**Theorem 3.1.4** (Saghin)**.** *Let $f\colon M \to M$ be a $C^1$-partially hyperbolic diffeomorphism such that there exists a closed u-form, $\omega$, which is non-degenerate on unstable bundle $E^u_f$. Then, $\chi_u(f) = \log sp(f_{*,u})$.*

The proof of Theorem 3.1.1 is as follows:

*Proof (of Theorem 3.1.1).* Now we show that the unique measure of maximal entropy is hyperbolic for each $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$. In fact, we will show that the sign of the center Lyapunov exponents, for both $f$ and $A$, are the same. We can assume that for $A$, the Anosov diffeomorphism, the center direction is expanding; otherwise, we would use the inverse map.

For $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$, we consider $\mu_f$ the unique measure of maximal entropy. By Theorem 3.1.3 we have:

$$h_{top}(A) = h_{top}(f) = h_{\mu_f}(f) \leq \lambda^c_f + \chi_u(f).$$

Because $\mathsf{PH}^0_A(\mathbb{T}^d) \ni A$, by Proposition 3.1.2 and Theorem 3.1.4 we also have

$$\chi_u(f) = \chi_u(A),$$

and since

$$h_{top}(A) = \chi_u(A) + \lambda^c(A),$$

we have that $\lambda^c_A \leq \lambda^c_f$. Now, if $f$ is $C^1$ the existence of hyperbolic sets with the required property follow from Katok's result (see Theorem 3.1.3). □

## 3.2 Theorem B

The examples that have been considered so far have the particularity that the topological entropy is equal to the homological entropy. In this section we construct an example where equality does not hold. Although the example does not rule out the possibility of having a measure of maximal entropy that has 'good index', hyperbolic measures having 'good index' are insured in order to approximate the homological entropy.

**Theorem 3.2.1.** *There exists an open subset* $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^4)$ *such that any* $f \in \mathcal{U}$ *is an absolutely partially hyperbolic diffeomorphism with* $\dim E^c_f = 2$*. Furthermore,*

1. *$f$ is homotopic to a hyperbolic linear map $A$ that has unstable index 2.*

2. *$f$ satisfies $h_{top}(f) \geq h_{top}(f \mid W^c) > h_H(f) = \log sp(A_{*,2})$.*

3. *Let $\mu$ be an ergodic measure such that $h_\mu(f) > \log sp(A_{*,1})$ then $\mu$ is a hyperbolic measure with index 2.*

4. *$f$ is topologically transitive.*

That on the isotopy class of a linear hyperbolic map of 2-torus, it is always possible to find a diffeomorphism with entropy as big as desired, it is ensured by the proposition below.

**Proposition 3.2.2.** *Let $A$ be a hyperbolic linear automorphism of the 2-torus. For every $K > 0$ there exists $f \colon \mathbb{T}^2 \to \mathbb{T}^2$, a diffeomorphism isotopic to $A$ such that $h_{top}(f) > h_{top}(A) + K$.*

*Proof.* The idea is as follows: if we start with an Anosov system going through an Anosov derivative, the claim follows by 'pasting a horseshoe', with large entropy, in a neighborhood of some attracting fixed point.

Let $g = DA$ be the derived from a hyperbolic toral automorphism $A$ (this is the classical derived from Anosov by modifying the map on a neighborhood of some fixed point, replacing it with a pair of saddles and a attractor). Consider $g(p) = p \in \mathbb{T}^2$ some attractor fixed point, and a small disk $D \ni p$ such that $\overline{g(D)} \subset \text{int}\,(D)$.

We consider $f\colon D_1 \to D_1$, where $D_1 = \overline{g(D)}$, such that $f$ maps the topological disk $D_1$ into itself and has the following properties:

$$\Omega(f) = \{p\} \cup H, \qquad H \subset D_1,$$
$$f(p) = p \qquad \text{is an attracting fixed point,}$$
$$f(H) = H \qquad \text{is conjugate to a } n\text{-shift.}$$

Here, $\mathbb{N} \ni n$ is taken so that $\log n > h_{top}(A) + K$. Finally, $f$ can be extended to all $\mathbb{T}^2$ so all other points of $D$ enter this topological disk under forward iterations, and such that $f = g$ over $\mathbb{T}^2 \setminus D$. □

*Remark* 3.2.1. Notice that because hyperbolic sets persist under small perturbations, if $g$ is $C^1$-close to $f$, there exists a hyperbolic invariant set $H_g$ close to $H$ such that $g|_{H_g}\colon H_h \to H_g$ and $f|_H\colon H \to H$ are topologically conjugates. Thus,

$$h_{top}(g) \geq h_{top}(g|_{H_g}) = h_{top}(f|_H) > h_{top}(A) + K.$$

The following proposition tells us that, for hyperbolic linear maps, the entropy 'does not look affected' when removing a fixed point of its domain, i.e,

**Proposition 3.2.3.** *Let $B$ be a hyperbolic linear automorphism of the 2-torus. Assume that $p \in \mathbb{T}^2$ is a fixed point of $B$. For every $\varepsilon > 0$, there exists $\delta > 0$, such that $h_{top}(B|_{\Lambda_\delta}) > h_{top}(B) - \varepsilon/2$ where $\Lambda_\delta$ the maximal invariant set contained in $\mathbb{T}^2 \setminus B_\delta(p)$.*

*Proof.* All we need to do is look at Markov partitions and ask what happens if one removes a symbol (Bowen proved that any hyperbolic toral automorphism on $\mathbb{T}^2$ has a Markov partition, see [3]).

Take a Markov partition, $\mathcal{R} = \{R_j\}_{j=0}^{\ell}$, for $B$ on $\mathbb{T}^2$ with a sufficiently small diameter. We set up the symbolic dynamic by letting $\sigma$ be the shift map on the full $k$-shift, $\Sigma = \{1,\dots,k\}^{\mathbb{Z}}$ and define $\sigma_A = \sigma|\Sigma_A\colon \Sigma_A \hookleftarrow$, where $A$ is the transition matrix. Let $h\colon \Sigma_A \to \mathbb{T}^2$ be a semiconjugacy from $\sigma_A$ to $B$.
We know that $h_{top}(B) = h_{top}(\sigma|\Sigma_A)$ and for a sub-shift of finite type topological entropy it is equal to the logarithm of the spectral radius of the transition matrix.

Now consider a matrix transition $\overline{A}$ obtained by removing row $j_0$ and column $j_0$ of matrix transition $A$, where $j_0 \in \{0,\dots,\ell\}$ such that $R_{j_0}$ contains the fixed point $p \in \mathbb{T}^2$. One can prove that $\Sigma_{\overline{A}} \subset \Sigma_A$ and $h_{top}(\sigma|\Sigma_{\overline{A}}) \approx h_{top}(\sigma|\Sigma_A)$, where accuracy is reached by taking the sufficiently small diameter of the partition. $\square$

In the next proposition we establish an invariance property of cones. To do so, assume that $L\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear map written as a matrix

$$L = \begin{pmatrix} B_N & X \\ 0 & Y \end{pmatrix}$$

of linear maps $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Where $Y$ is invertible and

$$B_N = \begin{pmatrix} \lambda^N & 0 \\ 0 & \lambda^{-N} \end{pmatrix}, \qquad \mathbb{R} \ni \lambda > 1.$$

Now consider $E_1 = \mathbb{R}e_1$, a linear subspace generated by $e_1 = (1,0,0,0)$ and a cone

$$\mathcal{C}^u(E_1,\gamma_1,\gamma_2) = \left\{ \upsilon \in \mathbb{R}^4\colon\ |\upsilon_2| \le \gamma_2 |\upsilon_1|, \quad |\upsilon_3|, |\upsilon_4| \le \gamma_1 |\upsilon_1| \right\}.$$

Notice that $E_1$ is invariant by $L$. Similarly, we define a cone $\mathcal{C}^s(E_2,\gamma_1,\gamma_2)$, where $e_2 = (0,1,0,0)$ and $E_2 = \mathbb{R}e_2$ is also $L$-invariant. The basic fact we want is

**Proposition 3.2.4.** *Let $N$ be a positive integer such that* $(\|Y\|+1) < 100^{-1}\lambda^N$ *and* $\|Y^{-1}\| < 100^{-1}\lambda^N$. *Consider* $0 < K \in \mathbb{R}$ *so that* $\|X\| < K$. *Then*

1. *There exist* $\gamma_1, \gamma_2 > 0$, *so that* $\mathcal{C}^u(E_1, \gamma_1, \gamma_2)$ *becomes a L-invariant cone.*

2. *There exists* $\lambda > 1$ *such that for all* $v \in \mathcal{C}^u$, $\|Lv\| \geq \lambda \|v\|$.

*Proof.* Assume that $v \in \mathcal{C}^u$, then $|v_2| \leq \gamma_2 |v_1|$ and $|v_3|, |v_4| \leq \gamma_1 |v_1|$. To simplify, consider $X_1, X_2$ the row vectors of matrix $X$. We have that $L(v) = w$, where

$$w_1 = \lambda^N v_1 + X_1(v_3, v_4) \qquad w_3 = Y_1(v_3, v_4),$$

$$w_2 = \lambda^N v_2 + X_2(v_3, v_4) \qquad w_4 = Y_2(v_3, v_4).$$

Thus, we need to show that $|w_2| \leq \gamma_2 |w_1|$ and $|w_3|, |w_4| \leq \gamma_1 |w_1|$. The first immediate estimates are:
$|w_1| \geq (\lambda^N - 2K\gamma_1)|v_1|$,
$|w_2| \leq (\lambda^{-N}\gamma_2 + 2K\gamma_1)|v_1|$,
$|w_3|, |w_4| \leq \|Y\| \gamma_1 |v_1|$.

Now, in order to find the condition on $\gamma_1, \gamma_2$, we just need to prove that is possible to have the following estimates:

**(a)** $\gamma_2(\lambda^N - 2K\gamma_1) > (\lambda^N\lambda_2 + 2K\gamma_1)$,

**(b)** $\lambda^N - 2K\gamma_1 > \|Y\|$,

for some $\gamma_1, \gamma_2 \in \mathbb{R}$. But, by hypothesis we have $\lambda^N > 100(\|Y\|+1)$ and therefore the item $(b)$ requires us to take $\gamma_1$ in such a way $\gamma_1 < 99\|Y\|/(2K)$, which is possible. Moreover, again by hypothesis we have $2K\gamma_1 < (\lambda^N - \lambda^{-N})$, and thus the condition $(a)$ is also possible because $\frac{\gamma_2}{1+\gamma_2} \to 1$, as $\gamma_2 \to \infty$. □

*Proof (of Theorem 3.2.1).* We start by considering a hyperbolic linear automorphism of the 2-torus $A$. Therefore, by Proposition 3.2.2 we have $f_t\colon \mathbb{T}^2 \to \mathbb{T}^2$ an isotopy between $f_0 = A$ and $f_1 = f$, where $h_{top}(f) > h_{top}(A) + \varepsilon$. Notice that we can think of $f_t$ as being in fact a diffeotopy.

Let $\lambda > 1$ be an eigenvalue corresponding to a linear hyperbolic automorphism, call it $B$, which has a fixed point $p \in \mathbb{T}^2$. Consider also $N > 0$ big enough in such a way $(\|D_y f_t(y)\|+1) < 100^{-1}\lambda^N$. Subsequently, take $\delta > 0$, given by Proposition 3.2.3, so that we have

$$h_{top}(B^N|_{\Lambda_\delta}) > h_{top}(B^N) - \frac{\varepsilon}{2}, \quad \text{where} \quad \Lambda_\delta = \mathbb{T}^2 \setminus B_\delta(p).$$

Finally, we define $F\colon \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ by

$$F(x,y) = (B^N x, f_{k(x)} y), \qquad \mathbb{T}^2 \ni x, y,$$

where the smooth function $k\colon \mathbb{T}^2 \to \mathbb{R}$ satisfies $k(x) = 0$ if $\|x - p\| > \delta$, and $k(x) = 1$ if $\|x - p\| \le \delta/2$.

The derivative of $F$ is the matrix

$$DF(x,y) = \begin{pmatrix} B^N & D_x(f_{k(x)} y) \\ 0 & D_y(f_{k(x)} y) \end{pmatrix},$$

and we have that $F$ is a diffeomorphism and meets the requirements of Proposition 3.2.4.

Hereafter, we consider an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^4)$ such that $\mathcal{U} \ni F$, any $G \in \mathcal{U}$ is partially hyperbolic and such that $G$ is isotopic to $F$. Remember that partial hyperbolicity is a $C^1$ open condition, and we know that if $G$ is $C^0$-close to $F$ then $G$ and $F$ are homotopic. First of all, $F$ is indeed homotopic to the linear hyperbolic automorphism $(B^N, A)$ for which we know that $h_H(F) = \log sp(F_{*,2})$. Also, we have

$$\begin{aligned} h_{top}(F) &\geq h_{top}(B^N|_{\Lambda_\delta}) + h_{top}(f) \\ &> h_{top}(B^N) - \frac{\varepsilon}{2} + h_{top}(A) + \varepsilon \\ &= h_{top}(B^N, A) + \frac{\varepsilon}{2}. \end{aligned}$$

So, the second item of theorem is proved.

Now, we consider $G \in \mathcal{U}$. Let us denote with $\lambda_N^- < 0 < \lambda_N^+$ the Lyapunov exponents of $B^N$, and by $\lambda_A^- < 0 < \lambda_A^+$ the corresponding to $A$. Let $\mu$ be an ergodic $G$-invariant measure with $h_\mu(G) > h_H(G)$, and for this measure we have the exponents $\lambda_N^-, \lambda_N^+, \gamma_1, \gamma_2 \in \mathbb{R}$. Now, by Ruelle's inequality it follows that

$$\lambda_A^+ + \lambda_N^+ = h_H(G) < h_\mu(G) \leq \lambda_N^+ + \max\{0, \gamma_1\} + \max\{0, \gamma_2\}.$$

Thus, because $\max\{0, \gamma_1\} + \max\{0, \gamma_2\} > 0$ we can suppose $\gamma_1 > 0$. On the other hand, we have $h_\mu(G^{-1}) = h_\mu(G)$ and $h_\mu(G^{-1}) \leq -\lambda_N^- + \max\{0, -\gamma_1\} + \max\{0, -\gamma_2\}$. Then we have necessarily $\gamma_2 < 0$. This implies that $\mu$ is actually a hyperbolic measure, with its index being the same as its linear part. Thus, the third requirement is established.

Let us now prove that if we adjust the initial set $\mathcal{U}$, then any $G \in \mathcal{U}$ is topologically transitive. We consider the following $F$-invariant laminations:

- Lamination by leaves homeomorphic to $\mathbb{R} \times \mathbb{T}^2$:

  $$\mathcal{F}^s = \left\{ \{L_x^s\}_{\mathbb{T}^2 \ni x} \mid L_x^s = W^s(x, B^N) \times \mathbb{T}^2 \right\},$$

  analogously we have the $\mathcal{F}^u$-lamination.

- Lamination by torus $\mathbb{T}^2$:

  $$\mathcal{G} = \left\{ \{T_x\}_{\mathbb{T}^2 \ni x} \mid T_x = \{x\} \times \mathbb{T}^2 \right\}.$$

These laminations are normally hyperbolic and $C^1$, therefore, they are plaque expansive for $F$, and as such we have $C^1$–persistence of such a laminations. This means that for every $C^1$–perturbation of the dynamic, call it $G$, there exist a $G$–invariant lamination $\mathcal{F}^s_G$, and a homeomorphism $\mathcal{H}^s_G\colon M \to M$ sending the original foliation equivariantly to $\mathcal{F}^s_F$. Restricted to each leaf, $\mathcal{H}^s_G$ is $C^1$. Likewise, we also have laminations $\mathcal{F}^u_G$, $\mathcal{G}_G$ and homeomorphisms $\mathcal{H}^u_G$, $\mathcal{H}_G$ mapping the respective leaves with each other. Robust transitivity is due to these properties, as well as because the dynamic induced on the space of the leaves remains the same. Namely, consider $\mathcal{U} \ni F$ the initial $C^1$–neighborhood and reduces it in such a way the above laminations persist. So for any $G \in \mathcal{U}$ we have the following properties,

**(a)** $G|_\Lambda$ is transitive on the torus $\Lambda = \mathcal{H}_G(\{p\} \times \mathbb{T}^2) \subset \mathbb{T}^4$,

**(b)** $\bigcup_{\Lambda \ni z} W^{ss}_G(z)$ is dense on $\mathbb{T}^4$.

In order to establish (b), by using $\mathcal{H}^s_G$ we just need to prove that:

$$\bigcup_{\mathbb{T}^2 \ni z} W^{ss}_F(p, z) = W^{ss}(p, B^N) \times \mathbb{T}^2.$$

On the other hand, for every $z \in \mathbb{T}^2$ and for small enough $\varepsilon > 0$, we have that

$$W^{ss}_\varepsilon(p, z) = W^{ss}_\varepsilon(p, B^N) \times \{z\},$$

this is because over $B_\delta(p) \times \mathbb{T}^2$, $F$ is equal to the Anosov $(B^N, A)$. Thus,

$$W^{ss}(p, z) \subset \bigcup_{n \geq 0} F^{-n}(W^{ss}_\varepsilon(p, B^N) \times \{z\}) \subset W^{ss}(p, B^N) \times \mathbb{T}^2.$$

Now, take $(x, z) \in W^{ss}(p, B^N) \times \mathbb{T}^2$. We need to prove that there exists $w \in \mathbb{T}^2$ such that $(x, z) \in W^{ss}(p, z)$. Take $m \in \mathbb{N}$ such that $B^{Nm}(x) \in W^{ss}_\varepsilon(p, B^N)$, thus we have

$F^m(x,z) = (B^{Nm}x, w)$ for some $w \in \mathbb{T}^2$, hence $(x,z) \in F^{-m}(W^{ss}_{\varepsilon}(p, B^N) \times \{w\}) \subset W^{ss}(p,w)$. This ends the proof of $(b)$. Note that (a) is easy to see because the restriction of $F$ to $\{p\} \times \mathbb{T}^2$ is equal to $A$, so by structural stability of $A$ the same holds for $G$.

To see the transitive property, take $U, V \subset \mathbb{T}^4$ open sets. Take $(p,q) \in \mathbb{T}^4$ a periodic point of $G$, and suppose $G^k(p,q) = (p,q)$, such that

$$W^{ss}(p,q) \cap U \neq \emptyset \quad \text{and} \quad W^{uu}(G^{\ell}(p,q)) \cap V \neq \emptyset,$$

where $0 \leq \ell < k$. Take $D \subset U$ a disk transverse to $W^{ss}(p,q)$, so by $\lambda$-lemma, $G^{nk}(D)$ converges in compact parts, thus, there exists $n > 0$ such that $G^{nk+\ell}(U) \cap V \neq \emptyset$. $\square$

## 3.3 Theorem C

In this section we shall consider an open set of diffeomorphism with similar properties to those obtained in the previous section. The steps of such construction follow some ideas that, originally, appear in [14] (the techniques were also used in [2]). As usual, the construction is done starting from a hyperbolic model.

**Theorem 3.3.1.** *There exists an open subset $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^4)$ such that any $f \in \mathcal{U}$ is a pointwise partially hyperbolic diffeomorphism whit $\dim E^c_f = 2$. Furthermore,*

1. *$f$ is homotopic to a hyperbolic linear map $A$ that has unstable index 1.*
2. *$f$ satisfies $h_{top}(f) \geq h_{top}(f \mid W^c) > h_H(f) = \log sp(A_{*,1})$.*
3. *Any ergodic measure $\mu$, such that $h_{\mu}(f) > \log sp(A_{*,1})$ is a hyperbolic measure with index 2. If $\Lambda \subset \mathbb{T}^4$ is a hyperbolic set with index 1 then $h_{top}(f|\Lambda) \leq h_{top}(A)$.*

*4. $f$ is topologically transitive.*

We start by considering $f_A\colon \mathbb{T}^4 \to \mathbb{T}^4$, a linear Anosov automorphism induced in $\mathbb{T}^4$ by a linear map $A\colon \mathbb{R}^4 \to \mathbb{R}^4$ with eigenvalues $\lambda_4 > 1 > \lambda_3 > \lambda_2 > \lambda_1 > 0$.

Let $T\mathbb{T}^4 = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ be the splitting associated to the eigenvalues. Take a fixed point $\mathbb{T}^4 \ni x_0$, a neighborhood $U \ni x_0$ and $\varphi\colon U \subset \mathbb{T}^4 \to D_1 \subset \mathbb{R}^4$, a chart diffeomorphism where $D_r = \{x \in \mathbb{R}^4 \mid \; \|x\| < r\}$, satisfying

$$\varphi \circ f_A \circ \varphi^{-1}(x) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \lambda_4 x_4)$$

for all $x \in D_{r_0}$, where $r_0$ is so small that $f_A \circ \varphi^{-1}(D_{r_0}) \subset D_1$. Take $U_0 = \varphi^{-1}(D_{r_0/2})$. We can suppose that $U \subset B_{\delta_0}(x_0)$, where $\delta_0$ is small enough so that two different lifts of $U_0$ are at a distance of at least $5000\delta_0$.

Because $f_A\colon \mathbb{T}^4 \to \mathbb{T}^4$ is an Anosov diffeomorphism, we have that $A$ is topologically stable in the strong sense, as proved by Walters, (see [21]). So given $0 < \varepsilon \ll \delta_0$, there exists $0 < \delta < \delta_0$ small with the property that any diffeomorphism $f$ with $\delta$-$C^0$-distance to $A$ is semiconjugated to $A$. The semiconjugacy map $h$ is at $\varepsilon$-$C^0$-distance from the identity map.

We shall modify $f_A$ inside a suitable ball such that we get a new diffeomorphism $f_0\colon \mathbb{T}^4 \to \mathbb{T}^4$. In fact, we choose $r_0$ small enough so that $U_0 \subset B_{\delta/2}(x_0)$. Finally, we define the map $f_0$ by:

$$f_0(x) = \begin{cases} f_A(x) & \text{when } \; x \notin U_0 \\ \varphi^{-1} \circ F \circ \varphi(x) & \text{when } \; x \in U_0 \end{cases}$$

where, if $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, $F$ is written as

$$F(x_1, y, x_2) = (\phi(x), h(x,y), k(x)),$$

and the involved functions $\phi$, $h$, $k$, will be specified below.

What we want to do is to construct an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^4), \mathcal{U} \ni f_0$, such that any $f \in \mathcal{U}$ is required to verify the following properties:

**(P1)** $f$ is dynamical coherent.

**(P2)** Let $\mathcal{C}^u$ the unstable cone field around the subspace $E_4$ which is preserved by $A$. Then, there exist $N \in \mathbb{N}$ and $\lambda > 1$ such that $D_x f^N(\mathcal{C}(x)) \subset \mathrm{int}\,(\mathcal{C}(f^N x))$ and for every $v \in \mathcal{C}(x) \setminus \{0\}$ we have $\|D_x f^N v\| > \lambda \|v\|$.

**(P3)** There exists a continuous and surjective map, $h_f \colon \mathbb{T}^4 \to \mathbb{T}^4$, $\quad h_f \circ f = f_A \circ h_f$. The semiconjugacy map $h_f$, is 1-1 restricted to the unstable manifold $W^u_f$, and sends $cs$-leaves of $f$ on $s$-leaves of $A$.

We shall construct $f_0$, making sure that it verifies the desired properties (P1), (P2), and (P3), and such that by perturbing $f_0$, the properties still remain.

First of all, we proceed detailing the construction of $f_0$; to this, we locally modify the linear map along the central direction $E_2 \oplus E_3$. The resulting application will be denoted by $h$. The entire construction of $h$ is summarized by saying 'pasting a horseshoe'. So, let $g \colon B_r \subset \mathbb{R}^2 \to B_r$ be a map such that

$$\begin{aligned} &\Omega(g) = \{0\} + H, \text{ where } H \subset B_\ell \text{ for } \ell \leq \tfrac{r}{3}, \\ &g(0) = 0 \qquad \text{is an attracting fixed point}, \\ &g(H) = H \quad \text{is conjugate to a } n\text{-shift}. \end{aligned}$$

Here, $B_r = \{x \in \mathbb{R}^2 \mid \|x\| < r\}$ and $n \in \mathbb{N}$ is taken so that $\log n > \log \lambda_1$. Now, consider an isotopy $g_t \colon B_r \to B_r$ between the maps $g_0(x_2, x_3) = (\lambda_2 x_2, \lambda_3 x_3)$ and $g_1 = g$ such that $g_t = g_0$ for all $t \in [0,1]$ and $x \notin B_{r/2}$. It is also considered a bump-function $\chi \colon \mathbb{R} \to [0,1]$ such that $\chi(t) = 0$ if $|t| \geq \delta_2$, $\chi(t) = 1$ if $|t| \geq \delta_1$, and

$\chi(-t) = \chi(t)$ for all $\mathbb{R} \ni t$. We define:

$$h(x, y) = g_{\chi(\|x\|)}(y).$$

Observe that from $g$, it is possible to construct a function $\widetilde{g}$ so that the topological entropy does not change while maintaining control on the growth of its derivative. To do so, just take a homothety and call it $\zeta$, and it is sufficient to consider the map $\zeta \circ g \circ \zeta^{-1}$. The map is well glued in the complement of a suitable ball, a region where $g$ is a linear map.

The second modification is done in the direction corresponding to the bundle $E_1 \oplus E_4$. With this, we attempt to maintain the dominance of the bundles. We consider a linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T(x, y) = (\lambda_1 x, \lambda_4 y)$, and real numbers $\hat{\lambda}_1 < \lambda_1,\ \hat{\lambda}_4 > \lambda_4$. The modification will lead to the maps $\phi$ and $k$, so that the application $(\phi(x, y), k(x, y))$ has strong $(\hat{\lambda}_1, \hat{\lambda}_4)$-contraction/expansion in a neighborhood of the origin, and so that outside a larger ball the map becomes equal to the linear map $T(x, y)$. We explain the construction of $k$ below. The other map, $\phi$, has its immediate rejoinder.

For $a_1 < a_2, b_1 < b_2 \in \mathbb{R}$, consider a rectangle $R(a_1, b_1) = [-a_1, a_1] \times [-b_1, b_1] \subset \mathbb{R}^2$, and similarly $R(a_2, b_2)$.

What we need to consider is a family of functions $\{k_x\}$, each defined over a vertical fiber $E_4$ such that $k_x = k_{-x}$ for all $x$, also $k_x = k_0$ for all $x \in [0, a_1]$ and $k_x = \lambda_4$ for $x \geq a_2$. This is because we can construct $k_0 \colon \mathbb{R} \to \mathbb{R}$ such that $k_0(y) = \hat{\lambda}_4 y$ if $y \in [-b_1, b_1]$, $k_0(y) = \lambda_4 y$ if $y \notin (-b_2, b_2)$, and the derivative $k_0'(y) > 0$ for all $y$. Then we can do an isotopy between $k_0$ and $k_{a_2} = \lambda_4$.

This results in a map $k(x, y) = k_x(y)$, in a similar way we have the map

$\phi(x,y) = \phi_y(x)$ such that

$$y \in [b_1, b_2] \quad \text{then} \quad \partial_y k(x, k_x^2(y)) = \lambda_4,$$

$$x \in [a_1, a_2] \quad \text{then} \quad \partial_y \phi(\phi_y^{-2}(x), y) = \lambda_1.$$

It can adapt $R(a_i, b_i)$ so that it remains in a domain as small as required, also ensuring that $\hat{\lambda}_1, \hat{\lambda}_4$ are chosen so that they meet the estimate: $\hat{\lambda}_1 < K^{-1}$, $\hat{\lambda}_4 > K$, where $K = \|Dg\|$.

To ensure that $f_0$, constructed as such, becomes partially hyperbolic, we can modify $f_0$ so that it verifies a cone-criterium. For this, consider $\varepsilon_0 > 0$ and a ball $D_{\varepsilon_0} \subset \mathbb{R}^4$. So, there exists $N_{\varepsilon_0} \in \mathbb{N}$ such that

$$x \in D_{\varepsilon_0}, \quad Fx \notin D_{\varepsilon_0} \quad \Longrightarrow \quad F^k x \notin D_{\varepsilon_0}, \quad \text{for all} \quad 1 \leq k \leq N_{\varepsilon_0}.$$

We have that $N_{\varepsilon_0} \to \infty$ as $\varepsilon_0 \to 0$.

Consider $\alpha > 0 \in \mathbb{R}$ such that for all $x$, $\angle(D_x F e_4, e_4^{\perp}) \geq \alpha$ where $e_j \in \mathbb{R}^4$ are the canonical vectors, and $e_4^{\perp} = \langle e_1, e_2, e_3 \rangle$. Let $\theta > 0$ be the cone angle, selected small enough such that

$$\angle(D_x F \upsilon, e_4^{\perp}) \geq \frac{\alpha}{2}, \quad \text{for all} \quad \upsilon \in \mathcal{C}_\theta^u(E_4(x)).$$

Let $N \in \mathbb{N}$ big enough so that

$$\angle(\mathcal{C}^u(E_4), e_4^{\perp}) \geq \alpha \quad \Longrightarrow \quad \varphi \circ f_A^N \circ \varphi^{-1} \mathcal{C}^u(E_4) \subset \mathcal{C}_\theta^u(E_4).$$

Now, take $\varepsilon_0 > 0$ small such that $N_{\varepsilon_0} > 2N$, and take an appropriate homothetic transformation $\zeta \colon B_r \to B_{\varepsilon_0}$. We can consider the new map $f_0$, by considering the conjugation $\zeta \circ F \circ \zeta^{-1}$. Of course, the condition $\angle(D_x F e_4, e_4^{\perp}) \geq \alpha$ remains for the new map.

The second condition required in the theorem is resolved in a way similar to the previous theorem.

We need to observe that the map $f_0$ is in fact dynamically coherent. This follows as an immediate consequence of [4], because by construction $f_0$ verifies the $SADC$ and *properness* conditions (see [4] for formal definitions) which are sufficient to achieve the integrability of $cs, cu$ bundles. It is also proved that such conditions remain valid when a small perturbation is added.

The properties (P2) and (P3) are persistent in a $C^1$-neighborhood of $f_0$. We just need to check that the semiconjugacy $h_f$ is 1-1 when restricted to unstable leaves; however, because the unstable manifold $E^u_A$ is one-dimensional, this is a consequence of a property that we have by looking at its lift: $\widetilde{h}_f(\widetilde{x}) = \widetilde{h}_f(\widetilde{y})$ if and only if there exists $K > 0$ such that $\|\widetilde{f}^n(\widetilde{x}) - \widetilde{f}^n(\widetilde{y})\| < K$ for every $n \in \mathbb{Z}$.

Let us now prove that every $f \in \mathcal{U}$ is topologically transitive. The whole idea in order to obtain this is to consider:

1. The existence of $L > 0$ such that every unstable arc with length bigger than $L$ intersects every $cs$-disc with internal radius $5\delta$.
2. The classical Mañé-Bonatti-Viana argument allows us to state that the backward iterates of any $cs$-disc will contain a disc of radius bigger than $5\delta$.

So, the transitive property is immediate, in fact, suppose that $U$, $V$ are open sets of $\mathbb{T}^4$. On one hand, we have that a forward iterated of $U$, $f^{m_0}(U)$, contains an arc-segment of length larger than $L$, and on the other hand there exists $n_0$ such that $f^{-n_0}(V)$ will contain a $cs$-disc of radius bigger that $5\delta$. Thus, $f^{m_0+n_0}(U) \cap V \neq \emptyset$.

As in [2] (see page 190), condition 1 will be satisfied if one chooses sufficiently narrow cone fields. By the lemma below, condition 2 also holds.

**Lemma 3.3.2.** *Consider $f \in \mathcal{U}$ and denote by $W^{cs}_{loc}(f)$ an arbitrary cs-disc. Then there exists $n_0 \in \mathbb{N}$ such that $f^{-n_0}(W^{cs}_{loc}(f))$ has cs-radius larger than $5\delta$.*

*Proof.* We consider an arbitrary *cs*-disc in $\mathbb{T}^4$. Since it contains a strong stable manifold, their negative iterates reach a size larger than $\varepsilon$. This implies that $h_f$, the semiconjugacy map, cannot collapse the disc and so the negative iterates of the disc grow exponentially in diameter. The classical Mañé's argument then implies that there exists a point $x_0$ in the disc and $n_0 \in \mathbb{N}$ such that $f^{-n}(x_0)$ does not belong to the perturbation region for any $n \geq n_0$. Since outside the perturbation region the dynamic is hyperbolic, one obtains that the negative iterates of the disc eventually reach the size of $5\delta$. □

## 3.4 Remarks

**1.** Notice that the connected component $\mathsf{PH}^0_A(\mathbb{T}^d)$ containing the linear Anosov diffeomorphism $A$ is an open set.

**2.** The example constructed in Theorem C can be done even in the 3-torus with splitting $T\mathbb{T}^3 = E^{cs} \oplus E^u$ by using the same tools and getting similar properties.

**3.** Far from good regularity, ($C^{1+\alpha}$ setting) the Pesin theory fails dramatically (see, for example [1]). However, even though it is not known if Katok's theorem fails in $C^1$ regularity, $C^{1+\alpha}$ assumption can be relaxed (just requiring the weaker $C^1$ differentiability hypothesis) when the domination condition is added, (see [6]).

# Chapter 4

# Considerations

**1.** It turns out that there exists an interesting question, which we have yet to mention. Notice that in principle, if there is $u_0 \in \mathbb{N}$ such that $h_H(f) = \log sp(f_{*,u_0})$, this $u_0$ may not necessarily be unique. However, in the setting of Theorem $A$, every partially hyperbolic diffeomorphism considered there associates a unique value $u_0$ which maximizes the homological entropy. This follows immediately from the uniqueness of the maximal entropy measure. The general case could be different, and should be studied in a deep way.

What one would expect is that the existence of hyperbolic sets with 'good index' still continues to exist under certain conditions.

For instance, consider a partially hyperbolic diffeomorphism $A \times Id\colon \mathbb{T}^3 \hookleftarrow$, where $A$ is a linear hyperbolic map on $\mathbb{T}^3$ and $Id$ the identity map on $\mathbb{S}^1$. Consider $f$ a small perturbation of $A \times Id$. We know that $f$ is also partially hyperbolic and has center fibers $W^c(x)$ homeomorphic to $\mathbb{S}^1$, where the dynamic induced on space of leaves, which is $\mathbb{T}^2$, is the same as the linear dynamic, $A$. We have that $h_H(f) = \log sp(f_{*,1})$ and also $h_H(f) = \log sp(f_{*,2})$. In [5] is proved that if $f$ has one maximizing hyperbolic measure with index 1, then there is at least another maximizing hyperbolic measure with index 2. Actually, in the context of dynamically coherence partially hyperbolic diffeomorphism with compact one dimensional center leaves, they form an open and dense subset.

**2.** Related to Theorem C, we ask if the following can be proved: there is not an example as in Theorem C which is absolutely partially hyperbolic.

**3.** Consider $f \in \mathsf{PH}(\mathbb{T}^3)$ with splitting $E^{cs}_f \oplus E^u_f$ isotopic to a linear Anosov $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $\dim E^s_A = 2$. Let $\mu$ be an ergodic invariant measure such that $h_\mu(f) > h(A)$. Where is such a measure supported? In Theorem C, we saw that the support of such a measure is on an invariant surface. Is that always the case?

**4.** Consider $f \in \mathsf{PH}^0_A(\mathbb{T}^d)$. Is $f$ transitive?.

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