Some aspects of group actions on one-dimensional manifolds

Joaquín Brum[1]

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Abstract

- We exhibit flexibility phenomena for some (countable) groups acting by order preserving homeomorphisms of the line. More precisely, we show that if a left orderable group admits an amalgam decomposition of the form $G = \mathbb{F}_n *_{\mathbb{Z}} \mathbb{F}_m$ where $n + m \geq 3$, then every faithful action of G on the line by order preserving homeomorphisms can be approximated by another action (without global fixed points) that is *not* semi-conjugated to the initial action. We deduce that $\mathcal{LO}(G)$, the space of left orders of G, is a Cantor set.

  In the special case where $G = \pi_1(\Sigma)$ is the fundamental group of a closed hyperbolic surface, we found finer techniques of perturbation. For instance, we exhibit a single representation whose conjugacy class in dense in the space of representations. This entails that the space of representations without global fixed points of $\pi_1(\Sigma)$ into $Homeo_+(\mathbb{R})$ is connected, and also that the natural conjugation action of $\pi_1(\Sigma)$ on $\mathcal{LO}(\pi_1(\Sigma))$ has a dense orbit.

- We prove that if Γ is a countable group without a subgroup isomorphic to $\mathbb{Z}^2$ that acts faithfully and minimally by orientation preserving homeomorphisms on the circle, then it has a free orbit. We give examples showing that this does not hold for actions by homeomorphisms of the line.

Abstract

- Mostramos fenómenos de flexibilidad para acciones en la recta por homeomorfismos que preservan orientación, de algunos grupos numerables. Más concretamente, mostramos que si un grupo ordenable admite una descomposición como producto amalgamado $G = \mathbb{F}_n *_{\mathbb{Z}} \mathbb{F}_m$ donde $n + m \geq 3$, cualquier acción de G en la recta por homeomorfismos que preservan orientación puede ser aproximada por otra acción (sin puntos fijos globales) que *no* es semi-conjugada a la acción original. Deducimos que $\mathcal{LO}(G)$, el espacio de órdenes invariantes a izquierda de G, es un conjunto de Cantor.

  En el caso especial en que $G = \pi_1(\Sigma)$ es el grupo fundamental de una superficie hiperbólica cerrada, encontramos técnicas de perturbación más finas. Por ejemplo, mostramos que existe una representación cuya clase de conjugación es densa en el espacio de representaciones. Esto permite probar que el espacio de representaciones sin puntos fijos globales de $\pi_1(\Sigma)$ en $Homeo_+(\mathbb{R})$ es conexo, y también que la acción natural por conjugación de $\pi_1(\Sigma)$ en $\mathcal{LO}(\pi_1(\Sigma))$ tiene una órbita densa.

- Probamos que si Γ es un grupo numerable sin subgrupos isomorfos a $\mathbb{Z}^2$, cualquier accion fiel y minimal de Γ en el círculo por homeomorfismos que preservan orientación, tiene una órbita libre. Damos ejemplos mostrando que esto no ocurre para acciones en la recta.

Table of contents

Chapter 1

Introduction

The theory of group actions on manifolds has relations with several areas of mathematics. In this thesis we treat two examples of this interplay, for the case where M is one-dimensional (that is M is either $\mathbb{R}$ or S^1). In the first case, we develop perturbation techniques that allow us to obtain results about the space of left-invariant orders of a family of groups. In the second, we prove some results concerning stabilizers of points for minimal actions on the circle and show its connections with the theory of foliations.

1.1 Flexibility and group orderings

This part of the thesis is a joint work with Juan Alonso (Udelar) and Cristóbal Rivas (Usach, Santiago de Chile).

Given a group G, the space $Rep(G, Homeo_+(M))$ of representations of G into the group of orientation preserving homeomorphisms of an orientable manifold M is a classical object that encompasses many different areas of mathematics (see for instance [16]). When the manifold M has dimension one (that is M is either $\mathbb{R}$ or S^1), allowing a faithful action on M has an algebraic counterpart in terms of left invariant (linear or circular) orders [7, 11, 14]. For instance, from any faithful action on the line of a group G one can induce a total left invariant linear order on G (*left order* for short), and conversely, from any left order on a (countable) group G one can produce a faithful action of G on the line by orientation preserving homeomorphisms which is unique up to conjugation. This is the so called *dynamical realization* of the left order. See §2.3 for details.

The counterpart of $Rep(G, Homeo_+(\mathbb{R}))$ is the *space of left orders on* G, here denoted $\mathcal{LO}(G)$, which is the set of all left orders of G endowed with a natural topology that makes it a Hausdorff, totally disconnected and compact space [34]. Linnell showed that this space is either finite or uncountable [24], hence contains a Cantor set when it is infinite. Although of very different nature (one is *continuous* while the other is totally disconnected) there are some relationships between these two spaces. For instance it is implicit in Navas's [31] that the dynamical realization of an isolated left order of G is a *locally rigid* action of G on the line, meaning that any sufficiently close representation is *semi-conjugated* to it (see §2 for definitions, and Proposition 2.3.1 for an explicit proof). In fact, recently in [26] a complete characterization of isolated order in terms of a strong form of rigidity was obtained.

A very natural question is if there is an algebraic characterization of groups allowing isolated left orders. To this day we only count with some partial results. Tararin obtained an algebraic classification of groups allowing only finitely many left orders, see [23, Theorem 5.2.1]. They all turn out to be solvable, and in [33] a classification of (virtually) solvable groups allowing isolated left orders was obtained: they all fit in Tararin's classification. On the other hand, free groups [28, 31] and more generally free product of left orderable groups [32] admits no isolated left orders, whereas -for instance- $F_2 \times \mathbb{Z}$ has infinitely many conjugacy classes of isolated left orders [26]. We generalize the above result for free products to groups allowing certain decompositions as amalgamated free products.

Let $G = \mathbb{F}_n *_{w_1=w_2} \mathbb{F}_m$, with $n \geq 2$, $m \geq 1$, be the amalgamated product of the free groups $\mathbb{F}_n$ and $\mathbb{F}_m$ identifying the cyclic subgroups $\langle w_1 \rangle \subseteq \mathbb{F}_n$ and $\langle w_2 \rangle \subseteq \mathbb{F}_m$. We will deduce our result about orders of G from a result of *flexibility* of its representations. Roughly speaking, a representation is *flexible* if it can be approximated by fixed-point-free representations that are not semi-conjugated to it (see Definition 2.1.3).

Theorem 1.1.1. *Let $G = \mathbb{F}_n *_{w_1=w_2} \mathbb{F}_m$ with $n \geq 2$, $m \geq 1$. Then, every representation $\rho : G \to Homeo_+(\mathbb{R})$ without global fixed points is flexible.*

Proposition 2.3.1 then entails

Corollary 1.1.2. *Let $G = \mathbb{F}_n *_{w_1=w_2} \mathbb{F}_m$ with $n \geq 2$. The space of left orders of G has no isolated points.*

Some comments are in order. The first one is that a group G as in Theorem 1.1.1 above is always left orderable since for this it suffices to have and order preserving isomorphisms between $\langle w_1 \rangle$ and $\langle w_2 \rangle$, see [4] (alternatively, they are one-relator and torsion free, hence left orderable [6]). For a general condition for orderability of amalgams see [5]. Secondly, prior to this work the amalgamated free product has been used to construct groups having an infinite space of left orders that *contains* isolated points, such as the groups $\langle a, b \mid a^n = b^m \rangle$ from [18, 29] and the groups constructed by Ito's iterative methods [19, 20]. In particular the condition $n+m \geq 3$ in Theorem 1.1.1 is sharp. Last but not least, Theorem 1.1.1 should be compared with the work of Mann [25], where she shows that some special representations of $\pi_1(\Sigma)$ into $Homeo_+(S^1)$ (the so called *geometric representations*) are *fully rigid*, meaning that their connected component inside $Rep(\pi_1(\Sigma), Homeo_+(S^1))$ is made of a single semi-conjugacy class. By contrast, Theorem 1.1.1 implies that the semi-conjugacy class of any representation of $\pi_1(\Sigma)$ (and more generally any group as in Theorem 1.1.1) into $Homeo_+(\mathbb{R})$) has empty interior.

We ask if every one-relator group with $k \geq 3$ generators has this flexibility property acting on $\mathbb{R}$.

Theorem 1.1.1 is deduced from a technical lemma involving perturbations of representations of the free group into $Homeo_+(\mathbb{R})$ under some conditions on the image of a specific element $w \in F_n$ (see Lemma 3.1.1). In the special case where w is a commutator of some generators of F_n, we obtain finer perturbations techniques (see Lemma 3.1.2 and Lemma 3.1.3) which allow us to show that $\mathcal{LO}(G)$ is a Cantor set whenever G is a countable left orderable group allowing a decomposition of the form $H *_{h=[a,b]} \mathbb{F}_2(a, b)$. See Theorem 3.3.1. But they provide much more! Indeed, when we restrict our attention to $G = \pi_1(\Sigma)$ the fundamental groups of an orientable, closed, hyperbolic surface, we found other results with a strong flexible flavor. For instance, we show how to perturb a given representation in order to blow up global fixed points. Precisely we prove

Theorem 1.1.3. *Let M be $\mathbb{R}$ or S^1, $\rho \in Rep(\pi_1(\Sigma), Homeo_+(M))$ and U a neighbourhood of ρ in the compact open topology. Then, there exists $\rho' \in U$ having no global fixed points.*

We also show how to build a very special representation of $\pi_1(\Sigma)$ into $Homeo_+(\mathbb{R})$, whose existence can be thought of as a strong form of flexibility. Actually, results such as Theorem 1.1.4 and Theorem 1.1.6 below, were only known to hold for the non-Abelian free groups [10, 32].

Theorem 1.1.4. *There is a representation of $\pi_1(\Sigma)$ into $Homeo_+(\mathbb{R})$ without global fixed point, whose conjugacy class under $Homeo_+(\mathbb{R})$ is dense in $Rep(\pi_1(\Sigma), Homeo_+(\mathbb{R}))$.*

Since conjugacy classes are path connected we immediately obtain (compare with Remark 2.1.5)

Corollary 1.1.5. *The space of representation without fixed points $Rep_{\#}(\pi_1(\Sigma), Homeo_+(\mathbb{R}))$* is connected.

The counterpart of Theorem 1.1.4 in the context of the group orders is the theorem below. For the statement recall that a group acts on its space of left orders by conjugation, see §2.3.

Theorem 1.1.6. *There is a left order on $\pi_1(\Sigma)$ whose orbit under the natural conjugation action is dense in $\mathcal{LO}(\pi_1(\Sigma))$.*

In §3.1 we state our three main lemmas and deduce all the theorems stated in this introduction from them. Lemma 3.1.1 is proved in §3.7.1 whereas Lemma 3.1.2 and Lemma 3.1.3 are proved in §3.7.2. The preliminary knowledge and definitions to carry out our study is given in §2.

Remark 1.1.7. After our first draft was released it was pointed out to us that Bonatti and Firmo had proved Theorem 1.1.3 in the category of C^∞ diffeomorphisms (Théorème 5.4 in [3]). Though their techniques are very similar to ours, we choose to provide self-contained proof of Theorem 1.1.3 for the sake of completeness.

1.2 Free orbits for minimal actions on S^1

This part of the thesis is a joint work with Matilde Martínez y Rafael Potrie.

Foliations of codimension one and groups of homeomorphisms of the circle are closely related.

A particular but illuminating example of a foliation can be obtained via the suspension construction, by which an action of a surface group on the circle gives rise to a foliation on a circle bundle over a surface (see Appendix 4.4). In this example, fundamental groups of leaves correspond to stabilizers of points under the action, so that simply connected leaves translate into free orbits.

When these foliations are minimal, either the generic leaf is simply connected or all leaves have non-finitely generated fundamental group (see [1]). With this motivation, it is natural to ask if a minimal and faithful action of the fundamental group of a surface on the circle must have some free orbit.

It turns out that this is true in some greater generality as we prove in the following result:

Theorem 1.2.1. Let Γ be a countable group without a subgroup isomorphic to $\mathbb{Z}^2$. If Γ acts faithfully and minimally by orientation preserving homeomorphisms on the circle, then there exists a free orbit.

Recall that a *free orbit* is the orbit of a point $x \in S^1$ such that for every $g \in \Gamma \setminus \{e\}$ one has that $gx \neq x$.

Minimality of the action is necessary as it is shown by an example in §4.2.1. For actions on $Homeo_+(\mathbb{R})$ the result is also false, see §4.2.2.

As a direct consequence of this results one deduces that if $f, g \in Homeo_+(S^1)$ are homeomorphisms such that f has a non-trivial interval of fixed points and g is conjugate to an irrational rotation, then the group generated by f and g inside $Homeo_+(S^1)$ is not free (and in particular contains a copy of $\mathbb{Z}^2$).

Remark 1.2.2. One can also see that $\mathbb{Z}^2$ itself does not admit faithful minimal actions on the circle without free orbits. In fact, any group admitting such an action must be non-abelian, as we will see in §4.3, where we give further conditions a group acting minimally and without free orbits must satisfy.

It is natural to wonder whether a similar result will hold in higher dimensions. For example, one can ask:

Is there a faithful and minimal action of the free group in two generators on a closed surface without free orbits?

For an excellent panoramic of the theory of group actions on the circle, see [17] or [30].

Chapter 2

Preliminaries

2.1 Flexibility and local rigidity in $Homeo_+(\mathbb{R})$

Throughout this work we will deal with the notion of local rigidity. To state it we first need to recall the definition of semi-conjugacy. We say that a non-decreasing map $c : \mathbb{R} \to \mathbb{R}$ is *proper* if c^{-1} maps compact sets into bounded sets[1]. Note that this is equivalent to demand that the non-decreasing map c satisfies that $c(\mathbb{R})$ is unbounded in both directions of the line.

Definition 2.1.1. We say that two representations $\rho_i : G \to Homeo_+(\mathbb{R})$, $i = 1, 2$, are semi-conjugated if there is a *monotone* (i.e. non-decreasing) map $c : \mathbb{R} \to \mathbb{R}$ which is *proper* and satisfies

$$c \circ \rho_1(g) = \rho_2(g) \circ c \qquad \text{for all } g \in G. \tag{2.1}$$

Traditionally (*e.g.* in [30]), one also insists on the continuity of c above. This has been a pity since that condition causes more inconveniences than the ones it solves. For instance without the continuity assumption one has

Proposition 2.1.2. *Semi-conjugacy is an equivalence relation.*

Proof. Reflexivity is obvious, transitivity is natural, we check symmetry.

Suppose (2.1) holds. Since c is proper we can make $c'(x) := \sup c^{-1}(x)$. Since c is monotone we have

$$c'(x) = \sup c^{-1}((-\infty, x]) = \sup\{y \mid c(y) \le x\}.$$

Since the above supremum is taken over a larger set, monotonicity of c' follows. To show that c' is proper, observe that by monotonicity of c, if $y > c(x)$, $c'(y) \ge x$ and if $y < c(x)$, $c'(y) \le x$.

[1]Please note that our definition of proper map is not the traditional one demanding that inverse image of compact sets are compact. If fact, a paradigmatic example that we want to consider as *proper* is the map $c : x \mapsto \max\{n \in \mathbb{Z} \mid n \le x\}$. For this map we have that $c^{-1}(0) = [0, 1)$.

Finally, covariance also follows since we have

$$\begin{aligned}
\rho_1(g)(c'(x)) &= \sup\{\rho_1(g)(y) \mid c(y) \le x\} \\
&= \sup\{z \mid c(\rho_1(g)^{-1}(z)) \le x\} \\
&= \sup\{z \mid \rho_2(g)^{-1}(c(z)) \le x\} \\
&= \sup\{z \mid c(z) \le \rho_2(g)(x)\} \\
&= c'(\rho_2(g)(x)).
\end{aligned}$$

□

We now let G be a countable and discrete group and M a locally compact oriented manifold (for our purpose it is enough to consider M as being the real line or the circle). The set $Rep(G, Homeo_+(M))$, of group representations from G to $Homeo_+(M)$, is endowed with the pointwise convergence. That is, ρ_n converges to ρ if and only if $\rho_n(g)$ converges to $\rho(g)$ for all $g \in G$, where the convergence $\rho_n(g) \to \rho(g)$ is given by the compact open topology: for every positive ε and for every compact set $K \subset M$ there is n_0 such that $n \ge n_0$ implies

$$\sup_{x \in K} |\rho_n(g)(x) - \rho(g)(x)| \le \varepsilon.$$

Given $\rho \in Rep(G, Homeo_+(M))$ we define

$$Fix(\rho) = \{x \in M : \rho(g)(x) = x, \ \forall g \in G\},$$

the set of global fixed points of ρ. The subset of representations without global fixed points, with the inherited topology, will be denoted by $Rep_\#(G, Homeo_+(M))$. In this work we will be mainly interested in understanding rigidity inside the space $Rep_\#(G, Homeo_+(\mathbb{R}))$.

Definition 2.1.3. We say that $\rho \in Rep_\#(G, Homeo_+(\mathbb{R}))$ is *locally rigid*, if there is a neighbourhood U of ρ such that every $\rho' \in U \cap Rep_\#(G, Homeo_+(\mathbb{R}))$ is semi-conjugated to ρ. If ρ is not locally rigid, then we say that ρ is *flexible*.

Remark 2.1.4. Observe that the convergence of $\rho_n \to \rho$ in $Rep(G, Homeo_+(\mathbb{R}))$ is equivalent to require that $\rho_n(g) \to \rho(g)$ for every g in a generating set of G. In particular, if we have a finite generating set for G with k elements, then $Rep(G, Homeo_+(\mathbb{R}))$ is homeomorphic to a subset of $Homeo_+(\mathbb{R})^k$. Since $Homeo_+(\mathbb{R})$ is metrizable and separable (see for instance [21]), it satisfies the second axiom of countability, and so does $Homeo_+(\mathbb{R})^k$ and any of its subsets (such as $Rep(G, Homeo_+(\mathbb{R}))$ and $Rep_\#(G, Homeo_+(\mathbb{R}))$).

Remark 2.1.5. Inside the space of all representation, $Rep(G, Homeo_+(\mathbb{R}))$, the so called *Alexander trick* can be performed both to retract the space to the trivial representation and/or to find non-semi-conjugated representations arbitrarily close to a given one. Indeed, for a representation $\rho : G \to Homeo_+(\mathbb{R})$, we can consider $f_t : \mathbb{R} \to \mathbb{R}$ a continuous path of continuous maps (homeomorphisms over its images for $t \neq 1$) with $f_0(x)$ the identity map, and f_1 a constant map, and construct

$$\rho_t(g)(x) = \begin{cases} f_t \rho(g) f_t^{-1}(x) & \text{if } x \in f_t(\mathbb{R}) \text{ and } t \neq 0 \\ x & \text{otherwise.} \end{cases}$$

These kind of tricks are not possible inside $Rep_{\#}(G, Homeo_+(\mathbb{R}))$. For instance, the space $Rep_{\#}(\mathbb{Z}, Homeo_+(\mathbb{R}))$ is not connected since the subset of representations satisfying $\rho(a)(x) > x$ for all x (where a is the generator of $\mathbb{Z}$) is open and closed in that space. A similar argument applies for groups of the form $G = \langle a, b | a^m = b^n \rangle$.

2.2 Conjugacy classes in $Homeo_+(\mathbb{R})$.

An important ingredient for proving our results involving commutators, is the description of conjugacy classes in $Homeo_+(\mathbb{R})$. Luckily, detecting when two given homeomorphisms of the line are conjugated is an easy task: *it is all encoded in the combinatorics of the homeomorphisms.* More precisely, if $\psi\phi_1\psi^{-1} = \phi_2$, then ψ maps bijectively the sets

$$\begin{aligned} Fix(\phi_1) = \{x \mid \phi_1(x) = x\} &\longleftrightarrow Fix(\phi_2), \\ Inc(\phi_1) = \{x \mid \phi_1(x) > x\} &\longleftrightarrow Inc(\phi_2), \\ Decr(\phi_1) = \{x \mid \phi_1(x) < x\} &\longleftrightarrow Decr(\phi_2), \end{aligned} \tag{2.2}$$

respectively. In fact, (2.2) characterizes when two homeomorphisms ϕ_1 and ϕ_2 are conjugated. If there is $\psi \in Homeo_+(\mathbb{R})$ which maps bijectively $Fix(\phi_i)$, $Inc(\phi_i)$ and $Decr(\phi_i)$ $(i = 1, 2)$, then there exist $\bar{\psi}$ such that $\bar{\psi}\phi_1\bar{\psi}^{-1} = \phi_2$. This motivates our next

Definition 2.2.1. For ψ, ϕ_1, ϕ_2 homeomorphisms of the real line, we will say that ψ is a *weak-conjugation* from ϕ_1 to ϕ_2 if

- $\psi(Fix(\phi_1)) = Fix(\phi_2)$ and
- $\psi(Inc(\phi_1)) = Inc(\phi_2)$.

Additionally, if for an interval I we have that $\psi\phi_1(x) = \phi_2\psi(x)$ for all $x \in I$ we will say that the weak conjugation ψ is *strong* on I.

Observe that conjugacy and weak-conjugacy classes are identical, but it is much easier to find/build weak conjugations rather than true conjugating elements. In order to pass from a weak conjugation to a conjugation, the following lemma (and its proof) will be useful. In its proof and throughout the text, the restriction of a function ϕ to a set C will be denoted by $\phi_{|C}$.

Lemma 2.2.2. *Let $\psi, \phi_1, \phi_2 \in Homeo_+(\mathbb{R})$. If ψ is a weak-conjugation from ϕ_1 to ϕ_2 that is strong on a interval I, then there exists a conjugation $\bar{\psi}$ from ϕ_1 to ϕ_2 such that:*

- *$\bar{\psi}(x) = \psi(x)$ for every $x \in I$ and*
- *$\bar{\psi}(x) = \psi(x)$ for every $x \in Fix(\phi_1)$.*

Moreover, $\bar{\psi}$ agrees with ψ over $I \cup \phi_1(I)$.

Proof. We will prove the lemma for the case in which $I = [u, v]$ is compact. The non-compact case is similar.

Since ψ is a weak conjugacy, every connected component C of $\mathbb{R} - (Fix(\phi_1))$ is sent by ψ to a connected component D of $\mathbb{R} - (Fix(\phi_2))$, and $C \subseteq Inc(\phi_1)$ if and only if

$D \subseteq Inc(\phi_2)$. We will define a conjugation on a component C of $\mathbb{R} - (Fix(\phi_1))$. Choose a point $p \in C$. We assume that $C \subseteq Inc(\phi_1)$, as the other case is analogous. Let $J = [p, \phi_1(p))$, $K = [\psi(p), \phi_2(\psi(p)))$, and take $\alpha : J \to K$ an orientation preserving homeomorphism. Notice that $C = \bigcup_{n\in\mathbb{Z}} \phi_1^n(J)$ and $D := \psi(C) = \bigcup_{n\in\mathbb{Z}} \phi_2^n(K)$.

Define, for $x \in C$, $\psi_C(x) = \phi_2^{-m}(\alpha(\phi_1^m(x)))$, where m is the only integer such that $\phi_1^m(x) \in J$. Then ψ_C is a homeomorphism between C and D that conjugates $\phi_{1|C}$ and $\phi_{2|D}$. Defining $\psi_0 \equiv \psi$ on $Fix(\phi_1)$, and $\psi_0 \equiv \psi_C$ on each component C of $\mathbb{R} - \text{Fix}(\phi_1)$ (for some choice of p and α) gives a conjugation from ϕ_1 to ϕ_2. However, ψ_0 may not agree with ψ over I. To solve this problem, on each component C that intersects I, we choose $p \in C$ such that the corresponding $J = [p, \phi_1(p))$ intersects I maximally (that is, $J \cap I$ is either J or I), and we choose α so that it agrees with ψ over $J \cap I$. Since ψ is strong on I, this ψ_0 is a conjugation from ϕ_1 to ϕ_2 that agrees with ψ over I.

To show the final claim, take $y = \phi_1(x)$ with $x \in I$. Then, $\psi(y) = \psi(\phi_1(x)) = \phi_2(\psi(x)) = \phi_2(\bar{\psi}(x)) = \bar{\psi}(\phi_1(x)) = \bar{\psi}(y)$ □

2.3 Group orders and dynamical realizations

Recall that a left order on a group G is a total order $\preceq$ satisfying that given $f, g, h \in G$ such that $f \preceq h$ then $gf \preceq gh$. If G admits a left order, then we say that G is *left orderable*. The reader unfamiliar with this notion may wish to consult [11], [14], [23].

A natural topology can be defined on the set of all left orders on G, here denoted $\mathcal{LO}(G)$, making it a compact and totally disconnected space. In this topology, a local base at a left order $\preceq \in \mathcal{LO}(G)$ is given by the sets

$$V_{g_1,\dots,g_n} := \{\preceq' \in \mathcal{LO}(G) \mid id \prec' g_i\},$$

where $\{g_1, \dots, g_n\}$ runs over over all finite subsets of $\preceq$-positive elements of G. In particular, a left order is isolated in $\mathcal{LO}(G)$ if there is a finite set $S \subset G$ such that $\preceq$ is the only left order satisfying

$$id \preceq s\,,\ \text{for every } s \in S.$$

When the group is countable this topology is metrizable [11], [14],[34]. For instance, if G is finitely generated, and B_n denotes the ball of radius n with respect to a finite generating set, then we can declare that $dist(\preceq_1, \preceq_2) = 1/n$, if B_n is the largest ball in which $\preceq_1$ and $\preceq_2$ coincide.

There is also a natural action of a group G on the space $\mathcal{LO}(G)$ by *conjugation* of the orders. Precisely, if $\preceq$ is a left order on G and $g \in G$, we can define the order $\preceq_g$ by

$$h \preceq_g k \ \Leftrightarrow\ ghg^{-1} \preceq gkg^{-1}.$$

This $\preceq_g$ is the result of acting on $\preceq$ by g, and it is easy to check that this defines a left action by homeomorphisms of $\mathcal{LO}(G)$.

When the group G is countable, for every left order $\preceq$ on G, one can attach a fixed-point-free action $\rho : G \to Homeo_+(\mathbb{R})$ that models the left translation action of G on $(G, \preceq)$, in the sense that

$$f \preceq g \Leftrightarrow \rho(f)(0) < \rho(g)(0). \tag{2.3}$$

This is the so called, *dynamical realization* of $\preceq$ (which is unique up to conjugation), and 0 is sometimes called the *base point*, see [11], [14], [17].

The action of G by conjugation can also be expressed nicely in terms of dynamical realizations. If ρ is a dynamical realization of $\preceq$, then

$$h \preceq_g k \;\Leftrightarrow\; \rho(h)\rho(g)^{-1}(0) < \rho(k)\rho(g)^{-1}(0).$$

So, a dynamical realization of $\preceq_g$ is the conjugation of ρ by $\rho(g)$. Alternatively, one can see the order $\preceq_g$ as the order induced by the representation ρ, but "based" at the point $\rho(g)^{-1}(0)$.

So far, only two techniques are know to approximate a given left order $\preceq$ on a group G. One is to approach it by its own conjugates $(\preceq_g)_{g\in G}$, see for instance [31], [33], and the other one, implicit in [31], [32], is by showing that the dynamical realization of $\preceq$ is not locally rigid. Indeed we have

Proposition 2.3.1. *Let G be a left orderable group and $\preceq \in \mathcal{LO}(G)$ an isolated order. Then its dynamical realization $\rho \in Rep_{\#}(G, Homeo_+(\mathbb{R}))$ is locally rigid.*

Proof. Take $F \subseteq G$ a finite set so that $\preceq$ is the only left order on G satisfying $id \preceq f$ for all $f \in F$. Let ρ be a dynamical realization of $\preceq$. Then, there is a neighbourhood $U \subset Rep_{\#}(G, Homeo_+(\mathbb{R}))$ of ρ so that for $\rho' \in U$ and for every $f \in F \setminus \{id\}$ we have $0 < \rho'(f)(0)$. Let $\preceq'$ be the partial left order defined by

$$g_1 \preceq' g_2 \text{ if and only if } \rho'(g_1)(0) \leq \rho'(g_2)(0).$$

Since $Stab_{\rho'(G)}(0)$ is left orderable, we can extend the partial order $\preceq'$ to a total left order, that we still call $\preceq'$. See for instance [14, §2.1]. As $f \preceq' id$, for all $f \in F$ and $\preceq$ is the only left order on G satisfying that set of inequalities, we must have $\preceq' = \preceq$. In particular, this means that $Stab_{\rho'(G)}(0)$ is trivial, since every non-trivial left orderable group has at least two different orders.

Therefore we have that $\rho'(g_1)(0) < \rho'(g_2)(0)$ if and only if $\rho(g_1)(0) < \rho(g_2)(0)$ for every $g_1, g_2 \in G$. Let $\mathcal{O}$ and $\mathcal{O}'$ be the orbits of 0 under ρ and ρ' respectively. Then $\rho(g)(0) \mapsto \rho'(g)(0)$ is a monotone and G-equivariant map, that we call $\varphi : \mathcal{O} \to \mathcal{O}'$. It can be extended to a semi-conjugacy $c : \mathbb{R} \to \mathbb{R}$ between ρ and ρ' by setting

$$c(x) = \sup\{\varphi(y) : y \in \mathcal{O}, y \leq x\}.$$

Indeed, the monotone map c is proper because both representations have no global fixed points. The covariance also follows since

$$\begin{aligned}
\rho'(g)(c(x)) &= \rho'(g)(\sup\{\varphi(y) : y \in \mathcal{O}, y \leq x\}) \\
&= \sup\{\rho'(g)(\varphi(y)) : y \in \mathcal{O}, y \leq x\} \\
&= \sup\{\varphi(\rho(g)(y)) : y \in \mathcal{O}, y \leq x\} \\
&= \sup\{\varphi(z) : z \in \mathcal{O}, z \leq \rho(g)(x)\} \\
&= c(\rho(g)(x)).
\end{aligned}$$

□

We refer the reader to [26] for more about orders and rigidity.

2.4 Groups of circle homeomorphisms

Here we present some classical concepts and results that we will need in §4.

Definition 2.4.1. If X is a topological space and $\rho \in Rep(G, Homeo(X))$ is a representation, we say that ρ is minimal if the only closed ρ-invariant subsets of X are $\emptyset$ and X itself. When $f \in Homeo(X)$ we say that f is minimal if its associated cyclic representation is minimal.

Now we review the construction of the rotation number of a circle homeomorphism, which is a key invariant in the theory of circle dynamics. For a complete treatment see [22]

Proposition 2.4.2. Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and $\tilde{f} : \mathbb{R} \to \mathbb{R}$ a lift to the line. Define $S(n,x) = \tilde{f}^n(x)/n$. Then, $\rho(\tilde{f}) := \lim_{n\to+\infty} S(n,x)$ exists and does not depend on x. Moreover, this number differs by an integer when we take another lift.

Proof. See [22] □

So we can define:

Definition 2.4.3. Let $f \in Homeo_+(S^1)$ and $\tilde{f} : \mathbb{R} \to \mathbb{R}$ a lift of f to the line. We define $\rho(f)$ as $\rho(\tilde{f})$ mod $\mathbb{Z}$.

Remark 2.4.4. The rotation number is invariant by conjugation. (see [22])

The main result about this invariant is the classification of circle homeomorphisms into two classes with very different behaviour, according to whether the rotation number is rational or irrational.

Proposition 2.4.5. Let $f \in Homeo_+(S^1)$.

- If $\rho(f) = p/q \in \mathbb{Q}$ with $g.c.d(p,q) = 1$, then there exists $x \in S^1$ with period q. In this case the non-wandering set of f consists of periodic orbits, all with period q.
- If $\rho(f) \notin \mathbb{Q}$ there is a unique minimal set K for f. This minimal set can be either the whole circle or a Cantor set. In both cases f is semiconjugated to the irrational rotation $R_{\rho(f)}$. That is, there exists a continuous surjective map (not one-to-one if K is a Cantor set) $h : S^1 \to S^1$ that verifies $h \circ f = R_{\rho(f)} \circ h$.

Proof. See [22] □

Given $f \in Homeo_+(\mathbb{R})$ define the centralizer of f as $Z(f) = \{g \in Homeo_+(S^1) : [f,g] = Id\}$. We finish this section with the following proposition describing centralizers of circle homeomorphisms.

Proposition 2.4.6. Let $f \in Homeo_+(\mathbb{R})$.

- If $\rho(f) = p/q$ then $Per(f) = \{x \in S^1 : f^q(x) = x\}$ is g-invariant for every $g \in Z(f)$.
- If $\rho(f) \notin \mathbb{Q}$ then every $g \in Z(f)$ preserves the unique minimal set for f.

Moreover, in the case that f is conjugated to an irrational rotation, $Z(f)$ is an abelian subgroup conjugated to the group of rigid rotations.

Proof. Suppose $\rho(f) = p/q \in \mathbb{Q}$. If $g \in Z(f)$ and $x \in Per(f)$, since $f^q \circ g = g \circ f^q$ we have that $g(x) \in Per(f)$ as desired.

If $\rho(f) \notin \mathbb{Q}$ and $g \in Z(f)$, note that g maps minimal sets of f into minimal sets of f. Since in the case of irrational rotation number there exists a unique minimal set, it must be preserved by any $g \in Z(f)$.

Note that if $h \in Homeo_+(\mathbb{R})$ then $Z(hfh^{-1}) = hZ(f)h^{-1}$, so to finish the proof it is enough to check that the centralizer of an irrational rotation coincides with the group of rigid rotations. Take an irrational rotation R_α and $g \in Z(R_\alpha)$, since g commutes with R_α, g commutes with every homeomorphism in the set $\overline{\{R_\alpha^n : n \in \mathbb{Z}\}}$ which coincides with the group of rigid rotations of the circle. Finally, $g(\beta) = g(R_\beta(0)) = R_\beta(g(0)) = R_{g(0)}(\beta)$ for every $\beta \in \mathbb{R} \bmod \mathbb{Z}$, therefore g is a rigid rotation.

□

Chapter 3

Flexibility and group orderings

3.1 Reduction to main lemmas

Let's fix some notation. If Γ is a group and $S \subseteq \Gamma$, we shall denote by $\langle S \rangle$ the subgroup generated by S. The set of elements of $\langle S \rangle$ that can be expressed as multiplications of at most n elements in $S \cup S^{-1}$ is denoted by $B_n(\langle S \rangle)$. We will work with $\Gamma = Homeo_+(\mathbb{R})$. In this case let $Fix(\langle S \rangle)$ be the set of global fixed points of the subgroup, which is the same as the common fixed points of the elements of S. If G is a group and $\rho \in Rep(G, Homeo_+(\mathbb{R}))$ then $Fix(\rho) = Fix(\rho(G))$.

We will deduce the theorems announced in §1.1 from three technical lemmas involving the level sets of the *word map*. The proof of these lemmas is postponed to §3.7.1 and §3.7.2.

Let $\mathbb{F}_n$ be the free group with basis $\{x_1, ..., x_n\}$. The word map associated to $w \in \mathbb{F}_n$ sends each $\rho \in Rep(\mathbb{F}_n, Homeo_+(\mathbb{R}))$ to $\rho(w) \in Homeo_+(\mathbb{R})$. Our first lemma can be seen as a weak form of semi-continuity of the level sets of the word map for a general $w \in \mathbb{F}_n$ under the mild dynamical assumption that $\rho(w)$ does not fix a neighborhood of ∞. Precisely, in §3.7.1 we show

Lemma 3.1.1. *Let $w \in \mathbb{F}_n$ be a cyclically reduced word, $\rho \in Rep(\mathbb{F}_n, Homeo_+(\mathbb{R}))$ $(n \geq 2)$ and $p \in \mathbb{R}$. Then, there exists $q_0 \in \mathbb{R}$ such that for every $q > q_0$ with $\rho(w)(q) \neq q$, and every $h \in Homeo_+(\mathbb{R})$ that satisfies*

- $h_{|(-\infty,q)} = \rho(w)_{|(-\infty,q)}$
- *$Fix(h) \cap (q, +\infty)$ consists of at most one point, and in case h fixes a point in (q, ∞), then (the graph of) h transverses the diagonal at that point,*

there exists $\rho^* \in Rep(\mathbb{F}_n, Homeo_+(\mathbb{R}))$ such that

- $\rho^*(x_i)_{|(-\infty,p)} = \rho(x_i)_{|(-\infty,p)}$
- $\rho^*(w) = h$
- *$Fix(\rho^*) \subseteq Fix(\rho)$.*

The next lemmas, that will be proved in §3.7.2, are for $w = [x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1} \in \mathbb{F}_2$. In this case, the level sets of the word map correspond to

$$\mathcal{V}_h = \{(f, g) \in Homeo_+(\mathbb{R}) : [f, g] = h\}.$$

In this case we obtain a stronger version of Lemma 3.1.1, as we do not require any (dynamical) condition on $[f, g]$.

Lemma 3.1.2. *Let f, g be homeomorphisms, let $h = [f, g]$ and let K be a proper closed interval. Then, for all h' coinciding with h over the convex closure of $K \cup f(K)$, there is $(f', g') \in \mathcal{V}_{h'}$ such that*

1. *(f', g') coincides with (f, g) over K*
2. *$Fix(\langle f', g' \rangle)$ is contained in $Fix(\langle f, g \rangle)$. Moreover, $Fix(\langle f', g' \rangle)$ is contained in $K \cup f(K)$.*

Our final lemma says it is possible to perturb a representation inside a fixed $\mathcal{V}_h$, changing its semi-conjugacy class, provided it satisfies the following condition:

If K is a proper closed interval of the line, we say that a pair of homeomorphisms (f, g) satisfies condition $(*_K)$ if the following holds:

$$(*_K) \qquad \begin{array}{c}\textit{There is a point } p_{(f,g,K)}, \textit{ not fixed by } [f,g], \\ \textit{that is greather than max } \{u(K) : u \in B_2(\langle f, g \rangle)\}.\end{array}$$

Lemma 3.1.3. *Let $K = (-\infty, k]$ be a closed proper interval of $\mathbb{R}$, and let (f, g) be a pair satisfying $(*_K)$. Let $h = [f, g]$. Then we can choose (f_1, g_1) and (f_2, g_2) in $\mathcal{V}_h$ agreeing with (f, g) over K, such that*

1. *$Fix(\langle f_i, g_i \rangle)$ is contained in $Fix(\langle f, g \rangle)$, for $i = 1, 2$*
2. *$g_1(x) > x$ and $g_2(x) < x$ for x large enough.*

Remark 3.1.4. Observe that condition $(*_K)$ is very natural. Indeed, it is satisfied by any action having no global fixed points and a free orbit. This is always the case for dynamical realizations of left orders on countable groups.

Dependency structure of results: Theorem 1.1.1 is derived from Lemma 3.1.1. We use Lemmas 3.1.2 and 3.1.3 to deduce two auxiliary lemmas (3.4.5 and 3.4.6) that yield Theorems 1.1.3, 1.1.4 and 1.1.6. Theorem 3.3.1 is deduced from Lemma 3.1.3.

3.2 Flexibility in amalgamated free products

In this section we show Theorem 1.1.1.

Let $G = \mathbb{F}_n *_{w_1=w_2} \mathbb{F}_m = \langle x_1, ..., x_{n+m} | w_1(x_1, ..., x_n) = w_2(x_{n+1}, ..., x_{n+m}) \rangle$, $n \geq 2$, $m \geq 1$. We can assume w_1 and w_2 are cyclically reduced, since conjugate words yield isomorphic amalgamated products. Take $\rho : G \to Homeo_+(\mathbb{R})$ a representation without global fixed points and $p \in \mathbb{R}$. We will construct ρ' another representation with no global fixed points so that $\rho'(x_i)$ and $\rho(x_i)$ coincide over $(-\infty, p]$ for each $i = 1, ..., n+m$, and such that $\rho'(w_1)$ and $\rho(w_1)$ are not semi-conjugated. Therefore we will get ρ' a perturbation of ρ not semi-conjugated to it.

Given $\rho \in Rep(G, Homeo_+(\mathbb{R}))$, define $\rho_1 \in Rep(\mathbb{F}_n, Homeo_+(\mathbb{R}))$ and $\rho_2 \in Rep(\mathbb{F}_m, Homeo_+(\mathbb{R}))$ as the restrictions of ρ to the first and second factors of the amalgam decomposition of G.

Case I: $m \geq 2$.

Take $q_0 > q$ as the maximum of the q_0 given by Lemma 3.1.1 for the representations ρ_1 and ρ_2, and the point p. We will distinguish two cases.

Subcase Ia: $Fix(\rho(w_1))$ does not contain a neighbourhood of $+\infty$.

Consider $h \in Homeo_+(\mathbb{R})$ and $q > q_0$ such that:

- $\rho(w_1)(q) \neq q$.
- h coincides with $\rho_1(w_1)$ on $(-\infty, q]$.
- $Fix(h) \cap (q, +\infty)$ consists of at most one point, and in case h fixes some point in (q, ∞), then h transverses the diagonal at that point. We also impose that

$$\begin{cases} \text{If } \rho(w_1)(x) > x \text{ for } x \text{ big enough} & h(x) < x \text{ for } x \text{ big enough.} \\ \text{If } \rho(w_1)(x) < x \text{ for } x \text{ big enough} & h(x) > x \text{ for } x \text{ big enough.} \\ \text{If } Fix(\rho(w_1)) \text{ accumulates at } +\infty & \text{no further condition on } h. \end{cases}$$

Observe that it is possible to choose such q since $\rho(w_1)$ has arbitrarily big points that are not fixed.

Since $n, m \geq 2$ we can apply Lemma 3.1.1 and obtain ρ_1^* and ρ_2^* such that

- $\rho_1^*(x_i)$ coincide with $\rho_1(x_i)$ over $(-\infty, p]$ for $i = 1, ..., n$
- $\rho_2^*(x_i)$ coincide with $\rho_2(x_i)$ over $(-\infty, p]$ for $i = n+1, ..., n+m$
- $\rho_1^*(w_1(x_1, ..., x_n)) = \rho_2^*(w_2(x_{n+1}, ..., x_{n+m})) = h$
- $Fix(\rho_1^*) \subseteq Fix(\rho_1)$ and $Fix(\rho_2^*) \subseteq Fix(\rho_2)$

Define $\rho' \in Rep(G, Homeo_+(\mathbb{R}))$ as $\rho'(x_i) = \rho_1^*(x_i)$ for $i = 1, ..., m$ and $\rho'(x_i) = \rho_2^*(x_i)$ for $i = n+1, ..., n+m$. Then ρ' satisfies the thesis of Theorem 1.1.1. □

Subcase Ib: $Fix(\rho(w_1))$ contains a neighbourhood of $+\infty$.

Assume first that either ρ_1 or ρ_2 has global fixed points accumulating on $+\infty$. Let's say ρ_1 does. Take $q \in Fix(\rho_1)$ with $q > p$ and $\rho_1(w_1)(x) = x$ for $x > q$. We can define ρ_1' that agrees with ρ_1 over $(-\infty, q]$, and on (q, ∞) we put an action without global fixed points but such that w_1 acts trivially. This can be done, for instance, by first sending $\mathbb{F}_n$ to an infinite cyclic homomorphic image where w_1 becomes trivial. In this way, the resulting action ρ_1' is certainly not semi-conjugate to the initial ρ_1.

If on the other hand both ρ_1 and ρ_2 have no global fixed points on a neighbourhood of $+\infty$, we use the Alexander trick (see Remark 2.1.5) on one factor. Concretely, Take

$q > p + 1$ such that $\rho_1(w_1)(x) = x$ for $x > q - 1$ and $Fix(\rho_2) \cap (q - 1, +\infty) = \emptyset$. Consider $\phi : (-\infty, q) \to \mathbb{R}$ and orientation preserving homeomorphism that restricts to the identity over $(-\infty, q - 1]$. Define ρ_1' as $\phi^{-1} \circ \rho_1 \circ \phi$ on $(-\infty, q]$ and as the trivial action on $[q, +\infty)$.

In both cases, we have $\rho_1'(w_1) = \rho_1(w_1)$, and thus can define $\rho' \in Rep(G, Homeo_+(\mathbb{R}))$ by exchanging ρ_1 for ρ_1' (leaving ρ_2 as it is). This gives a representation that is not semi-conjugated to ρ, since we changed the behaviour near $+\infty$ of the global fixed points of the first factor. By construction we have $Fix(\rho') = \emptyset$ in both cases. □

Case II: $m = 1$.

Now we have $w_2 = x_{n+1}^k$ for some $k \geq 1$. Notice that $Fix(\rho) = Fix(\rho_1)$ in this case. We take $\rho_1^* \in Rep(\mathbb{F}_n, Homeo_+(\mathbb{R}))$ a representation that is not semi-conjugate to ρ_1, but satisfies that $\rho_1^*(x_i)$ and $\rho_1^*(w_1)$ agrees with $\rho(x_i)$ and $\rho(w_1)$ over $(-\infty, p]$ respectively. This can certainly be constructed by perturbing ρ very close to infinity. We can also demand that ρ_1^* has no global fixed points on $(p, +\infty)$ (see [17]). Now let f be a k-th root of $\rho_1^*(w_1)$ that agrees with $\rho(x_{n+1})$ on $(-\infty, p]$. We let $\rho' \in Rep(G, Homeo_+(\mathbb{R}))$ be defined as $\rho'(x_i) = \rho_1^*(x_i)$ for $i = 1, ..., n$ and $\rho'(x_{n+1}) = f$. The representation ρ' satisfies the thesis of Theorem 1.1.1. □

3.3 Orderings on groups with 'handle' decomposition

As announced in §1.1, in this section we show

Theorem 3.3.1. *Suppose G is a countable left orderable group admitting a decomposition of the form $H *_{h=w} \mathbb{F}_2$, where $\mathbb{F}_2 = \langle a, b\rangle$, $w = [a, b]$ and $h \neq id$. Then the space of left orders of G has no isolated points.*

Proof. Suppose $\preceq$ is a left order on a countable group G admitting a decomposition of the form $H *_{h=w} \mathbb{F}_2$ as stated. To show that $\preceq$ is non-isolated, we just need to show that its dynamical realization ρ is flexible by Proposition 2.3.1. Let ρ_1 and ρ_2 be the restrictions of ρ to the factors H and $\mathbb{F}_2$ respectively. Since ρ is a dynamical realization, the pair $(\rho(a), \rho(b))$ satisfies condition $(*_K)$ for every closed proper interval K. Given such a proper interval K, we use Lemma 3.1.3 to produce $\rho_2' \in Rep(\mathbb{F}_2, Homeo_+(\mathbb{R}))$ that is not semi-conjugated to ρ_2, and such that

- $\rho_2'([a, b]) = \rho_2([a, b])$,
- $\rho_2'(a)$ and $\rho_2'(b)$ agree with $\rho_2(a)$ and $\rho_2(b)$ over K,
- $Fix(\rho_2') \subseteq Fix(\rho_2)$.

Since $\rho_2'([a, b]) = \rho_2([a, b]) = \rho_1(h)$, we can build a representation $\rho' \in Rep(G, Homeo_+(\mathbb{R}))$ that restricts to ρ_1 on H and ρ_2' on $\mathbb{F}_2$. It has no global fixed points since $Fix(\rho_2') \subseteq Fix(\rho_2)$, so it lies in $Rep_\#(G, Homeo_+(\mathbb{R}))$. It is clear that ρ and ρ' are not semi-conjugated, and that ρ' belongs to a neighbourhood of ρ that can be taken arbitrarily small as K gets big. □

3.4 Removing fixed points

In this section we give the proof of Theorem 1.1.3. Throughout this section, M will denote either the real line or the circle. As customary, for $a, b \in S^1$, the *interval* (a, b) is the set of points $p \in S^1$ such that (a, p, b) is clockwise oriented.

Let $q \geq 2$, and Σ_q be a genus q closed (orientable) surface. Our preferred presentation for $\pi_1(\Sigma_q)$ we will be $\langle a_1, b_1, \ldots, a_q, b_q \mid [a_1, b_1] = w_1 \rangle$, where $w_1 = \Pi_{j \neq 1}[a_j, b_j]$. This is the presentation induced from the amalgam decomposition

$$\pi_1(\Sigma_q) \simeq \mathbb{F}_2 *_{[a_1,b_1]=w_1} \mathbb{F}_{2(q-1)}.$$

The following definitions about fixed points will be central to the proof.

Definition 3.4.1. Let $p \in M$ and $\phi \in Homeo_+(M)$. We will say that fixed point p of ϕ is of *hyperbolic type* if there exists a neighbourhood V of p such that $V - \{p\}$ has two connected components, and such that either ϕ^n or ϕ^{-n} shrinks V to $\{p\}$ as $n \to +\infty$.

In dynamical terms, that is to say that p is either an *attracting* or *repelling* fixed point of ϕ.

Definition 3.4.2. Let $\rho \in Rep(\pi_1(\Sigma_q), Homeo_+(M))$ and $p \in Fix(\rho)$. We say that ρ is *tame* on p if p is both an isolated fixed point of $\rho([a_1, b_1])$ and a fixed point of hyperbolic type for $\rho(b_1)$. In this case, if V is a convex neighbourhood of p with $Fix(\rho([a_1, b_1])) \cap V = Fix(\rho(b_1)) \cap V = \{p\}$, we say that ρ is tame on p over V.

The skeleton of the proof is the following: First we will show that any representation of $\pi_1(\Sigma_q)$ on $Homeo_+(M)$ can be approximated by representations whose global fixed points are isolated. Next we will approximate a representation with isolated global fixed points by one that is tame on each of them. Finally we show how to remove tame global fixed points by small perturbations.

Definition 3.4.3. Let $\mathcal{F}$ be a family of closed intervals. We will say that $\mathcal{F}$ is *locally finite* if given a compact set K, only finitely many intervals in $\mathcal{F}$ intersect K.

Lemma 3.4.4 (Isolating)**.** *Given $\rho \in Rep(\pi_1(\Sigma_q), Homeo_+(M))$, and U a neighbourhood of ρ in the compact open topology, there exists $\rho' \in U$ such that $Fix(\rho')$ consists of isolated points.*

Proof. Let $\rho \in Rep(\pi_1(\Sigma_q), Homeo_+(M))$. Given $\epsilon > 0$, consider a locally finite family $\mathcal{F}$ of closed intervals of diameter less than ϵ that satisfy

- If $I, J \in \mathcal{F}$ are different then its interiors are disjoint.
- If $x \in Fix(\rho)$ then there exists $I \in \mathcal{F}$ that contains x.
- The endpoints of every $I \in \mathcal{F}$ are global fixed points of ρ.

Now, for each $I \in \mathcal{F}$ consider $\rho_I \in Rep(\pi_1(\Sigma_q), Homeo_+(I))$ a representation without global fixed points (except for the endpoints of I) and define $\rho' \in Rep(\pi_1(\Sigma_q), Homeo_+(M))$ as $\rho'(g)(x) = \rho(g)(x)$ if $x \notin \cup_{I \in \mathcal{F}} I$, and $\rho'(g)(x) = \rho_I(g)(x)$ if $x \in I$ for some $I \in \mathcal{F}$. Note that if ϵ is sufficiently small then $\rho' \in U$.

Finally, the local-finiteness of $\mathcal{F}$ implies that $Fix(\rho')$ is a discrete set. □

Lemma 3.4.5 ((Taming). *Let $\rho \in Rep(\pi_1(\Sigma_q), Homeo_+(M))$, $p \in Fix(\rho)$ and V a neighbourhood of p in M such that $Fix(\rho) \cap V = \{p\}$. Then there exists $\rho' \in Rep(\pi_1(\Sigma_q), Homeo_+(M))$ such that*

- $\rho'(a_i)_{|V^c} = \rho(a_i)_{|V^c}$ *and* $\rho'(b_i)_{|V^c} = \rho(b_i)_{|V^c}$ *for* $i = 1, ..., q$, *where* V^c *denotes the complement of* V,
- ρ' *is tame on* p,
- $Fix(\rho') \subseteq Fix(\rho)$.

Proof. Consider (α, p) a subset of M homeomorphic to $\mathbb{R}$ where either $\alpha = -\infty$ or $\alpha \in Fix(\rho)$, and such that $Fix(\rho) \cap (\alpha, p) = \emptyset$. Take $\Phi\colon \mathbb{R} \to (\alpha, p)$ an orientation preserving homeomorphism. We will consider the representation $\theta \in Rep(\pi_1(\Sigma_q), Homeo_+(\mathbb{R}))$ defined as $\theta(g)(x) = \Phi^{-1}\rho(g)\Phi(x)$. This is equivalent to consider ρ acting on the interval (α, p).

Take $k \in \mathbb{R}$ such that $\Phi([k, +\infty)) \subseteq V$. Apply Lemma 3.1.2 to get a perturbation θ_1 such that $\theta_1(a_i)_{|(-\infty,k)} = \theta(a_i)_{|(-\infty,k)}$ and $\theta_1(b_i)_{|(-\infty,k)} = \theta(b_i)_{|(-\infty,k)}$ for $i = 1, \ldots, q$, and also that $\theta_1([a_1, b_1])(x) = x + 1$ for x big enough and $Fix(\theta_1) = \emptyset$.

Since $\theta_1([a_1, b_1])(x) = x + 1$ for x large enough, the pair $(\theta_1(a_1), \theta_1(b_1))$ satisfies condition $(*_{(-\infty,k]})$. Therefore we can apply Lemma 3.1.3 and find f_1 and g_1 perturbations of $\theta_1(a_1)$ and $\theta_1(b_1)$ supported on $(k, +\infty)$, such that $[f_1, g_1] = \theta_1([a_1, b_1])$, $Fix(\langle f_1, g_1\rangle) \subseteq Fix(\langle \theta_1(a_1), \theta_1(b_1)\rangle)$ and $g_1(x) > x$ for x large enough.

Since $[f_1, g_1] = \theta_1([a_1, b_1])$ we can define a representation $\theta' \in Rep(\pi_1(\Sigma_q), Homeo_+(\mathbb{R}))$ by $\theta'(a_1) = f_1$, $\theta'(b_1) = g_1$, $\theta'(a_i) = \theta_1(a_i)$ and $\theta'(b_i) = \theta_1(b_i)$ for $i = 2, \ldots, q$. By construction, we have that there is a neighbourhood of $+\infty$ where $\theta'(b_1) = g_1$ is increasing and $\theta'([a_1, b_1]) = \theta_1([a_1, b_1])$ has no fixed points.

Finally, since $Fix(\langle f_1, g_1\rangle) \subseteq Fix(\langle \theta_1(a_1), \theta_1(b_1)\rangle)$ we have

$$Fix(\theta') = Fix(\langle f_1, g_1\rangle) \cap Fix(\langle \theta_1(a_2), \theta_1(b_2), \ldots, \theta_1(a_q), \theta_1(b_q)\rangle) \subseteq Fix(\theta_1) = \emptyset.$$

We define $\bar{\rho}$ by $\bar{\rho}(g)(x) = \rho(g)(x)$ if $x \notin (\alpha, p)$ and $\bar{\rho}(g)(x) = \Phi\theta'(g)\Phi^{-1}(x)$ if $x \in (\alpha, p)$. Note that $\bar{\rho}(a_i)_{|V^c} = \rho(a_i)_{|V^c}$ and $\bar{\rho}(b_i)_{|V^c} = \rho(b_i)_{|V^c}$ for $i = 1, ..., q$. By this construction we have that $Fix(\bar{\rho}) \subseteq Fix(\rho)$, and also that p has a neighbourhood V' so that in the left component of $V' - \{p\}$ there are no fixed points of $\bar{\rho}([a_1, b_1])$ and $\bar{\rho}(b_1)$ is increasing.

We repeat the same procedure on the other side of p and get ρ'. We will have that p is an isolated point of $Fix(\rho'([a_1, b_1]))$. Moreover, taking the right choice when applying Lemma 3.1.3, we will have that p is of hyperbolic type for $\rho'(b_1)$ and so ρ' is tame on p. Finally, note that $Fix(\rho') \subseteq Fix(\bar{\rho}) \subseteq Fix(\rho)$. □

The following Lemma shows how to remove tame global fixed points

Lemma 3.4.6 (Removing). *Let $\rho \in Rep(\pi_1(\Sigma_q), Homeo_+(M))$ and $p \in Fix(\rho)$ such that ρ is tame on p over V. Then we can construct $\bar{\rho} \in Rep(\pi_1(\Sigma_q), Homeo_+(M))$ such that*

- $\bar{\rho}(a_i)_{|V^c} = \rho(a_i)_{|V^c}$ *and* $\bar{\rho}(b_i)_{|V^c} = \rho(b_i)_{|V^c}$ *for* $i = 1, ..., q$
- $Fix(\bar{\rho}) \cap V = \emptyset$

Proof. Since ρ is tame over V we can construct g a perturbation of $\rho(b_1)$ supported on V such that the graph of $g_{|V}$ transverses the graphs of the identity and of $\rho([a_1, b_1]^{-1})$ only once, at different points p_1 and p_2 respectively. These points are then the only fixed points of g and $\rho([a_1, b_1])g$ on the interval V. In particular $Fix(g) \cap Fix(\rho([a_1, b_1])) \cap V = \emptyset$.

Notice that a homeomorphism ψ with $\psi_{|V^c} = \rho(a_1)_{|V^c}$ and $\psi(p_1) = p_2$ is a weak conjugation from g to $\rho([a_1, b_1])g$ that is strong on each component of V^c. Since these components are separated by a fixed point of g, the arguments for Lemma 2.2.2 also work in this case. So we get $f \in Homeo_+(\mathbb{R})$ such that $f_{|V^c} = \rho(a_1)_{|V^c}$ and that conjugates g to $\rho([a_1, b_1])g$.

Finally, we define $\bar{\rho}$ as

- $\bar{\rho}(a_i) = \rho(a_i)$ and $\bar{\rho}(b_i) = \rho(b_i)$ for $i = 2, ..., q$
- $\bar{\rho}(a_1) = f$ and $\bar{\rho}(b_1) = g$

□

Now we are in position to finish the

Proof of Theorem 1.1.3. Let $\rho \in Rep(\pi_1(\Sigma_q), Homeo_+(M))$ and U a neighbourhood of ρ in the compact open topology. First we apply Lemma 3.4.4 to find $\rho_1 \in U$ with isolated global fixed points.

For each $p \in Fix(\rho_1)$ take a neighbourhood V_p so that p is the only global fixed point of ρ_1 on it. We can also assume they are pairwise disjoint. On each V_p we apply the perturbation of Lemma 3.4.5 followed by that of Lemma 3.4.6. We can do this recursively (for some order of $Fix(\rho_1)$) and take the limit. This will be the representation ρ' in the statement of Theorem 1.1.3. It is clear that $Fix(\rho') = \emptyset$. Finally, notice that by taking the V_p small enough we can guarantee that $\rho' \in U$. □

3.5 A dense orbit in $Rep_\#(\pi_1(\Sigma), Homeo_+(\mathbb{R}))$

In this section we give the proof of Theorem 1.1.4.

Fix an orientation preserving homeomorphism $\Phi : \mathbb{R} \to (0, 1)$, and for each $n \in \mathbb{Z}$ let $\Phi_n(x) = \Phi(x) + n$.

We will write $Rep_\# := Rep_\#(\pi_1(\Sigma_q), Homeo_+(\mathbb{R}))$. Notice that $Rep_\#$ is separable (by Remark 2.1.4), so we can consider $Q \subseteq Rep_\#$ a dense countable subset. Let $\{\rho_n : n \in \mathbb{Z}\}$ be a sequence in Q that repeats every element infinitely often. We will define $\theta_0 \in Rep(\pi_1(\Sigma_q), Homeo_+(\mathbb{R}))$ as follows:

- Each $n \in \mathbb{Z}$ is a global fixed point.
- On the interval $(n, n+1)$, define $\theta_0(g) = \Phi_n \rho_n(g) \Phi_n^{-1}$ for all $g \in \pi_1(\Sigma_q)$.

Since $\rho_n \in Rep_\#$, we see that $Fix(\theta_0) = \mathbb{Z}$.

For each $n \in \mathbb{Z}$ we consider the interval $V_n = (n - 2^{-|n|-1}, n + 2^{-|n|-1})$. Then V_n is a convex neighbourhood of n with $diam V_n = 2^{-|n|}$, disjoint with any other V_m. As in the proof of Theorem 1.1.3, we apply Lemma 3.4.5 followed by Lemma 3.4.6 on each V_n, obtaining a representation θ without global fixed points.

We claim that the conjugacy class of θ is dense in $Rep_{\#}$. To show this, it is enough to prove that for every $\rho \in Q$ and every $m > 0$ there is a conjugate $\bar{\theta}$ of θ so that $\bar{\theta}(a_i)_{|[-m,m]} = \rho(a_i)_{|[-m,m]}$ and $\bar{\theta}(b_i)_{|[-m,m]} = \rho(b_i)_{|[-m,m]}$ for $i = 1, \ldots, q$.

Take $n \in \mathbb{Z}$ so that $\rho_n = \rho$ and $\Phi_n([-m, m])$ is disjoint with $V_n \cup V_{n+1}$. This is possible since the sequence $\{\rho_n : n \in \mathbb{Z}\}$ repeats ρ infinitely many times, and $diam V_n$ goes to 0 as $|n| \to +\infty$. Take $\psi \in Homeo_+(\mathbb{R})$ that agrees with Φ_n on $[-m, m]$. Then $\psi^{-1}\theta(a_i)\psi_{|[-m,m]} = \rho(a_i)_{|[-m,m]}$ and $\psi^{-1}\theta(b_i)\psi_{|[-m,m]} = \rho(b_i)_{|[-m,m]}$ for $i = 1, \ldots, q$.

Finally, applying Theorem 1.1.3 we get that $Rep_{\#}$ is dense in Rep and therefore the conjugacy class of θ is dense in $Rep(\pi_1(\Sigma), Homeo_+(\mathbb{R}))$. □

3.6 A dense orbit in $\mathcal{LO}(\pi_1(\Sigma))$

In this section we give the proof of Theorem 1.1.6. The construction follows closely the one for Theorem 1.1.4.

Take Q a countable dense subset of $\mathcal{LO}(\pi_1(\Sigma))$ (this certainly exists since the space of left orders of countable groups is compact and metrizable, therefore separable, see §2.3). Take also $\{\rho_n : n \in \mathbb{Z} - \{0\}\}$ a sequence of dynamical realizations of the orders in Q, repeating each representation infinitely often. Let ρ_0 be a representation of $\pi_1(\Sigma)$ by translations with dense orbits (e.g. translations by lengths that are linearly independent over $\mathbb{Q}$).

Now let $\Phi : \mathbb{R} \to (-1, 1)$ be an orientation preserving homeomorphism with $\Phi(0) = 0$, and for each $n \in \mathbb{Z}$ let $\Phi_n(x) = \Phi(x) + 2n$. We define the representation θ_0 as follows:

- Each odd integer is a global fixed point.
- On the interval $(2n - 1, 2n + 1)$, define $\theta_0(g) = \Phi_n \rho_n(g) \Phi_n^{-1}$ for all $g \in \pi_1(\Sigma_q)$.

For each odd integer n we take a convex neighbourhood V_n with $diam V_n < 2^{-|n|}$, and we use Lemmas 3.4.5 and 3.4.6 as in the proof of Theorem 1.1.3 to remove the global fixed points with a perturbation supported on the V_n. Let θ be the representation thus obtained.

We check next that if V_1 and V_{-1} are small enough, then the orbit of 0 under θ is dense.

Let S be a generating set of $\pi_1(\Sigma)$ (say, the one given at the beginning of §3.4). For $\alpha > 0$ and $\rho \in Rep(\pi_1(\Sigma), Homeo_+(\mathbb{R}))$ we consider the *local orbit* of 0 on $[-\alpha, \alpha]$, that is the set $L_\alpha(\rho)$ of points of the form $\rho(g)(0)$ where $g = s_1 \cdots s_k$ with $s_j \in S$ and so that $\rho(s_i \cdots s_k)(0) \in [-\alpha, \alpha]$ for all $i = 1, \ldots, k$. Notice that a perturbation of ρ that only changes the $\rho(s_i)$ outside of $[-\alpha, \alpha]$, does not change the local orbit L_α.

By our choice of ρ_0, we have that for α big enough the closure of $L_\alpha(\rho_0)$ contains a neighbourhood of 0. So by taking V_1 and V_{-1} small enough (disjoint from $\Phi([-\alpha, \alpha])$) we get that there is a neighbourhood of 0 contained in the closure of some local orbit of 0 under θ. Therefore the closure of the orbit of 0 under θ is both open and closed, so this orbit is dense.

Define $\prec$ by

$$g_1 \preceq g_2 \text{ if and only if } \theta(g_1)(0) \leq \theta(g_2)(0)$$

As in the proof of Proposition 2.3.1, $\prec$ is really a partial left order if 0 has non trivial stabilizer, but it can be extended to a total left order. We will show that any such extension has a dense orbit under conjugation.

Let $\tilde{\prec}$ be any element of $\mathcal{LO}(\pi_1(\Sigma))$. Take a finite subset F of $\pi_1(\Sigma)$. Enlarging it if necessary, we can assume F is *closed under prefix*, i.e. whenever $f = s_1 \cdots s_k \in F$ we have that $s_i \cdots s_k \in F$ for all $i = 1, \ldots, k$. Take $\prec' \in Q$ an element that agrees with $\tilde{\prec}$ on F, and let ρ' be its dynamical realization. Consider $\alpha > 0$ so that $\rho'(f)(0) \in [-\alpha, \alpha]$ for $f \in F$. Notice that $\rho'(f)(0) \in L_\alpha(\rho')$ for $f \in F$, since F is closed under prefix. There are infinitely many repetitions of ρ' in the sequence $\{\rho_n : n \in \mathbb{Z} - \{0\}\}$, so we can take one with n big enough so that $\Phi_n([-\alpha, \alpha])$ is disjoint with $V_{2n-1} \cup V_{2n+1}$. Applying the conjugation by Φ_n we get that

$$f_1 \prec' f_2 \text{ if and only if } \theta(f_1)(2n) < \theta(f_2)(2n) \quad \text{for } f_1, f_2 \in F$$

noticing that $2n = \Phi_n(0)$, and that ρ' and ρ_n have the same local orbit of 0 on $[-\alpha, \alpha]$.

We also get that the map $f \to \theta(f)(2n)$ for $f \in F$ is injective, and since F is finite there is a neighbourhood U of $2n$ so that the sets $\theta(f)(U)$ are disjoint. Thus, for any $p \in U$ and $f_1, f_2 \in F$, we have that $f_1 \prec' f_2$ if and only if $\theta(f_1)(p) < \theta(f_2)(p)$. Recall that the orbit of 0 under θ is dense, so we can take $g \in \pi_1(\Sigma)$ with $\theta(g)^{-1}(0) \in U$. It follows that $\prec_g$ agrees with $\prec'$, and therefore with $\tilde{\prec}$, on F as desired. □

3.7 Proof of Technical Lemmas

3.7.1 Proof of Lemma 3.1.1

Let $w \in \mathbb{F}_n = \langle x_1, ..., x_n \rangle$ be a cyclically reduced word. Write $w = a_m...a_1$ with $a_i \in \{x_1^{\pm 1}, ..., x_n^{\pm 1}\}$. We define $w_0 = e$ and $w_j = a_j...a_1$ for $0 < j \leq m$. If $\rho \in Rep(\mathbb{F}_n, Homeo_+(\mathbb{R}))$ and $x \in \mathbb{R}$ we will be interested in the sequence $S(\rho, w, x) = (\rho(w_0)(x), ..., \rho(w_m)(x))$. For each generator x_i we will look at the minimum point from which we can perturb $\rho(x_i)$ without changing the sequence $S(\rho, w, x)$. With this in mind, for a general sequence $S = (s_0, ..., s_m)$ and $i \in \{1, ..., n\}$ we define $D_w(S, i) = \{s_j : a_{j+1} = x_i \textit{ or } a_j = x_i^{-1}\}$ and $d_w(S, i) = max D_w(S, i)$. (Figure 3.1 provides an example).

Recall p and ρ from the statement of Lemma 3.1.1, and let $f_i = \rho(x_i)$ for $i = 1, ..., n$. We take q_0 such that $max\{\rho(u)(p) : u \in B_1(\langle x_1, ..., x_n \rangle)\}$ is less than every point in $S(\rho, w, q_0)$ and $S(\rho, w^{-1}, q_0)$.

Take $q > q_0$ and h an homeomorphism as in the statement of Lemma 3.1.1. Let $d_i = d_w(S(\rho, w, q), i)$ (See figure 3.1). We will first define $\overline{\rho} \in Rep(\mathbb{F}_n, Homeo_+(\mathbb{R}))$ such that $\overline{\rho}(w)$ is conjugated to h and $\overline{\rho}(x_i) = g_i$ agrees with f_i over $(-\infty, d_i]$ for $i = 1, ..., n$. We will do this by defining each g_i on a discrete subset of $(d_i, +\infty)$ and then extend by interpolation.

Lemma 3.7.1. There exists $\overline{\rho} \in Rep(\mathbb{F}_n, Homeo_+(\mathbb{R}))$ such that:

- $\overline{\rho}(x_i)$ agrees with $\rho(x_i)$ over $(-\infty, d_i]$ for $i = 1, ..., n$.
- $\overline{\rho}(w)$ is weakly conjugated to h, by a map that coincides with the identity on $(-\infty, q]$. Moreover, this weak conjugation is strong on $(-\infty, q]$.
- $Fix(\overline{\rho}) \subseteq Fix(\rho)$.

Proof. **Case Ia:** $h(q) > q$ and $Fix(h) \cap (q, +\infty) = \emptyset$.

Take r_1 so that $q < r_1 < h(q)$.

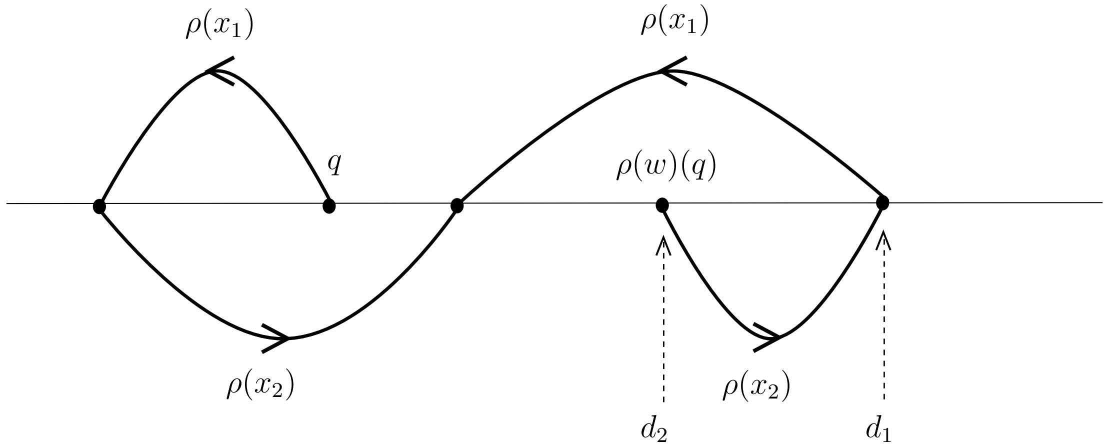

Figure 3.1: The picture shows a possible example of a sequence $S(\rho, w, q)$ for $w = x_2^{-1}x_1^{-1}x_2x_1$. We denote $d_i = d_w(S(\rho, w, q), i)$.

Construct $S_1 = (s_{1,0}, \dots, s_{1,m})$ as follows: Put $s_{1,0} = r_1$. Inductively, set $s_{1,j} = \rho(w_j)(r_1)$ for as long as $s_{1,j-1} \in (-\infty, d_i]$ if $a_j = x_i$ or $s_{1,j-1} \in (-\infty, f_i(d_i)]$ if $a_j = x_i^{-1}$. We get to $s_{1,k}$, the last element we can define by that process. We must have $k \leq m - n$: otherwise $d_w(S(\rho, w, r_1), i) \leq d_i$ for some i, which is not possible since $d_w(S(\rho, w, x), i)$ is increasing on x (since it is a maximum of increasing homeomorphisms). Choose $s_{1,k+1} > max\{S(\rho, w, q)\}$, and then set $s_{1,j+1} = s_{1,j} + 1$ for every $j \geq k + 1$. (See figure 3.7.1).

Notice that the sequence S_1 defines maps g_i on the sets $D_w(S_1, i)$, by taking $g_i(s_{1,j-1}) = s_{1,j}$ if $a_j = x_i$ and $g_i(s_{1,j}) = s_{1,j-1}$ if $a_j = x_i^{-1}$. Define each g_i on $(-\infty, d_i] \cup D_w(S_1, i)$ so that it agrees with f_i on $(-\infty, d_i]$.

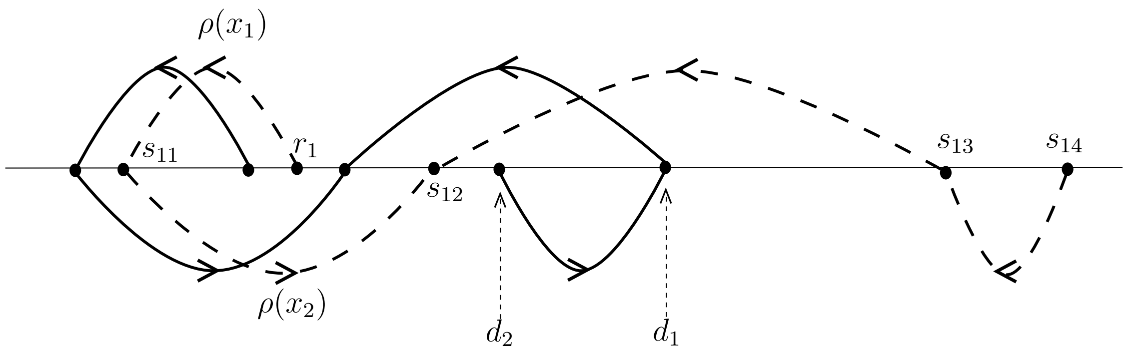

Figure 3.2: Here we draw the construction of the sequence S_1 for the example in Figure 3.1.

Claim 3.7.2. The maps $g_i : (-\infty, d_i] \cup D_w(S_1, i) \to \mathbb{R}$ are increasing.

Proof of the Claim It is clear that g_i is increasing on $(-\infty, d_i]$. Notice next that $D_w(S_1, i) \setminus (-\infty, d_i] \subseteq \{s_{1,k}, ..., s_{1,m}\}$. This makes it easy to check that g_i is increasing on that set.

It only remains to show that if $s_{1,l} \in D_w(S_1, i)$ with $l \geq k$ then $g_i(s_{1,l}) > g_i(d_i)$. We distinguish two cases:

- Case A: $a_{l+1} = x_i$.

  By construction of S_1 we have $g_i(s_{1,l}) = s_{1,l+1} \geq s_{1,k+1} > max\{S(\rho, w, q)\} \geq f_i(d_i) = g_i(d_i)$ as desired.

- Case B: $a_l = x_i^{-1}$.

  If $l > k+1$, then $g_i(s_{1,l}) = s_{1,l-1} \geq s_{1,k+1} > max\{S(\rho, w, q)\} \geq f_i(d_i) = g_i(d_i)$. If $l = k+1$, we notice that $s_{1,k} \notin (-\infty, f_i(d_i)]$: Otherwise, following our construction we would have to set $s_{1,k+1}$ as $f_i^{-1}(s_{1,k})$, but that contradicts the definition of k. Therefore $g_i(s_{1,k+1}) = s_{1,k} > f_i(d_i) = g_i(d_i)$.

  Finally, notice that in this case $l \neq k$: If we suppose that $s_{1,k} \notin (-\infty, d_i]$, then $f_i(s_{1,k}) = s_{1,k-1} \notin (-\infty, f_i(d_i)]$ which also contradict the definition of k.

This concludes the proof of the claim. $\Diamond$

Next we will continue extending the g_i in a similar fashion.

Take r_2 with $s_{1,m-1} < r_2 < s_{1,m}$, and define $S_2 = (s_{2,0}, \ldots, s_{2,m})$ by $s_{2,0} = r_2$ and $s_{2,j+1} = s_{2,j} + 1$ for $j = 0, \ldots, m-1$. As in the previous case, this extends g_i to $D_w(S_2, i)$. To check that this extension is increasing observe that w is cyclically reduced, so $a_1 \neq a_m^{-1}$. This ensures that there is no problem at the first step, taking $s_{2,0}$ to $s_{2,1}$.

Inductively, construct $S_l = (s_{l,0}, \ldots, s_{l,m})$ from S_{l-1} as we did for S_2 from S_1. This defines each g_i on $(-\infty, d_i] \cup \bigcup_{l>0} D_w(S_l, i)$ as an increasing and proper map, that agrees with f_i on $(-\infty, d_i]$. Thus each g_i can be extended to $\mathbb{R}$ as an homeomorphism. These extensions can be chosen so that two different g_{i_1} and g_{i_2} do not have any common fixed points after $min\{d_{i_1}, d_{i_2}\}$. Therefore $\bigcap_i Fix(g_i) \subset \bigcap_i Fix(f_i) \cap (-\infty, d]$ for $d = min\{d_1, \ldots, d_n\}$.

We define $\overline{\rho}$ by setting $\overline{\rho}(x_i) = g_i$, as discussed before. Since $\overline{\rho}(w)(s_{l,0}) = s_{l,m} > s_{l,0}$ we get that $\overline{\rho}(w)$ has no fixed points in $[s_{l,0}, s_{l,m}]$ for every $l > 0$. Recalling that $\overline{\rho}(w)(q) = h(q) > q$, we get that $\overline{\rho}(w)$ has no fixed points in $[q, +\infty) = [q, h(q)] \cup \bigcup_{l>0}[s_{l,0}, s_{l,m}]$.

Notice that $\overline{\rho}(w)$ and h are weakly-conjugated by the identity, which is strong on $(-\infty, q]$. This finishes the proof in the case Ia.

Case Ib: $h(q) < q$ and $Fix(h) \cap (q, +\infty) = \emptyset$.

Note that $S(\rho, w, q) = S(\rho, w^{-1}, h(q))$. Therefore, we can exchange w, h and q by w^{-1}, h^{-1} and $h(q)$, and repeat the construction in Case Ia.

Case IIa: $h(q) > q$ and $Fix(h) \cap (q, +\infty)$ consists of a single point where h transverses the diagonal.

Repeat the process in case Ia to construct $\rho' \in Rep(\mathbb{F}_n, Homeo_+(\mathbb{R}))$ so that $\rho'(x_i)$ agrees with $\rho(x_i)$ on $(-\infty, d_i]$ for each i, and that $\rho'(w)(x) > x$ for every $x > q$. Moreover, we can ask each $\rho'(x_i)$ to be piecewise linear on $(d_i, +\infty)$. We will define $\overline{\rho}$ as a perturbation of ρ', making it so that $\overline{\rho}(w)$ is conjugated to h.

Take $z > q$ so that every point in $S(\rho', w, z)$ is bigger than $max\{d_i\}$. Let $d_i' = d_w(S(\rho', w, z), i)$. We take $z' > max\{d_i'\} + m$ and define $S' = (z', z'-1, ..., z'-m)$. Define ρ'' so that $\rho''(x_i)$ is a piecewise linear interpolation that extends $\rho'(x_i)_{|(-\infty, d_i']}$ and the map defined on $D_w(S', i)$ as in Case Ia.

Let $\overline{h} = \rho''(w)$. Since $\overline{h}(z) > z$ and $\overline{h}(z') < z'$ we see that $\overline{h}$ must have a fixed point in (z, z'). Let $y = min\{Fix(\overline{h}) \cap (z, z')\}$. For our argument we will need y to be a transverse fixed point of $\overline{h}$. If it is not, we will perform an additional perturbation that we turn to describe now.

Assume y is not transverse. Then y must be a break point of $\overline{h}$, and since it is the first fixed point in (z, z') its left derivative is smaller than 1. Let $S'' = S(\rho'', w, y) = (y_0, y_1, ..., y_m)$ and $\epsilon > 0$ so that the intervals $(y_j - \epsilon, y_j + \epsilon)$ are either disjoint or identical. For each y_j in $D_w(S'', i)$ we make a small perturbation of $\rho''(x_i)$ supported on $[y_j, y_j + \epsilon)$ so that y_j is no longer a break point (if it was one). This perturbation remains piecewise linear, introducing two new break points on $(y_j, y_j + \epsilon)$. The new $\overline{h}$ has a transverse fixed point at y, since now y is not a break point, and $\overline{h}'(y) < 1$ as the left derivative at y has not changed.

We must ensure that there are no new fixed points on (z, y). If ϵ is small enough, $\overline{h}$ does not change on $(y - \delta, y]$ for some $\delta > 0$. This is because for $t \in (y - \delta, y]$ we have that $S(\rho'', w, t)$ is disjoint from the supports of the perturbations. On the other hand, the perturbations are C^0 and $\overline{h}$ is away from the diagonal on $[z, y - \delta]$, thus they can be made small enough not to introduce new fixed points on $[z, y - \delta]$.

Having produced the representation ρ'' so that $\overline{h} = \rho''(w)$ has a transverse fixed point at y, we take $y' > y$ so that y is the only fixed point of $\overline{h}$ on (z, y'). Next we will proceed as in Case I, redefining each $\rho''(x_i)$ so that $\rho''(w)$ is unchanged on $(-\infty, y']$ and $\rho''(w)(x) < x$ for every $x > y'$. More precisely, we are in the situation of Case Ib and each $\rho''(x_i)$ gets redefined from the point $d_w(S(\rho'', w^{-1}, \overline{h}(y')), i)$.

This new ρ'' works as $\overline{\rho}$ in the statement of the lemma (3.7.1), as we will check now. It is clear from the construction that h and $\overline{\rho}(w)$ are weakly-conjugated by a map that coincides with Id on $(-\infty, q]$, and that each $\overline{\rho}(x_i)$ coincides with $\rho(x_i)$ on $(-\infty, d_i]$. We check that $Fix(\overline{\rho}) \subseteq Fix(\rho)$: By Case Ia we have $Fix(\rho') \subseteq Fix(\rho)$. The piecewise linear interpolation for ρ'' can be performed without introducing any new global fixed points, and so can the final perturbation following Case Ib.

Case IIb: $h(q) < q$ and $Fix(h) \cap (q, +\infty)$ consists of a single point where h transverses the diagonal.

This case is analogous to IIa, also exchanging w, h and q by w^{-1}, h^{-1} and $h(q)$. □

Let $\bar{\rho}$ be the representation given by Lemma 3.7.1. By lemma 2.2.2 there exists $\varphi \in Homeo_+(\mathbb{R})$ such that $\varphi^{-1} \circ h \circ \varphi = \overline{\rho}(w)$ and φ equals Id on $(-\infty, q]$. Define ρ^* as $\varphi \circ \overline{\rho} \circ \varphi^{-1}$. Now $\rho^*(w) = h$ and by construction of q_0 we obtain that $\rho^*(x_i)$ and $\rho(x_i)$ coincide over $(-\infty, p)$ as desired.

3.7.2 On Commutator varieties

In this section we prove Lemmas 3.1.2 and 3.1.3 from §3.1. Our arguments are based on the analysis of the commutator variety

$$\mathcal{V}_h := \{(f, g) \in Homeo_+(\mathbb{R}) \times Homeo_+(\mathbb{R}) \mid [f, g] = h\},$$

of a given homeomorphism of the line h. Though very simple, the key observation (and main difference with the strategy for proving Lemma 3.1.1) is that the equation $[f, g] = h$ can be rewritten as the equation

$$fgf^{-1} = hg.$$

This rewriting provides us the insight that "f is conjugating g to hg". The idea will be to modify g outside a large compact set, in a way that keeps g and hg conjugated by an element close to f.

In order to control conjugacy class of hg we observe that (see §2.2 for definitions)

- $Inc(hg) = \{x \in \mathbb{R} : g(x) > h^{-1}(x)\}$
- $Decr(hg) = \{x \in \mathbb{R} : g(x) < h^{-1}(x)\}$
- $Fix(hg) = \{x \in \mathbb{R} : g(x) = h^{-1}(x)\}$

Because of this, in the same way as the conjugacy class of g is determined by the combinatorics of its graph's crossings against the diagonal, we think the conjugacy class of hg as the combinatorics of the crossings of the graph of g against the graph of h^{-1}.

It will be handy to have

Definition 3.7.3. For ϕ_1, ϕ_2 (partial) homeomorphisms of the line, we define the combinatorics of (ϕ_1, ϕ_2) as $\mathcal{C}(\phi_1, \phi_2)(x) := sign(\phi_1(x) - \phi_2(x)) \in \{1, -1, 0\}$.

With this language, $Inc(\phi) = \mathcal{C}(\phi, id)^{-1}(1)$, and ψ is a weak-conjugation from ϕ_1 to ϕ_2 if and only if $\mathcal{C}(\phi_1, id) = \mathcal{C}(\phi_2, id) \circ \psi$. Observe that if $\psi \in Homeo_+(\mathbb{R})$ then $\mathcal{C}(\psi\phi_1, \psi\phi_2) = \mathcal{C}(\phi_1, \phi_2)$. This implies that f is a weak conjugation from g to hg if and only if $\mathcal{C}(g, id) = \mathcal{C}(g, h^{-1}) \circ f$.

3.7.3 Proof of Lemma 3.1.3

Let $K = (-\infty, k]$ be a closed proper interval of the line, and suppose (f, g) is a pair satisfying condition $(*_K)$. We will denote $p_{(f,g,K)}$ from condition $(*_K)$ simply by p.

We begin by proving the lemma in a simple case, that will play an important role in the general proof.

Toy case: Assume that $f(k) > k$ and that $Fix(g) \cap (k, f(k)] = \emptyset$.

In this case, the perturbation of g will be supported on $(f(k), +\infty)$ and the perturbation of f on $(k, +\infty)$. Assume that $g(f(k)) < f(k)$, the complementary case (i.e. $g(f(k)) > f(k)$) can be treated identically.

We focus first on the construction of g_2: From the fact that f conjugates g to hg and that $g(x) < x$ for every $x \in (k, f(k)]$, we get that $hg(f(k)) \leq f(k)$. This implies that $g(f(k)) \leq min\{f(k), h^{-1}(f(k))\}$, and therefore we can define g_2 satisfying $g_2(x) < min\{id, h^{-1}\}(x)$ for every $x > f(k)$.

Now we build f_2: By the construction of g_2 and the assumptions of the Toy Case, we have that both $\mathcal{C}(g_2, id)_{|(k,+\infty)}$ and $\mathcal{C}(g_2, h^{-1})_{|(f(k),+\infty)}$ are constant -1. So there exists ψ a perturbation of f supported on $(k, +\infty)$ that weakly conjugates g_2 and hg_2. Since all the perturbations are supported outside $(-\infty, k)$, we know that ψ is strong on $(-\infty, min\{k, f^{-1}(k)\})$. Therefore we can apply Lemma 2.2.2 to "promote" ψ to a conjugation f_2 between g_2 and hg_2 such that $f_{2|(-\infty,k]} = f_{|(-\infty,k]}$.

We turn to the construction of g_1: Since $h^{-1}(p) \neq p$ we can pick (by continuity) a point $p_1 > p$ so that $\mathcal{C}(h^{-1}, id)_{|[p,p_1]}$ is constant (p and p_1 are in the same "bump" of the graph of h^{-1}). Recall that $g(f(k)) \leq min\{f(k), h^{-1}(f(k))\}$. So we can define g_1 on $[f(k), p)$ so that $g_1(x) < min\{x, h^{-1}(x)\}$ and $g_1(p) = min\{p, h^{-1}(p)\}$. On (p, p_1) we define g_1 so that $min\{x, h^{-1}(x)\} < g_1(x) < max\{x, h^{-1}(x)\}$ and $g_1(p_1) = max\{p_1, h^{-1}(p_1)\}$. Finally we can define g_1 on $(p_1, +\infty)$ so that $g_1(x) > max\{x, h^{-1}(x)\}$.

Now we build f_1: Observe that both $\mathcal{C}(g_1, id)_{|(k,+\infty)}$ and $\mathcal{C}(g_1, h^{-1})_{|(f(k),+\infty)}$ have a single sign change, that is of the form -1 to $+1$. This implies that we can construct ψ a perturbation of f supported on $(k, +\infty)$ such that ψ weakly-conjugates g_1 and hg_1. Again, applying Lemma 2.2.2 we finish the construction.

Finally, notice that $Fix(\langle f_i, g_i\rangle) \subseteq Fix(\langle f, g\rangle)$. Indeed, g_2 has no fixed points on the support of the perturbation, while g_1 has a single fixed point (either p or p_1) and that point is not fixed by $h = [f_2, g_2]$. $\Diamond$

Observe that the Toy Case is analogous to the case where $f(k) < k$ and $Fix(hg) \cap (f(k), k] = \emptyset$. The case $f(k) = k$ is even simpler.

In general, since f is a conjugation from g to hg, we know that the combinatorial information of g on $(-\infty, k]$ coincides with the combinatorial information of hg on $(-\infty, f(k)]$, that is

$$\mathcal{C}(g, id)(x) = \mathcal{C}(g, h^{-1}) \circ f(x) \quad \text{for } x \leq k.$$

The Toy Case was easy because we assumed that the combinatorics of g (namely $\mathcal{C}(g, id)$) was constant on $[k, f(k)]$. In general, we will need to make a previous perturbation in order to attain a similar situation.

Claim 3.7.4. (Local perturbation) *There are $q > p$, a homeomorphism $\psi : (-\infty, q] \to (-\infty, q]$, and a homeomorphism over its image $\bar{g} : (-\infty, q] \to \mathbb{R}$ such that*

1. *$\mathcal{C}(\bar{g}, id) = \mathcal{C}(\bar{g}, h^{-1}) \circ \psi$, that is, ψ is a weak conjugation from $\bar{g}$ to $h\bar{g}$ on $(-\infty, q]$.*
2. *the pair $(\psi, \bar{g})$ agrees with (f, g) on $(-\infty, k]$.*
3. *$Fix(\langle \psi, \bar{g}\rangle) \subseteq Fix(\langle f, g\rangle) \cap (-\infty, q]$, and $\bar{g}(q) \neq q$.*

Proof of the Claim. The proof is easy for the Toy Case ($f(k) > k$ and $Fix(g) \cap (k, f(k)] = \emptyset$), its analogue ($f(k) < k$ and $Fix(hg) \cap (f(k), k] = \emptyset$), and the case with $f(k) = k$. For instance, in the Toy Case we can set $\bar{g} = g_2$ and ask the ψ in the construction of f_2 to fix a point $q > p$. Nevertheless, the claim is not needed for these cases.

The remaining situations can be split according to whether $f(k) < k$ (and $Fix(hg) \cap (f(k), k] \neq \emptyset$) or $f(k) > k$ (and $Fix(g) \cap (k, f(k)] \neq \emptyset$). We focus first on the construction of the maps ψ and $\bar{g}$, and we shall check later that they satisfy the conditions.

Case I: $f(k) < k$ and $Fix(hg) \cap (f(k), k] \neq \emptyset$.

Recall the properties of p from condition $(*_K)$ for the pair (f, g). For this case we will only need to know that $p > g(k)$, $p > k$ and $h^{-1}(p) \neq p$ (which is guaranteed by $(*_K)$).

Subcase Ia: $f(k) \notin Fix(hg)$.

We start by defining $\bar{g}$ over $(-\infty, p]$. We set $\bar{g} = g$ on $(-\infty, k]$ and then we extend it over $[k, p]$ satisfying $Fix(\bar{g}) \cap [k, p] = p$. This is possible because $k \notin Fix(g)$. Let $s_1 = min\ Fix(hg) \cap (f(k), p]$ and $s_2 = max\ Fix(hg) \cap [f(k), p]$. Choose $\epsilon > 0$ and define ψ over $(-\infty, p + \epsilon]$ satisfying: $\psi_{|(-\infty,k]} = f_{|(-\infty,k]}$, $\psi(p) = s_1$ and $\psi(p + \epsilon) = s_2$. Now we continue extending $\bar{g}$. Define $\bar{g}$ over $[p, p+\epsilon]$ as $\psi^{-1}hg\psi$. Notice that $\bar{g}$ takes $[p, p+\epsilon]$ to itself. Since $h^{-1}p \neq p$ we can take ϵ small enough so that $Fix(h\bar{g}) \cap [p, p+\epsilon] = \emptyset$ (i.e. the graph of $\bar{g}$ does not meet that of h^{-1} over $[p, p+\epsilon]$). Take $q > p+\epsilon$ and define $\bar{g}$ over $[p+\epsilon, q]$ satisfying $Fix(\bar{g}) \cap [p+\epsilon, q] = \{p+\epsilon\}$, $Fix(h\bar{g}) \cap [p+\epsilon, q] = \emptyset$ and so that $\mathcal{C}(\bar{g}, id)$ and $\mathcal{C}(\bar{g}, h^{-1})$ agree on $(p+\epsilon, q]$. (Visually, this amounts to draw the graph of $\bar{g}$ on $(p+\epsilon, q]$ avoiding the diagonal and the graph of h^{-1}, and leaving both on the same side of the graph of $\bar{g}$. That is possible since $\bar{g}$ fixes $p+\epsilon$). Finally, we extend ψ to $(-\infty, q]$ so that $\psi(q) = q$. It is straightforward form this construction that $\mathcal{C}(\bar{g}, id)(x) = \mathcal{C}(\bar{g}, h^{-1})(\psi(x))$.

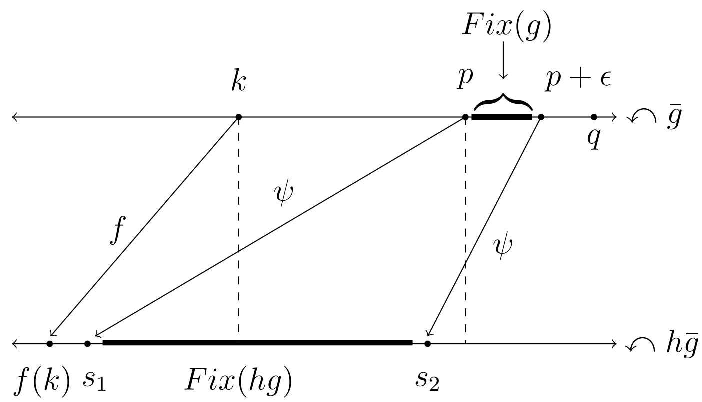

Figure 3.3: Local perturbation, subcase Ia

Subcase Ib: $f(k) \in Fix(hg)$.

Here we begin by defining ψ over $(-\infty, p]$. Take $s \in Fix(hg) \cap (f(k), k]$. Let $\psi : (-\infty, p] \to (-\infty, s]$ be any homeomorphism with $\psi_{|(-\infty,k]} = f_{|(-\infty,k]}$ and $\psi(p) = s$. We define $\bar{g}$ over $(-\infty, p]$, agreeing with g on $(-\infty, k]$, and with $\psi^{-1}hg\psi$ over $(k, p]$. Notice this is well defined since $\psi((k, p]) \subseteq (-\infty, k]$. This extension is continuous because $g(k) = k$ and $h\bar{g}(f(k)) = f(k)$. Then we proceed as in subcase Ia, with s as s_1.

Case II: $f(k) > k$, and $Fix(g) \cap (k, f(k)] \neq \emptyset$.

Recall (from condition $(*_K)$ for the pair (f, g)) that $p > g(f(k))$ and $h^{-1}(p) \neq p$. Then there exists $p_1 > p$ such that $h^{-1}(p_1) > p$ and $h^{-1}(p_1) \neq p_1$. This p_1 can be taken as an iterate of p by $h^{\pm 1}$. Notice that p_1 satisfies the conditions for being $p_{(f,g,K)}$ in condition $(*_K)$, and also satisfies $g(f(k)) < p < h^{-1}(p_1)$. Thus we can redefine $p := p_1$ and assume that $g(f(k)) < h^{-1}(p)$.

Subcase IIa: $k \notin Fix(g)$.

We begin by defining $\bar{g}$ on $(-\infty, p]$. It must agree with g on $(-\infty, f(k)]$. We define it over $[f(k), p]$ so that $Fix(h\bar{g}) \cap [f(k), p] = p$. This can be done because $f(k) \notin Fix(hg)$ and $g(f(k)) < h^{-1}(p)$. (Draw the graph of $\bar{g}$ avoiding that of h^{-1} over $(f(k), p)$, and meeting it at p).

Now consider $s_1 = min\ Fix(\bar{g}) \cap (k, +\infty)$ and $s_2 = max\ Fix(\bar{g}) \cap [k, p]$. Now we build ψ on $(-\infty, s_2]$, extending $f_{|(-\infty,k]}$. Take $\epsilon > 0$ and define ψ on $[k, s_2]$ satisfying $\psi(s_1) = p$ and $\psi(s_2) = p+\epsilon$. To continue $\bar{g}$ beyond p, we define an auxiliary function $\phi\colon [p, p+\epsilon] \to [p, p+\epsilon]$ as $\phi(x) = \psi\bar{g}\psi^{-1}(x)$. This is well defined since ψ^{-1} takes $[p, p+\epsilon]$ into $(-\infty, p]$. Then we define $\bar{g}$ over $[p, p+\epsilon]$ as $h^{-1}\phi$. Notice that $\bar{g}$ is well defined on p since $h^{-1}\phi(p) = h^{-1}(p)$, and we had from before that $p \in \text{Fix}(h\bar{g})$. Since $h^{-1}(p) \neq p$, we can show as in case I that if ϵ is small enough then $Fix(\bar{g}) \cap [p, p+\epsilon] = \emptyset$. Finally, extend $\bar{g}$ over $[p+\epsilon, q]$ satisfying $Fix(\bar{g}) \cap (p+\epsilon, q] = Fix(h\bar{g}) \cap (p+\epsilon, q] = \emptyset$ and $\mathcal{C}(\bar{g}, id)_{|(p+\epsilon,q]} = \mathcal{C}(\bar{g}, h^{-1})_{|(p+\epsilon,q]}$. (That is analogous to Case I, but with $(p+\epsilon, \bar{g}(p+\epsilon))$ on the graph of h^{-1} instead of the diagonal). Then we extend ψ over $[s_2, q]$ so that $\psi(q) = q$. We get again that $\mathcal{C}(\bar{g}, id)(x) = \mathcal{C}(\bar{g}, h^{-1})(\psi(x))$.

Subcase IIb: $k \in Fix(g)$.

We can make the construction as in case Ib, with the same modifications we did for case IIa. (Namely, interchanging the roles of $\bar{g}$ and $h\bar{g}$).

It only remains to check that $Fix(\langle \psi, \bar{g} \rangle) \subseteq Fix(\langle f, g \rangle)$. For $x \leq min\ \{k, f(k)\}$ that inclusion is trivial, and for $x \in [min\ \{k, f(k)\}, q]$ observe that ψ does not fix any point in $Fix(\bar{g})$. □

Now we derive Lemma 3.1.3 from the local perturbation Claim 3.7.4.

We suppose that $\mathcal{C}(\bar{g}, id)(q) = 1 = \mathcal{C}(\bar{g}, h^{-1})(q)$. The other case, when $\mathcal{C}(\bar{g}, id)(q) = -1 = \mathcal{C}(\bar{g}, h^{-1})(q)$, can be treated analogously.

We begin by defining the g_i, for $1 = 1, 2$: With our assumption, we can just define g_1 as an extension of $\bar{g}$ such that $g_1(x) > max\{x, h^{-1}(x)\}$ for every $x \geq q$. In order to construct g_2, we need to make a small modification on the construction of $\bar{g}$ and ψ in the proof of Claim 3.7.4. The change on $\bar{g}$ occurs in the interval $(p+\epsilon, q]$, where we ask the new graph of $\bar{g}$ to meet the diagonal and the graph of h^{-1}, each transversally and on a single point. This can be done just as in the Toy Case. (The fact that $h^{-1}(p+\epsilon) \neq p+\epsilon$ implies that we can make these two transversal intersections occur at different points. So we do not create any new global fixed point). Now we can redefine ψ over $[s_2, q]$ to obtain $\mathcal{C}(\bar{g}, id)(x) = \mathcal{C}(\bar{g}, h^{-1})(\psi(x))$ for all $x \leq q$ and still have $\psi(q) = q$. With this modification we have $\mathcal{C}(\bar{g}, id)(q) = -1 = \mathcal{C}(\bar{g}, h^{-1})(q)$, and we define g_2 as an extension of $\bar{g}$ that satisfies $g_2(x) < min\ \{x, h^{-1}(x)\}$ for every $x \geq q$.

Now we construct the f_i, for $i = 1, 2$: For each $i = 1, 2$ we can extend the map ψ used in the construction of g_i to the whole line, satisfying $\mathcal{C}(g_i, id)(x) = \mathcal{C}(g_i, h^{-1})(\psi(x))$ for every $x \in \mathbb{R}$. Therefore ψ is a weak-conjugation between g_i and hg_i that is strong on $(-\infty, min\{k, f^{-1}(k)\}]$. Applying Lemma 2.2.2 we obtain f_i extending $f_{|(-\infty,k]}$ and conjugating g_i with hg_i as desired.

It remains to check that $Fix(\langle f_i, g_i \rangle) \subseteq Fix(\langle f, g \rangle)$. On $(-\infty, q]$ this follows from the Claim 3.7.4 (even with the modification for g_2), because $f_{i|Fix(\bar{g})} = \psi_{|Fix(\bar{g})}$. On $[q, +\infty)$ there are no fixed points of g_i. □

3.7.4 Proof of Lemma 3.1.2

We point out to the reader that we will be relying heavily on the techniques developed in the previous section, §3.7.3.

Let K' be the convex closure of $K \cup f(K)$.

First we will assume $K = (-\infty, k]$. As in Lemma 3.1.3, we distinguish two main cases:

Case I : $f(k) < k$.

Here $K' = (-\infty, k]$.

Subcase Ia : hg **has no fixed points in** $(f(k), k]$**.** Then, as in the Toy Case of Lemma 3.1.3, $g(k)$ is not between k and $h^{-1}(k)$. So we can define g' on $(k, +\infty)$ making it either less than $min\{id, h'^{-1}\}$, or greater than $max\{id, h'^{-1}\}$ (depending on whether $hg(k) < k$ or $hg(k) > k$). Then f' also follows the construction in Lemma 3.1.3, and conjugates g' to $h'g'$.

Subcase Ib : **There are fixed points of** hg **in** $(f(k), k]$**.** Let $s_1 = min\ Fix(hg) \cap [f(k), k]$ and $s_2 = max\ Fix(hg) \cap [f(k), k]$. If $f(k) < s_1$ we find a point $k_1 > k$ so that $k_1 > g(k)$, and define g' over $(k, k_1]$ so that k_1 is the only fixed point of $g'_{|(k,k_1]}$. If $f(k) = s_1$, let $k_1 = k$. Now let $\psi : (-\infty, k_1 + 1] \to (-\infty, s_2]$ an homeomorphism agreeing with f on $(-\infty, k]$, and with $\psi(s_1) = k_1$ and $\psi(s_2) = k_1 + 1$. Extend g' over $(k_1, k_1 + 1]$ as $\psi^{-1} \circ hg \circ \psi$. (Notice h' agrees with h on $(-\infty, s_2]$).

Now, if $h'g'$ has no fixed points in $(s_2, k_1 + 1]$, proceed as in Subcase Ia. If there are such fixed points let $s_3 = max\ Fix(h'g') \cap (s_2, k_1 + 1]$, and we will extend g' and ψ over $(k_1 + 1, k_1 + 2]$ as follows. ψ will take $(k_1 + 1, k_1 + 2]$ homeomorphically onto $(s_2, s_3]$ and define g' over $(k_1 + 1, k_1 + 2]$ as $\psi^{-1} \circ h'g' \circ \psi$. (Notice g' was defined already on $(s_2, s_3]$).

We proceed inductively. The process stops if we fall in Subcase Ia in any instance. Otherwise, notice that $s_{n+3} > k_1 + n$ ($s_3 > k$, $s_4 > k_1 + 1$ and so on, since $h'g'$ has no fixed points in $(s_{n+2}, k_1 + n)$). So the ψ obtained is a homeomorphism of the line, that weakly conjugates g' and $h'g'$. We finish the construction applying Lemma 2.2.2.

We need to check that $Fix(\langle f', g' \rangle) \subseteq Fix(\langle f, g \rangle)$ and that $Fix(\langle f', g' \rangle) \cap (k, +\infty) = \emptyset$. The first statement is clear since (f', g') agrees with (f, g) over $(-\infty, k)$. For the second one, notice that if $x > k$ is a fixed point of g', then x is not fixed by ψ in the previous construction. Therefore, by Lemma 2.2.2, it is not fixed by f' either.

Case II : $f(k) > k$.

Here $K' = (-\infty, f(k)]$.

Follow the same scheme as in Case I, with the following modifications (similar to those in Case II of the Local Perturbation Claim 3.7.4): k_1 will be taken so that $k_1 > f(k)$ and $h'^{-1}(k_1) > g(f(k))$ in the case $f(k)$ is not fixed by hg, and as $f(k)$ otherwise. On each step of the extension as in Subcase Ib, we extend $h'g'$ over $[k_1 + n, k_1 + n + 1]$ as $\phi = \psi^{-1} \circ g' \circ \psi$ and define $g' = h'^{-1}\phi$ over $[k_1 + n, k_1 + n + 1]$. $\Diamond$

It remains the case $f(k) = k$ but it is a simple modification of Subcase Ia.

The proof for $K = [k, +\infty)$ is analogous. For a compact $K = [u, v]$, we iterate the case for semi-infinite intervals: Write $K' = [u', v']$ and consider $h_1 \in Homeo_+(\mathbb{R})$ that agrees with h on $(-\infty, v')$ and with h' on $[v', +\infty)$. We first apply the lemma for h_1 on $(-\infty, v]$. Next we apply it again to the perturbations just obtained for h' on $[u, +\infty)$. $\square$

Chapter 4

Free orbits for minimal actions on S^1

4.1 Proof of our main Theorem on Stabilizers

We recall the statement of Theorem 1.2.1, that will be proved in this section:

Let Γ be a countable group without a subgroup isomorphic to $\mathbb{Z}^2$. If Γ acts faithfully and minimally by orientation preserving homeomorphisms on the circle, then there exists a free orbit.

The following remark works for general countable groups.

Remark 4.1.1. For each $g \in \Gamma \backslash \{e\}$, consider the set $Fix(g) = \{x \in S^1 : \ gx = x\}$ of its fixed points. The points with free orbit are exactly those in

$$\bigcap_{g \in \Gamma \backslash \{e\}} \mathrm{Fix}(g)^c.$$

If a countable group Γ acts minimally on the circle and the action has no free orbit, then the following holds:

1. By Baire's Category Theorem there must exist $g \in \Gamma \backslash \{e\}$ such that $Fix(g)$ has non-empty interior.

2. Since the Γ-action is minimal on S^1, for every $x \in S^1$ there exists $h \in \Gamma \backslash \{e\}$ such that x is an interior point of $Fix(h)$.

Notice that the fact that Γ is countable is crucial for the proof of this remark as it uses Baire's category theorem. It is likely that arguments in the lines of the ones presented in [2] may help construct a non-countable group for which Theorem 1.2.1 fails, however, we could not construct such and example and believe that this would exceed the purposes of this note. The main difficulty we encountered in approaching this problem can be summarized in the following question:

Is it possible to construct a map $\varphi : S^1 \to \mathrm{Homeo}_+(S^1)$ such that the group generated by the elements in the image of φ is free?

We return to the proof of the Theorem. The following lemma will be the tool to obtain abelian subgroups.

Lemma 4.1.2. Let f and g be two nontrivial orientation-preserving homeomorphisms of the circle. If $Fix(f) \neq Fix(g)$ and $Fix(f) \cup Fix(g) = S^1$, then the subgroup of $Homeo_+(S^1)$ generated by f and g is isomorphic to $\mathbb{Z}^2$.

Proof: Let $H \subset Homeo_+(S^1)$ the subgroup generated by f and g. We will begin by proving that H is abelian.

Notice that since $Fix(f) \cup Fix(g) = S^1$, we know that any point is either fixed by f or fixed by g. Let $x \in S^1$. Without loss of generality, assume that $x \in Fix(g)$. Therefore $[f, g](x) = fgf^{-1}(x)$. If $x \in \mathrm{Fix}(f)$, then x is fixed by both f and g and therefore by $[f, g]$. Otherwise, $f^{-1}(x)$ is not fixed by f and is therefore fixed by g, so $[f, g](x) = x$. This implies that every point is fixed by $[f, g]$ and therefore $[f, g] = \mathrm{id}$ showing that f and g commute.

Next, remark that since $Fix(f) \neq Fix(g)$ the group H cannot be cyclic. Due to the classification of abelian groups, all we have to see is that H is torsion-free. Since the sets $Fix(f)$ and $Fix(g)$ are closed they cannot be disjoint, so any element of H must have fixed points. This means that H does not contain a non trivial element of finite order.

□

In order to prove Theorem 1.2.1, we will consider a countable group Γ acting faithfully and minimally on S^1. Assuming that the action has no free orbit, we will prove that Γ contains a subgroup isomorphic to $\mathbb{Z}^2$.

We will only use that Γ is countable in order to use Remark 4.1.1 so that there is an element whose fixed point set has non-empty interior. Under this assumption, the result does not further use countability of Γ.

Proof: [Proof of Theorem 1.2.1] For every $x \in S^1$, consider the set

$$A_x = \{I : \ I \text{ is an open interval in } S^1 \text{ and } x \in I \subset Fix(g) \text{ for some } g \in \Gamma \backslash \{e\}\}.$$

Remark 4.1.1 guarantees that A_x is non-empty for every $x \in S^1$. We fix an orientation in S^1. The orientation induces a total order on any interval I, and we can therefore write $I = (I_-, I_+)$. In particular, the interval $S^1 \backslash \{x\}$ has an order, which allows us to consider suprema and infima of subsets of $S^1 \backslash \{x\}$.

Assume that for a given $x \in S^1$ the set $\mathcal{A}_x = \{I_+ : \ I \in A_x\}$ is unbounded above in the total order of $S^1 \backslash \{x\}$. Consider $f \in \Gamma \backslash \{e\}$ such that x is an interior point of $Fix(f)$. Since $\mathcal{A}_x$ is unbounded, there exists $g \in \Gamma$ whose set of fixed points contains an interval I such that $I \cup Fix(f) = S^1$. In particular, $Fix(f) \cup Fix(g) = S^1$, and Lemma 4.1.2 implies that Γ contains a free abelian group of rank 2.

Otherwise, $\mathcal{A}_x$ must be bounded for all $x \in S^1$. In this case, we can define

$$h : S^1 \to S^1, \quad h(x) = \sup \mathcal{A}_x.$$

The map h has the following properties which follow directly from its definition:

1. it is monotonically increasing, (i.e.: any lift of h to the line is a monotone map)

2. it is equivariant, meaning that for every $g \in \Gamma$ and $x \in S^1$ one has $gh(x) = h(gx)$.

Let us now show that h is an homeomorphism. By equivariance, it follows that the image of h is invariant by the Γ action, therefore, by minimality it must be dense as otherwise

h would have a proper closed invariant subset. Now we check that h has to be strictly monotonous. Consider $V = \bigcup_{x/int(h^{-1}(x))\neq\emptyset} int(h^{-1}(x))$. V is an open, proper, Γ-invariant subset. The minimality of the action implies that V is empty. Finally, since h is strictly monotonous and has dense image, one obtains that $h \in Homeo_+(S^1)$.

Now, we distinguish cases according to the rotation number of h (see §2.4).

If $\rho(h)$ is irrational, then h must be either a Denjoy counterexample or conjugated to an irrational rotation. In the former case, h has a unique minimal set that is strictly contained in the circle, and must be Γ-invariant since h is equivariant (see Proposition 2.4.6). This is inconsistent with the minimality of the Γ-action. Therefore, h is conjugated to an irrational rotation, and Γ is isomorphic to a subgroup of the centralizer of h in $Homeo_+(S^1)$, but the centralizer of an irrational rotation does not have non-trivial elements with fixed points (Proposition 2.4.6), which also gives a contradiction.

We can therefore assume that $\rho(h)$ is rational. We first claim that h has to be conjugate to a rigid rotation (e.g. there exists $n > 0$ so that $h^n = \mathrm{id}$): indeed, if it were not the case then the closed set of periodic points would be a proper closed invariant set for Γ contradicting minimality.

So, take n the smallest positive integer such that $h^n = \mathrm{id}$. Note that by equivariance and definition of h, we must have that $h(x) < h^2(x) < ... < h^{n-1}(x)$ on $S^1\backslash\{x\}$. We will find g and g' in Γ whose set of fixed points is different and whose union is S^1. For this, consider $x \in S^1$ and $g \in \Gamma \setminus \{\mathrm{id}\}$ such that $\mathrm{Fix}(g)$ contains x in its interior. It follows that $\mathrm{Fix}(g)$ contains at least n connected components each containing respectively $x, h(x), \ldots, h^{n-1}(x)$. Choose a point y in $(x, h(x))$ which is in the interior of the component of $\mathrm{Fix}(g)$ containing x. Therefore, on $S^1\backslash\{x\}$ we have $y < h(x) < h(y) < h^2(x) < ... < h^{n-1}(x) < h^{n-1}(y)$.

Now, we will see that there exists $g' \in \Gamma \setminus \{\mathrm{id}\}$ such that $\mathrm{Fix}(g) \cup \mathrm{Fix}(g') = S^1$ and both g and g' are not the identity. Indeed, by definition of h we can find $g' \neq \mathrm{id}$ so that its set of fixed points contains y in its interior and has a connected component containing y and a point arbitrarily close to $h(y)$, in particular, we can assume it covers the set of points not fixed by g in the interval $(x, h(x)]$. This implies that the set of $\mathrm{Fix}(g) \cup \mathrm{Fix}(g')$ covers the whole interval $[x, h(x)]$ and by equivariance it covers the whole S^1.

This allows one to apply Lemma 4.1.2 to conclude the proof.

□

4.2 Counterexamples

4.2.1 A surface group action on S^1 without free orbits

We construct here a faithful action of the fundamental group of a surface on the circle with no free orbits.

Let Γ be the fundamental group of an oriented compact surface of genus greater than one. It does not contain any subgroup isomorphic to $\mathbb{Z}^2$. Surface groups are known to be ω-residually free (see [9]), which means that for any finite subset X of Γ there exists a homomorphism from Γ to a free group whose restriction to X is injective.

Consider a free subgroup F of $Homeo_+(\mathbb{R})$. Write $\Gamma = \cup_{n=0}^{\infty} X_n$ as an increasing union of finite subsets, and for each n let $\varphi_n : \Gamma \to F$ be an homomorphism that sends X_n injectively into F. Notice that, since Γ is non-free, the Nielsen-Schreier theorem (see, for example, Section 2.2.4 of [35]) implies that φ_n must have a non-trivial kernel. Take an increasing sequence of points $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ which does not accumulate in $\mathbb{R}$. Taking S^1 to be $\mathbb{R} \cup \{\infty\}$ and setting $x_0 = \infty$, the circle is the union of the intervals $[x_n, x_{n+1}]$, for $n \geq 0$. We will identify each open interval (x_n, x_{n+1}) with the real line, so that φ_n can be seen as a representation of Γ in $Homeo_+(x_n, x_{n+1})$.

We will define

$$\varphi : \Gamma \to Homeo_+(S^1)$$

as follows:

* for any $g \in \Gamma$, $\varphi(g)$ fixes $\{x_n,\ n \geq 0\}$,
* restricted to (x_n, x_{n+1}), $\varphi(g)$ coincides with $\varphi_n(g) \in Homeo_+((x_n, x_{n+1}))$.

It is clear that φ is a faithful representation, since each φ_n is injective on X_n. We will see that the Γ-action defined by φ has no free orbits. Let $x \in S^1$. If x is not fixed by Γ, it belongs to (x_n, x_{n+1}) for some n, and it is therefore fixed by the non-trivial subgroup $\ker(\varphi_n)$.

4.2.2 The main theorem does not hold in $\mathbb{R}$

The following example shows that Theorem 1.2.1 is not true if we consider actions on the line. We will construct a faithful action of the free group $\mathbb{F}_2 = \langle a, b \rangle$ on the line that is minimal and such that every point is stabilized by some non trivial element.

We will start by defining three different $\mathbb{F}_2$ actions and later we will "glue" them.

Consider

- $\phi_1 \colon \mathbb{F}_2 \to \mathbb{R}$ such that $\phi_1(a)(x) = x$ and $\phi_1(b)(x) = x + 1$.
- $\phi_2 \colon \mathbb{F}_2 \to \mathbb{R}$ such that $\phi_2(a)(x) = x + \alpha$ and $\phi_2(b)(x) = x + \beta$ for α and β linearly independent over $\mathbb{Q}$. We also ask that $0 < \alpha < 1$ and $0 < \beta < 1$
- $\phi_3 \colon \mathbb{F}_2 \to \mathbb{R}$ any action with a free orbit and without global fixed points.

Take $p > 4$ such that $\phi_3(a)(p) > 4 + \alpha$ and $\phi_3(b)(p) > 4 + \beta$. Define $f \in Homeo_+(\mathbb{R})$ satisfying $f(x) = \phi_1(a)(x)$ if $x < 0$, $f(x) = \phi_2(a)(x)$ if $x \in [1, 4]$ and $f(x) = \phi_3(a)(x)$ if $x > p$. Now we define $g \in Homeo_+(\mathbb{R})$ satisfying $g(x) = \phi_1(b)(x)$ if $x < 0$, $g(x) = \phi_2(b)(x)$ if $x \in [1, 4]$ and $g(x) = \phi_3(b)(x)$ if $x > p$. Finally define f and g over $[0, 1] \cup [4, p]$ so that $Fix(f) \cap Fix(g) = \emptyset$.

Consider $\psi \colon \mathbb{F}_2 \to Homeo_+(\mathbb{R})$ defined as $\psi(a) = f$ and $\psi(b) = g$. Given $w \in \mathbb{F}_2$ define p_w such that $\psi(w)(x) = \phi_3(w)(x)$ for every $x \geq p_w$. Since ϕ_3 has no global fixed points, every ϕ_3-orbit accumulates on $+\infty$. Otherwise $sup\{\phi_3(g)(x) : g \in \mathbb{F}_2\}$ would be a global fixed point. So, given $w \in \mathbb{F}_2$, we can take $x \in \mathbb{R}$ with ϕ_3-free orbit and $g \in \mathbb{F}_2$ such that $\phi_3(g)(x) > p_w$. Now, $\psi(w)(\phi_3(g)(x)) = \phi_3(w)(\phi_3(g)(x)) \neq \phi_3(g)(x)$. This implies that ψ is a faithful action.

Again, since ψ has no global fixed points, given $x \in \mathbb{R}$ there exists $w \in \mathbb{F}_2$ so that $\psi(w)(x) < 0$, and therefore $\psi(w^{-1}aw)(x) = x$ which proves that ψ has no free orbit.

It remains to check the minimality of ψ. Observe that given any $x \in [1, 2]$ it is clear that the ψ orbit of x is dense on $[1, 2]$. Now, since $\psi(a)([1, 2]) \cap [1, 2] \neq \emptyset$ and $\psi(b)([1, 2]) \cap [1, 2] \neq \emptyset$

we can deduce that $\mathbb{F}_2.[1,2]$, the union of the ψ orbits of points in $[1,2]$, is a connected set. Also, since ψ has no global fixed points $\mathbb{F}_2.[1,2]$ is unbounded in both directions and therefore $\mathbb{F}_2.[1,2] = \mathbb{R}$. Finally, any orbit accumulates on $[1,2]$ and therefore on $\mathbb{R}$ as claimed.

Remark 4.2.1. Since any action on $\mathbb{R}$ can be seen as an action on $S^1 = \mathbb{R} \cup \{\infty\}$ with a global fixed point, this is also an example of how Theorem 1.2.1 can fail when the action is not minimal.

4.3 Further properties of minimal actions without free orbits

Remark 4.3.1. If Γ is a non-cyclic group acting minimally and faithfully without free orbits con the circle, then it is non-abelian.

To see this, consider an element $f \in \Gamma \backslash \{e\}$ such that $Fix(f)$ is non-empty. If Γ were abelian, the set $Fix(f)$ would be invariant by all elements of Γ, so the action would not be minimal.

Proposition 4.3.2. If Γ is a countable group acting minimally and faithfully without free orbits on the circle, then it contains a free group in two generators.

Proof: A result conjectured by Ghys and later proved by Margulis (see [27] or [30]), states that any group of circle homeomorphisms either preserves a probability measure on S^1 or contains a free group in two generators. If Γ acts without free orbits, it must be non-abelian.

Suppose there is a Γ-invariant probability measure μ. Since the action is minimal, it must have full support and no atoms. There is an homeomorphism sending μ to the Lebesgue measure. This means Γ must be conjugated to a group of rotations, and therefore abelian, which gives a contradiction. □

Proposition 4.3.3. If Γ is a countable group acting minimally and faithfully without free orbits on the circle, then it contains free abelian groups of arbitrarily large rank.

Proof: This follows by further inspection on the proof of theorem 1.2.1. We just sketch the proof.

First, notice that h is defined by contradiction. If it cannot be constructed, it means that for every $x \in S^1$ there are elements for which there exist arbitrarily large intervals of fixed points containing x (they contain the complement of arbitrarily small neighbourhoods of x). Therefore, given $n > 0$ we can find $\gamma_1, \ldots, \gamma_n$ and $V_1, ..., V_n$ open intervals so that $Fix(\gamma_i)^c \subseteq V_i$ for every i and $V_i \cap V_j = \emptyset$ if $i \neq j$. Lemma 4.1.2 implies that the group generated by $\gamma_1, ..., \gamma_n$ is abelian. Note that if we write $w = \gamma_1^{k_1} ... \gamma_n^{k_n}$ with $k_i \neq 0$ for some i then $w_{|V_i} \neq Id$ and therefore $\langle \gamma_1, ..., \gamma_n \rangle \equiv \mathbb{Z}^n$ as desired.

On the other side, if we can construct h its rotation number must be rational (see Theorem 1.2.1). Again in this case, we can find for every $n > 0$, $\gamma_1, \ldots, \gamma_n \in \Gamma$ whose supports are pairwise disjoint. □

4.4 Appendix: The suspension of an action.

In this appendix we work with right actions because holonomy actions of foliated bundles are naturally right actions. Also, since we have to simultaneously deal with left and right actions, we find that action notation ($\rho : G \times X \to X$ and $\rho : X \times G \to X$) is more suitable than representation notation.

A *foliation* of dimension M on a manifold M is a partition of M into m-dimensional submanifolds that is locally "nice". For the general theory of foliations see [8].

Definition 4.4.1. Let M be a differentiable manifold of dimension n. A foliation of dimension m on M is a maximal atlas $\mathcal{F} = \{(U_i, \phi_i)\}$ with the following properties:

- If $(U, \varphi) \in \mathcal{F}$ then $\varphi(U) = U_1 \times U_2 \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m}$ where U_1 and U_2 are open discs in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ respectively.
- If (U, φ) and $(V, \psi) \in \mathcal{F}$ are such that $U \cap V \neq \emptyset$ then the coordinate change $\psi \circ \phi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is of the form $\psi \circ \phi^{-1}(x, y) = (h_1(x, y), h_2(y))$ with $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.

We call the pair $(M, \mathcal{F})$ a foliated manifold.

A subset of the form $\phi(\mathbb{R}^m \times \{y\})$ for some $(U, \varphi) \in \mathcal{F}$ is called a *plaque*. *Leaves* are the classes of the equivalence relation on M generated by being in the same plaque.

Foliated Bundles are special types of foliations that are strongly related with group actions on manifolds.

Definition 4.4.2. Let M be a compact manifold, $p : M \to B$ a fiber bundle and $\mathcal{F}$ a foliation on M. We say that $(p : M \to B, \mathcal{F})$ is a foliated bundle if for every $x \in M$ we have $T_xM = T_x\mathcal{F}_x \bigoplus KerD_xp$ where $\mathcal{F}_x$ is the leaf through x.

Given a foliated bundle with fiber F and base B one can construct a right action of $\pi_1(B)$ on F. We call this action the *holonomy action* associated to the foliated bundle. For this we need the following :

Definition 4.4.3. Let $(p : M \to B, \mathcal{F})$ be a foliated bundle, $\tilde{x_0} \in M$ and $x_0 = p(\tilde{x_0})$. Given $\gamma : [0, 1] \to B$ a continuous curve with $\gamma(0) = x_0$ we define the lift of γ subordinated to $\mathcal{F}$ starting at $\tilde{x_0}$, as the unique continuous curve $\tilde{\gamma} : [0, 1] \to M$ that is contained in a leaf, and satisfies $\tilde{\gamma}(0) = \tilde{x_0}$ and $p \circ \tilde{\gamma} = \gamma$.

Remark 4.4.4. Notice that lifts of curves subordinated to $\mathcal{F}$ exist by compactness of M.

It can be shown (see [8]) that if γ_0 is another curve homotopic with fixed ends to γ and $\tilde{\gamma_0}$ is its lift subordinated to $\mathcal{F}$ starting at $\tilde{x_0}$, then $\tilde{\gamma_0}(1) = \tilde{\gamma}(1)$.

Given $x_0 \in B$, this allows us to define a holonomy action $\rho_* : p^{-1}(x_0) \times \pi_1(B, x_0) \to p^{-1}(x_0)$ as $\rho_*(y, [\gamma]) = \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the lift of γ starting at y subordinated to $\mathcal{F}$.

The following proposition is the main result of this section.

Proposition 4.4.5. Given compact manifolds B, F and a right action $\rho : F \times \pi_1(B) \to F$, there exists $(p : M \to B, \mathcal{F})$ a foliated bundle whose holonomy action is conjugated to ρ.

We call this foliated bundle the *suspension* of ρ.

Proof. Let $q : \tilde{B} \to B$ the universal covering projection of B and $\tau : \pi_1(B) \times \tilde{B} \to \tilde{B}$ its associated action by deck transformations. Define $\sigma : \pi_1(B) \times \tilde{B} \times F \to \tilde{B} \times F$ as $\sigma(g, x, y) = (\tau(g, x), \rho(y, g^{-1}))$. Since σ is a product action whose first coordinate is τ, we conclude that σ is free and properly discontinuous. So if $p_1 : \tilde{B} \times F \to M$ is the quotient projection associated to the action σ, we get that M is a manifold and p_1 is a covering map.

Since $\sigma(g, \{x\} \times F) = \{\tau(g, x)\} \times F$ we can define $p : M \to B$ a fiber bundle with fiber F and base B. On the other hand, since $\sigma(g, \tilde{B} \times \{y\}) = \tilde{B} \times \{\rho(y, g^{-1})\}$ we obtain $\mathcal{F}$ a foliation on M whose leaves are the projections of the horizontals $\tilde{B} \times \{y\}$ under p_1. Notice that the leaves of $\mathcal{F}$ are transverse to the fibers of p.

Finally, we recover ρ (up to conjugation) by considering the holonomy action associated to the suspension. Take $x_0 \in B$ and $\overline{x_0}$ with $q(\overline{x_0}) = x_0$. Define $i : F \to \tilde{B} \times F$ as $i(y) = (\overline{x_0}, y)$ and define the homeomorphism $\psi := p_1 \circ i : F \to p^{-1}(x_0)$. Let ρ_* be the holonomy action of the suspension $(p : M \to B, \mathcal{F})$.

We now show that ρ and ρ_* are conjugated by ψ. Take $y \in p^{-1}(x_0)$ and $[\gamma] \in \pi_1(B, x_0)$. By construction $\rho_*(y, [\gamma]) = \overline{\gamma}_0(1)$ where $\overline{\gamma}_0$ is the lift of γ subordinated to $\mathcal{F}$ and starting at y. Consider $\overline{\gamma}_1$ the lift of $\overline{\gamma}_0$ to the covering p_1, starting at $(\overline{x_0}, \psi^{-1}(y))$. Notice that $\overline{\gamma}_0 = p_1 \circ \overline{\gamma}_1$. Now, $\rho_*(y, [\gamma]) = \overline{\gamma}_0(1) = p_1(\overline{\gamma}_1(1))$. By the lift definition, we get that $\overline{\gamma}_1(1) = (\tau([\gamma], \overline{x_0}), \psi^{-1}(y)))$.

So $\rho_*(y, [\gamma]) = p_1(\tau([\gamma], \overline{x_0}), \psi^{-1}(y))$ and since p_1 is σ-invariant, acting with $[\gamma]^{-1}$ we get $p_1(\tau([\gamma], \overline{x_0}), \psi^{-1}(y)) = p_1(\overline{x_0}, \rho(\psi^{-1}(y), [\gamma])) = \psi(\overline{x_0}, \rho(\psi^{-1}(y), [\gamma]))$.

Putting all together we obtain

$$\rho_*(y, [\gamma]) = \psi(\rho(\psi^{-1}(y), [\gamma]))$$

as desired. □

A Theorem in [8] shows that every foliated bundle is isomorphic to the suspension of its holonomy action.

Now we translate some concepts between actions and foliations.

Given $x \in M$ we note $\mathcal{F}_x$ the leaf of $\mathcal{F}$ that contains x. If $A \subseteq M$, we will call the saturation of A to the set $sat(A) = \bigcup_{x \in A} L_x$ and say that $B \subseteq M$ is invariant if $B = sat(B)$.

We say that $(M, \mathcal{F})$ is *minimal* if every closed invariant subset is either $\emptyset$ or the whole M.

Indeed, we can show that the holonomy action $\rho : p^{-1}(x_0) \times \pi_1(B) \to p^{-1}(x_0)$ is minimal if and only if $\mathcal{F}$ is minimal. By definition of the holonomy action, invariant subsets of M intersect $p^{-1}(x_0)$ on ρ-invariant subsets. To see the other direction we need some notations.

Take $<, >$ a Riemannian metric on M. Given a leaf $L \subseteq M$ let d_L be the Riemannian distance on L defined by the inclusion. If $x \in M$ set $D_r(x) = \{y \in L_x : d_L(x, y) \leq r\}$. Since M is compact, if $r > 0$ is big enough, we have that for every $x \in M$ we get $p(D_r(x)) = B$. Given a ρ-invariant subset $A \subseteq p^{-1}(x_0)$, we can consider $C = \bigcup_{x \in A} D_{2r}(x)$ which is invariant, closed, and satisfies $C \cap p^{-1}(x_0) = A$.

Now, we point out the relation between the fundamental groups of the leaves of $\mathcal{F}$ and the stabilizers of points for ρ. Indeed, take $x \in M$ and $y \in \mathcal{F}_x \cap p^{-1}(x_0)$. Notice that $p_x := p_{|\mathcal{F}_x} : \mathcal{F}_x \to B$ is a covering map and that $(p_x)_*(\pi_1(\mathcal{F}_x, y)) = \{g \in \pi_1(B, x_0) :$

$\rho(y, g) = y\}$. Therefore $\pi_1(\mathcal{F}_x, y) \cong Stab_\rho(y)$. In particular, free orbits of ρ correspond to simply-connected leaves of $\mathcal{F}$.

Finally we translate Theorem 1.2.1 to the language of Foliated Bundles.

Theorem 4.4.6. Let $(p : M \to \Sigma, \mathcal{F})$ be a foliated bundle with Σ a closed hyperbolic surface, p a fibration with S^1-fibers and $\mathcal{F}$ a minimal foliation. Then one and only one of the following hold:

- There is a simply connected leaf
- The holonomy action $\rho_* : \pi_1(\Sigma) \to Homeo_+(S^1)$ has non-trivial kernel.

Proof. First, we observe that the minimality of $\mathcal{F}$ implies the minimality of ρ_*. So, if $x_0 \in \Sigma$, the action $\rho_* : p^{-1}(x_0) \times \pi_1(\Sigma, x_0) \to p^{-1}(x_0)$ is a minimal action of $\pi_1(\Sigma, x_0)$ on the circle. Since $\pi_1(\Sigma)$ contains no subgroup isomorphic to $\mathbb{Z}^2$, Theorem 1.2.1 implies that if ρ_* is faithfull then ρ_* must have a free orbit.

On the other hand, simply-connected leaves of $\mathcal{F}$ correspond to free orbits of ρ_*. So we conclude that if $\mathcal{F}$ has no simply-connected leaves then ρ_* has non-trivial kernel.

Clearly both conditions cannot be satisfied simultaneously. □

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Joaquin Brum
Fac. Ingenieria, Universidad de la Republica Uruguay
joaquinbrum@fing.edu.uy