

Handel's fixed point theorem revisited

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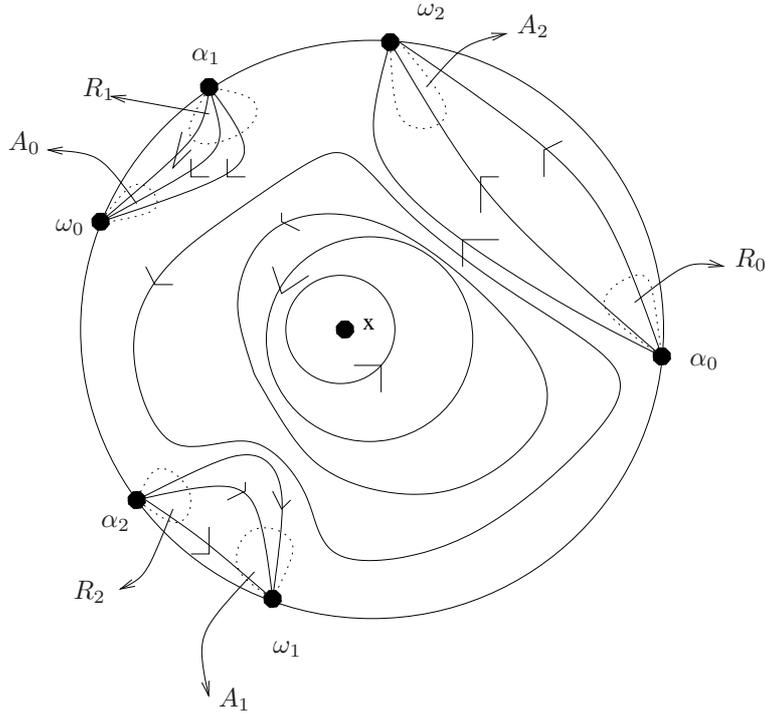
Je porte toute mon admiration envers l'école mathématique uruguayenne, qui fait des choses énormes malgré le fait qu'elle soit petite. Mes remerciements chaleureux à tous mes collègues là - bas qui me manquent tellement.

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Introduction

Dans ce manuscrit, on considère des homéomorphismes préservant l'orientation du disque $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. On motivera notre travail avec l'exemple qui suit.

Soit $f : \mathbb{D} \rightarrow \mathbb{D}$ le temps un du flot dont les orbites sont dessinées sur la figure ci-dessous.



L'homéomorphisme f a les propriétés suivantes :

1. **Prolongement au bord:** f se prolonge en un homéomorphisme du disque fermé $\overline{\mathbb{D}}$;
2. **Configuration Répulseur/Attracteur elliptique:** il existe des sous-ensembles de \mathbb{D} connexes et disjoints deux-à-deux $R_i, A_i, i \in \mathbb{Z}/3\mathbb{Z}$, tels que pour tout i :
 - (a) $\overline{R_i} \cap \partial\mathbb{D} \neq \emptyset$ et $\overline{A_i} \cap \partial\mathbb{D} \neq \emptyset$;
 - (b) si C est un ensemble connexe contenant A_{i-1} et A_i , et C' est un ensemble connexe contenant R_{i+1} et $X \in \cup_j (R_j \cup A_j) \setminus R_{i+1}$, alors $C \cap C' \neq \emptyset$,
 - (c) $f^{-1}(R_i) \subset \text{Int}(R_i)$ et $f(A_i) \subset \text{Int}(A_i)$.

(Ces ensembles sont dessinés en lignes pointillées sur la figure).

3. **Existence d'une courbe fermée simple d'indice 1:** il existe une courbe fermée simple $C \subset \mathbb{D}$ ne contenant pas de point fixe de f et dont l'indice est 1.

Nous rappelons que si $\beta : S^1 \rightarrow \mathbb{D}$ est un paramétrage de C , l'indice $i(f, \beta)$ est égal au degré de l'application

$$s \mapsto \frac{f(\beta(s)) - \beta(s)}{|f(\beta(s)) - \beta(s)|}.$$

Il n'est pas difficile de voir (et ça sera expliqué plus tard dans ce manuscrit) que l'on peut perturber f en un homéomorphisme g qui satisfait les propriétés 1. à 3. au dessus, et en plus les propriétés:

4. **Réalisation d'un cycle d'enlacements elliptique:** il existe $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}, n \geq 3$, points distincts de $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ tels que pour tout i :

- (a) le seul, parmi les points $\{\alpha_i\}, \{\omega_i\}$, qui se trouve sur l'intervalle ouvert du cercle orienté S^1 joignant ω_{i-1} à ω_i est α_{i+1} ;
- (b) il existe $z_i \in \mathbb{D}$ tel que $\lim_{k \rightarrow -\infty} g^{-k}(z_i) = \alpha_i$, et $\lim_{k \rightarrow \infty} g^k(z_i) = \omega_i$.

(Dans notre exemple $n = 3$).

5. **Connexion de R_i à A_i :** pour tout i il existe une chaîne de pseudo-disques de R_i à A_i .

Un *pseudo-disque* est une partie U de \mathbb{D} telle que pour tout $x, y \in U$, il existe une courbe simple $\gamma : [0, 1] \rightarrow \mathbb{D}$ qui va de x à y , telle que $\gamma(]0, 1[) \subset \text{Int}(U)$.

Une *chaîne de pseudo-disques* pour g est une suite $(U_i)_{i=1, \dots, k}$ de pseudo-disques tels que:

- 1. les *intérieurs* des pseudo-disques U_i sont disjoints deux à deux;
- 2. chaque pseudo-disque U_i est *libre* (c'est-à-dire que $g(U_i) \cap U_i = \emptyset$);
- 3. il existe des entiers $n_i > 0$ tels que $g^{n_i}(U_i) \cap U_{i+1} \neq \emptyset$ pour $i = 1, \dots, k-1$.

On dit qu'une chaîne de pseudo-disques va d'une partie X à une partie Y si $U_0 \subset X$ et $U_k \subset Y$.

(En effet, dans notre exemple il y a une g -orbite de R_i à A_i pour tout i , mais c'est cette propriété plus faible qui nous intéresse.)

On dira que l'homéomorphisme f est *récurrent* s'il existe une chaîne *périodique* de pseudo-disques pour f (c'est-à-dire avec $U_k = U_0$).

Le théorème de point fixe de Handel [8] établit que les conditions de prolongement au bord (1.), et la réalisation d'un cycle d'enlacements elliptique (4.) sont des conditions suffisantes pour avoir un point fixe. De plus, Le Calvez [11] a raffiné ce résultat en affaiblissant l'hypothèse de prolongement, en obtenant la conclusion plus forte d'existence d'une courbe fermée simple d'indice 1 (3.).

Plus précisément:

Théorème 1 ([11]). *Soit f un homéomorphisme de \mathbb{D} qui préserve l'orientation et n un entier supérieur ou égal à 3. On suppose que les hypothèses suivantes sont satisfaites:*

- 1. *il existe une famille $(z_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ dans \mathbb{D} et deux familles $(\alpha_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ et $(\omega_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ dans S^1 , telles que $\lim_{k \rightarrow -\infty} f^{-k}(z_i) = \alpha_i$, et $\lim_{k \rightarrow \infty} f^k(z_i) = \omega_i$, pour tout $i \in \mathbb{Z}/n\mathbb{Z}$;*

2. les $2n$ points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$, sont distincts;
3. le seul, parmi ces points, qui se trouve sur l'intervalle ouvert du cercle orienté S^1 joignant ω_{i-1} à ω_i est α_{i+1} ;
4. f se prolonge en un homéomorphisme de $\mathbb{D} \cup (\cup_{i \in \mathbb{Z}/n\mathbb{Z}} \{\alpha_i, \omega_i\})$.

Il existe alors une courbe fermée simple $C \subset \mathbb{D}$ ne contenant pas de point fixe de f et dont l'indice est 1.

La première partie de ce manuscrit consiste à énoncer et montrer un théorème établissant que l'existence d'une configuration Répulseur/Attracteur elliptique avec connexions entre R_i et A_i (2. et 5.) est une condition suffisante pour avoir existence d'une courbe fermée d'indice 1 (3.). De plus, sous les quatre hypothèses du théorème de Le Calvez, on montre que soit il y a de la récurrence (3.), soit on peut construire une configuration Répulseur/Attracteur elliptique satisfaisant 5. En particulier, en raisonnant par contradiction, on obtient une nouvelle preuve (qui se trouve plus simple) du Théorème 1.

On illustrera dans la suite de l'introduction l'importance de ce théorème de point fixe pour l'étude de la dynamique des surfaces, ce qui nous permettra d'introduire naturellement les autres résultats établis dans ce manuscrit.

Tout d'abord on a le résultat suivant, qui est l'outil-clé de ce travail:

Proposition 1 ([6], [13]). *Soit f un homéomorphisme du plan préservant l'orientation pour lequel il existe une chaîne de pseudo-disques $(U_i)_{i=1, \dots, k}$ avec $U_k = U_0$. Il existe alors une courbe fermée simple $C \subset \mathbb{D}$ ne contenant pas de point fixe de f et dont l'indice est 1.*

Le résultat précédent entraîne en particulier qu'un homéomorphisme du plan préservant l'orientation avec un point périodique non-fixe est récurrent, et en conséquence possède toujours un point fixe.

On appellera anneau ouvert toute surface homéomorphe à $S^1 \times]0, 1[$, et anneau fermé toute surface homéomorphe à $S^1 \times [0, 1]$. Quand on dira anneau cela signifiera anneau ouvert ou anneau fermé. Rappelons maintenant la définition de nombre de rotation $\rho(z, F)$ d'un point périodique z d'un homéomorphisme préservant l'orientation de l'anneau $f : A \rightarrow A$, quand on se fixe un relèvement $F : \tilde{A} \rightarrow \tilde{A}$ au revêtement universel \tilde{A} de A . Soit q la période de z , et $\tilde{z} \in \tilde{A}$ un relevé de z . Il existe alors $p \in \mathbb{Z}$ tel que $F^q(\tilde{z}) = T^p(\tilde{z})$, où $T : \tilde{A} \rightarrow \tilde{A}$ est la translation horizontale d'une unité à droite. Cet entier p est le *nombre de rotation* $\rho(z, F)$ de z pour le relèvement F . La proposition 1 entraîne facilement que pour un homéomorphisme de l'anneau sans point fixe, tout point périodique doit "tourner autour" de cet anneau:

Proposition 2. *Soit f un homéomorphisme de l'anneau préservant l'orientation et z un point périodique. Si f n'a pas de point fixe, alors $\rho(z, F) \neq 0$, pour tout relèvement F de f au revêtement universel de l'anneau.*

Dans le même esprit, on peut se poser la question suivante. On considère maintenant une sphère trouée avec au moins trois trous; est ce qu'en l'absence de point fixe, tout point périodique est obligé de tourner autour de l'un des trous de la surface? La réponse affirmative est une conséquence du théorème de

point fixe de Handel, comme on l'expliquera dans notre prochaine proposition. Ce résultat a été utilisé par Franks [5] pour montrer qu'un homéomorphisme de l'anneau préservant l'orientation ainsi que l'aire, et qui possède au moins un point périodique doit avoir une infinité de points périodiques intérieurs.

Si $(f_t)_{t \in [0,1]}$ est une isotopie issue de l'identité sur une surface M , on peut définir la *trajectoire* $\gamma_z : t \mapsto f_t(z)$ de tout point $z \in M$. Si z est un point périodique, de période $q \geq 1$, on obtient un lacet $\prod_{i=0}^{q-1} \gamma_{f^i(z)}$ en assemblant les trajectoires, dont on note $\rho(z) \in H_1(M, \mathbb{Z})$ sa classe d'homologie.

Le résultat qui suit est dû à Bestvina et Handel, et la preuve que nous donnons est extraite de [11].

Proposition 3. *Soit $M = S^2 \setminus \{x_1, \dots, x_n\}$, $n \geq 3$, et $(f_t)_{t \in [0,1]}$ une isotopie issue de l'identité dans M . Si $f = f_1$ n'a pas de point fixe, alors $\rho(z) \neq 0$ pour tout point périodique $z \in M$.*

Remarque 1. *Non seulement le lacet $\prod_{i=0}^{q-1} \gamma_{f^i(z)}$ n'est pas homotope à zéro, ce qui se déduirait de la proposition 1, mais mieux: il n'est pas homologué à zéro.*

Preuve. On peut munir M d'une structure riemannienne complète de courbure -1 et identifier le revêtement universel \tilde{M} au disque de Poincaré \mathbb{D} . Si $(F_t)_{t \in [0,1]}$ est le relèvement de l'isotopie $(f_t)_{t \in [0,1]}$ issue de l'identité dans \tilde{M} , on sait alors que $F = F_1$ se prolonge en un homéomorphisme de $\overline{\mathbb{D}}$ qui fixe tout point de S^1 . Supposons qu'il existe un point périodique $z \in M$ avec $\rho(z) = 0$, et montrons que dans ce cas il y a un point fixe de f . Si le lacet $\prod_{i=0}^{q-1} \gamma_{f^i(z)}$ est homotope à zéro, F a un point périodique et en conséquence un point fixe. Sinon, soit Γ l'unique géodésique fermée qui est librement homotope à $\prod_{i=0}^{q-1} \gamma_{f^i(z)}$. Tout relèvement de Γ dans \mathbb{D} est une géodésique de \mathbb{D} qui joint un point $\alpha \in S^1$ à un point $\omega \in S^1$. De plus, il existe un antécédent \tilde{z} de z tel que $\lim_{k \rightarrow -\infty} F^{-k}(\tilde{z}) = \alpha$, et $\lim_{k \rightarrow \infty} F^k(\tilde{z}) = \omega$. Puisque Γ est homologué à zéro dans M , il existe alors une fonction $\Lambda : M \setminus \Gamma \rightarrow \mathbb{Z}$ localement constante, uniquement définie par les propriétés suivantes:

- $\Lambda(x) = 0$ si x appartient à une composante connexe non relativement compacte (trouée) de $M \setminus \Gamma$;
- $\Lambda(x) - \Lambda(x')$ représente le nombre d'intersection algébrique $\Gamma \wedge \Gamma'$ entre Γ et une courbe quelconque Γ' joignant x et x' .

L'un au moins des deux nombres $\max \Lambda$ ou $\min \Lambda$ doit être non nul. Supposons par exemple que $\max \Lambda \neq 0$ et fixons une composante connexe U de $M \setminus \Gamma$ où Λ atteint son maximum. Alors, U est un disque ouvert, et sa frontière est une courbe simple réunion de p sous-segments de Γ , et U est localement à gauche de chacun de ses segments. Toute composante connexe de la préimage de U dans \mathbb{D} est l'intersection de p demi-plans géodésiques, à gauche des géodésiques relevant Γ . Parmi ces demi-plans, on choisit une famille dont l'intersection est relativement compacte dans \mathbb{D} , et qui est minimale pour cette propriété. Les hypothèses du théorème de point fixe de Handel sont alors vérifiées, et F a un point fixe dans \mathbb{D} .

Ce résultat n'est pas sans lien avec la version feuilletée équivariante du théorème de translation de Brouwer [9], comme on l'expliquera plus tard.

On appellera *droite* tout plongement topologique propre de \mathbb{R} dans \mathbb{R}^2 . Si γ est une droite orientée de \mathbb{R}^2 on notera $D(\gamma)$ (resp. $G(\gamma)$) la composante connexe de $\mathbb{R}^2 \setminus \gamma$ qui se trouve à droite (resp. à gauche) de γ .

On dira qu'une droite γ est une *droite de Brouwer* si

$$f(\gamma) \subset G(\gamma), f^{-1}(\gamma) \subset D(\gamma).$$

Théorème 2 ([9]). *Soit G un groupe discret d'homéomorphismes du plan préservant l'orientation, agissant librement et proprement. Soit f un homéomorphisme du plan préservant l'orientation, qui n'a pas de point fixe et qui commute avec les éléments de G . Il existe alors un feuilletage topologique orienté du plan, invariant sous l'action de G , dont toute feuille est une droite de Brouwer de f .*

On va utiliser ce résultat pour donner une autre preuve de la proposition 3, ce qui motivera la suite de notre travail.

On dira qu'un arc $\gamma : [0, 1] \rightarrow M$ est *positivement transverse* à un feuilletage \mathcal{F} défini sur une surface M si pour tout $t_0 \in [0, 1]$ il existe un voisinage U de $\gamma(t_0)$ dans M et un homéomorphisme préservant l'orientation h entre U et un ouvert V de \mathbb{R}^2 qui envoie le feuilletage \mathcal{F} sur le feuilletage vertical, orienté suivant les y croissants, et tel que l'application $t \rightarrow p_1(h(\gamma(t)))$ soit strictement croissante au voisinage de t_0 , où p_1 désigne la première projection.

En passant au revêtement universel, on peut énoncer le théorème sous la forme équivalente suivante:

Théorème 3 ([9]). *Soit M une surface et $(f_t)_{t \in [0,1]}$ une isotopie dans M joignant l'identité à un homéomorphisme f . Pour tout $z \in M$, on définit l'arc $\gamma_z : t \mapsto f_t(z)$. On suppose que f n'a pas de point fixe contractile, c'est-à-dire de point fixe z tel que γ_z soit un lacet homotope à 0. Il existe alors un feuilletage topologique orienté \mathcal{F} sur M et pour tout $z \in M$, un arc positivement transverse à \mathcal{F} joignant z à $f(z)$ qui est homotope, à extrémités fixées, à l'arc γ_z .*

Nous dirons que le feuilletage \mathcal{F} et l'isotopie $(f_t)_{t \in [0,1]}$ sont transverses.

Nous allons voir comment montrer la proposition 3 comme conséquence du théorème 3:

Preuve. [9] Considérons un feuilletage \mathcal{F} sur M transverse à l'isotopie $(f_t)_{t \in [0,1]}$ donné par le théorème 3. Si la conclusion du théorème est fautive, on peut trouver un lacet Γ positivement transverse au feuilletage qui est homologue à zéro. Il existe alors une fonction $\Lambda : M \setminus \Gamma \rightarrow \mathbb{Z}$ localement constante, uniquement définie par les propriétés suivantes:

- $\Lambda(x) = 0$ si x appartient à une composante connexe non relativement compact (trouée) de $M \setminus \Gamma$;
- $\Lambda(x) - \Lambda(x')$ représente le nombre d'intersection algébrique $\Gamma \wedge \Gamma'$ entre Γ et une courbe quelconque Γ' joignant x et x' .

Quitte à perturber Γ , on peut supposer que cette courbe vérifie les propriétés suivantes:

- elle n'a pas de point triple ou d'ordre supérieur;

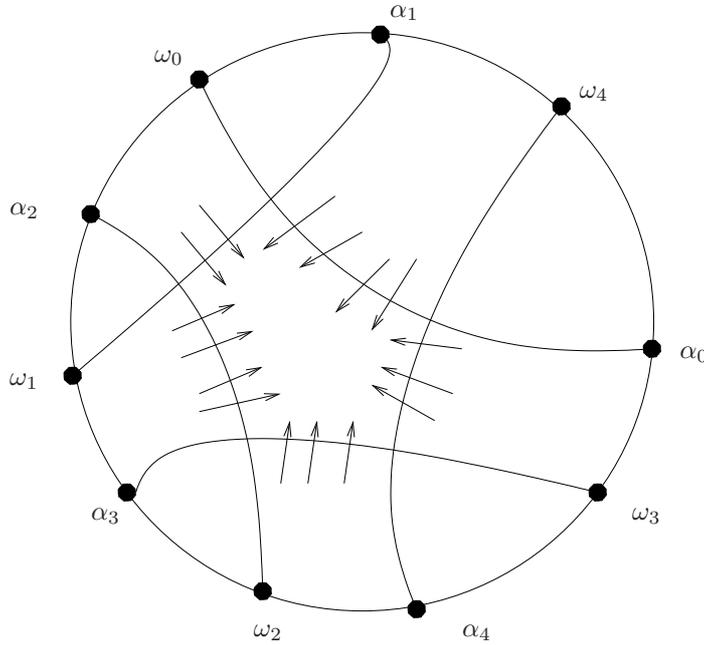
- elle a un nombre fini de points doubles avec des auto-intersections transverses.

La fonction n'est pas constante et on doit avoir $\max \Lambda > 0$ ou $\min \Lambda < 0$. Plaçons nous dans le premier cas. Toute composante connexe U de $M \setminus \Gamma$ où Λ atteint son maximum est relativement compacte et c'est donc un disque. Sa frontière est une courbe fermée simple transverse au feuilletage. Ceci est bien sûr impossible puisque le feuilletage est non singulier, par la formule de Poincaré-Hopf.

On pourrait essayer de faire une preuve du théorème de point fixe de Handel en utilisant cette méthode. En effet, plaçons nous sous les hypothèses de prolongement au bord et réalisation d'un cycle d'enlacements elliptiques (conditions 1. et 4. dans l'exemple au début de cette introduction). On veut montrer que ces deux hypothèses entraînent l'existence d'un point fixe. Si ce n'est pas vrai, on peut trouver un feuilletage topologique orienté \mathcal{F} de \mathbb{D} , et pour tout $z \in \mathbb{D}$ et $k \in \mathbb{Z}$, un arc γ_z^k joignant $f^k(z)$ et $f^{k+1}(z)$ positivement transverse à \mathcal{F} . De plus, si l'on définit $\Gamma_i = \cup_{k \in \mathbb{Z}} \gamma_{z_i}^k$, chaque Γ_i^k est un plongement propre de \mathbb{R} (sinon, on pourrait trouver un lacet positivement transverse au feuilletage, ce qui donnerait une singularité du feuilletage, toujours par la formule de Poincaré-Hopf).

Supposons, de plus que pour tout i , l'on a la propriété suivante:

(P) les courbes $\gamma_{z_i}^k$ convergent dans la topologie de Hausdorff vers $\{\omega_i\}$ quand $k \rightarrow +\infty$ et vers $\{\alpha_i\}$ quand $k \rightarrow -\infty$.



Dans ce cas, chaque $\overline{\Gamma_i}$ serait un arc joignant α_i et ω_i et on aurait

$$\Gamma_i \cap \Gamma_{i+1} \neq \emptyset,$$

pour tout i (voir la figure au dessus). On pourrait donc trouver une courbe fermée simple positivement transverse au feuilletage, ce qui contredirait là encore la formule de Poincaré - Hopf.

Malheureusement, on ne peut pas faire marcher cet argument. Sans aucune hypothèse sur le comportement de f près du bord, il n'y a aucune raison pour que la propriété (P) soit vérifiée.

Cependant, on peut penser à la question suivante. Considérons la courbe Γ obtenue par assemblage des trajectoires d'un point périodique pour un homéomorphisme isotope à l'identité d'une surface M (comme dans les preuves de la proposition 3).

Y a-t-il une généralisation du théorème de Handel correspondant à des composantes connexes de $S^2 \setminus \Gamma$ où l'orientation sur le bord est d'indice négatif? Soyons plus précis. Considérons une isotopie issue de l'identité $(f_t)_{t \in [0,1]}$ sur une surface M . Supposons de plus qu'il existe un point périodique $z \in M$, de période q , et que $f = f_1$ n'a pas de point fixe. Alors, $\prod_{i=0}^{q-1} \gamma_{f^i(z)}$ n'est pas homotope à zéro dans M (voir proposition 1) et il existe un feuilletage topologique orienté \mathcal{F} sur M transverse à f . Considérons (dans l'esprit de la première preuve qu'on a donné de la proposition 3) la géodésique Γ librement homotope à $\prod_{i=0}^{q-1} \gamma_{f^i(z)}$, et Γ' dans la même classe d'homotopie, positivement transverse au feuilletage (comme dans notre deuxième preuve de la Proposition 3). D'après le théorème de Handel, on obtient un point fixe s'il existe une composante connexe U du complémentaire de Γ telle que:

- U est homéomorphe à un disque;
- le bord orienté de U est un assemblage d'un nombre fini de sous-arcs de Γ héritant l'orientation de Γ ou l'orientation opposée à celle de Γ .

D'après la formule de Poincaré-Hopf, ceci est vrai aussi pour la courbe Γ' . On dira qu'une composante connexe U du complémentaire d'une courbe fermée dans M est *contractile* si U est homéomorphe à un disque. Les composantes connexes du complémentaire de la courbe Γ' transverse au feuilletage semblent donner plus d'information que celles de Γ . En effet, soit U une composante connexe contractile du complémentaire de Γ' où le bord orienté s'écrit $\partial U = \prod_{0 \leq i < 2r-1} \beta_i$ où β_i est un assemblage d'un nombre fini de sous-arcs de Γ' héritant de l'orientation de Γ' si i est pair, et héritant l'orientation opposée si i est impair. Alors, $\sum_{z \in \text{Fix}(F_1)} i(\mathcal{F}, z) = 1 - r$. On dira que ces composantes du complémentaire de Γ' sont *d'indice négatif*. Donc, pour un lacet transverse au feuilletage, l'existence d'une composante connexe contractile d'indice négatif du complémentaire du lacet, entraîne l'existence d'un point fixe pour f .

La deuxième partie de ce manuscrit consiste à montrer que ce fait est aussi vrai pour les géodésiques (notamment pour n'importe quelle courbe dans la classe d'homotopie de $\prod_{i=0}^{q-1} \gamma_{f^i(z)}$). Plus précisément, considérons $P \subset \mathbb{D}$ un polygone à n côtés compact et convexe. Soit $\{v_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ l'ensemble des sommets et pour tout $i \in \mathbb{Z}/n\mathbb{Z}$, soit a_i l'arête joignant v_i et v_{i+1} . On suppose chaque a_i munie d'une orientation, et on peut donc dire si P se trouve à gauche ou à droite de a_i . On dira que les orientations de a_i et a_j *coïncident* si P se trouve à la fois à droite (ou à gauche) de a_i et a_j , $i, j \in \mathbb{Z}/n\mathbb{Z}$.

On définit *l'indice* de P par

$$i(P) = 1 - \frac{1}{2} \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \delta_i,$$

où $\delta_i = 0$ si les orientations de a_{i-1} et a_i coïncident, et $\delta_i = 1$ sinon.

On note α_i le premier point où la droite Δ_i contenant a_i et héritant son orientation intersecte $\partial\mathbb{D}$, et ω_i le dernier point où cette droite intersecte $\partial\mathbb{D}$. On remarque que ces points ne sont pas nécessairement tous différents.

On dira qu'un homéomorphisme $f : \mathbb{D} \rightarrow \mathbb{D}$ réalise P s'il existe une famille $(z_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ de points de \mathbb{D} telle que pour tout $i \in \mathbb{Z}/n\mathbb{Z}$

$$\lim_{k \rightarrow -\infty} f^k(z_i) = \alpha_i, \quad \lim_{k \rightarrow +\infty} f^k(z_i) = \omega_i.$$

Théorème 4. *Soit $f : \mathbb{D} \rightarrow \mathbb{D}$ un homéomorphisme préservant l'orientation qui réalise un polygone compact et convexe $P \subset \mathbb{D}$ et qui se prolonge en un homéomorphisme de $\mathbb{D} \cup (\cup_{i \in \mathbb{Z}/n\mathbb{Z}} \{\alpha_i, \omega_i\})$.*

Si $i(P) \neq 0$, alors $\text{Fix}(f) \neq \emptyset$. De plus, si $i(P) = 1$, alors il existe une courbe fermée simple $C \subset \mathbb{D}$ ne contenant pas de point fixe de f et dont l'indice est 1.

Bien entendu, l'hypothèse " $f : \mathbb{D} \rightarrow \mathbb{D}$ réalise P " n'est autre chose qu'une hypothèse sur l'ordre cyclique dans S^1 des points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$. Notre théorème est plus général; on a choisi de présenter cet énoncé ici pour faciliter la lecture.

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Chapter 1

Introduction

Handel's fixed point theorem [8] has been of great importance for the study of surface homeomorphisms. It guarantees the existence of a fixed point for an orientation preserving homeomorphism f of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ provided that it can be extended to the boundary $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and that it has points whose orbits form an oriented cycle of links at infinity. More precisely, there exist n points $z_i \in \mathbb{D}$ such that

$$\lim_{k \rightarrow -\infty} f^k(z_i) = \alpha_i \in S^1, \quad \lim_{k \rightarrow +\infty} f^k(z_i) = \omega_i \in S^1,$$

$i = 1, \dots, n$, where the $2n$ points $\{\alpha_i\}, \{\omega_i\}$ are different points in S^1 and satisfy the following order property:

(*) α_{i+1} is the only one among these points that lies in the open interval in the oriented circle S^1 from ω_{i-1} to ω_i .

(Although this is not Handel's original statement, it is an equivalent one as already pointed out in [11]).

Le Calvez gave an alternative proof of this theorem [11], relying only in Brouwer theory and plane topology, which allowed him to obtain a sharper result. Namely, he weakened the extension hypothesis by demanding the homeomorphism to be extended just to $\mathbb{D} \cup (\cup_{i \in \mathbb{Z}/n\mathbb{Z}} \{\alpha_i, \omega_i\})$ and he strengthened the conclusion by proving the existence of a simple closed curve of index 1.

We give a new, simpler proof of this improved version of the theorem and we generalize it to non-oriented cycles of links at infinity; that is, we relax the order property (*) as follows.

A *cycle of links of order $n \geq 3$* is a family of pairs of points on the circle S^1 ,

$$\mathcal{L} = ((\alpha_i, \omega_i))_{i \in \mathbb{Z}/n\mathbb{Z}}$$

such that for all $i \in \mathbb{Z}/n\mathbb{Z}$:

1. $\alpha_i \neq \omega_i$,
2. α_{i+1} and ω_{i+1} belong to different connected components of $S^1 \setminus \{\alpha_i, \omega_i\}$.

If \mathcal{L} is a cycle of links, we define the set

$$\ell = \{\alpha_i, \omega_i : i \in \mathbb{Z}/n\mathbb{Z}\} \subset S^1$$

of points in the circle which belong to a pair in the cycle.

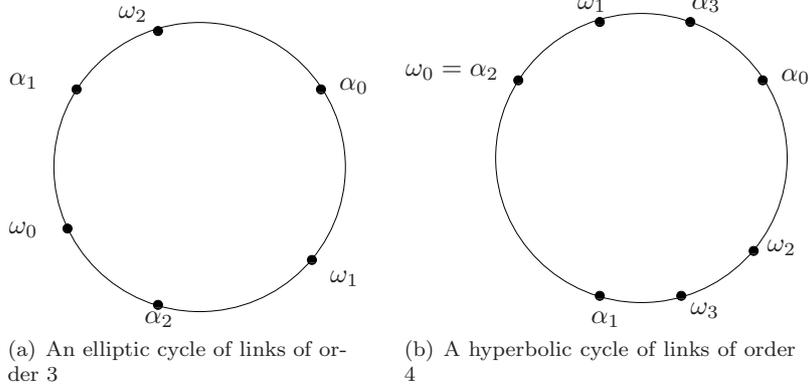
If $a, b \in \ell$, we note $a \rightarrow b$ if b follows a in the natural (positive) cyclic order on S^1 , and $a \xrightarrow{=} b$ if either $a = b$ or $a \rightarrow b$.

We say that a cycle of links \mathcal{L} is *elliptic* if for all $i \in \mathbb{Z}/n\mathbb{Z}$:

$$\omega_{i-1} \xrightarrow{=} \alpha_{i+1} \rightarrow \omega_i.$$

We say it is *hyperbolic* if $n = 2k, k \geq 2$ and for all $i \in \mathbb{Z}/n\mathbb{Z}, i = 0 \pmod 2$:

$$\alpha_i \rightarrow \alpha_{i-1} \xrightarrow{=} \omega_{i+1} \rightarrow \omega_i \xrightarrow{=} \alpha_{i+2}.$$



We say that \mathcal{L} is *non-degenerate* if:

$$(\alpha_i, \omega_i) \in \mathcal{L} \Rightarrow (\omega_i, \alpha_i) \notin \mathcal{L}.$$

Of course, we say it is *degenerate*, if this condition is not satisfied. An example is illustrated in Figure 1.1.

We say that a homeomorphism $f : \mathbb{D} \rightarrow \mathbb{D}$ *realizes* \mathcal{L} if there exists a family $(z_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ of points in \mathbb{D} such that for all $i \in \mathbb{Z}/n\mathbb{Z}$,

$$\lim_{k \rightarrow -\infty} f^k(z_i) = \alpha_i, \quad \lim_{k \rightarrow +\infty} f^k(z_i) = \omega_i.$$

The following result is the main theorem of this article.

Theorem 1.1. *Suppose that $f : \mathbb{D} \rightarrow \mathbb{D}$ is an orientation preserving homeomorphism which realizes a cycle of links \mathcal{L} and can be extended to a homeomorphism of $\mathbb{D} \cup \ell$.*

If \mathcal{L} is either elliptic or hyperbolic, then f has a fixed point. Furthermore, if \mathcal{L} is non-degenerate and elliptic, then there exists a simple closed curve $C \subset \mathbb{D}$ of index 1.

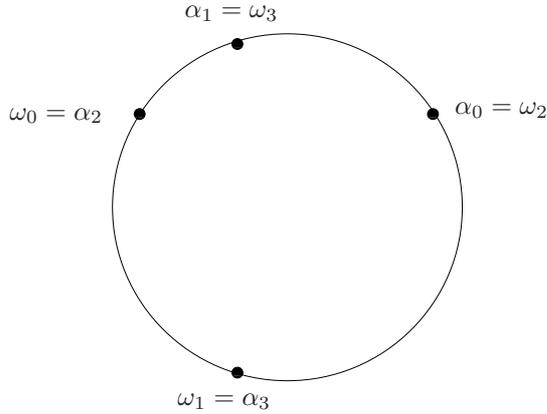


Figure 1.1: A degenerate cycle of links

The elliptic non-degenerate case contains Le Calvez’s improvement of Handel’s theorem.

Indeed, if the points in ℓ are all different, \mathcal{L} is non-degenerate. As the following example shows, our theorem is more general even in this case.

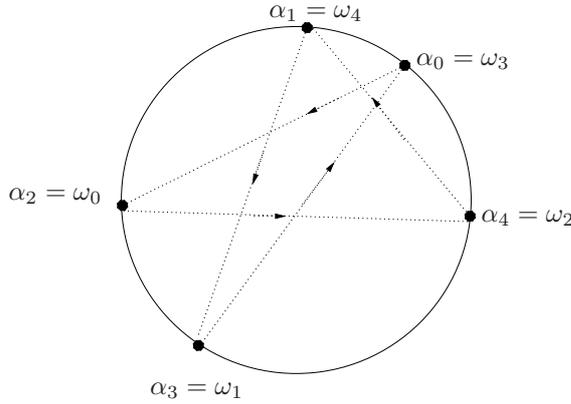


Figure 1.2: A non-degenerate elliptic cycle with coincidences among the points in ℓ .

The extension hypothesis is needed.

Consider a North-South dynamics $T : \mathbb{D} \rightarrow \mathbb{D}$ with a source $N \in \partial\mathbb{D}$ and a sink $S \in \partial\mathbb{D}$. One can construct a homeomorphism $h : \mathbb{D} \rightarrow \mathbb{D}$ such that hTh^{-1} realizes an elliptic cycle of links. Indeed, take any elliptic cycle of links \mathcal{L} of order n , and take any n points $z_i \in \mathbb{D}$, $i \in \mathbb{Z}/n\mathbb{Z}$, in different T -orbits. We can construct inductively simple paths γ_i^k , $i \in \mathbb{Z}/n\mathbb{Z}$, $k \in \mathbb{Z}$, such that:

- the paths γ_i^k , $i \in \mathbb{Z}/n\mathbb{Z}$, $k \in \mathbb{Z}$, are pairwise disjoint,

- for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $k \geq 0$, $\gamma_i^k(0) = T^k(z_i)$ and $\gamma_i^k(1) = \omega_i$,
- for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $k < 0$, $\gamma_i^k(0) = \alpha_i$ and $\gamma_i^k(1) = T^k(z_i)$,
- $\gamma_i^k \cap T^p(z_j) \neq \emptyset$ if and only if $i = j$ and $k = p$.

For all $i \in \mathbb{Z}/n\mathbb{Z}$ and $k \in \mathbb{Z}$ we construct open disks $U_i^k \subset \mathbb{D}$, such that:

- the disks U_i^k , $i \in \mathbb{Z}/n\mathbb{Z}$, $k \in \mathbb{Z}$, are pairwise disjoint,
- for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $k \geq 0$, $\gamma_i^k([0, 1)) \subset U_i^k$,
- for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $k < 0$, $\gamma_i^k((0, 1]) \subset U_i^k$,
- $U_i^k \cap T^p(z_j) \neq \emptyset$ if and only if $i = j$ and $k = p$.

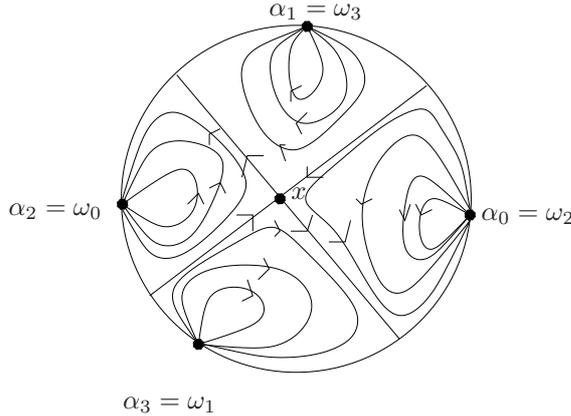
And finally, for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $k \in \mathbb{Z}$ we construct homeomorphisms $h_i^k : \mathbb{D} \rightarrow \mathbb{D}$, such that:

- each h_i^k , $i \in \mathbb{Z}/n\mathbb{Z}$, $k \in \mathbb{Z}$, is the identity outside U_i^k ,
- for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $k \geq 0$, $h_i^k(T^k(z_i)) = \gamma_i^k(1 - 1/k)$,
- for all $i \in \mathbb{Z}/n\mathbb{Z}$ and $k < 0$, $h_i^k(T^k(z_i)) = \gamma_i^k(1/k)$.

We define $h : \mathbb{D} \rightarrow \mathbb{D}$ as being the identity outside $\cup_{i \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{Z}} U_i^k$, and $h = h_i^k$ in each disk U_i^k . Then, the homeomorphism hTh^{-1} realizes \mathcal{L} with the family of points $\{h(z_i)\}$, and hTh^{-1} is clearly fixed-point free in \mathbb{D} .

Non-degeneracy is needed for obtaining the index result.

Let f_1 be the time-one map of the flow whose orbits are drawn in the figure below.



As we will explain below, one can perturb f_1 in a homeomorphism f such that:

- $\text{Fix}(f) = \text{Fix}(f_1) = \{x\}$,
- $f = f_1$ in a neighbourhood of x ,

- f realizes $\mathcal{L} = ((\alpha_i, \omega_i))_{i \in \mathbb{Z}/4\mathbb{Z}}$.

We say that the set X is *free* if $f(X) \cap X = \emptyset$.

One can find (by means of a transverse foliation, for example), free and pairwise disjoint simple paths β_i and γ_i , $i \in \mathbb{Z}/4\mathbb{Z}$ such that :

- β_i joins z_i and z'_i , where $\lim_{k \rightarrow \infty} f_1^{-k}(z_i) = \alpha_i$ and $\lim_{k \rightarrow \infty} f_1^k(z'_i) = \alpha_{i^*}$, where $i^* = i + 1$ for even values of i , and $i^* = i - 1$ for odd values of i ,
- γ_i joins $f_1^{p_i}(z'_i)$ and z''_i , where $p_i > 0$ and $\lim_{k \rightarrow \infty} f_1^k(z''_i) = \omega_i$,
- γ_i and β_i are disjoint from the f_1 -orbits of every z_j, z'_j, z''_j with $i \neq j$.

By thickening the paths $\{\beta_i\}$ and $\{\gamma_i\}$, one can find free, pairwise disjoint open disks $\{D'_i\}$ and $\{D''_i\}$ such that the disks D'_i and D''_i are disjoint from the f_1 -orbits of the points z_j, z'_j , and z''_j , for $i \neq j$.

We construct a homeomorphism $h : \mathbb{D} \rightarrow \mathbb{D}$ such that:

- $h = \text{Id}$ outside $\cup_{i \in \mathbb{Z}/4\mathbb{Z}} D'_i \cup D''_i$,
- $h(z_i) = z'_i$,
- $h(f_1^{p_i}(z'_i)) = z''_i$.

So, if we define $f = h \circ f_1$, we obtain

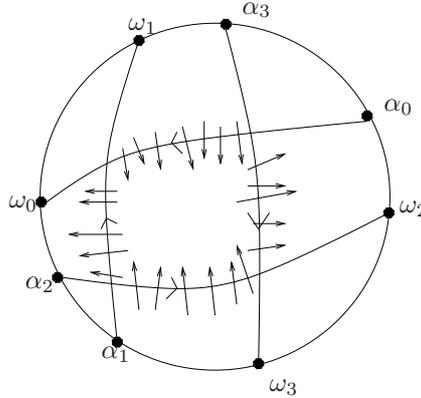
$$\lim_{k \rightarrow \infty} f^{-k}(z_i) = \alpha_i, \quad \lim_{k \rightarrow \infty} f^k(z_i) = \omega_i,$$

for all $i \in \mathbb{Z}/4\mathbb{Z}$. Clearly we can make this construction in such a way that $f = f_1$ in a neighbourhood of x . Moreover, as the disks $\{D'_i\}$ and $\{D''_i\}$ are free,

$$\text{Fix}(f) = \text{Fix}(f_1) = \{x\}.$$

So, f realizes the elliptic cycle \mathcal{L} , but there is no simple closed curve of index 1.

No negative-index fixed point is guaranteed by hyperbolicity.

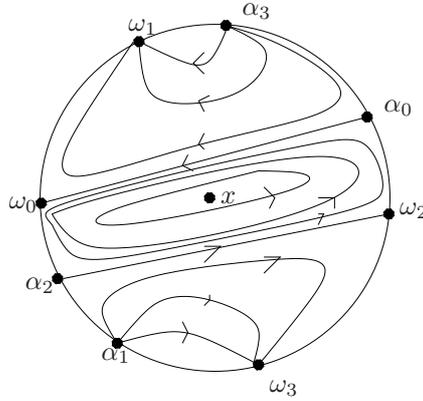


One could think that when \mathcal{L} is hyperbolic, a negative-index fixed point should be obtained. For example, this would be the case if one had an oriented foliation \mathcal{F} in $\mathbb{D} \setminus \text{Fix}(f)$ whose leaves are Brouwer lines for f and simple paths γ_i , $i \in \mathbb{Z}/n\mathbb{Z}$ joining α_i and ω_i such that:

- each γ_i is positively transverse to \mathcal{F} ,
- the paths $\{\gamma_i\}$ bound a compact disc in \mathbb{D} .

(See the figure above.) Indeed, in this case, the Poincaré-Hopf formula would give a singularity x of the foliation for which $i(\mathcal{F}, x) < 0$. So, $x \in \text{Fix}(f)$ and by a result of Le Calvez ([12]) one has $i(f, x) = i(\mathcal{F}, x) < 0$.

However, this is not the case, as the following example shows. Let f_1 be the time-one map of the flow whose orbits are drawn in the figure below.



In the same fashion we did in our preceding example, one can perturb f_1 in a homeomorphism f such that:

- $\text{Fix}(f) = \text{Fix}(f_1) = \{x\}$,
- $f = f_1$ in a neighbourhood of x ,
- f realizes $\mathcal{L} = ((\alpha_i, \omega_i))_{i \in \mathbb{Z}/4\mathbb{Z}}$.

So, f realizes the hyperbolic cycle \mathcal{L} , but there is no fixed point of negative index.

A visual corollary.

Let us explain the hypothesis of Theorem 1.1 with the following corollary.

Let $P \subset \mathbb{D}$ be a compact convex n -gon. Let $\{v_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ be its set of vertices and for each $i \in \mathbb{Z}/n\mathbb{Z}$, let e_i be the edge joining v_i and v_{i+1} . We suppose that each e_i is endowed with an orientation, so that we can tell whether P is to the right or to the left of e_i . We say that the orientations of e_i and e_j

coincide if P is to the right (or to the left) of both e_i and e_j , $i, j \in \mathbb{Z}/n\mathbb{Z}$. We define the *index* of P by

$$i(P) = 1 - \frac{1}{2} \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \delta_i,$$

where $\delta_i = 0$ if the orientations of e_{i-1} and e_i coincide, and $\delta_i = 1$ otherwise.

We will note α_i and ω_i the first, and respectively the last, point where the straight line Δ_i containing e_i and inheriting its orientation intersects $\partial\mathbb{D}$. We do not require all of these points to be different; some of them may coincide. Then, $\mathcal{L} = ((\alpha_i, \omega_i))_{i \in \mathbb{Z}/n\mathbb{Z}}$ is a cycle of links. We say that it is the cycle *induced* by P .

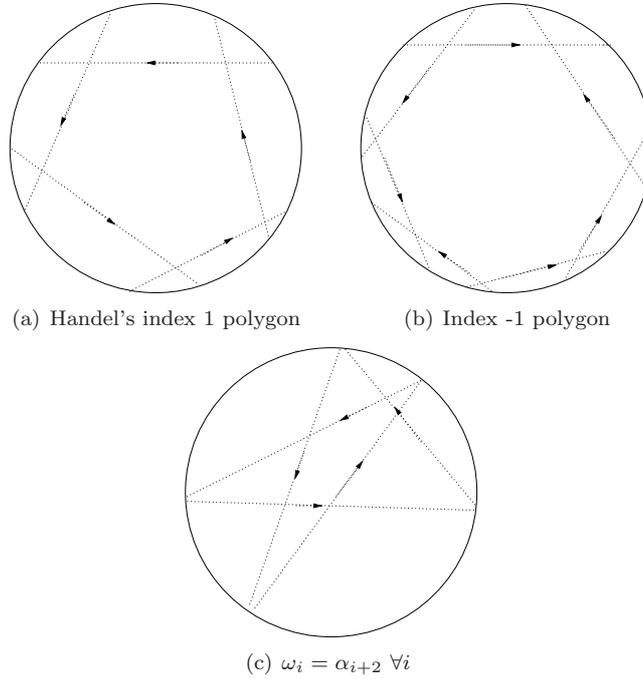


Figure 1.3: Polygons of different indices.

We say that the homeomorphism $f : \mathbb{D} \rightarrow \mathbb{D}$ *realizes* P if f realizes the cycle induced by P .

As a consequence of Theorem 1.1, we will show:

Corollary 1.2. *Suppose that $f : \mathbb{D} \rightarrow \mathbb{D}$ is an orientation preserving homeomorphism which realizes a compact convex polygon $P \subset \mathbb{D}$ and can be extended to a homeomorphism of $\mathbb{D} \cup (\cup_{i \in \mathbb{Z}/n\mathbb{Z}} \{\alpha_i, \omega_i\})$.*

If $i(P) \neq 0$, then f has a fixed point. Furthermore, if $i(P) = 1$, then there exists a simple closed curve $C \subset \mathbb{D}$ of index 1.

The three polygons appearing in Figure 1.3 satisfy the hypothesis of this corollary. Note, however, that the situation illustrated in (b) is not contained in

the hypothesis of Theorem 1.1, as the order of the points $\{\alpha_i\}, \{\omega_i\}$ is neither elliptic, nor hyperbolic.

The structure of this work is the following. In Chapter 2 we will recall the notion of brick decompositions (the main tool of this article), and relate them to the existence of simple closed curves of index 1. We also state the results we use from [11]. In Chapter 3 we use brick decompositions to define and study configurations of “repellers and attractors at infinity”, with orbits connecting repeller/attractor pairs. We prove that the existence of configurations of this kind guarantees the existence of a fixed point, or even a simple closed curve of index 1. Chapter 4 is devoted to give a quick and easy proof of Le Calvez’s refinement of the classic Handel’s theorem; this proof is contained in Proposition 3.9 and Proposition 4.1. In Chapter 5 we prove that whenever an elliptic or hyperbolic cycle of links is realized, either one can construct one of the configurations studied in Chapter 3, or there exists a simple closed curve of index 1. Finally, in Chapter 6 we give a proof of Corollary 1.2.

I am indebted to Patrice Le Calvez. Not only he suggested me to study possible generalizations of Handel’s theorem, but he guided my research through a great number of discussions. His careful reading and correction of thousands of versions of this paper led to its final form.

Chapter 2

Preliminaries

2.1 Brick decompositions

A *brick decomposition* \mathcal{D} of an orientable surface M is a 1- dimensional singular submanifold $\Sigma(\mathcal{D})$ (the *skeleton* of the decomposition), with the property that the set of singularities V is discrete and such that every $\sigma \in V$ has a neighborhood U for which $U \cap (\Sigma(\mathcal{D}) \setminus V)$ has exactly three connected components. We have illustrated two brick decompositions in Figure 2.1. The *bricks* are the closure of the connected components of $M \setminus \Sigma(\mathcal{D})$ and the *edges* are the closure of the connected components of $\Sigma(\mathcal{D}) \setminus V$. We will write E for the set of edges, B for the set of bricks and finally $\mathcal{D} = (V, E, B)$ for a brick decomposition.

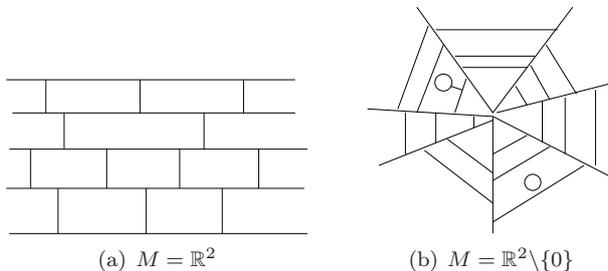


Figure 2.1: Brick decompositions

Let $\mathcal{D} = (V, E, B)$ be a brick decomposition of M . We say that $X \subset B$ is connected if given two bricks $b, b' \in X$, there exists a sequence $(b_i)_{0 \leq i \leq n}$, where $b_0 = b, b_n = b'$ and such that b_i and b_{i+1} have non empty intersection, $i \in \{0, \dots, n-1\}$. Whenever two bricks b and b' have no empty intersection, we say that they are *adjacent*. Moreover, we say that a brick b is *adjacent to a subset* $X \subset B$ if $b \notin X$, but b is adjacent to one of the bricks in X . We say that $X \subset B$ is adjacent to $X' \subset B$ if X and X' have no common bricks but there exists $b \in X$ and $b' \in X'$ which are adjacent.

From now on we will identify a subset X of B with the closed subset of M formed by the union of the bricks in X . By making so, there may be ambiguities (for instance, two adjacent subsets of B have empty intersection in B and

nonempty intersection in M), but we will point it out when this happens. We remark that ∂X is a one-dimensional topological manifold and that the connectedness of $X \subset B$ is equivalent to the connectedness of $X \subset M$ and to the connectedness of $\text{Int}(X) \subset M$ as well. We say that the decomposition \mathcal{D}' is a *subdecomposition* of \mathcal{D} if $\Sigma(\mathcal{D}') \subset \Sigma(\mathcal{D})$.

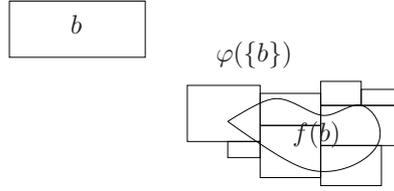
If $f : M \rightarrow M$ is a homeomorphism, we define the application $\varphi : \mathcal{P}(\mathcal{B}) \rightarrow \mathcal{P}(\mathcal{B})$ as follows:

$$\varphi(X) = \{b \in B : f(X) \cap b \neq \emptyset\}.$$

We remark that $\varphi(X)$ is connected whenever X is.

We define analogously an application $\varphi_- : \mathcal{P}(\mathcal{B}) \rightarrow \mathcal{P}(\mathcal{B})$:

$$\varphi_-(X) = \{b \in B : f^{-1}(X) \cap b \neq \emptyset\}.$$



We define the *future* $[b]_{\geq}$ and the *past* $[b]_{\leq}$ of a brick b as follows:

$$[b]_{\geq} = \bigcup_{k \geq 0} \varphi^k(\{b\}), \quad [b]_{\leq} = \bigcup_{k \geq 0} \varphi_-^k(\{b\}).$$

We also define the *strict future* $[b]_{>}$ and the *strict past* $[b]_{<}$ of a brick b :

$$[b]_{>} = \bigcup_{k > 0} \varphi^k(\{b\}), \quad [b]_{<} = \bigcup_{k > 0} \varphi_-^k(\{b\}).$$

We say that a set $X \subset B$ is an *attractor* if it verifies $\varphi(X) \subset X$; this is equivalent in M to the inclusion $f(X) \subset \text{Int}(X)$. A *repeller* is any set which verifies $\varphi_-(X) \subset X$. In this way, the future of any brick is an attractor, and the past of any brick is a repeller. We observe that $X \subset B$ is a repeller if and only if $B \setminus X$ is an attractor.

Remark 2.1. The following properties can be deduced from the fact that $X \subset B$ is an attractor if and only if $f(X) \subset \text{Int}(X)$:

1. If $X \subset B$ is an attractor and $b \in X$, then $[b]_{\geq} \subset X$; if $X \subset B$ is a repeller and $b \in X$, then $[b]_{\leq} \subset X$,
2. if $X \subset B$ is an attractor and $b \notin X$, then $[b]_{\leq} \cap X = \emptyset$; if $X \subset B$ is a repeller and $b \notin X$, then $[b]_{\geq} \cap X = \emptyset$,

3. if $b \in B$ is adjacent to the attractor $X \subset B$, then $[b]_{>} \cap X \neq \emptyset$; if $b \in B$ is adjacent to the repeller $X \subset B$, then $[b]_{<} \cap X \neq \emptyset$;
4. two attractors are disjoint as subsets of B if and only if they are disjoint as subsets of M ; in other words, two disjoint (in B) attractors cannot be adjacent; respectively two disjoint (in B) repellers cannot be adjacent;

The following conditions are equivalent:

$$b \in [b]_{>}, [b]_{>} = [b]_{\geq}, b \in [b]_{<}, [b]_{<} = [b]_{\leq}, [b]_{<} \cap [b]_{\geq} \neq \emptyset, [b]_{\leq} \cap [b]_{>} \neq \emptyset.$$

The existence of a brick $b \in B$ for which any of these conditions is satisfied is equivalent to the existence of a *closed chain of bricks*, i.e a family $(b_i)_{i \in \mathbb{Z}/r\mathbb{Z}}$ of bricks such that for all $i \in \mathbb{Z}/r\mathbb{Z}$, $\cup_{k \geq 1} f^k(b_i) \cap b_{i+1} \neq \emptyset$.

In general, a *chain* for $f \in \text{Homeo}(M)$ is a family $(X_i)_{0 \leq i \leq r}$ of subsets of M such that for all $0 \leq i \leq r-1$, $\cup_{k \geq 1} f^k(X_i) \cap X_{i+1} \neq \emptyset$. We say that the chain is closed if $X_r = X_0$.

We say that a subset $X \subset M$ is *free* if $f(X) \cap X = \emptyset$.

We say that a brick decomposition $\mathcal{D} = (V, E, B)$ is *free* if every $b \in B$ is a free subset of M . If f is fixed point free it is always possible, taking sufficiently small bricks, to construct a free brick decomposition.

We recall the definition of *maximal free decomposition*, which was introduced by Sauzet in his doctoral thesis [15]. Let f be a fixed point free homeomorphism of a surface M . We say that \mathcal{D} is a maximal free decomposition if \mathcal{D} is free and any strict subdecomposition is no longer free. Applying Zorn's lemma, it is always possible to construct a maximal free subdecomposition of a given brick decomposition \mathcal{D} .

2.2 Brouwer Theory background.

We say that $\Gamma : [0, 1] \rightarrow \overline{\mathbb{D}}$ is an *arc*, if it is continuous and injective. We say that an arc Γ joins $x \in \overline{\mathbb{D}}$ to $y \in \overline{\mathbb{D}}$, if $\Gamma(0) = x$ and $\Gamma(1) = y$. We say that an arc Γ joins $X \subset \overline{\mathbb{D}}$ to $Y \subset \overline{\mathbb{D}}$, if Γ joins $x \in X$ to $y \in Y$.

Fix $f \in \text{Homeo}^+(\mathbb{D})$. An arc γ joining $z \notin \text{Fix}(f)$ to $f(z)$ such that $f(\gamma) \cap \gamma = \{z, f(z)\}$ if $f^2(z) = z$ and $f(\gamma) \cap \gamma = \{f(z)\}$ otherwise, is called a *translation arc*.

Proposition 2.2. (Brouwer's translation lemma [1], [2], [4] or [7]) *If any of the two following hypothesis is satisfied, then there exists a simple closed curve of index 1:*

1. *there exists a translation arc γ joining $z \in \text{Fix}(f^2) \setminus \text{Fix}(f)$ to $f(z)$;*
2. *there exists a translation arc γ joining $z \notin \text{Fix}(f^2)$ to $f(z)$ and an integer $k \geq 2$ such that $f^k(\gamma) \cap \gamma \neq \emptyset$.*

If $z \notin \text{Fix}(f)$, there exists a translation arc containing z (this is not difficult to prove once you have that the connected components of $\mathbb{D} \setminus \text{Fix}(f)$ are invariant; for this last fact you can see [10] for an easy proof in dimension 2, or [3] for a general result in dimension n). We deduce:

Corollary 2.3. *If $\text{Per}(f) \setminus \text{Fix}(f) \neq \emptyset$, then there exists a simple closed curve of index 1.*

Proposition 2.4. (Franks' lemma [6]) *If there exists a closed chain of free, open and pairwise disjoint disks for f , then there exists a simple closed curve of index 1.*

Following Le Calvez [11], we will say that f is *recurrent* if there exists a closed chain of free, open and pairwise disjoint disks for f .

The following proposition is a refinement of Franks' lemma due to Guillou and Le Roux (see [13], page 39).

Proposition 2.5. *Suppose there exists a closed chain $(X_i)_{i \in \mathbb{Z}/r\mathbb{Z}}$ for f of free subsets whose interiors are pairwise disjoint and which verify the following property: given any two points $z, z' \in X_i$ there exists an arc γ joining z and z' such that $\gamma \setminus \{z, z'\} \subset \text{Int}(X_i)$. Then, f is recurrent.*

We deduce:

Proposition 2.6. *Let $\mathcal{D} = (V, E, B)$ be a free brick decomposition of $\mathbb{D} \setminus \text{Fix}(f)$. If there exists $b \in B$ such that $b \in [b]_{>}$, then f is recurrent.*

2.3 Previous results.

Fix $f \in \text{Homeo}^+(\mathbb{D})$, different from the identity map and *non-recurrent*. We will make use of the following two propositions from [11] (both of them depend on the non-recurrent character of f). The first one (Proposition 2.2 in [11]) is a refinement of a result already appearing in [15]; the second one is Proposition 3.1 in [11].

Proposition 2.7. *[15],[11] Let $\mathcal{D} = (V, E, B)$ be a free maximal brick decomposition of $\mathbb{D} \setminus \text{Fix}(f)$. Then, the sets $[b]_{\geq}$, $[b]_{>}$, $[b]_{\leq}$ and $[b]_{<}$ are connected. In particular every connected component of an attractor is an attractor, and every connected component of a repeller is a repeller.*

Proposition 2.8. *[11] If f satisfies the hypothesis of Theorem 1.1, then for all $i \in \mathbb{Z}/n\mathbb{Z}$ we can find a sequence of arcs $(\gamma_i^k)_{k \in \mathbb{Z}}$ such that:*

- each γ_i^k is a translation arc from $f^k(z_i)$ to $f^{k+1}(z_i)$,
- $f(\gamma_i^k) \cap \gamma_i^{k'} = \emptyset$ if $k' < k$,
- the sequence $(\gamma_i^k)_{k \leq 0}$ converges to $\{\alpha_i\}$ in the Hausdorff topology,
- the sequence $(\gamma_i^k)_{k \geq 0}$ converges to $\{\omega_i\}$ in the Hausdorff topology.

This result is a consequence of Brouwer's translation lemma and the hypothesis on the orbits of the points $(z_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$. In particular, the extension hypothesis of Theorem 1.1 is used. Proposition 2.8 allows us to construct a particular brick decomposition suitable for our purposes:

Lemma 2.9. For every $i \in \mathbb{Z}/n\mathbb{Z}$, take U_i^- a neighbourhood of α_i in $\overline{\mathbb{D}}$ and U_i^+ a neighbourhood of ω_i in $\overline{\mathbb{D}}$ such that $U_i^- \cap U_i^+ = \emptyset$. There exists two families $(b_i^l)_{i \in \mathbb{Z}/n\mathbb{Z}, l \geq 1}$ and $(b_i^l)_{i \in \mathbb{Z}/n\mathbb{Z}, l \leq -1}$ of closed disks in \mathbb{D} , and a family of integers $(l_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ such that:

1. each b_i^l is free and contained in U_i^- ($l \leq -1$) or in U_i^+ ($l \geq 1$),
2. $\text{Int}(b_i^l) \cap \text{Int}(b_i^{l'}) = \emptyset$, if $l \neq l'$,
3. for every $k > 1$ the sets $(b_i^l)_{1 \leq l \leq k}$ and $(b_i^l)_{-k \leq l \leq -1}$ are connected,
4. for all $i \in \mathbb{Z}/n\mathbb{Z}$, $\partial \cup_{l \in \mathbb{Z} \setminus \{0\}} b_i^l$ is a one dimensional submanifold,
5. if $x \in \mathbb{D}$, then x belongs to at most two different disks in the family $(b_i^l)_{l \in \mathbb{Z} \setminus \{0\}}$, $i \in \mathbb{Z}/n\mathbb{Z}$,
6. for all $i \in \mathbb{Z}/n\mathbb{Z}$ $f^{l_i+1}(z_i) \in \text{Int}(b_i^{l_i+1})$ for all $l \geq 0$, and $f^{-l_i-1}(z_i) \in \text{Int}(b_i^{-l_i-1})$ for all $l \geq 0$,
7. $f^k(z_j) \in b_i^l$ if and only if $j = i$ and $k = l_i + l - 1$,
8. the sequence $(b_i^l)_{l \geq 1}$ converges to $\{\omega_i\}$ in the Hausdorff topology and the sequence $(b_i^l)_{l \leq -1}$ converges to $\{\alpha_i\}$ in the Hausdorff topology.

The idea is to construct trees $T_i^- \subset U_i^-$, $T_i^+ \subset U_i^+$, $i \in \mathbb{Z}/n\mathbb{Z}$ by deleting the loops of the curves $\prod_{k \geq -1} \gamma_i^k \cap U_i^-$ and $\prod_{k \leq 1} \gamma_i^k \cap U_i^+$ respectively, and then thickening these trees to obtain the families $(b_i^l)_{i \in \mathbb{Z}/n\mathbb{Z}, l \geq 1}$ and $(b_i^l)_{i \in \mathbb{Z}/n\mathbb{Z}, l \leq -1}$. We refer the reader to [11] for details. We have illustrated these families in Figure 2.2.

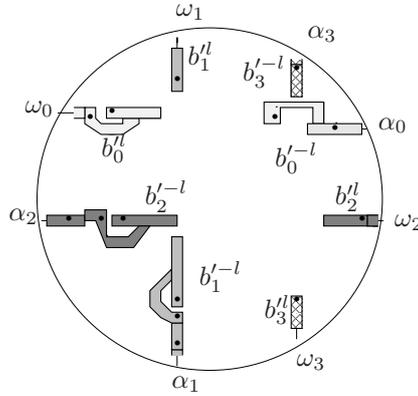


Figure 2.2: The families b_i^l

Remark 2.10. The fact that the sequence $(b_i^l)_{l \geq 1}$ converges in the Hausdorff topology to ω_i , implies that we can find an arc $\Gamma_i^+ : [0, 1] \rightarrow \text{Int}(\cup_{l \geq 0} b_i^l) \cup \{\omega_i\}$ such that $\Gamma_i^+(1) = \omega_i$, $i \in \mathbb{Z}/n\mathbb{Z}$. Similarly, we can find an arc $\Gamma_i^- : [0, 1] \rightarrow \text{Int}(\cup_{l \leq 0} b_i^l) \cup \{\alpha_i\}$ such that $\Gamma_i^-(1) = \alpha_i$, $i \in \mathbb{Z}/n\mathbb{Z}$.

Remark 2.11. If the points α_i, ω_i , $i \in \mathbb{Z}/n\mathbb{Z}$, are all different, the bricks b_i^l , $i \in \mathbb{Z}/n\mathbb{Z}$, $l \in \mathbb{Z} \setminus \{0\}$ can be constructed as to have pairwise disjoint interiors.

Corollary 2.12. *If the points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$, are all different, there exists a free brick decomposition (V, E, B) of $\mathbb{D} \setminus \text{Fix}(f)$ such that for all $i \in \mathbb{Z}/n\mathbb{Z}$ and all $l \in \mathbb{Z} \setminus \{0\}$, there exists $b_i^l \in B$ such that $b_i^l \subset b_i^l$.*

We will make use of proposition 2.7 in the next chapter. Propositions 2.8 and 2.9 will not be used until chapter 5.

Chapter 3

Repeller/Attractor configurations at infinity

3.1 Cyclic order at infinity.

Let $(a_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ be a family of non-empty, pairwise disjoint, closed, connected subsets of \mathbb{D} , such that $\bar{a}_i \cap \partial\mathbb{D} \neq \emptyset$ and $U = \mathbb{D} \setminus (\cup_{i \in \mathbb{Z}/n\mathbb{Z}} a_i)$ is a connected open set. As U is connected, and its complementary set in \mathbb{C}

$$\{z \in \mathbb{C} : |z| \geq 1\} \cup \cup_{i \in \mathbb{Z}/n\mathbb{Z}} a_i$$

is connected, U is simply connected.

With these hypotheses, there is a natural cyclic order on the sets $\{a_i\}$. Indeed, U is conformally isomorphic to the unit disc via the Riemann map $\varphi : U \rightarrow \mathbb{D}$, and one can consider the Carathéodory's extension of φ ,

$$\hat{\varphi} : \hat{U} \rightarrow \overline{\mathbb{D}},$$

which is a homeomorphism between the prime ends completion \hat{U} of U and the closed unit disk $\overline{\mathbb{D}}$. The set \hat{J}_i of prime ends whose impression is contained in a_i is open and connected. It follows that the images $J_i = \hat{\varphi}(\hat{J}_i)$ are pairwise disjoint open intervals in S^1 , and are therefore cyclically ordered following the positive orientation in the circle.

We devote this section to explaining the preceding paragraph to the reader unfamiliarized with prime end theory.

We say that $z \in \partial U$ is *accessible* if there exists an arc $\gamma : [0, 1] \rightarrow U \cup \{z\}$ such that $\gamma(1) = z$.

Lemma 3.1 ([14]). *If $z \in \partial U$ is accessible, then $\lim_{t \rightarrow 1} \varphi(\gamma(t)) = x \in S^1$.*

Lemma 3.2 ([14]). *The set of accessible points in ∂U is dense in ∂U , and the set H of points in S^1 that are images of accessible points in ∂U is dense in S^1 .*

A *crosscut* in U is an arc $\gamma : [0, 1] \rightarrow \overline{U}$ such that $\gamma \cap \partial U = \{\gamma(0), \gamma(1)\}$. Any crosscut separates U in two connected components. We choose a basepoint $b_0 \in U$, and for any crosscut γ which is disjoint from b_0 , we define the neighbourhood $N(\gamma)$ as to be the connected component of $U \setminus \gamma$ which does not contain b_0 .

A *fundamental chain* is a sequence of disjoint crosscuts $(\gamma_k)_{k \in \mathbb{N}}$ such that

$$N(\gamma_1) \supset N(\gamma_2) \supset \dots,$$

and such that $\bigcap_{k \in \mathbb{N}} \overline{N(\gamma_k)}$ does not contain two different accessible points. The *impression* of a fundamental chain is

$$\bigcap_{k \in \mathbb{N}} \overline{N(\gamma_k)}.$$

Lemma 3.3 ([14]). *If $(\gamma_k)_{k \in \mathbb{N}}$ is a fundamental chain, then $\bigcap_{k \in \mathbb{N}} \overline{\varphi(N(\gamma_k))}$ contains a unique point $x \in S^1$.*

A fundamental chain $(\gamma_k)_{k \in \mathbb{N}}$ is said to determine a *prime end* of the domain U , and the point x in the preceding lemma is called the *image of the prime end* under φ . Prime ends are said to be different if they have distinct images, and the same if they have the same image.

Lemma 3.4 ([14]). *If $x \in S^1$, then there exists a fundamental chain $(\gamma_k)_{k \in \mathbb{N}}$ such that x is the image of the prime end it determines.*

The two preceding lemmas establish a one-to-one correspondence between the set of prime ends in U and points in the circle S^1 .

We define the *impression* of a prime end ξ as to be the impression of any representative chain. We will note $I(\xi)$ the impression of the prime end ξ . For all $i \in \mathbb{Z}/n\mathbb{Z}$, we define \hat{J}_i as to be the set of prime ends whose impression is contained in a_i . We define $J_i \subset S^1$ as the set of images of prime ends in \hat{J}_i .

Lemma 3.5. *Each J_i is open and connected.*

Proof. We will first prove that J_i is open. Take a point $x \in J_i$, and a fundamental chain $(\gamma_k)_{k \in \mathbb{N}}$ defining the prime end ξ corresponding to x (see Lemmas 3.3, 3.4). As $x \in J_i$, the impression of $I(\xi)$ is contained in a_i . Take an open neighbourhood V of $I(\xi)$ in \mathbb{D} such that $V \cap \partial U \subset a_i$. So, if k is big enough $\overline{N(\gamma_k)} \subset V$. In particular, there exists $k \in \mathbb{N}$ such that the whole interval $\overline{\varphi(N(\gamma_k))} \cap S^1$ is contained in J_i . So, x is an interior point of J_i .

To prove that J_i is connected, let us suppose that it has two different connected components I_1 and I_2 . So, there exists two points $x_1, x_2 \in S^1 \setminus J_i$ such that any arc joining I_1 and I_2 separates x_1 from x_2 in $\overline{\mathbb{D}}$.

Lemma 3.2 and the fact that each I_l is an open interval, $l = 1, 2$ imply that we can find two points point $z_1 \in I_1 \cap H$ and $z_2 \in I_2 \cap H$. We can find an arc $\beta \subset \mathbb{D} \cup \{z_1, z_2\}$ joining z_1 and z_2 such that $\beta' = \overline{\varphi^{-1}(\beta \cap \mathbb{D})}$ is an arc joining two points z'_1 and z'_2 in ∂U . Then, both z'_1 and z'_2 belong to a_i . Let C_1 and C_2 be the connected components of $U \setminus \beta'$. One of this components, say C_1 has the property that $\overline{C_1} \cap \partial U \subset a_i$. Otherwise, a_i would not be connected, as we could find an embedding Γ of \mathbb{R} such that

$$\Gamma \wedge \beta' = 1, \text{ and } \Gamma \cap a_i = \emptyset$$

The closure of the corresponding connected component of $\overline{\mathbb{D}} \setminus \beta$ contains one of the points x_1 or x_2 . But then, $I(\xi) \subset a_i$, where ξ is the prime end corresponding to this point. This contradiction finishes the proof. \square

We deduce that $(J_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ is a family of pairwise disjoint open intervals in S^1 . This allows us to define a cyclic order on the sets $(a_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$. We say that a_j follows a_i in the cyclic order induced by the orientation of \mathbb{D} (and we note $a_i \rightarrow a_j$), if and only if J_i follows J_j in the natural (positive) cyclic order on S^1 .

Lemma 3.6. *If $z \in a_i$ is an accessible point, $z = \gamma(1)$, $\gamma : [0, 1] \rightarrow U \cup \{z\}$, then $\lim_{t \rightarrow 1} \varphi(\gamma(t)) = x \in J_i$.*

Proof. The point x corresponds to a prime end ξ whose impression contains z , and as z is accessible, it is the only accessible point on this impression. If $I(\xi)$ is not contained in a_i , as $I(\xi)$ is a connected set containing z , and the sets $\{a_j\}$ are pairwise disjoint, there exists $x \in \partial D \cap I(\xi)$. As the set of accessible points is dense in ∂U , it follows that there must be an accessible point in $I(\xi) \setminus \{z\}$, a contradiction. \square

Lemma 3.7. *If $(x_n)_{n \in \mathbb{N}} \subset \mathbb{D}$ is such that $\lim_{n \rightarrow \infty} x_n = x \in J_i$, then*

$$\overline{\{\varphi^{-1}(x_n) : n \in \mathbb{N}\}} \cap \partial U \subset a_i.$$

Proof. As $x \in J_i$, x is the image of a prime end ξ such that $I(\xi) \subset a_i$. Let Take a neighbourhood V of $I(\xi)$ such that $V \cap \partial U \subset a_i$, and a fundamental chain $(\gamma_k)_{k \in \mathbb{N}}$ defining ξ . So, if k is large enough, $N(\gamma_k) \subset V$. Besides, $\varphi(N(\gamma_k))$ is a neighbourhood of x in $\overline{\mathbb{D}}$. So, if n is large enough, $\varphi^{-1}(x_n) \in V$, and $\overline{\{\varphi^{-1}(x_n) : n \in \mathbb{N}\}} \cap \partial U \subset a_i$. \square

Lemma 3.8. *If γ is a crosscut from a_i to a_j , then:*

1. $\gamma' = \overline{\varphi(\gamma \cap U)}$ is an arc joining J_i and J_j ,
2. if γ' separates J_k and J_l , then $a_i \cup \gamma \cup a_j$ separates a_k from a_l .

Proof. 1. Lemma 3.1 implies that γ' is an arc, and Lemma 3.6 implies that it joins J_i and J_j .

2. If a_k and a_l belong to the same connected component of $\mathbb{D} \setminus (a_i \cup \gamma \cup a_j)$, we can find a crosscut α from a_k to a_l such that $\alpha \cap \gamma = \emptyset$. Then, $\alpha' = \overline{\varphi(\alpha \cap U)}$ is an arc joining J_k and J_l , and such that $\alpha' \cap \gamma' = \emptyset$. This contradicts that γ' separates J_k and J_l . \square

3.2 Repeller/Attractor configurations.

We fix $f \in \text{Homeo}^+(\mathbb{D})$ together with a free maximal decomposition in bricks $\mathcal{D} = (V, E, B)$ of $\mathbb{D} \setminus \text{Fix}(f)$.

Let $(R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ and $(A_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ be two families of connected, pairwise disjoint subsets of B such that :

1. For all $i \in \mathbb{Z}/n\mathbb{Z}$:
 - (a) R_i is a repeller and A_i is an attractor;

- (b) there exists non-empty, closed, connected subsets of \mathbb{D} , $r_i \subset \text{Int}(R_i)$, $a_i \subset \text{Int}(A_i)$ such that $\overline{r_i} \cap \partial\mathbb{D} \neq \emptyset$ and $\overline{a_i} \cap \partial\mathbb{D} \neq \emptyset$,

2. $\mathbb{D} \setminus (\cup_{i \in \mathbb{Z}/n\mathbb{Z}} (a_i \cup r_i))$ is a connected open set.

We say that the pair $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ is a *Repeller/Attractor configuration of order n* .

We will note

$$\mathcal{E} = \{R_i, A_i : i \in \mathbb{Z}/n\mathbb{Z}\}.$$

Property 2 in the previous definition allows us to give a cyclic order to the sets $r_i, a_i, i \in \mathbb{Z}/n\mathbb{Z}$ (see the previous section).

We say that a Repeller/Attractor configuration of order $n \geq 3$ is an *elliptic configuration* if :

1. the cyclic order of the sets $r_i, a_i, i \in \mathbb{Z}/n\mathbb{Z}$, satisfies the *elliptic order property*:

$$a_0 \rightarrow r_2 \rightarrow a_1 \rightarrow \dots \rightarrow a_i \rightarrow r_{i+2} \rightarrow a_{i+1} \rightarrow \dots \rightarrow a_{n-1} \rightarrow r_1 \rightarrow a_0.$$

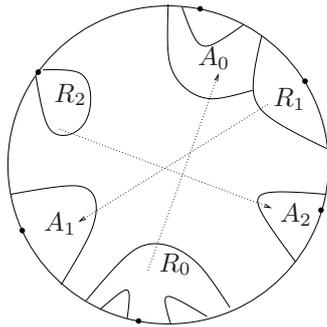
2. for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exists a brick $b_i \in R_i$ such that $b_i \cap A_i \neq \emptyset$;

We say that a Repeller/Attractor configuration is a *hyperbolic configuration* if:

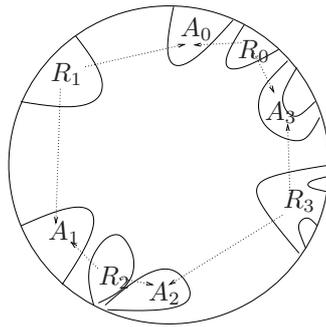
1. the cyclic order of the sets $r_i, a_i, i \in \mathbb{Z}/n\mathbb{Z}$, satisfies the *hyperbolic order property*:

$$r_0 \rightarrow a_0 \rightarrow r_1 \rightarrow a_1 \rightarrow \dots \rightarrow r_i \rightarrow a_i \rightarrow r_{i+1} \rightarrow a_{i+1} \rightarrow \dots \rightarrow r_{n-1} \rightarrow a_{n-1} \rightarrow r_0.$$

2. for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exists two bricks $b_i^i, b_i^{i-1} \in R_i$ such that $[b]_{i>}^i \cap A_i \neq \emptyset$, and $[b]_{i>}^{i-1} \cap A_{i-1} \neq \emptyset$;



(a) An elliptic configuration



(b) A hyperbolic configuration

We will show:

Proposition 3.9. *If there exists an elliptic configuration of order $n \geq 3$, then f is recurrent.*

Proposition 3.10. *If there exists a hyperbolic configuration of order $n \geq 2$, then $\text{Fix}(f) \neq \emptyset$.*

One could think that Proposition 3.10 should give a negative-index fixed point, as the example that comes to mind is that of a saddle point (see the figure below).

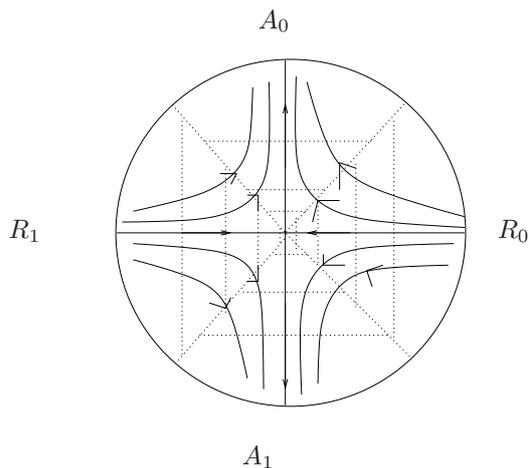


Figure 3.1: A hyperbolic configuration arising from a saddle point.

However, this is not the case, as the following example shows. (Compare with the remarks following the statement of Theorem 1.1.)

Example 1. Let f_1 be the time-one map of the flow whose orbits are drawn in the following figure:

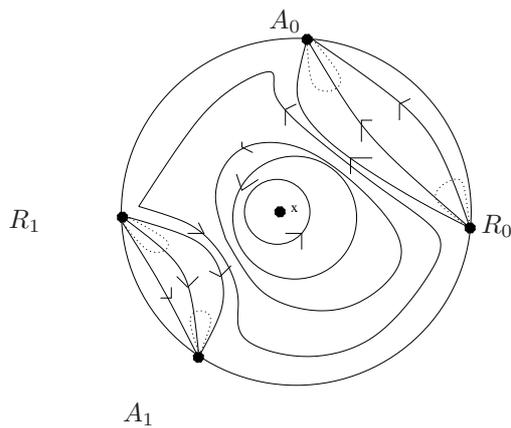


Figure 3.2: A hyperbolic configuration without a fixed point of negative index.

One can perturb f_1 in a homeomorphism f such that:

1. $\text{Fix}(f) = \text{Fix}(f_1) = \{x\}$,

2. $f = f_1$ in a neighbourhood of x ,
3. $f = f_1$ in a neighbourhood of S^1 (and so f preserves the repellers and attractors drawn in dotted lines),
4. there is an f -orbit from R_0 to A_1 ,
5. there is an f -orbit from R_1 to A_0 .

So, $((R_i)_{i \in \mathbb{Z}/2\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/2\mathbb{Z}})$ is a hyperbolic configuration for f , but the only fixed point f has is an index-one fixed point.

We define an order relationship in the set of Repeller/Attractor configurations of order n :

$$((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}}) \leq ((R'_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A'_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$$

if and only if for all $i \in \mathbb{Z}/n\mathbb{Z}$

$$A_i \subseteq A'_i \text{ and } R_i \subseteq R'_i.$$

As the union of attractors (resp. repellers) is an attractor (resp. repeller), the existence of an elliptic (resp. hyperbolic) Repeller/Attractor configuration implies the existence of a maximal elliptic (resp. hyperbolic) Repeller/Attractor configuration by Zorn's lemma.

Example 2. The hyperbolic configuration in Figure 3.1 is maximal.

We will assume for the rest of this chapter that f is non-recurrent. In particular, for any brick $b \in B$, the sets $[b]_{\geq}$, $[b]_{>}$, $[b]_{\leq}$ and $[b]_{<}$ are connected (see Proposition 2.7).

The following lemma is an immediate consequence of the maximality of configurations:

Lemma 3.11. *Let $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ be a maximal configuration (either elliptic or hyperbolic), and consider a brick $b \in B \setminus \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$. If b is adjacent to R_i , then there exists, $j \neq i$, such that $[b]_{<} \cap R_j \neq \emptyset$ in B . If b is adjacent to A_i , then there exists, $j \neq i$, such that $[b]_{>} \cap A_j \neq \emptyset$ in B .*

Proof. Let $b \in B \setminus \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ be adjacent to R_i . As both R_i and $[b]_{\leq}$ are connected and they intersect, it follows that the repeller $R = [b]_{\leq} \cup R_i$ is connected. As our configuration is maximal and $R_i \subsetneq R$, there exists $X \in \mathcal{E} \setminus \{R_i\}$, such that $R \cap X \neq \emptyset$ (in B). As the sets in \mathcal{E} are pairwise disjoint, and b does not belong to X , this implies that $[b]_{<} \cap X \neq \emptyset$ (in B). So, $X = R_j$ for some $j \neq i$, because $[b]_{\leq}$ cannot intersect any attractor (see Remark 2.1, item 2). The second statement in the lemma is proved analogously. \square

We say that a brick $b \in B$ is a *connexion brick* from R_j to A_j if:

1. $b \in B \setminus \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$,

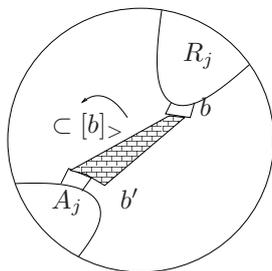


Figure 3.3: A connexion brick.

2. b is adjacent to R_j and
3. $[b]_{>}$ contains a brick $b' \in B \setminus \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ which is adjacent to A_j .

Lemma 3.12. *Let $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ be a maximal elliptic or hyperbolic configuration. The following two conditions guarantee the existence of a connexion brick from R_i to A_i :*

1. *There exists a brick $b \notin \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ which is adjacent to both R_i and A_i ,*
2. *R_i is not adjacent to A_i .*

Proof. 1. Let $b' \notin \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ be adjacent to both R_i and A_i . As a subset of B , the repeller $[b']_{<}$ meets a repeller R_j different from R_i (Lemma 3.11), meets R_i because b' is adjacent to R_i (Remark 2.1, item 3), and does not meet any A_j , $j \in \mathbb{Z}/n\mathbb{Z}$ (Remark 2.1, item 2). As it is connected, $[b']_{<}$ contains a brick b which is adjacent to R_i , which implies that $b \notin \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ (Remark 2.1, item 4). As $b' \in [b]_{>}$, and b' is adjacent to A_i , b is a connexion brick from R_i to A_i .

2. Assume that R_i is not adjacent to A_i . We know there exists $b_i \in R_i$ such that $[b_i]_{\geq} \cap A_i \neq \emptyset$. As $[b_i]_{\geq}$ is connected, it contains a brick b' adjacent to A_i . This brick b' is not contained in R_i ; otherwise, R_i would be adjacent to A_i . Neither it is contained in any attractor or in any repeller other than R_i (Remark 2.1, items 2 and 4). Therefore, $b' \notin \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$.

As $b_i \in [b']_{\leq}$ and $[b']_{\leq}$ is connected, $[b']_{\leq}$ contains a brick b adjacent to R_i . If $b \in [b']_{<}$, then b is a connexion brick from R_i to A_i (again, $b \notin \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ by Remark 2.1, items 2 and 4). If $b = b'$, then b is adjacent to both R_i and A_i and we are done by the previous item. □

Remark 3.13. Connexion bricks do not always exist; Figure 3.1 exhibits an example. Of course, none of the conditions of Lemma 3.12 is satisfied. Indeed, in this example $\cup_{i \in \mathbb{Z}/2\mathbb{Z}} (R_i \cup A_i) = B$ and R_i is adjacent to A_i for all $i \in \mathbb{Z}/2\mathbb{Z}$.

3.3 The elliptic case.

The following consequences of the elliptic order property will be used in the proof of Proposition 3.9:

Lemma 3.14. *Let $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ be an elliptic configuration.*

1. *If $C \subset B$ is a connected set containing both R_i and A_i , and $C \cap (R_{i+1} \cup A_{i+1}) = \emptyset$ in B , then R_{i+1} and A_{i+1} belong to different connected components of $\mathbb{D} \setminus \text{Int}(C)$; in particular $R_{i+1} \cap A_{i+1} = \emptyset$ in \mathbb{D} .*
2. *If $C \subset B$ is a connected set containing both R_i and R_{i+1} , and $C \cap (R_{i-1} \cup A_{i-1}) = \emptyset$ in B , then R_{i-1} and A_{i-1} belong to different connected components of $\mathbb{D} \setminus \text{Int}(C)$; in particular $R_{i-1} \cap A_{i-1} = \emptyset$ in \mathbb{D} .*
3. *If $C \subset B$ is a connected set containing every repeller R_i , and disjoint (in B) from every attractor A_i , then the n attractors $\{A_i\}$ belong to n different connected components of $\mathbb{D} \setminus \text{Int}(C)$.*

Proof. 1. First we remark that $C \cap (R_{i+1} \cup A_{i+1}) = \emptyset$ in B implies $\text{Int}(R_{i+1}) \cap \text{Int}(C) = \emptyset$ and $\text{Int}(A_{i+1}) \cap \text{Int}(C) = \emptyset$. Besides, $\text{Int}(C)$ is a connected set containing both r_i and a_i . So, the elliptic order property implies that r_{i+1} and a_{i+1} belong to different connected components of $\mathbb{D} \setminus \text{Int}(C)$. Now, $\text{Int}(R_{i+1})$ and $\text{Int}(A_{i+1})$ belong to different connected components of $\mathbb{D} \setminus \text{Int}(C)$. As each connected component of $\mathbb{D} \setminus \text{Int}(C)$ is closed (in \mathbb{D}), we obtain that R_{i+1} and A_{i+1} belong to different connected components of $\mathbb{D} \setminus \text{Int}(C)$; in particular $R_{i+1} \cap A_{i+1} = \emptyset$ in \mathbb{D} .

2. As before, we know that $\text{Int}(R_{i-1}) \cap \text{Int}(C) = \emptyset$ and $\text{Int}(A_{i-1}) \cap \text{Int}(C) = \emptyset$. Besides, $\text{Int}(C)$ is a connected set containing both r_i and r_{i+1} . So, the elliptic order property implies that r_{i-1} and a_{i-1} belong to different connected components of $\mathbb{D} \setminus \text{Int}(C)$. It follows that $\text{Int}(R_{i-1})$ and $\text{Int}(A_{i-1})$ belong to different connected components of $\mathbb{D} \setminus \text{Int}(C)$, and we conclude as in the preceding item.

3. As before, we know that $\text{Int}(A_i) \cap \text{Int}(C) = \emptyset$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Furthermore, $\text{Int}(C)$ is a connected set containing r_i for all $i \in \mathbb{Z}/n\mathbb{Z}$. So, the elliptic order property implies that each a_i , $i \in \mathbb{Z}/n\mathbb{Z}$ belong to a different connected component of $\mathbb{D} \setminus \text{Int}(C)$. It follows that each $\text{Int}(A_i)$, $i \in \mathbb{Z}/n\mathbb{Z}$, belong to a different connected component of $\mathbb{D} \setminus \text{Int}(C)$, and we conclude as in the preceding item. □

Lemma 3.15. *Let $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ be a maximal elliptic configuration. Then, for some $i \in \mathbb{Z}/n\mathbb{Z}$ there exists a connexion brick from R_i to A_i .*

Proof. Because of lemma 3.12, it is enough to show that for some $i \in \mathbb{Z}/n\mathbb{Z}$, R_i is not adjacent to A_i .

If R_i is adjacent to A_i , then $C = R_i \cup A_i$ is a connected set containing R_i and A_i . Besides, $C \cap (R_{i+1} \cup A_{i+1}) = \emptyset$ in B , because the sets in \mathcal{E} are pairwise disjoint. So, item 1 of the preceding lemma tells us that $R_{i+1} \cap A_{i+1} = \emptyset$ in \mathbb{D} . In particular, R_{i+1} cannot be adjacent to A_{i+1} . □

The following lemma tells us that it is enough to prove Proposition 3.9 for configurations of order $n = 3$:

Lemma 3.16. *Let $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ be an elliptic configuration of order $n > 3$. Then, there exists an elliptic configuration $((R'_i)_{i \in \mathbb{Z}/(n-1)\mathbb{Z}}, (A'_i)_{i \in \mathbb{Z}/(n-1)\mathbb{Z}})$ of order $n - 1$.*

Proof. We claim that there exists a brick $b \in R_0$ such that $[b]_{\geq} \cap A_1 \neq \emptyset$. Indeed,

$$(R_0 \cup [b_0]_{\geq} \cup A_0) \cap R_1 = \emptyset \text{ in } B,$$

by Remark 2.1, item 2 (we recall that for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exists $b_i \in R_i$ such that $[b_i]_{\geq} \cap A_i \neq \emptyset$). So, Lemma 3.14, item 1 implies that either

$$(R_0 \cup [b_0]_{\geq} \cup A_0) \cap A_1 \neq \emptyset \text{ in } B,$$

or $\text{Int}(R_0 \cup [b_0]_{\geq} \cup A_0)$ separates R_1 from A_1 (recall that $b_0 \in R_0$, $[b_0]_{\geq} \cap A_0 \neq \emptyset$, and that the future of any brick is connected). In the first case, necessarily

$$[b_0]_{\geq} \cap A_1 \neq \emptyset \text{ in } B,$$

and we take $b = b_0$. In the second case, we obtain

$$(R_0 \cup [b_0^-]_{\geq} \cup A_0) \cap (R_1 \cup [b_1^+]_{\leq} \cup A_1) \neq \emptyset \text{ in } B,$$

where $b_1^+ \in [b_1]_{\geq} \cap A_1$. By Remark 2.1, item 2, we know that $[b_0]_{\geq} \cap R_1 = \emptyset$ and $[b_1^+]_{\leq} \cap A_0 = \emptyset$. So, in fact

$$(R_0 \cup [b_0]_{\geq}) \cap ([b_1^+]_{\leq} \cup A_1) \neq \emptyset \text{ in } B.$$

If $R_0 \cap [b_1^+]_{\leq} \neq \emptyset$ in B , we take any brick $b \in R_0 \cap [b_1^+]_{\leq}$; if $[b_0]_{\geq} \cap ([b_1^+]_{\leq} \cup A_1) \neq \emptyset$ in B , we take $b = b_0$. (Note that $b \in [b_1^+]_{\leq}$ implies $b_1^+ \in [b]_{\geq} \cap A_1$). This finishes the proof of our claim.

Now, by defining

$$R'_0 = R_0, R'_i = R_{i+1} \text{ for } 1 \leq i \leq n - 2,$$

$$A'_i = A_{i+1} \text{ for } 0 \leq i \leq n - 2,$$

we are done. □

We are now ready to prove Proposition 3.9 :

Proof. Because of the previous lemma, we can suppose that there exists an elliptic configuration of order $n = 3$ and take a maximal one

$$((R_i)_{i \in \mathbb{Z}/3\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/3\mathbb{Z}}).$$

We will show that our assumption that f is not recurrent contradicts the maximality of this configuration. Lemma 3.15 allows us to consider a connexion brick b from R_i to A_i , for some $i \in \mathbb{Z}/3\mathbb{Z}$, and there is no loss of generality in supposing $i = 0$. Let $b' \in B \setminus \cup_{i \in \mathbb{Z}/3\mathbb{Z}} (R_i \cup A_i)$ be adjacent to A_0 and such that $b' \in [b]_{>}$. We will first show that $[b]_{<}$ meets every repeller and no attractor in the configuration. Then, by defining A'_i as to be the connected

component of $B \setminus (\cup_{i \in \mathbb{Z}/3\mathbb{Z}} R_i \cup [b]_{<})$ containing A_i , we will be able to show that $((R_i)_{i \in \mathbb{Z}/3\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/3\mathbb{Z}})$ is an elliptic configuration strictly bigger than the initial configuration, due to the fact that $b' \in A'_0 \setminus A_0$.

Indeed, we know by Lemma 3.11 that $[b]_{\leq} \cap R_j \neq \emptyset$ for some $j \in \{1, 2\}$. We will suppose $[b]_{\leq} \cap R_1 \neq \emptyset$; the proof is analogous in the other case. We claim that this implies $[b]_{\leq} \cap R_2 \neq \emptyset$. To see this, note that item 2 of Lemma 3.14 implies

$$R \cap (R_2 \cup [b_2]_{\geq} \cup A_2) \neq \emptyset,$$

where

$$R = R_0 \cup [b]_{\leq} \cup R_1.$$

So, actually

$$[b]_{\leq} \cap [b_2]_{\geq} \neq \emptyset,$$

which implies $[b]_{\leq} \cap R_2 \neq \emptyset$.

We have obtained that $R' = \cup_{i \in \mathbb{Z}/3\mathbb{Z}} R_i \cup [b]_{\leq}$ is a connected repeller disjoint (in B) from every attractor A_i , $i \in \mathbb{Z}/3\mathbb{Z}$ (Remark 2.1, item 2). Let A'_j be the connected component of $B \setminus R'$ containing A_j for all $j \in \mathbb{Z}/3\mathbb{Z}$. Then, the sets A'_j , $j \in \mathbb{Z}/3\mathbb{Z}$ are pairwise disjoint (in \mathbb{D}) by the elliptic order property. We know that $b' \in B \setminus R'$; otherwise, we would have $b' \in [b]_{\leq}$ as $b' \notin \cup_{i \in \mathbb{Z}/3\mathbb{Z}} (R_i \cup A_i)$, which is impossible because $b' \in [b]_{>}$ and we are supposing that f is non-recurrent. So, A_0 is strictly contained in A'_0 and we deduce that $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A'_i)_{i \in \mathbb{Z}/3\mathbb{Z}})$ is an elliptic configuration strictly greater than $((R_i)_{i \in \mathbb{Z}/3\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/3\mathbb{Z}})$, contradicting the maximality of the configuration. \square

3.4 The hyperbolic case.

In what follows, we deal with the hyperbolic case. The proof of the following lemma is analogous to that of Lemma 3.14, substituting of course the elliptic order property by the hyperbolic order property.

Lemma 3.17. *Let $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ be a hyperbolic configuration.*

If $C \subset B$ is a connected set containing R_i and R_{i+1} , and $C \cap A_m = \emptyset$ in B for all $m \in \mathbb{Z}/n\mathbb{Z}$, then $\text{Int}(C)$ separates (in \mathbb{D}) A_i from any A_j , $j \neq i$.

Lemma 3.18. *Let $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ be a hyperbolic configuration. If $X \in \mathcal{E}$, then there is only one connected component of $B \setminus X$ containing sets in \mathcal{E} .*

Proof. We will suppose that $X = R_j$, $j \in \mathbb{Z}/n\mathbb{Z}$; the proof is analogous for any $X \in \mathcal{E}$. We will show that the connected component C of $B \setminus R_j$ containing A_j contains every $X \in \mathcal{E}$, $X \neq R_j$. As $B \setminus R_j$ is an attractor, and there is a brick in R_{j+1} whose (connected) future intersects A_j , we have that $R_{j+1} \subset C$ (we recall that every connected component of an attractor is an attractor, see Proposition 2.7). As there is also a brick in R_{j+1} whose future intersects A_{j+1} , the same argument shows that $A_{j+1} \in C$. By induction, we get that every $X \in \mathcal{E} \setminus \{R_j\}$ belongs to C . \square

Lemma 3.19. *Let $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ be a maximal hyperbolic configuration. One of the following is true:*

1. $\text{Fix}(f) \neq \emptyset$,
2. there exists a connexion brick from R_j to A_j for some $j \in \mathbb{Z}/n\mathbb{Z}$.

Proof. We will show that if $\text{Fix}(f) = \emptyset$, then there exists a connexion brick from R_j to A_j for some $j \in \mathbb{Z}/n\mathbb{Z}$. By Lemma 3.12, we can suppose that R_i is adjacent to A_i for all $i \in \mathbb{Z}/n\mathbb{Z}$. If R_i is adjacent to A_i , either there is one connected component γ of ∂R_i which is also a connected component of ∂A_i or there is a point $x \in R_i \cap A_i \cap \partial(R_i \cup A_i)$. If $\text{Fix}(f) = \emptyset$, then every connected component of ∂X is an embedded line in \mathbb{D} , for any $X \in \mathcal{E}$. So, if there were one connected component γ of ∂R_i which is also a connected component of ∂A_i , γ would separate \mathbb{D} into two connected components C_1 and C_2 , containing $\text{Int}(A_i)$ and $\text{Int}(R_i)$ respectively. Then, Lemma 3.18 would imply that every set in $\mathcal{E} \setminus R_i$ belongs to C_1 , and that every set in $\mathcal{E} \setminus A_i$ belongs to C_2 , which is clearly impossible.

We are left with the case where there is a point $x \in R_i \cap A_i \cap \partial(R_i \cup A_i)$. This point x is necessarily a vertex of $\Sigma(\mathcal{D})$. It belongs to three bricks: one that belongs to R_i , another one which belongs to A_i , and a third one which is adjacent to both R_i and A_i . This third brick does not belong to any repeller or attractor, as it is adjacent to both R_i and A_i (see Remark 2.1, item 4). So, by Lemma 3.12, item 1, there exists a connexion brick from R_i to A_i . \square

We will prove Proposition 3.10 by induction on the order of the configuration. We begin by the case $n = 2$:

Proposition 3.20. *If there exists a hyperbolic configuration of order 2, then $\text{Fix}(f) \neq \emptyset$.*

Proof. Suppose there exists such a configuration and take a maximal one

$$((R_i)_{i \in \mathbb{Z}/2\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/2\mathbb{Z}}).$$

Because of Lemma 3.19, we can suppose that there exists a connexion brick b from R_j to A_j for some $j \in \mathbb{Z}/2\mathbb{Z}$, and there is no loss of generality in supposing $j = 0$. We take a brick b' such that $b' \in [b]_{>}$, $b' \in B \setminus \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ and b' is adjacent to A_0 . Here again, we will first show that $[b]_{<}$, the strict past of b , meets every repeller and no attractor in the configuration. Then, by defining A'_i as the connected component of $B \setminus (\cup_{i \in \mathbb{Z}/2\mathbb{Z}} R_i \cup [b]_{<})$ containing A_i , we will be able to show that $((R_i)_{i \in \mathbb{Z}/2\mathbb{Z}}, (A'_i)_{i \in \mathbb{Z}/2\mathbb{Z}})$ is a hyperbolic configuration strictly greater than the original one, due to the fact that $b' \in A'_0 \setminus A_0$.

Because of Lemma 3.11 we know that $[b]_{<} \cap R_1 \neq \emptyset$ in B . So,

$$R = R_0 \cup b_{\leq} \cup R_1$$

is connected and disjoint from every attractor in the configuration (see Remark 2.1, item 2). It follows that $\text{Int}(R)$ separates A_0 from A_1 , this being the content of Lemma 3.17. Let A'_i be the connected component of $B \setminus R$ containing A_i , $i \in \mathbb{Z}/2\mathbb{Z}$. Then, $A'_0 \cap A'_1 = \emptyset$. We know that $b' \notin R$, because $b' \in [b]_{>}$, and otherwise f would be recurrent. So, b' belongs to $A'_0 \setminus A_0$, contradicting the maximality of $((R_i)_{i \in \mathbb{Z}/2\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/2\mathbb{Z}})$. \square

Now we are ready to prove Proposition 3.10:

Proof. We will show that given a maximal hyperbolic configuration of order $n > 2$

$$((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}}),$$

we can construct a new hyperbolic configuration whose order is strictly smaller than n (and yet greater or equal to 2). We can suppose there exists a connexion brick b from R_0 to A_0 . We take a brick $b' \in [b]_>$ such that $b' \in B \setminus \cup_{i \in \mathbb{Z}/n\mathbb{Z}} (R_i \cup A_i)$ and b' is adjacent to A_0 . By Lemma 3.11,

$$[b]_{\leq} \cap R_i \neq \emptyset \text{ for some } i \neq 0.$$

We can suppose that $i \neq 1$; otherwise, we could use the same argument we used for the case $n = 2$. Indeed, Lemma 3.17 would imply that $R_0 \cup R_1 \cup [b]_{\leq}$ is a connected repeller which separates A_0 from any other A_j , $j \neq 0$. So, by replacing A_0 by A'_0 , the connected component of $B \setminus (R_0 \cup R_1 \cup [b]_{\leq})$ containing A_0 , we would have a hyperbolic configuration strictly bigger than the original one.

So, we may suppose that

$$i = \min\{j \in \{1, \dots, n-1\} : [b]_{\leq} \cap R_j \neq \emptyset\} \neq 1.$$

We define

$$R = R_0 \cup [b]_{\leq} \cup R_i,$$

which is a connected repeller.

If we set $R'_0 = R$, $R'_j = R_j$ for all $1 \leq j \leq i-1$, and $A'_j = A_j$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, $0 \leq j \leq i-1$. Then, $((R'_j)_{j \in \mathbb{Z}/i\mathbb{Z}}, (A'_j)_{j \in \mathbb{Z}/i\mathbb{Z}})$ is a hyperbolic configuration of order i , $2 \leq i < n$.

□

3.5 Applications.

We finish this section giving applications of Propositions 3.9 and 3.10 respectively, that will be used in the proof of Theorem 1.1. We will introduce two technical lemmas that will not be used until Chapter 5. In particular, Chapter 4 is independent of these lemmas. The reader interested in the proof of the classic Handel's theorem can skip what follows and go directly to the next chapter.

We recall that we have fixed $f \in \text{Homeo}^+(\mathbb{D})$ together with a free maximal decomposition in bricks $\mathcal{D} = (V, E, B)$ of $\mathbb{D} \setminus \text{Fix}(f)$, and that we are supposing that f is non-recurrent.

Let a_i , $i \in \mathbb{Z}/n\mathbb{Z}$, be non-empty, pairwise disjoint, closed, connected subsets of \mathbb{D} , such that $\bar{a}_i \cap \partial\mathbb{D} \neq \emptyset$, for all $i \in \mathbb{Z}/n\mathbb{Z}$, and $U = \mathbb{D} \setminus (\cup_{i \in \mathbb{Z}/n\mathbb{Z}} a_i)$ is a connected open set. We consider the Riemann map $\varphi : U \rightarrow \mathbb{D}$, and the open intervals on the circle J_i , $i \in \mathbb{Z}/n\mathbb{Z}$ defined in 3.1. We recall that the interval J_i correspond to the prime ends in U whose impression is contained in a_i .

Let $(I_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ be the connected components of $S^1 \setminus (\cup_{i \in \mathbb{Z}/n\mathbb{Z}} J_i)$. So, each I_i is a closed interval, that maybe reduced to a point.

Remark 3.21. One can cyclically order the sets $(a_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$, $(r_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$, where $(r_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$ is any family of closed, connected and pairwise disjoint subsets of U satisfying:

1. $\overline{r_j} \cap \partial U \neq \emptyset$, $j \in \mathbb{Z}/m\mathbb{Z}$,
2. for all $j \in \mathbb{Z}/m\mathbb{Z}$, there exists $i_j \in \mathbb{Z}/n\mathbb{Z}$ such that $\overline{\varphi(r_j)} \cap S^1 \subset I_{i_j}$,
3. the correspondence $j \rightarrow i_j$ is injective.

Lemma 3.22. *We suppose that:*

1. *the cyclic order of the sets a_i , $i \in \mathbb{Z}/n\mathbb{Z}$, is the following:*

$$a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_i \rightarrow a_{i+1} \rightarrow \dots \rightarrow a_{n-1} \rightarrow a_0.$$

2. *for all $i \in \mathbb{Z}/n\mathbb{Z}$ there exists $b_i^+ \in B$, such that $a_i \subset [b_i^+]_>$,*
3. *there exists three bricks $(b_s^-)_{s \in \mathbb{Z}/3\mathbb{Z}}$ such that*

- (a) *for all $s \in \mathbb{Z}/3\mathbb{Z}$ and for all $i \in \mathbb{Z}/n\mathbb{Z}$, one has $b_s^- \subset [b_i^+]_<$ (and so $[b_s^-]_< \subset U$),*
- (b) $\overline{[b_s^-]_<} \cap \partial U \neq \emptyset$ *for all $s \in \mathbb{Z}/3\mathbb{Z}$,*
- (c) *for all $s \in \mathbb{Z}/3\mathbb{Z}$ there exists $i_s \in \mathbb{Z}/n\mathbb{Z}$ such that $\overline{\varphi([b_s^-]_<)} \cap S^1 \subset I_{i_s}$,*

Then, the correspondence $s \rightarrow i_s$ is not injective.

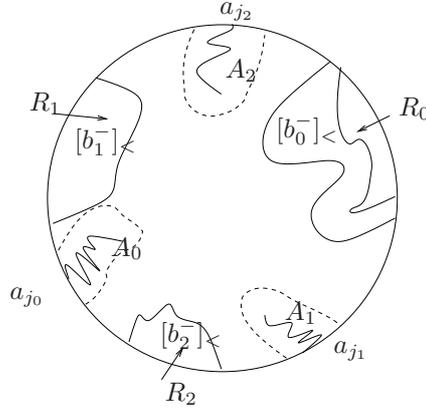


Figure 3.4: Lemma 3.22

Proof. We will prove that if the correspondence $s \rightarrow i_s$ is injective, we can construct an elliptic configuration of order 3. As we are assuming f is not recurrent, this is not possible by Proposition 3.9.

We begin by proving that $[b_s^-]_< \cap [b_r^-]_< \neq \emptyset$ implies $i_s = i_r$. Indeed, if $[b_s^-]_< \cap [b_r^-]_< \neq \emptyset$, then $[b_s^-]_< \cup [b_r^-]_<$ is a connected set and $\overline{\varphi([b_s^-]_< \cup [b_r^-]_<)}$ intersects both I_{i_s} and I_{i_r} . If $i_s \neq i_r$, then there exists $j_0, j_1 \in \mathbb{Z}/n\mathbb{Z}$ such that

any arc joining J_{j_0} and J_{j_1} separates I_{i_r} from I_{i_s} in $\overline{\mathbb{D}}$. Our hypothesis 3.(a) allows us to take a crosscut γ from a_{j_0} to a_{j_1} such that $\gamma \cap U \subset [b_s^-]_>$. So, $\overline{\varphi(\gamma \cap U)}$ is an arc joining J_{j_0} and J_{j_1} (see Lemma 3.8), and

$$\overline{\varphi(\gamma \cap U)} \cap \varphi([b_s^-]_< \cup [b_r^-]_<) \neq \emptyset.$$

This gives us

$$([b_s^-]_< \cup [b_r^-]_<) \cap [b_s^-]_> \neq \emptyset,$$

and as we are supposing that f is not recurrent,

$$[b_r^-]_< \cap [b_s^-]_> \neq \emptyset.$$

So,

$$[b_s^-]_< \subset [b_r^-]_< ,$$

which implies

$$\overline{\varphi([b_s^-]_<)} \cap S^1 \subset I_{i_s} \cap I_{i_r} ,$$

a contradiction.

So, if the correspondence $s \rightarrow i_s$ is injective, the sets $[b_s^-]_<$ are pairwise disjoint, and one can cyclically order the $n+3$ sets $a_i, [b_s^-]_<, i \in \mathbb{Z}/n\mathbb{Z}, s \in \mathbb{Z}/3\mathbb{Z}$ (see Remark 3.21). We may suppose without loss of generality that

$$[b_0^-]_< \rightarrow [b_1^-]_< \rightarrow [b_2^-]_< \rightarrow [b_0^-]_< .$$

For all $s \in \mathbb{Z}/3\mathbb{Z}$, we can take $j_s \in \mathbb{Z}/3\mathbb{Z}$ such that

$$[b_0^-]_< \rightarrow a_{j_2} \rightarrow [b_1^-]_< \rightarrow a_{j_0} \rightarrow [b_2^-]_< \rightarrow a_{j_1} \rightarrow [b_0^-]_<$$

(see Figure 3.4).

For all $s \in \mathbb{Z}/3\mathbb{Z}$, we define:

$$R_s = [b_s^-]_<, \quad A_s = [b_{j_s}^+]_> .$$

We want to show that

$$((R_s)_{s \in \mathbb{Z}/3\mathbb{Z}}), (A_s)_{s \in \mathbb{Z}/3\mathbb{Z}},$$

is an elliptic configuration. It is enough to show that the sets $A_s, R_s, s \in \mathbb{Z}/3\mathbb{Z}$, are pairwise disjoint, because of the cyclic order of these sets, and our hypothesis 3.(a). We already know that the sets $R_s, s \in \mathbb{Z}/3\mathbb{Z}$, are pairwise disjoint. As we are supposing that f is not recurrent, and $b_{j_s}^+ \in [b_s^-]_>$ for every pair of indices s, s' in $\mathbb{Z}/3\mathbb{Z}$ (3.(a)), we know that

$$[b_{j_s}^+]_> \cap [b_{s'}^-]_< = \emptyset$$

for all s, s' in $\mathbb{Z}/3\mathbb{Z}$. So, the sets $\{A_s\}$, are disjoint from the sets $\{R_s\}$, and we just have to show that the sets $\{A_s\}$ are pairwise disjoint to finish the proof of the lemma.

Because of the symmetry of the problem it is enough to show that

$$A_0 \cap A_1 = \emptyset.$$

If this is not so,

$$A_0 \cup A_1 = [b_{j_0}^+]_{>} \cup [b_{j_1}^+]_{>}$$

would be a connected set containing both a_{j_1} and a_{j_0} , and the cyclic order would imply that

$$([b_{j_0}^+]_{>} \cup [b_{j_1}^+]_{>}) \cap [b_{j_0}^+]_{<} \neq \emptyset,$$

by our hypothesis 3.(a). As we are supposing that f is not recurrent, we have

$$[b_{j_1}^+]_{>} \cap [b_{j_0}^+]_{<} \neq \emptyset.$$

But this implies that $[b_{j_1}^+]_{>}$ is a connected set containing both a_{j_1} and a_{j_0} . Once again our hypothesis 3.(a) and the cyclic order gives us

$$[b_{j_1}^+]_{>} \cap [b_{j_1}^+]_{<} \neq \emptyset,$$

and we are done. □

For our next lemma, we keep the assumption on the cyclic order of the sets $a_i, i \in \mathbb{Z}/n\mathbb{Z}$:

$$a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_i \rightarrow a_{i+1} \rightarrow \dots \rightarrow a_{n-1} \rightarrow a_0.$$

We define I_i , as to be the connected component of $S^1 \setminus \cup_{j \in \mathbb{Z}/n\mathbb{Z}} J_j$ that follows J_{i-1} in the natural cyclic order on S^1 , so that we have:

$$J_{i-1} \rightarrow I_i \rightarrow J_i,$$

for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Lemma 3.23. *If for all $i \in \mathbb{Z}/n\mathbb{Z}$:*

1. *there exists $b_i^+ \in B$, such that $a_i \subset [b_i^+]_{>}$,*
2. *there exists $b_i^- \in B$ such that $b_i^- \subset [b_j^+]_{<}$, $j \in \{i-1, i\}$,*
3. *$[b_i^-]_{<} \subset U$, and $\overline{[b_i^-]_{<}} \cap \partial U \neq \emptyset$,*
4. *$\varphi(\overline{[b_i^-]_{<}}) \cap S^1 \subset I_i$,*

then $\text{Fix}(f) \neq \emptyset$.

Proof. By Proposition 3.10 it is enough to show that we can construct a hyperbolic configuration.

We begin by proving that the sets $\{[b_i^-]_{<}\}$, are pairwise disjoint. Otherwise, there exists $i \neq j$, such that

$$[b_i^-]_{<} \cap [b_j^-]_{<} \neq \emptyset.$$

Then, $[b_i^-]_{<} \cup [b_j^-]_{<}$ is a connected set and $\overline{\varphi([b_i^-]_{<} \cup [b_j^-]_{<})}$ intersects both I_i and I_j . The cyclic order implies that any arc joining J_{i-1} and J_i separates I_i from I_j , $i \neq j$.

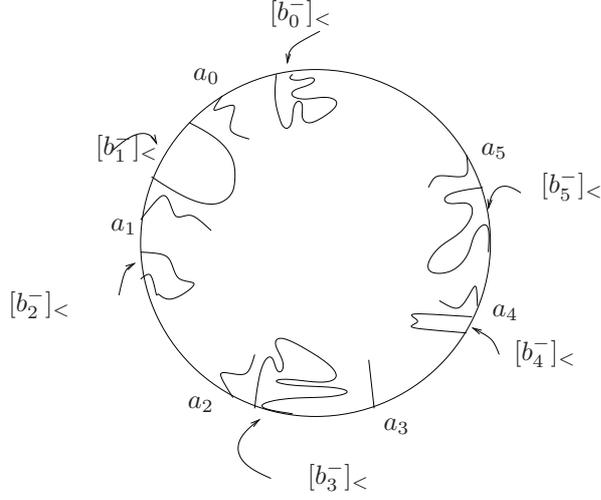


Figure 3.5: Lemma 3.23 with $n = 6$

Our hypothesis 2. allows us to take a crosscut γ from a_{i-1} to a_i such that

$$\gamma \cap U \subset [b_i^-]_>.$$

So, $\overline{\varphi(\gamma \cap U)}$ is an arc joining J_{i-1} and J_i , and

$$\overline{\varphi(\gamma \cap U)} \cap \varphi([b_i^-]_< \cup [b_j^-]_<) \neq \emptyset.$$

This gives us

$$([b_i^-]_< \cup [b_j^-]_<) \cap [b_i^-]_> \neq \emptyset,$$

and as we are supposing that f is not recurrent,

$$[b_j^-]_< \cap [b_i^-]_> \neq \emptyset.$$

So, $[b_i^-]_< \subset [b_j^-]_<$, which implies

$$\overline{\varphi([b_i^-]_<)} \cap S^1 \subset I_i \cap I_j,$$

a contradiction.

So, we can cyclically order the $2n$ sets $a_i, [b_i^-]_<, i \in \mathbb{Z}/n\mathbb{Z}$ (see Remark 3.21). Moreover, for all $i \in \mathbb{Z}/n\mathbb{Z}$,

$$a_{i-1} \rightarrow [b_i^-]_< \rightarrow a_i.$$

Define $A_i = [b_i^+]_>$ and $R_i = [b_i^-]_<$, for $i \in \mathbb{Z}/n\mathbb{Z}$. To finish the proof of the lemma, it is enough to show that the sets $R_i, A_i, i \in \mathbb{Z}/n\mathbb{Z}$, are pairwise disjoint. Indeed, if this is true, our previous remark on the cyclic order, and our hypothesis 2. imply that $((R_i)_{i \in \mathbb{Z}/n\mathbb{Z}}, (A_i)_{i \in \mathbb{Z}/n\mathbb{Z}})$ is a hyperbolic configuration.

We have already proved that the sets $R_i, i \in \mathbb{Z}/n\mathbb{Z}$ are pairwise disjoint. We will also show that $[b_i^-]_< \cap [b_j^+]_> = \emptyset$ for any $j \in \mathbb{Z}/n\mathbb{Z}$. By hypothesis 2., $[b_i^-]_< \cap [b_i^+]_> = \emptyset$, as we are supposing that f is not recurrent. If $[b_i^-]_< \cap [b_i^+]_> \neq \emptyset$

for some $j \neq i$, then $[b_j^+]_{<} \subset [b_i^-]_{<}$, $j \neq i$. Therefore, $\overline{\varphi([b_j^+]_{<})} \cap S^1 \subset I_i$, $j \neq i$, which contradicts our hypothesis 4..

We have proved that the sets R_i are disjoint from the sets A_i , $i \in \mathbb{Z}/n\mathbb{Z}$. So, in order to finish, we only have to prove that the sets A_i , $i \in \mathbb{Z}/n\mathbb{Z}$ are pairwise disjoint.

If this is not the case, there would exist $i \neq j$, such that $[b_i^+]_{>} \cap [b_j^+]_{>} \neq \emptyset$. So, $[b_i^+]_{>} \cup [b_j^+]_{>}$ is a connected set containing $a_i \cup a_j$, and must therefore intersect $[b_i^+]_{<}$, because of the cyclic order and hypothesis 2. We may of course assume that $[b_j^+]_{>} \cap [b_i^+]_{<} \neq \emptyset$. Now, we have that $[b_j^+]_{>}$ is a connected set containing $a_j \cup a_i$ and must therefore intersect $[b_j^+]_{<}$. This contradiction proves our claim. \square

Chapter 4

A simple proof of Handel's fixed point theorem

In this short chapter we include a simple proof of Le Calvez's improvement [11] of the classic fixed point theorem of Handel [8].

Proposition 4.1. *Suppose that $f : \mathbb{D} \rightarrow \mathbb{D}$ is an orientation preserving homeomorphism which realizes a cycle of links \mathcal{L} and can be extended to a homeomorphism of $\mathbb{D} \cup (\cup_{i \in \mathbb{Z}/n\mathbb{Z}} \{\alpha_i, \omega_i\})$.*

If \mathcal{L} is elliptic, and the points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$, are all different, then f is recurrent.

Remark 4.2. With these assumptions, the order of the points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$ at the circle at infinity satisfies:

$$\omega_0 \rightarrow \alpha_2 \rightarrow \omega_1 \rightarrow \dots \rightarrow \omega_i \rightarrow \alpha_{i+2} \rightarrow \omega_{i+1} \rightarrow \dots \rightarrow \omega_{n-1} \rightarrow \alpha_1 \rightarrow \omega_0.$$

From now on, we suppose that f is not recurrent. We apply Lemma 2.9 and obtain a family of closed disks $(b_i^l)_{l \in \mathbb{Z} \setminus \{0\}, i \in \mathbb{Z}/n\mathbb{Z}}$. The hypothesis on the points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$, allows us to suppose that all the disks $(b_i^l)_{l \in \mathbb{Z} \setminus \{0\}, i \in \mathbb{Z}/n\mathbb{Z}}$ have pairwise disjoint interiors (see Remark 2.11).

Remark 4.3. The sets $\Gamma_i^- \cap \mathbb{D}, \Gamma_i^+ \cap \mathbb{D}$ defined in Remark 2.10 satisfy the elliptic order property (see Remark 4.2).

By Corollary 2.12, we can construct a free brick decomposition (V, E, B) such that for all $i \in \mathbb{Z}/n\mathbb{Z}$ and for all $l \in \mathbb{Z} \setminus \{0\}$, there exists $b_i^l \in B$ such that $b_i^l \subset b_i^l$. Moreover, one can suppose that this decomposition is maximal.

Remark 4.4. As $\cup_{l > 0} [b_i^l]_{\leq}$ is a connected set whose closure contains both α_i and ω_i , if $\Gamma : [0, 1] \rightarrow \overline{\mathbb{D}}$ is an arc that separates α_i from ω_i , then $\Gamma \cap \cup_{l > 0} [b_i^l]_{\leq} \neq \emptyset$.

Lemma 4.5. *If for some $k > 0, m > 0$ and $j \in \mathbb{Z}/n\mathbb{Z}$, both b_j^k and b_{j+1}^k are contained in $[b_i^{-m}]_{>}$, then there exists $l > 0$ such that $b_{j+2}^l \in [b_i^{-m}]_{>}$.*

Proof. If b_j^k and b_{j+1}^k are contained in $[b_i^{-m}]_{>}$, then b_j^p and b_{j+1}^p are contained in $[b_i^{-m}]_{>}$ for all $p \geq k$ (note that $[b_i^{-m}]_{>}$ is an attractor, and that Lemma 2.9,

item 6. implies that $b_j^p \subset [b_j^k]_{\geq}$ for all $p \geq k$). So, as $[b_i^{-m}]_{>}$ is connected, we can find an arc

$$\Gamma : [0, 1] \rightarrow [b_i^{-m}]_{>} \cup \{\omega_j, \omega_{j+1}\}$$

joining ω_j and ω_{j+1} (see Remark 2.10). Then, Γ separates α_{j+2} from ω_{j+2} in $\overline{\mathbb{D}}$ (see Remark 4.2). By Remark 4.4, we obtain

$$\Gamma \cap (\cup_{l>0} [b_{j+2}]_{<}^l) \neq \emptyset.$$

So,

$$[b_i^{-m}]_{>} \cap (\cup_{l>0} [b_{j+2}]_{<}^l) \neq \emptyset,$$

from which one gets (as the future of any brick is an attractor) that there exists $l > 0$ such that $b_{j+2}^l \in [b_i^{-m}]_{>}$. \square

Lemma 4.6. (Domino effect) *There exists $k > 0$ such that for all $i, j \in \mathbb{Z}/n\mathbb{Z}$, $[b_i^{-k}]_{>}$ contains b_j^k .*

Proof. Fix $i \in \mathbb{Z}/n\mathbb{Z}$. There exists an arc

$$\Gamma : [0, 1] \rightarrow \cup_{l>0} [b_i^{-l}]_{>} \cup \{\alpha_i, \omega_i\}$$

joining α_i and ω_i (see Remark 2.10). Then, Γ separates α_{i+1} from ω_{i+1} in $\overline{\mathbb{D}}$ (see Remark 4.2). So, Remark 4.4 gives us

$$\Gamma \cap (\cup_{l>0} [b_{i+1}]_{<}^l) \neq \emptyset.$$

So,

$$(\cup_{l>0} [b_i^{-l}]_{>}) \cap (\cup_{l>0} [b_{i+1}]_{<}^l) \neq \emptyset,$$

from which one immediately gets that there exists $l_i, m_i > 0$ such that $b_{i+1}^{l_i} \in [b_i^{-m_i}]_{>}$. As $b_i^{l_i} \in [b_i^{-m_i}]_{>}$ as well, the previous lemma tells us that there exists $l > 0$ such that $b_{i+2}^l \in [b_i^{-m_i}]_{>}$. We finish the proof of the lemma by induction, and then taking $k > 0$ large enough. \square

We are now ready to prove Proposition 4.1:

Proof. We will show that $(([b_i^{-k}]_{<})_{i \in \mathbb{Z}/n\mathbb{Z}}, ([b_i^k]_{>})_{i \in \mathbb{Z}/n\mathbb{Z}})$ is an elliptic configuration, where $k > 0$ is given by the preceding lemma. This contradicts our assumption that f is not recurrent, by Proposition 3.9.

We define $r_i = \Gamma_i^- \cap \cup_{m \geq k} b_i^{-m}$, and $a_i = \Gamma_i^+ \cap \cup_{m \geq k} b_i^m$, $i \in \mathbb{Z}/n\mathbb{Z}$; we may suppose that the sets $r_i, a_i, i \in \mathbb{Z}/n\mathbb{Z}$ are arcs (the sets $\Gamma_i^- \cap \mathbb{D}$, $\Gamma_i^+ \cap \mathbb{D}$ were defined in Remark 2.10). These arcs $a_i, r_i, i \in \mathbb{Z}/n\mathbb{Z}$ satisfy the elliptic order property (see Remark 4.3). Besides, for all $i \in \mathbb{Z}/n\mathbb{Z}$,

- $r_i \subset [b_i^{-k}]_{<}$,
- $a_i \subset [b_i^k]_{>}$, and
- $b_i^k \in [b_i^{-k}]_{>}$.

So, we only have to show that the sets $\{[b_i^{-k}]_{<}\}, \{[b_j^k]_{>}\}$, are pairwise disjoint. As we are supposing that f is not recurrent, the preceding lemma gives us that for any pair of indices i, j in $\mathbb{Z}/n\mathbb{Z}$:

$$[b_i^{-k}]_{<} \cap [b_j^k]_{>} = \emptyset.$$

Let us show that for any pair of different indices i, j in $\mathbb{Z}/n\mathbb{Z}$ one has

$$[b_i^{-k}]_{<} \cap [b_j^{-k}]_{<} = \emptyset.$$

Otherwise, there would exist $i \neq j$ such that $[b_i^{-k}]_{<} \cup [b_j^{-k}]_{<}$ is a connected set containing r_i and r_j . As $[b_i^{-k}]_{>}$ is a connected set containing a_j for all $j \in \mathbb{Z}/n\mathbb{Z}$ (again by the preceding lemma), the elliptic order property tells us:

$$([b_i^{-k}]_{<} \cup [b_j^{-k}]_{<}) \cap [b_i^{-k}]_{>} \neq \emptyset.$$

We deduce (as f is not recurrent) that

$$[b_j^{-k}]_{<} \cap [b_i^{-k}]_{>} \neq \emptyset,$$

but then $[b_j^{-k}]_{<}$ is a connected set containing both r_j and r_i , and once again the preceding lemma and the elliptic order property imply

$$[b_j^{-k}]_{<} \cap [b_j^{-k}]_{>} \neq \emptyset,$$

a contradiction. To prove that for any pair of different indices i, j in $\mathbb{Z}/n\mathbb{Z}$ one also has

$$[b_i^k]_{>} \cap [b_j^k]_{>} = \emptyset,$$

it is enough to interchange the roles of $<$ and $>$, k and $-k$ in the proof we just did. \square

Chapter 5

Our fixed point theorem

This chapter is devoted to the proof of Theorem 1.1.

We fix an orientation preserving homeomorphism $f : \mathbb{D} \rightarrow \mathbb{D}$ which realizes a cycle of links $\mathcal{L} = ((\alpha_i, \omega_i))_{i \in \mathbb{Z}/n\mathbb{Z}}$. We recall that this means that there exists a family $(z_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ of points in \mathbb{D} such that for all $i \in \mathbb{Z}/n\mathbb{Z}$

$$\lim_{k \rightarrow -\infty} f^k(z_i) = \alpha_i, \quad \lim_{k \rightarrow +\infty} f^k(z_i) = \omega_i.$$

We also recall that

$$\ell = \{\alpha_i, \omega_i : i \in \mathbb{Z}/n\mathbb{Z}\} \subset S^1,$$

and that we suppose that f can be extended to a homeomorphism of $\mathbb{D} \cup \ell$.

5.1 The elliptic case.

Let us state our first proposition:

Proposition 5.1. *If \mathcal{L} is elliptic, then $\text{Fix}(f) \neq \emptyset$. Moreover, one of the following holds:*

1. f is recurrent,
2. \mathcal{L} is a degenerate cycle.

As the proof is long, we will first describe our strategy. The first part of the work consists in constructing a brick decomposition which is suitable for our purposes. Once this done, we study the “domino effect” of the elliptic order property; that is, we prove an analogue of Lemma 4.6 in the previous chapter. Then, we show that if f is not recurrent, this “domino effect” gives rise to constraints on the order of the cycle of links \mathcal{L} . We will show (as a consequence of Lemma 3.22) that the only possibility for the order of \mathcal{L} is $n = 4$. The case $n = 4$ is special, as degeneracies may occur (see Figure 1.1, and Chapter 1 where we explain that non-degeneracy is needed for obtaining the index result).

For $n = 4$ we prove that $\text{Fix}(f) \neq \emptyset$, and that if f is not recurrent, then \mathcal{L} is degenerate.

I. Construction of the brick decomposition.

We consider cycles of links where the points $\{\alpha_i\}, \{\omega_i\}$, are not necessarily different. In particular, we have that $n > 3$ (if $n = 3$, the definition of cycle of links implies automatically that these points are all different). As we are dealing with the elliptic case, the only possible coincidences among the points $\{\alpha_i\}, \{\omega_i\}$, are of the form $\omega_{i-2} = \alpha_i$. In particular, the points $\{\omega_i\}$ are all different and for all $i \in \mathbb{Z}/n\mathbb{Z}$ we can take a neighbourhood U_i^+ of ω_i in \mathbb{D} in such a way that $U_i^+ \cap U_j^+ = \emptyset$ if $i \neq j$. We define $U_i^- = U_{i-2}^+$ if $\alpha_i = \omega_{i-2}$, and for all $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\alpha_i \neq \omega_{i-2}$ we take a neighbourhood U_i^- of α_i in \mathbb{D} in such a way that $U_i^- \cap U_j^+ = \emptyset$ for all $j \in \mathbb{Z}/n\mathbb{Z}$ and $U_i^- \cap U_j^- = \emptyset$ for all $i \neq j$.

We suppose from now on that f is not recurrent.

We apply Lemma 2.9 and obtain families of closed disks $(b_i^l)_{l \in \mathbb{Z} \setminus \{0\}, i \in \mathbb{Z}/n\mathbb{Z}}$. So, the disks in the family $(b_i^l)_{l \geq 1, i \in \mathbb{Z}/n\mathbb{Z}}$, have pairwise disjoint interiors.

Let I_{reg} be the set of $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\alpha_i \neq \omega_{i-2}$, or such that $\alpha_i = \omega_{i-2}$ but there exists $K > 0$ such that

$$\cup_{k > K} \text{Int}(b_{i-2}^k) \cap \cup_{k > K} \text{Int}(b_i^{-k}) = \emptyset.$$

Let I_{sing} be the complement of I_{reg} in $\mathbb{Z}/n\mathbb{Z}$.

After discarding a finite number of disks, we can suppose that the disks b_i^l with $l \geq 1, i \in \mathbb{Z}/n\mathbb{Z}$, and b_i^{-l} with $l \geq 1, i \in I_{\text{reg}}$, have pairwise disjoint interiors.

If $i \in I_{\text{sing}}$, then $\alpha_i = \omega_{i-2}$ and for all $k > 0$ there exists $k' > k, j' > k$, such that $\text{Int}(b_{i-2}^{k'}) \cap \text{Int}(b_i^{-j'}) \neq \emptyset$.

In the following lemma we refer to the family of integers $(l_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ constructed in Lemma 2.9.

Lemma 5.2. *If $i \in I_{\text{sing}}$, we can find sequences of free closed disks $(c_i^m)_{m \geq 0}$, such that:*

1. $c_i^m \subset U_{i-2}^+ = U_i^-$,
2. there exists an increasing sequence $(k_i^m)_{m \geq 0}$ such that $b_{i-2}^{k_i^m} \cap c_i^m \neq \emptyset$ for all $m \geq 0$,
3. $(b_{i-2}^{k_i^p} \cup c_i^p) \cap (b_{i-2}^{k_i^m} \cup c_i^m) = \emptyset$ for all $p \neq m$,
4. there exists an increasing sequence $(j_i^m)_{m \geq 0}$ such that $f^{-l_i - j_i^m + 1}(z_i) \in c_i^m$ for all $m \geq 0$,
5. the sequence $(c_i^m)_{m \geq 0}$ converges in the Hausdorff topology to $\omega_{i-2} = \alpha_i$.
6. $b_{i-2}^{k_i^m} \cap c_i^m$ is an arc for all $m \geq 0$ (so, $c_i^m \cup b_{i-2}^{k_i^m}$ is a topological closed disk),
7. $\partial(\cup_{k \geq 1} b_{i-2}^k \cup \cup_{m \geq 0} c_i^m)$ is a one dimensional submanifold,
8. if $x \in \mathbb{D}$, then x belongs to at most two different disks in the family $\{b_{i-2}^k, c_i^m : k \geq 1, m \geq 0\}$

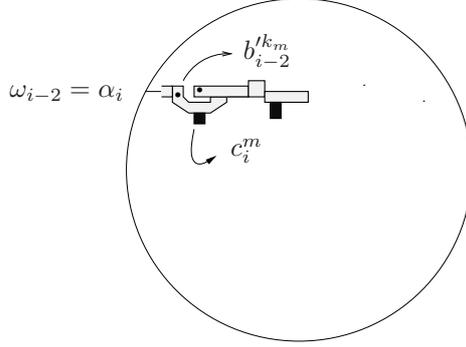


Figure 5.1: The disks $b_{i-2}^{k_m}$ and c_i^m

Proof. Take $i \in I_{\text{sing}}$ and consider the family of closed disks $(b_{i-2}^k)_{k \geq 1} \subset U_{i-2}^+$. As $i \in I_{\text{sing}}$, there exists $j_i^0 > 1$, such that

$$\text{Int}(\cup_{k \geq 1} b_{i-2}^k) \cap \text{Int}(b_i'^{-j_i^0}) \neq \emptyset.$$

By Lemma 2.9, item 7, $f^{(-l_i - j_i^0 + 1)}(z_i) \in \text{Int}(b_i'^{-j_i^0}) \setminus (\cup_{l \geq 1} b_{i-2}^l)$. We take an arc

$$\gamma_i^0 \subset \text{Int}(b_i'^{-j_i^0}) \setminus \text{Int}(\cup_{l \geq 1} b_{i-2}^l)$$

joining $f^{(-l_i - j_i^0 + 1)}(z_i)$ and a point $x_i^0 \in \partial(\cup_{l \geq 1} b_{i-2}^l)$. We define $k_i^0 \geq 1$ by

$$x_i^0 \in b_{i-2}^{k_i^0}.$$

We define inductively for $m \geq 0$:

1. $U_m \subset U_{i-2}^+ = U_i^-$ a neighbourhood of $\omega_{i-2} = \alpha_i$ in $\overline{\mathbb{D}}$ such that

$$\overline{U_m} \cap (\text{Int}(b_{i-2}^{k_i^m}) \cup \text{Int}(b_i'^{-j_i^m})) = \emptyset,$$

2. $K_m > 0$ such that for all $k \geq K_m$ $b_{i-2}^{k_i^m} \cup b_i'^{-k} \subset U_m$,
3. $j_i^{m+1} > K_m$, such that $\text{Int}(\cup_{k \geq K_m} b_{i-2}^{k_i^m}) \cap \text{Int}(b_i'^{-j_i^{m+1}}) \neq \emptyset$,
4. $\gamma_i^{m+1} \subset \text{Int}(b_i'^{-j_i^{m+1}}) \setminus (\cup_{l \geq K_m} b_{i-2}^l)$ an arc joining $f^{(-l_i - j_i^{m+1} + 1)}(z_i)$ and a point $x_i^{m+1} \in \partial(\cup_{k \geq K_m} b_{i-2}^{k_i^m})$,
5. $k_i^{m+1} > K_m$ by

$$x_i^{m+1} \in b_{i-2}^{k_i^{m+1}}.$$

The existence of K_m comes from the fact that both sequences $(b_{i-2}^{l_i})_{l \geq 1}$, $(b_{i-2}^l)_{l \geq 1}$ converge in de Hausdorff topology to $\alpha_i = \omega_{i-2}$; that of j_i^{m+1} from the fact that $i \in I_{\text{sing}}$; that of γ_i^{m+1} from the choice of j_i^{m+1} and the fact that $f^{(-l_i - j_i^{m+1} + 1)}(z_i) \in \text{Int}(b_i'^{-j_i^{m+1}}) \setminus (\cup_{l \geq K_m} b_{i-2}^l)$, and that of x_i^{m+1} and k_i^{m+1} follows from the choice of j_i^{m+1} .

By thickening these arcs $\{\gamma_i^m\}$, we can construct disks $\{c_i^m\}$ verifying all the conditions of the lemma. \square

The proposition above allows us to construct a free brick decomposition (V, E, B) such that:

1. for all $i \in \mathbb{Z}/n\mathbb{Z}$ and for all $l \geq 1$, there exists $b_i^l \in B$ such that $b_i^l \subset b_i^1$,
2. for all $i \in I_{\text{reg}}$ and for all $l \geq 1$, there exists $b_i^{-l} \in B$ such that $b_i^{-l} \subset b_i^{-1}$,
3. for all $m \geq 0$ and for all $i \in I_{\text{sing}}$ there exists $b_i^{-j_i^m} \in B$ such that $c_i^m \subset b_i^{-j_i^m}$.

Remark 5.3. The main difference between this brick decomposition and the one we were able to construct when the points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$, were all different, is that for $i \in I_{\text{sing}}$ we do NOT necessarily have

$$\cup_{l \leq j_i^0} b_i^{-l} \subset [b_i^{-j_i^0}]_{\leq}.$$

In particular, we may not be able to construct a curve

$$\Gamma : [0, 1] \rightarrow \cup_{m \geq 0} [b_i^{-j_i^m}]_{>} \cup \{\alpha_i, \omega_i\}$$

joining α_i and ω_i (see the proof of Lemma 4.6 in the previous Chapter).

II. The “domino effect” of the elliptic order property.

Lemma 5.4. *Take two indices i, j in $\mathbb{Z}/n\mathbb{Z}$, and two integers k and N . If b_j^k and b_{j+2}^k are contained in $[b_i^N]_{>}$, then there exists $k' \in \mathbb{Z}$ such that $b_i^{k'}$ is contained in $[b_i^N]_{>}$ for all $l \in \mathbb{Z}/n\mathbb{Z}$.*

Proof. We will show that if b_j^k and b_{j+2}^k are contained in $[b_i^N]_{>}$, then there exists k'' such that both $b_{j+1}^{k''}$ and $b_{j+3}^{k''}$ are contained in $[b_i^N]_{>}$. If b_j^k and b_{j+2}^k are contained in $[b_i^N]_{>}$, b_j^l and b_{j+2}^l are contained in $[b_i^N]_{>}$ for all $l \geq k$. By Remark 2.10, we can find an arc

$$\gamma : [0, 1] \rightarrow [b_i^N]_{>} \cup \{\omega_j, \omega_{j+2}\}$$

joining ω_j and ω_{j+2} . As $n > 3$, and the coincidences are of the form $\alpha_i = \omega_{i-2}$, we know that the points $\alpha_{j+1}, \omega_j, \alpha_{j+3}, \omega_{j+2}$ are all different. So, γ separates both α_{j+1} from ω_{j+1} and α_{j+3} from ω_{j+3} . So, by Remark 4.4 there exists $k'' > 0$ such that $[b_{j+1}^{k''}]_{\leq} \cap [b_i^N]_{>} \neq \emptyset$ and $[b_{j+3}^{k''}]_{\leq} \cap [b_i^N]_{>} \neq \emptyset$. We are done by induction, and by taking k' large enough. \square

In the following lemma we make reference to the sequences $(k_i^m)_{m \geq 0}$ and $(j_i^m)_{m \geq 0}$ defined in Lemma 5.2.

Lemma 5.5. *For every $i \in I_{\text{sing}}$, there exists $N > 0$ such that $[b_i^{-j_i^N}]_{\geq}$ contains $b_{i-2}^{k_i^N}$.*

Proof. We will prove the following stronger statement which implies immediately that $[b_i^{-j_i^N}]_{\geq}$ contains $b_{i-2}^{k_i^N}$: there exists $N > 0$ such that $f(c_i^N) \cap b_{i-2}^{k_i^N} \neq \emptyset$.

I. Let us begin by studying the local dynamics of the brick decomposition at $\alpha_i = \omega_{i-2}$, $i \in I_{\text{sing}}$. We define for all $m \geq 0$,

$$X_m = b_{i-2}^{k_i^m} \cup c_i^m,$$

and we recall that every X_m is a closed disk (see Lemma 5.2). Then, for all $m \geq 0$,

$$f^{l_{i-2}+k_i^m-1}(z_{i-2}) \cup f^{-l_i-j_i^m-j_i^m}(z_i) \in X_m.$$

So, given any two positive integers $m > p$, one has:

$$\cup_{k \geq 1} f^k(X_p) \cap X_m \neq \emptyset$$

and

$$\cup_{k \geq 1} f^k(X_m) \cap X_p \neq \emptyset.$$

Besides, $X_m \cap X_p = \emptyset$ and X_m and X_p are topological closed disks. Therefore, if we can find $m > p \geq 0$ such that both X_p and X_m are free sets, f would be recurrent by Proposition 2.5. So, we can suppose that for all $m \geq 0$ the set X_m is not free. So, as for all $m \geq 0$ both $b_{i-2}^{k_i^m}$ and c_i^m are free sets, then either $f(b_{i-2}^{k_i^m}) \cap c_i^m \neq \emptyset$, or $f(c_i^m) \cap b_{i-2}^{k_i^m} \neq \emptyset$. If there exists $m > 0$ such that $f(c_i^m) \cap b_{i-2}^{k_i^m} \neq \emptyset$, we are done. So, we may assume that for all $m \geq 0$, $f(b_{i-2}^{k_i^m}) \cap c_i^m \neq \emptyset$. Then, $f(b_{i-2}^{k_i^m}) \cap b_{i-2}^{-j_i^m} \neq \emptyset$ for all $m \geq 0$. In particular, $[b_{i-2}^{k_i^m}]_{>}$ contains b_{i-2}^l for all $l > 0$ and for all $m \geq 0$.

II. We will show that this implies that f is recurrent. As $[b_{i-2}^{k_i^m}]_{>}$ contains b_{i-2}^k and b_{i-2}^l , for $k > l$, Lemma 5.4 implies that for all $m \geq 0$ there exists $l_m > 0$ such that $[b_{i-2}^{k_i^m}]_{>}$ contains b_{i-2}^l for all $l \in \mathbb{Z}/n\mathbb{Z}$ and for all $l \geq l_m$.

In particular, Remark 2.10 tells us that for all $m \geq 0$ there exists an arc

$$\Gamma_m : [0, 1] \rightarrow [b_{i-2}^{k_i^m}]_{>} \cup \{\omega_{i-2}, \omega_{i-4}\}$$

joining ω_{i-2} and ω_{i-4} , which implies that Γ_m separates α_{i-1} from α_{i-3} in $\overline{\mathbb{D}}$ (see Figure 5.2 (a) and observe that as $n > 3$ the points $\alpha_{i-3}, \omega_{i-4}, \alpha_{i-1}, \omega_{i-2}$ are all different). As we are assuming that f is not recurrent, we obtain that the closure of $[b_{i-2}^{k_i^m}]_{\leq}$ cannot contain both points α_{i-1} and α_{i-3} .

We will suppose that for all $m \geq 0$, the closure of $[b_{i-2}^{k_i^m}]_{\leq}$ does not contain one of the points α_{i-1} and α_{i-3} , and obtain a contradiction. As $m > p$ implies

$$[b_{i-2}^{k_i^p}]_{\leq} \subset [b_{i-2}^{k_i^m}]_{\leq},$$

one of the points α_{i-1} or α_{i-3} is not contained in the closure of any of the sets $[b_{i-2}^{k_i^m}]_{\leq}$, $m \geq 0$. Let us suppose that α_{i-3} is not contained in $[b_{i-2}^{k_i^m}]_{\leq}$ for any $m \geq 0$ (the proof is analogous in the other case). In particular, for all $m \geq 0$, $[b_{i-2}^{k_i^m}]_{\leq}$ does not contain any of the bricks containing the orbit of z_{i-3} . We take a neighbourhood U of α_{i-3} in $\overline{\mathbb{D}}$ such that $U \cap [b_{i-2}^{k_i^0}]_{\leq} = \emptyset$ and such that $U \cap \cup_{l > k_i^0} b_{i-2}^l = \emptyset$. We take $j > 0$ such that $f^{-j}(z_{i-3}) \in U$, and

an arc $\beta : [0, 1] \rightarrow U$ joining α_{i-3} and $f^{-j}(z_{i-3})$. Take a brick $b \in B$ such that $f^{-j}(z_{i-3}) \in b$. As $\cup_{l \geq 1} b_{i-3}^l \subset [b]_{\geq}$, Remark 2.10 allows us to take an arc $\gamma : [0, 1] \rightarrow [b]_{\geq} \cup \omega_{i-3}$ joining $f^{-j}(z_{i-3})$ and ω_{i-3} .

So, $\beta \cdot \gamma$ separates α_{i-2} from ω_{i-2} in $\overline{\mathbb{D}}$ and

$$\beta \cdot \gamma \cap (\cup_{l > k_0} b_{i-2}^l \cup [b_{i-2}^{k_0}]_{\leq}) \neq \emptyset,$$

which implies

$$\gamma \cap (\cup_{l > k_0} b_{i-2}^l \cup [b_{i-2}^{k_0}]_{\leq}) \neq \emptyset,$$

because of our choice of U (see Figure 5.2 (b)). So,

$$b_{\geq} \cap \cup_{l > 0} [b_{i-2}^l]_{<} \neq \emptyset,$$

which implies that for some $m \geq 0$,

$$[b]_{\geq} \cap [b_{i-2}^m]_{<} \neq \emptyset.$$

So, $b \in [b_{i-2}^{k_i^m}]_{\leq}$, and $[b_{i-2}^{k_i^m}]_{\leq}$ contains a brick containing one point of the orbit of z_{i-3} .

This contradiction finishes the proof of the lemma. □

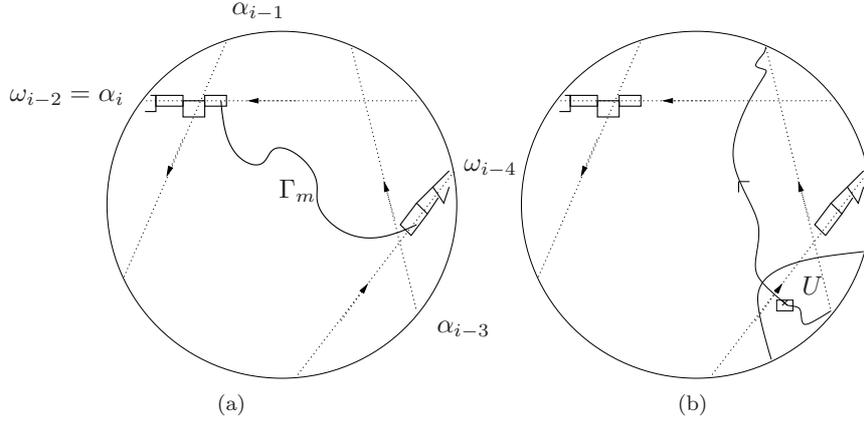


Figure 5.2: The proof of lemma 5.5

Lemma 5.6. *There exists $k > 0$ such that for any pair of indices i, j in $\mathbb{Z}/n\mathbb{Z}$, the attractor $[b_i^{-k}]_{>}$ contains b_j^k .*

Proof. For all $i \in I_{\text{reg}}$, we know that $\cup_{l \geq 1} b_i'^{-l} \subset \cup_{l > 0} [b_i^{-l}]_{>}$ (note that this is not necessarily the case if $i \in I_{\text{sing}}$). So, by Remark 2.10, there exists an arc

$$\Gamma_i : [0, 1] \rightarrow \cup_{l > 0} [b_i^{-l}]_{>} \cup \{\alpha_i, \omega_i\}$$

joining α_i and ω_i . So, Γ_i separates both α_{i-1} from ω_{i-1} and α_{i+1} from ω_{i+1} in $\overline{\mathbb{D}}$. By Remark 4.4, there exists $m > 0$ such that $[b_i^{-m}]_{>}$ contains both b_{i+1}^m and b_{i-1}^m . By Lemma 5.4, $[b_i^{-m}]_{>}$ contains b_j^l for all $j \in \mathbb{Z}/n\mathbb{Z}$, and l large enough.

For all $i \in I_{\text{sing}}$, the previous lemma tells us that there exists $N > 0$ such that $[b_i^{-j_i^N}]_{\geq}$ contains $b_{i-2}^{k_i^N}$. Clearly, $[b_i^{-j_i^N}]_{\geq}$ also contains $b_i^{k_i^N}$ and so once again, Lemma 5.4 implies that $[b_i^{-j_i^N}]_{\geq}$ contains b_j^l , for all $j \in \mathbb{Z}/n\mathbb{Z}$, and l large enough. We finish by taking k sufficiently large. \square

III. Constraints on the order of the cycle of links \mathcal{L} .

We fix $k > 0$ such that for any pair of indices i, j in $\mathbb{Z}/n\mathbb{Z}$, $[b_i^{-k}]_{>}$ contains b_j^k . We define

$$a_i = (\cup_{m \geq k} b_i^m) \cap \Gamma_i^+, \quad i \in \mathbb{Z}/n\mathbb{Z}$$

(see Remark 2.10 for the definition of Γ_i^+). We may suppose that

$$U = \mathbb{D} \setminus \cup_{i \in \mathbb{Z}/n\mathbb{Z}} a_i$$

is simply connected. As $a_i \subset \cup_{m \geq k} b_i^m$, and we are supposing that f is not recurrent, we know that $[b_i^{-k}]_{<} \subset \bar{U}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Let $\varphi : U \rightarrow \mathbb{D}$ be the Riemann map and consider the intervals $J_i, i \in \mathbb{Z}/n\mathbb{Z}$ defined in 3.1. We define I_i as to be the connected component of $S^1 \setminus \cup_{l \in \mathbb{Z}/n\mathbb{Z}} J_l$ following J_{i-2} in the natural (positive) cyclic order on S^1 . So, each I_i is a closed interval, and we have:

$$J_{i-2} \rightarrow I_i \rightarrow J_{i-1}$$

for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Lemma 5.7. *For all $i \in \mathbb{Z}/n\mathbb{Z}$,*

1. *there exists $j_i \in \mathbb{Z}/n\mathbb{Z}$ such that $\overline{\varphi([b_i^{-k}]_{<})} \cap S^1 \subset I_{j_i}$,*
2. *$j_i \in \{i-1, i\}$,*
3. *if $\alpha_i \neq \omega_{i-2}$, then $j_i = i$.*

Proof. 1. If there exists $x \in \overline{\varphi([b_i^{-k}]_{<})} \cap J_j$ for some $j \in \mathbb{Z}/n\mathbb{Z}$, then $\overline{[b_i^{-k}]_{<}} \cap a_j \neq \emptyset$ (see Lemma 3.7). As $[b_i^{-k}]_{<}$ is closed in \mathbb{D} , and as $a_j \subset \mathbb{D}$, we obtain $[b_i^{-k}]_{<} \cap a_j \neq \emptyset$, a contradiction. So, $\overline{\varphi([b_i^{-k}]_{<})} \subset \cup_{j \in \mathbb{Z}/n\mathbb{Z}} I_j$. If $\varphi([b_i^{-k}]_{<})$ intersects I_j and I_k , $k \neq j$, then there exists two different indices i_0 and i_1 in $\mathbb{Z}/n\mathbb{Z}$ such that any arc joining J_{i_0} and J_{i_1} separates I_j from I_k . We take a crosscut γ from a_{i_1} to a_{i_2} such that $\gamma \subset [b_i^{-k}]_{>}$. So, Lemma 3.6 implies

$$\varphi(\gamma \cap U) \cap \varphi([b_i^{-k}]_{<}) \neq \emptyset,$$

and consequently

$$[b_i^{-k}]_{>} \cap [b_i^{-k}]_{<} \neq \emptyset,$$

which contradicts our assumption that f is not recurrent.

2. Take a crosscut $\gamma \subset [b_i^{-k}]_{>}$ from a_{i-3} to a_{i-1} . Then, the elliptic order property implies that α_i belongs to the closure of only one of the two connected components of $U \setminus \gamma$; the one to the right of γ . We use here the fact that $\alpha_i \notin \{\omega_{i-3}, \omega_{i-1}\}$. So, $[b_i^{-k}]_{<}$ also belongs to the connected component of $U \setminus \gamma$ which is to the right of γ . Consequently, $\varphi([b_i^{-k}]_{<})$ belongs to the connected component of $\mathbb{D} \setminus \varphi(\gamma \cap U)$ which is to the right

of $\varphi(\gamma \cap U)$. As $\overline{\varphi(\gamma \cap U)}$ is an arc from J_{i-3} to J_{i-1} (see Lemma 3.6), the closure of this connected component only contains I_i and I_{i-1} . So, we obtain $j_i \in \{i-1, i\}$.

3. If $\alpha_i \neq \omega_{i-2}$, we can apply exactly the same argument than in the preceding item, but using a crosscut γ from a_{i-2} to a_{i-1} , obtaining $j_i = i$. \square

Remark 5.8. If we set $b_i^- = b_i^{-k}$, and $b_i^+ = b_i^k$, the bricks b_i^- , $i \in \{i_0, i_1, i_2\}$ satisfy all the hypothesis of Lemma 3.22, where i_0, i_1, i_2 are any three different indices $\in \mathbb{Z}/n\mathbb{Z}$. Indeed, k is chosen so that 2. and 3. (a), hold, 3.(b) is granted since $\alpha_i \subset [b_i^-]_<$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, and 3. (c) is the content of item 1. in the preceding lemma.

The second item in the preceding lemma gives us:

Corollary 5.9. *If $|i-l| \geq 2$, then $j_i \neq j_l$.*

The constraints on the order \mathcal{L} follows.

Lemma 5.10. *The order of \mathcal{L} is either 4 or 5.*

Proof. If $n \geq 6$, the sets $\{i, i-1\}$, $i \in \{0, 2, 4\}$ are pairwise disjoint, and so the three indices j_0, j_2, j_4 given by Lemma 5.7 are different. This contradicts Lemma 3.22. \square

Lemma 5.11. *We have $n = 4$.*

Proof. We show that $n = 5$ also contradicts Lemma 3.22. If j_0, j_2, j_3 are all different, we are done because of Lemma 3.22. Otherwise, the only possibility is that $j_2 = j_3 = 2$ (see Lemma 5.7). But then, j_1, j_3 and j_4 are different. \square

Lemma 5.12. *\mathcal{L} is degenerate.*

Proof. We will show that if $n = 4$ and \mathcal{L} is non-degenerate, we can also find a triplet i_0, i_1, i_2 in $\mathbb{Z}/n\mathbb{Z}$ such that the correspondent j_{i_s} , $s \in \{0, 1, 2\}$ are different.

For a non-degenerate cycle of links, there can be at most two coincidences of the type $\alpha_i = \omega_{i-2}$. Furthermore, if $\alpha_i = \omega_{i-2}$ and $\alpha_j = \omega_{j-2}$ for some $i \neq j$, then $|i-j| = 1$. Indeed, the points in ℓ are ordered as follows:

$$\omega_0 \xrightarrow{=} \alpha_2 \rightarrow \omega_1 \xrightarrow{=} \alpha_3 \rightarrow \omega_2 \xrightarrow{=} \alpha_0 \rightarrow \omega_3 \xrightarrow{=} \alpha_1 \rightarrow \omega_0,$$

and non-degeneracy means that we cannot have both $\omega_i = \alpha_{i+2}$ and $\omega_{i+2} = \alpha_i$, for some $i \in \mathbb{Z}/4\mathbb{Z}$. So, there exists $l \in \mathbb{Z}/4\mathbb{Z}$ such that $\alpha_l \neq \omega_{l-2}$ and $\alpha_{l+1} \neq \omega_{l-1}$. We can suppose without loss of generality that $\alpha_0 \neq \omega_2$, and $\alpha_1 \neq \omega_3$ (see Figure 5.3). Items 2. and 3. in Lemma 5.7 imply that j_0, j_1 , and j_3 are different, and we are done. \square

The following lemma finishes the proof of Proposition 6.2.

Lemma 5.13. *If $n = 4$, then $\text{Fix}(f) \neq \emptyset$.*

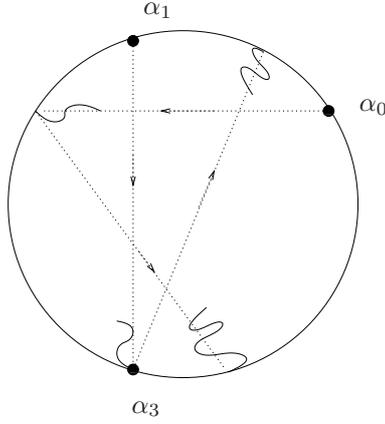


Figure 5.3: The case $n = 4$

Proof. We will be done by constructing a hyperbolic Repeller/Attractor configuration of order 2. We define

$$R_0 = [b_0^{-k}]_<, R_1 = [b_2^{-k}]_<, A_0 = [b_3^k]_>, A_1 = [b_1^k]_>.$$

By the choice of k , there exists two bricks c_i^i, c_i^{i-1} , contained in $R_i, i \in \mathbb{Z}/2\mathbb{Z}$ such that $[c_i^j]_> \cap A_j \neq \emptyset$, if $j \in \{i, i-1\}$.

Besides, the cyclic order of these sets is the following:

$$R_0 \rightarrow A_0 \rightarrow R_1 \rightarrow A_1 \rightarrow R_0.$$

Indeed, we know that $j_0 \in \{0, 3\}, j_2 \in \{2, 1\}$, and the cyclic order of the intervals $J_i, I_i, i \in \mathbb{Z}/4\mathbb{Z}$ is:

$$I_0 \rightarrow J_3 \rightarrow I_1 \rightarrow J_0 \rightarrow I_2 \rightarrow J_1 \rightarrow I_3 \rightarrow J_2 \rightarrow I_0.$$

So, we just have to show that the sets $R_i, A_i, i \in \mathbb{Z}/2\mathbb{Z}$ are pairwise disjoint. The choice of k implies that $[b_i^{-k}]_< \cap [b_j^k]_> = \emptyset$ for all i, j in $\mathbb{Z}/4\mathbb{Z}$. As a consequence, we just have to check $R_0 \cap R_1 = \emptyset$, and $A_0 \cap A_1 = \emptyset$.

If this is not the case, $[b_0^{-k}]_< \cup [b_2^{-k}]_<$ is a connected set separating $[b_1^k]_>$ and $[b_3^k]_>$. Again by the choice of k we have:

$$([b_0^{-k}]_< \cup [b_2^{-k}]_<) \cap [b_0^{-k}]_> \neq \emptyset,$$

and as we are supposing that f is not recurrent,

$$[b_2^{-k}]_< \cap [b_0^{-k}]_> \neq \emptyset.$$

But then,

$$[b_2^{-k}]_< \cap [b_2^{-k}]_> \neq \emptyset,$$

because $[b_2^{-k}]_<$ contains $[b_0^{-k}]_<$ and therefore separates $[b_1^k]_>$ and $[b_3^k]_>$, both of which are contained in $[b_2^{-k}]_>$. \square

5.2 The hyperbolic case.

Our next proposition finishes the proof of Theorem 1.1:

Proposition 5.14. *If \mathcal{L} is hyperbolic, then $\text{Fix}(f) \neq \emptyset$.*

We recall that the order of a hyperbolic cycle of links is an even number. That is, from now on $n = 2m$, $m \geq 2$.

To illustrate the ideas, we include a proof for the case where the points $\{\alpha_i\}, \{\omega_i\}$, are all different. We did this for the elliptic case in Chapter 4.

Proposition 5.15. *If \mathcal{L} satisfy the additional hypothesis:*

$$(H) \text{ the points } \alpha_i, \omega_i, i \in \mathbb{Z}/2m\mathbb{Z} \text{ are all different,}$$

then $\text{Fix}(f) \neq \emptyset$.

Remark 5.16. With these assumptions, the cyclic order of the points $\{\alpha_i\}, \{\omega_i\}$, at the circle at infinity satisfies:

$$\alpha_i \rightarrow \alpha_{i-1} \rightarrow \omega_{i+1} \rightarrow \omega_i \rightarrow \alpha_{i+2}$$

for all even values of $i \in \mathbb{Z}/2m\mathbb{Z}$.

We apply Lemma 2.9 and obtain a family of closed disks $(b_i^l)_{l \in \mathbb{Z} \setminus \{0\}, i \in \mathbb{Z}/2m\mathbb{Z}}$. The hypothesis (H) allows us to suppose that all the bricks $(b_i^l)_{l \in \mathbb{Z} \setminus \{0\}, i \in \mathbb{Z}/2m\mathbb{Z}}$ have pairwise disjoint interiors (see Remark 2.11). We construct a maximal free brick decomposition (V, E, B) such that for all $i \in \mathbb{Z}/2m\mathbb{Z}$ and for all $l \in \mathbb{Z} \setminus \{0\}$, there exists $b_i^l \in B$ such that $b_i^l \subset b_i^l$ (see Corollary 2.12).

We will suppose that f is not recurrent, and we will show that we can construct a hyperbolic configuration.

Lemma 5.17. (Hyperbolic domino effect) *There exists $k > 0$ such that for all even values of $i \in \mathbb{Z}/2m\mathbb{Z}$, both attractors $[b_i^{-k}]_>$ and $[b_{i-1}^{-k}]_>$ contain b_i^k for all $l \in \{i-2, i-1, i, i+1\}$.*

Remark 5.18. Note that for all $i = 0 \pmod 2$:

$$\omega_{i-1} \rightarrow \omega_{i-2} \rightarrow \alpha_i \rightarrow \alpha_{i-1} \rightarrow \omega_{i+1} \rightarrow \omega_i.$$

So, the “future indices” $\{i-2, i-1, i, i+1\}$ are those coming immediately before and immediately after the “past indices” $\{i, i-1\}$ in the cyclic order.

Proof. By Remark 2.10, we can find an arc

$$\Gamma : [0, 1] \rightarrow \cup_{l \geq 1} [b_i^{-l}]_> \cup \{\alpha_i, \omega_i\}$$

joining α_i and ω_i . So, Γ separates α_{i-1} from ω_{i-1} and α_{i+1} from ω_{i+1} (in $\overline{\mathbb{D}}$). So, there exists $l > 0$ such that $[b_i^{-l}]_> \cap [b_{i-1}^l]_< \neq \emptyset$ and $[b_i^{-l}]_> \cap [b_{i+1}^l]_< \neq \emptyset$. So,

$$(\cup_{k \geq l} b_{i-1}^k) \cap (\cup_{k \geq l} b_{i+1}^k) \subset [b_i^{-l}]_>.$$

Using Remark 2.10 again, we can find an arc

$$\Gamma' : [0, 1] \rightarrow [b_i^{-l}]_> \cup \{\omega_{i+1}, \omega_{i-1}\}$$

joining ω_{i+1} and ω_{i-1} . The cyclic order at S^1 of the points $\{\alpha_i\}, \{\omega_i\}$, implies that Γ' separates ω_{i-2} from α_{i-2} in $\overline{\mathbb{D}}$. So,

$$\Gamma' \cap \cup_{k \geq 1} [b_{i-2}^k]_{<} \neq \emptyset,$$

which implies that there exists $j > 0$ such that $b_{i-2}^j \in [b_i^{-l}]_{>}$. By taking $m > 0$ large enough, we obtain that for all $l \in \{i-2, i-1, i, i+1\}$, $b_l^m \in [b_i^{-m}]_{>}$. Analogously we obtain $b_l^p \in [b_{i-1}^{-p}]_{>}$ for all $l \in \{i-2, i-1, i, i+1\}$, for a suitable $p > 0$. We finish by taking $k \geq \max\{m, p\}$ \square

We are now ready to prove Proposition 5.15:

Proof. We will show that $(([b_i^{-k}]_{<})_{i=0 \bmod 2}, ([b_i^k]_{>})_{i=0 \bmod 2})$ is a hyperbolic configuration, where $k > 0$ is given by Lemma 5.17 (the choice of even indices is arbitrary; we may as well have chosen the odd indices).

By Remark 5.16 and Lemma 5.17, we just have to show that the sets $[b_i^{-k}]_{<}, [b_i^k]_{>}$, for i even, are pairwise disjoint. Lemma 5.17 also gives us,

$$[b_i^{-k}]_{<} \cap [b_{i-2}^k]_{>} = \emptyset,$$

for i even. If $[b_i^{-k}]_{<} \cap [b_j^k]_{>} \neq \emptyset$ for an even j other than $i-2$, then we can find an arc $\Gamma : [0, 1] \rightarrow [b_i^{-k}]_{<} \cup \{\alpha_i, \alpha_j\}$ joining α_i and α_j . The cyclic order at S^1 of the points $\{\alpha_i\}, \{\omega_i\}$ implies that Γ separates ω_i from ω_{i-2} in $\overline{\mathbb{D}}$. As $[b_i^{-k}]_{>}$ is a connected set whose closure contains both ω_i and ω_{i-2} (by the previous lemma), one gets

$$[b_i^{-k}]_{>} \cap \Gamma \neq \emptyset$$

and so

$$[b_i^{-k}]_{>} \cap [b_i^{-k}]_{<} \neq \emptyset,$$

which implies that f is recurrent. So, we have:

$$[b_i^{-k}]_{<} \cap [b_j^k]_{>} = \emptyset,$$

for any pair of even indices i, j . We will show that

$$[b_i^{-k}]_{<} \cap [b_j^{-k}]_{<} = \emptyset$$

for any two different even indices i, j . Otherwise, we could find an arc

$$\Gamma : [0, 1] \rightarrow [b_i^{-k}]_{<} \cup [b_j^{-k}]_{<} \cup \{\alpha_i, \alpha_j\}$$

joining α_i and α_j , from which we deduce again using the preceding lemma that

$$([b_i^{-k}]_{<} \cup [b_j^{-k}]_{<}) \cap [b_i^{-k}]_{>} \neq \emptyset.$$

So, as f is not recurrent, we have

$$[b_j^{-k}]_{<} \cap [b_i^{-k}]_{>} \neq \emptyset.$$

But now we can find an arc $\Gamma : [0, 1] \rightarrow [b_j^{-k}]_{<} \cup \{\alpha_i, \alpha_j\}$ joining α_i and α_j , which implies

$$[b_j^{-k}]_{<} \cap [b_j^{-k}]_{>} \neq \emptyset,$$

contradicting that f is not recurrent. The proof of the fact that $[b_i^k]_{>} \cap [b_j^k]_{>} = \emptyset$ for any two different even indices i, j , is completely analogous. \square

In what follows, we will deal with the general case; that is, we consider cycles of links where the points $\{\alpha_i\}, \{\omega_i\}$, are not necessarily different. By the hyperbolic order property, the only possible coincidences among the points $\alpha_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z}$ are of the form $\omega_{i-2} = \alpha_i$, for even values of i , or $\omega_{i+2} = \alpha_i$, for odd values of i .

As the points $\{\omega_i\}$ are all different, we can take a neighbourhood U_i^+ of ω_i in \mathbb{D} in such a way that $U_i^+ \cap U_j^+ = \emptyset$ if $i \neq j$. For even values of i , we define $U_i^- = U_{i-2}^+$ if $\alpha_i = \omega_{i-2}$, and if $\alpha_i \neq \omega_{i-2}$ we take a neighbourhood U_i^- of α_i in \mathbb{D} in such a way that $U_i^- \cap U_j^+ = \emptyset$ for any j , and $U_i^- \cap U_j^- = \emptyset$ if $j \neq i$. Similarly, for odd values of i , we define $U_i^- = U_{i+2}^+$ if $\alpha_i = \omega_{i+2}$, and if $\alpha_i \neq \omega_{i+2}$ we take a neighbourhood U_i^- of α_i in \mathbb{D} in such a way that $U_i^- \cap U_j^+ = \emptyset$ for any j , and $U_i^- \cap U_j^- = \emptyset$ if $j \neq i$.

We keep the assumption that f is not recurrent.

We apply Lemma 2.9 and obtain families of closed disks $(b_i^l)_{l \in \mathbb{Z} \setminus \{0\}, i \in \mathbb{Z}/2m\mathbb{Z}}$. So, the disks in the family $(b_i^l)_{l \geq 1, i \in \mathbb{Z}/2m\mathbb{Z}}$ have pairwise disjoint interiors.

Let I_{reg} be the set of even $i \in \mathbb{Z}/2m\mathbb{Z}$ such that $\alpha_i \neq \omega_{i-2}$, or such that $\alpha_i = \omega_{i-2}$ but there exists $K > 0$ such that $\cup_{k > K} b_{i-2}^{k'} \cap \cup_{k > K} b_i^{-k} = \emptyset$, together with the set of odd $i \in \mathbb{Z}/2m\mathbb{Z}$ such that $\alpha_i \neq \omega_{i+2}$, or such that $\alpha_i = \omega_{i+2}$ but there exists $K > 0$ such that $\cup_{k > K} b_{i+2}^{k'} \cap \cup_{k > K} b_i^{-k} = \emptyset$. Let I_{sing} be the complementary set of I_{reg} in $\mathbb{Z}/2m\mathbb{Z}$.

We can suppose that all the disks in the families $(b_i^l)_{l \geq 1, i \in \mathbb{Z}/2m\mathbb{Z}}, (b_i^{-l})_{l \geq 1, i \in I_{\text{reg}}}$ have disjoint interiors.

We define $i^* = i - 2$ if i is even, and $i^* = i + 2$ if i is odd.

Lemma 5.19. *If $i \in I_{\text{sing}}$, we can find sequences of free closed disks $(c_i^n)_{n \geq 0}$, satisfying :*

1. $c_i^n \subset U_{i^*}^+ = U_i^-$,
2. *there exists an increasing sequence $(k_i^n)_{n \geq 0}$ such that $b_{i^*}^{k_i^n} \cap c_i^n \neq \emptyset$ for all $n \geq 0$,*
3. $(b_{i^*}^{k_i^n} \cup c_i^n) \cap (b_{i^*}^{k_i^p} \cup c_i^p) = \emptyset$ for all $n \neq p$,
4. *there exists an increasing sequence $(j_i^n)_{n \geq 0}$ such that $f^{-j_i^n}(z_i) \in c_i^n$,*
5. *the sequence $(c_i^n)_{n \geq 0}$ converge in the Hausdorff topology to $\omega_{i^*} = \alpha_i$,*
6. $b_{i^*}^{k_i^n} \cap c_i^n$ is an arc for all $n \geq 0$,
7. $\partial(\cup_{k \geq 1} b_{i^*}^{k'} \cup \cup_{n \geq 0} c_i^n)$ is a one dimensional submanifold,
8. *if $x \in \mathbb{D}$, then x belongs to at most two different disks in the family $\{b_{i^*}^{k'}, c_i^n : k \geq 1, n \geq 0\}$.*

Proof. Note that the local dynamics in a neighbourhood of a point $\alpha_i, i \in I_{\text{sing}}$ is exactly the same as that in the elliptic case. So, the same proof we did for Lemma 5.2 works here as well. □

We construct a maximal free brick decomposition (V, E, B) such that:

1. for all $i \in \mathbb{Z}/2m\mathbb{Z}$ and for all $l \geq 1$, there exists $b_i^l \in B$ such that $b_i^l \subset b_i^l$,
2. for all $i \in I_{\text{reg}}$ and for all $l \geq 1$, there exists $b_i^{-l} \in B$ such that $b_i^{-l} \subset b_i^{-l}$,
3. for all $n \geq 0$ and for all $i \in I_{\text{sing}}$ there exists $b_i^{-j_i^n} \in B$ such that $c_i^n \subset b_i^{-j_i^n}$.

Lemma 5.20. *If $i \in I_{\text{sing}}$, then there exists $N > 0$ such that $[b_i^{-j_i^N}]_{\geq}$ contains $b_{i^*}^{k_i^N}$.*

Proof. Fix an even index $i \in I_{\text{sing}}$ (the proof for odd indices is analogous). The first part of the proof is identical to part I. in the proof of Lemma 5.5. Indeed, this proof is local, that is, it does not depend on how the rest of the point in ℓ are ordered. So, there are two possibilities: either $f(c_i^N) \cap b_{i-2}^{k_i^N} \neq \emptyset$ or $f(b_{i-2}^{k_i^N}) \cap c_i^N \neq \emptyset$. In the first case we are done, as it implies immediately the statement of the lemma. As a consequence, we may assume that for all $n \geq 0$, $[b_{i-2}^{k_i^n}]_{>}$ contains b_i^l for all $l > 0$. We will show that this contradicts the fact that f is not recurrent.

With this assumption, for all $n \geq 0$ there exists an arc

$$\Gamma_n : [0, 1] \rightarrow [b_{i-2}^{k_i^n}]_{>} \cup \{\omega_{i-2}, \omega_i\}$$

joining ω_{i-2} and ω_i (see Remark 2.10). So, the arc Γ_n separates α_{i-1} from α_{i-3} in $\overline{\mathbb{D}}$ for all $n > 0$ (see Figure 5.4, and note that the points $\alpha_{i-1}, \alpha_{i-3}, \omega_{i-2}, \omega_i$ are all different).

We deduce (as we are supposing that f is not recurrent) that for any $n > 0$ $[b_{i-2}^{k_i^n}]_{\leq}$ cannot contain both α_{i-1} and α_{i-3} . So, one of the points α_{i-1} or α_{i-3} is not contained in any of the sets $[b_{i-2}^{k_i^n}]_{\leq}$, $n > 0$. We will suppose that for all $n > 0$, $\alpha_{i-1} \notin [b_{i-2}^{k_i^n}]_{\leq}$ (the proof is analogous in the other case). We fix $n > 0$ and consider the connected set

$$K = \cup_{l \geq k_i^n} b_{i-2}^l \cup [b_{i-2}^{k_i^n}]_{\leq}.$$

We choose a neighbourhood U of α_{i-1} in $\overline{\mathbb{D}}$ such that $U \cap K = \emptyset$. Then, we take $j > 0$, such that $f^{-j}(z_{i-1}) \in U$ and $b \in B$ such that $f^{-j}(z_{i-1}) \in b$. We take an arc $\gamma \subset U$ joining α_{i-1} and $f^{-j}(z_{i-1})$, and an arc $\beta \subset [b]_{\geq} \cup \omega_{i-1}$ joining $f^{-j}(z_{i-1})$ and ω_{i-1} . We deduce that $\gamma \cdot \beta \cap K \neq \emptyset$, and as $\gamma \subset U$, we have $\beta \cap K \neq \emptyset$. So, there exists $l \geq k_i^n$ such that $b \in [b_{i-2}^l]_{\leq}$, and consequently $\alpha_{i-1} \in [b_{i-2}^l]_{\leq}$. This contradiction finishes the proof of the lemma. \square

Lemma 5.21. *There exists $k > 0$ such that for all even values of $i \in \mathbb{Z}/2m\mathbb{Z}$, both attractors $[b_i^{-k}]_{>}$ and $[b_{i-1}^{-k}]_{>}$ contain b_i^k for all $l \in \{i-2, i-1, i, i+1\}$.*

Proof. If $i \in I_{\text{sing}}$, the previous lemma tells us that there exists $N > 0$ such that $[b_i^{-j_i^N}]_{\geq}$ contains $b_{i-2}^{k_i^N}$. So, we can find an arc

$$\Gamma : [0, 1] \rightarrow [b_i^{-j_i^N}]_{\geq} \cup \{\omega_{i-2}, \omega_i\}$$

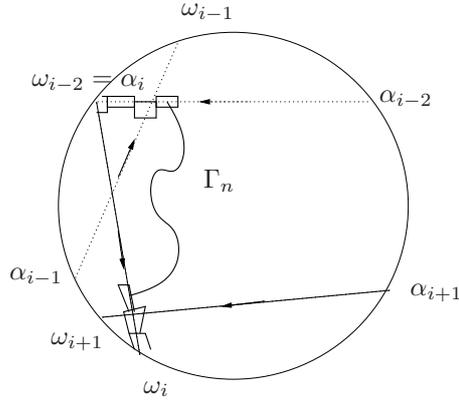


Figure 5.4: The proof of lemma 5.20

joining ω_{i-2} and ω_i . This arc separates both α_{i-1} from ω_{i-1} , and α_{i+1} from ω_{i+1} in $\overline{\mathbb{D}}$ (see Figure 13). As a consequence, both $\cup_{k \geq 1} [b_{i-1}^k]_{\leq}$ and $\cup_{k \geq 1} [b_{i+1}^k]_{\leq}$ intersect Γ , and so there exists $k > 0$ such that b_{i-1}^k and b_{i+1}^k belong to $[b_i^{-j_i^N}]_{>}$. If $i-1 \in I_{\text{sing}}$, we can show analogously that $[b_{i-1}^{-j_{i-1}^N}]_{>}$ contains b_i^k for all $l \in \{i-2, i-1, i, i+1\}$ and some $k > 0$.

If $i \in I_{\text{reg}}$, we can find an arc

$$\Gamma : [0, 1] \rightarrow \cup_{l > 0} [b_i^{-l}]_{>} \cup \{\alpha_i, \omega_i\}$$

joining α_i and ω_i . So, Γ separates (in $\overline{\mathbb{D}}$) both α_{i+1} from ω_{i+1} and α_{i-1} from ω_{i-1} . So, both $\cup_{k \geq 1} [b_{i-1}^k]_{\leq}$ and $\cup_{k \geq 1} [b_{i+1}^k]_{\leq}$ intersect Γ , and there exists $k, N > 0$ such that $[b_i^{-N}]_{>} \cap [b_{i-1}^k]_{\leq} \neq \emptyset$ and $[b_i^{-N}]_{>} \cap [b_{i+1}^k]_{\leq} \neq \emptyset$. Once b_{i-1}^k and b_{i+1}^k belong to $[b_i^{-N}]_{>}$, we can find an arc

$$\Gamma' : [0, 1] \rightarrow [b_i^{-N}]_{>} \cup \{\omega_{i-1}, \omega_{i+1}\}$$

joining ω_{i-1} and ω_{i+1} . So, Γ' separates α_{i-2} from ω_{i-2} in $\overline{\mathbb{D}}$, and one obtains $b_{i-2}^k \in [b_i^{-N}]_{>}$, for some $k > 0$. We obtain the result by sufficiently enlarging k . \square

We fix $k > 0$ as in Lemma 5.21.

Lemma 5.22. *There exists $p > k$ such that $[b_i^{-k}]_{<} \cap b_j^l = \emptyset$ for all i, j in $\mathbb{Z}/2m\mathbb{Z}$ and $l \geq p$.*

Proof. Fix $i \in \mathbb{Z}/2m\mathbb{Z}$ even. There exists an arc

$$\gamma_i : [0, 1] \rightarrow [b_i^{-k}]_{>} \cup \{\omega_{i+1}, \omega_{i-1}\}$$

joining ω_{i+1} and ω_{i-1} . As the three points α_i, ω_{i+1} , and ω_{i-1} are different, γ_i separates α_i from any ω_j $j \notin \{i-2, i-1, i+1\}$ (in $\overline{\mathbb{D}}$).

So, there exists $l_i > k$ such that γ_i separates $[b_i^{-k}]_{<}$ from any b_j^l with $l > l_i$ and $j \notin \{i-2, i-1, i+1\}$. Besides, we already know that $b_{i-1}^{l_i} \cap b_{i+1}^{l_i} = \emptyset$ if

$j \in \{i-2, i-1, i+1\}$, because $b_{i>}^{-l_i}$ contains $b_j^{l_i}$. In particular, $b_{i<}^{-l_i} \cap b_j^{l_i} = \emptyset$ for $l \geq l_i$ and $j \in \{i-2, i-1, i+1\}$.

If i is odd, we can do the same argument with an arc

$$\gamma_{i-1} : [0, 1] \rightarrow [b_i^{-k}]> \cup \{\omega_i, \omega_{i-2}\}$$

joining ω_i and ω_{i-2} .

We finish by taking $p = \max\{l_i, i \in \mathbb{Z}/2m\mathbb{Z}\}$.

□

Thanks to the two preceding lemmas we may fix $k > 0$ such that:

1. both attractors $[b_i^{-k}]>$ and $[b_{i-1}^{-k}]>$ contains b_l^k for all even values of i , and for all $l \in \{i-2, i-1, i, i+1\}$,
2. $[b_i^{-k}]< \cap b_j^{l_i} = \emptyset$ for all i, j in $\mathbb{Z}/2m\mathbb{Z}$, and $l \geq k$.

We define

$$a_i = \Gamma_i^+ \cap \cup_{l \geq k} b_i^{l_i}$$

for all $i \in \mathbb{Z}/2m\mathbb{Z}$. The cyclic order of the sets $\{a_i\}$ satisfies:

$$a_{i-2} \rightarrow a_{i+1} \rightarrow a_i,$$

for all even values of i . We may suppose that each a_i is an arc, and so $U = \mathbb{D} \setminus \cup_{i \in \mathbb{Z}/2m\mathbb{Z}} a_i$ is simply connected. Let $\varphi : U \rightarrow \mathbb{D}$ be the Riemann map and consider the intervals $\{J_i\}$ defined in 3.1.

For all even i , we define I_i as to be the connected component of $S^1 \setminus \cup_{l \in \mathbb{Z}/2m\mathbb{Z}} J_l$ following J_{i-2} in the natural (positive) cyclic order on S^1 . We define I_{i+1} , as to be the connected component of $S^1 \setminus \cup_{l \in \mathbb{Z}/2m\mathbb{Z}} J_l$ following I_i . So, for all even i we have:

$$J_{i-2} \rightarrow I_i \rightarrow J_{i+1} \rightarrow I_{i+1} \rightarrow J_i.$$

Lemma 5.23. *For all $i \in \mathbb{Z}/2m\mathbb{Z}$,*

1. $[b_i^{-k}]< \subset U$,
2. if i is even, then $\overline{\varphi([b_i^{-k}]<) \cap S^1} \subset I_i \cup I_{i-1}$, and $\overline{\varphi([b_{i-1}^{-k}]<) \cap S^1} \subset I_i \cup I_{i+1}$,
3. there exists j_i such that $\overline{\varphi([b_i^{-k}]<) \cap S^1} \subset I_{j_i}$ (so, if i is even, $j_i \in \{i, i-1\}$, $j_{i-1} \in \{i, i+1\}$).

Proof. 1. This is trivial because of the choice of $k > 0$.

2. First, we show that $\overline{\varphi([b_i^{-k}]<) \cap S^1} \subset \cup_{j \in \mathbb{Z}/2m\mathbb{Z}} I_j$. Otherwise, there exists $x \in \overline{\varphi([b_i^{-k}]<) \cap S^1} \cap J_j$ for some $j \in \mathbb{Z}/2m\mathbb{Z}$. So, $[b_i^{-k}]<$ contains a point in a_j (see Lemma 3.7). As $[b_i^{-k}]<$ is a closed subset of \mathbb{D} , and $a_j \subset \mathbb{D}$ we obtain $[b_i^{-k}]< \cap a_j \neq \emptyset$, contradicting the previous item.

Fix if $i \in \mathbb{Z}/2m\mathbb{Z}$ even. Take a crosscut $\gamma \subset [b_i^{-k}]>$ from ω_{i-1} to ω_{i+1} . So, α_i belongs to the closure of only one of the connected components of $\overline{\mathbb{D} \setminus \gamma}$; the one to the right of γ . So, $\varphi([b_i^{-k}]<)$ belongs to the connected component of $\mathbb{D} \setminus \varphi(\gamma \cap U)$ which is to the right of $\varphi(\gamma \cap U)$. As $\overline{\varphi(\gamma \cap U)}$ is

an arc joining J_{i-1} and J_{i+1} , the cyclic order implies that $\overline{\varphi([b_i^{-k}]_<)} \cap S^1 \subset I_i \cup I_{i-1}$.

The statement for $i - 1$ is proved analogously.

3. Suppose i is even (as before, the other case is analogous). The previous item implies that if $\overline{\varphi([b_i^{-k}]_<)}$ intersects I_j and I_l , $j \neq l$, then $\{j, l\} = \{i, i - 1\}$.

Take a crosscut $\gamma \subset [b_i^{-k}]_>$ from ω_{i-1} to ω_{i-2} . Then, $\overline{\varphi(\gamma \cap U)}$ separates in \mathbb{D} I_{i-1} from I_i . This gives us

$$[b_i^{-k}]_< \cap [b_i^{-k}]_> \neq \emptyset,$$

a contradiction. □

Remark 5.24. If we set $a'_i = a_{2i}$, $b_i^- = b_{2i}^-$, and $b_i^+ = b_{2i}^k$ for all $i \in \mathbb{Z}/m\mathbb{Z}$, then $a'_i, b_i^-, b_i^+, i \in \mathbb{Z}/m\mathbb{Z}$, satisfy hypothesis 1. to 3. of Lemma 3.23. So, if we prove that $j_{2i} = 2i$ for all $i \in \mathbb{Z}/m\mathbb{Z}$, then $\text{Fix}(f) \neq \emptyset$. Indeed, the sets $a'_i, i \in \mathbb{Z}/m\mathbb{Z}$ are cyclically ordered as follows:

$$a'_0 \rightarrow a'_1 \rightarrow a'_2 \rightarrow \dots \rightarrow a'_{m-2} \rightarrow a'_{m-1} \rightarrow a'_0.$$

Besides, if we set $J'_i = J_{2i}$, for all $i \in \mathbb{Z}/m\mathbb{Z}$, we have:

$$J'_{i-1} \rightarrow I_{2i} \rightarrow J'_i,$$

for all $i \in \mathbb{Z}/2m\mathbb{Z}$, and so $j_{2i} = 2i$ is exactly hypothesis 4. of Lemma 3.23.

We are now ready to prove Proposition 5.14:

Proof. Because of the previous remark, it is enough to show that $j_{2i} = 2i$ for all $i \in \mathbb{Z}/m\mathbb{Z}$. We will show that if this is not the case, we contradict Lemma 3.22. Lemma 5.23, tells us that $j_{2i} \in \{2i, 2i - 1\}$. Let us assume that $j_{2i} = 2i - 1$. This implies that j_{2i-2}, j_{2i-1} , and j_{2i} are different. Indeed, by Lemma 5.23 $j_{2i-2} \in \{2i - 3, 2i - 2\}$, $j_{2i-1} \in \{2i, 2i + 1\}$, and by assumption $j_{2i} = 2i - 1$. Besides, we have:

- $[b_{2i}^{-k}]_>$ contains b_{2i}^k, b_{2i-1}^k , and b_{2i-2}^k ,
- $[b_{2i-1}^{-k}]_>$ contains b_{2i}^k, b_{2i-1}^k , and b_{2i-2}^k ,
- $[b_{2i-2}^{-k}]_>$ contains both b_{2i-2}^k and b_{2i-1}^k .

So, as j_{2i-2}, j_{2i-1} , and j_{2i} are different, if we show that $[b_{2i-2}^{-k}]_>$ also contains b_{2i}^k , we contradict Lemma 3.22. Take a crosscut $\gamma \subset [b_{2i-2}^{-k}]_>$ from a_{2i-2} to a_{2i-4} . Then, $\overline{\varphi(\gamma \cap U)}$ separates I_{2i-1} from J_{2i} . On the other hand, $\overline{\varphi([b_{2i}^k]_<)}$ joins this both sets, as we are assuming $j_{2i} = 2i - 1$, and by definition of J_{2i} . So,

$$\varphi([b_{2i}^k]_<) \cap \varphi(\gamma \cap U) \neq \emptyset,$$

and we are done. □

Chapter 6

Polygonal cycles

In this section we prove Corollary 1.2. We fix an orientation preserving homeomorphism $f : \mathbb{D} \rightarrow \mathbb{D}$ which realizes a compact convex polygon $P \subset \mathbb{D}$, and can be extended to a homeomorphism of $\mathbb{D} \cup \ell$. We suppose that $i(P) \neq 0$, and we will show that either f is recurrent, or we can construct an elliptic or hyperbolic Repeller/Attractor configuration.

Some polygons can be simplified, due to the fact that they may have “extra” edges. More precisely, we will say that the polygon P is minimal if for every $i \in \mathbb{Z}/n\mathbb{Z}$, the lines $\{\Delta_j : j \neq i\}$ do not bound a compact convex polygon. The following lemma tells us that it is enough to deal with minimal polygons.

Lemma 6.1. *The map f realizes a minimal polygon P' such that $i(P') = i(P)$, or a triangle T such that $i(T) = 1$.*

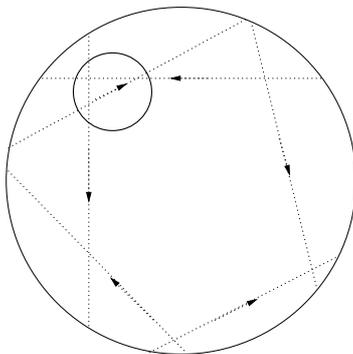


Figure 6.1: A non-minimal hexagon of index -2 presenting an index 1 subtriangle.

Proof. If P is not minimal, then there exists $i \in \mathbb{Z}/n\mathbb{Z}$ such that the straight lines $\{\Delta_j : j \neq i\}$ bound a compact polygon $P' \subset \mathbb{D}$. The line Δ_i intersects in \mathbb{D} both Δ_{i-1} and Δ_{i+1} ; it follows that necessarily

$$\Delta_{i-1} \cap \Delta_{i+1} \cap \mathbb{D} \neq \emptyset.$$

So, the lines Δ_{i-1} , Δ_i and Δ_{i+1} bound a triangle $T \subset \mathbb{D}$. Moreover,

$$i(P') = i(P) + i(T),$$

and the only possibilities for the index of a triangle are 0 or 1.

If $i(T) = 1$, we are done. Otherwise, $i(P') = i(P)$. If P' is minimal, we are done. If not, we apply the same procedure as before. We continue like this until we obtain an index 1 triangle, or a minimal polygon with the same index as P . \square

Let us state our first proposition:

Proposition 6.2. *If $i(P) = 1$, then f is recurrent.*

Proof. We observe that lemma 6.1 allows us to suppose that P is minimal; we will also suppose that the boundary of P is positively oriented. With these assumptions, the order of the points $\{\alpha_i\}, \{\omega_i\}$, satisfy the elliptic order property. Moreover, the cycle induced by P is non degenerate. We are now done by Theorem 1.1. \square

Our next proposition finishes the proof of Corollary 1.2:

Proposition 6.3. *If $i(P) < 0$, then $\text{Fix}(f) \neq \emptyset$.*

By Lemma 6.1 and Proposition 6.2, we can suppose that P is minimal. We would also like to suppose that $\delta_i = 1$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, so as to fix the cyclic order of the points $\{\alpha_i\}, \{\omega_i\}$, at the circle at infinity. For this reason, we introduce the following lemma.

Lemma 6.4. *If $\delta_i = 0$ for some $i \in \mathbb{Z}/n\mathbb{Z}$, then there exists $g \in \text{Homeo}^+(\mathbb{D})$ such that :*

1. $\text{Fix}(g) = \text{Fix}(f)$;
2. $g = f$ on the orbits of the points z_j , $j \notin \{i-1, i\}$,
3. there exists $z \in \mathbb{D}$ such that $\lim_{k \rightarrow -\infty} g^k(z) = \alpha_{i-1}$ and $\lim_{k \rightarrow +\infty} g^k(z) = \omega_i$.

We will need the following lemma, which is nothing but an adaptation of Franks' Lemma (see 2.2).

Lemma 6.5. *Let $(D_i)_{0 \leq i \leq p}$ be a chain of free, open and pairwise disjoint disks for f , and take two points $x \in D_0$ and $y \in D_p$.*

Then, there exists $g \in \text{Homeo}^+(\mathbb{D})$ and an integer $q \geq p$ such that:

- $\text{Fix}(g) = \text{Fix}(f)$,
- $g = f$ outside $\cup_{i=0}^p D_i$,
- $g^q(x) = f(y)$.

Proof. Take $z_i \in D_i$ and $k_i > 0$ the smallest positive integer such that $f^{k_i}(z_i) \in D_{i+1}$, $i \in \{0, \dots, p-1\}$. We may suppose that the chain $(D_i)_{0 \leq i \leq p}$ is of minimal length; that is, every $f^k(z_i)$, $0 < k < k_i$ is outside $\cup_{j=0}^p D_j$. We construct a homeomorphism h_0 which is the identity outside D_0 and such that $h_0(x) = z_0$, and a homeomorphism h_p which is the identity outside D_p and such that $h_p(f^{k_{p-1}}(z_{p-1})) = y$. For $i \in \{1, \dots, p-1\}$, we construct homeomorphisms h_i such that:

- h_i is the identity outside D_i ,
- $h_i(f^{k_{i-1}}(z_{i-1})) = z_i$

Finally, we construct a homeomorphism h which is the identity outside $\cup_{j=0}^p D_j$ and identical to h_i in D_i , $i \in \{0, \dots, p\}$.

So, as the disks $\{D_i\}$ are free, $g = f \circ h$ satisfy all the conditions of the lemma. □

The proof of Lemma 6.4 follows.

Proof. We will first construct a brick decomposition that suits our purposes. As the points $\alpha_{i-1}, \alpha_i, \omega_{i-1}, \omega_i$ are all different and f is not recurrent, we can construct families of closed disks $(b_i^k)_{k \in \mathbb{Z} \setminus \{0\}}$, $(b_{i-1}^k)_{k \in \mathbb{Z} \setminus \{0\}}$ as in Lemma 2.9 with the property that the interiors of the bricks in these families are pairwise disjoint.

Let $O = \cup_{i \in \mathbb{Z}/n\mathbb{Z}, k \in \mathbb{Z}} f^k(z_i)$. Here again we construct a maximal free brick decomposition such that for all $l \in \mathbb{Z} \setminus \{0\}$, there exists $b_i^l, b_{i-1}^l \in B$ such that $b_i^l \subset b_i^l$ and $b_{i-1}^l \subset b_{i-1}^l$. Furthermore, we may suppose that for all $x \in O$ there exists $b_x \in B$ such that $x \in \text{Int}(b_x)$.

If $\delta_i = 0$ for some $i \in \mathbb{Z}/n\mathbb{Z}$, then P is either to the right of both Δ_i and Δ_{i-1} or either to the left of both Δ_i and Δ_{i-1} . We will suppose that P is to the left of both lines, as the other case is analogous. By Remark 2.10, we can find an arc

$$\Gamma : [0, 1] \rightarrow \cup_{l>0} [b_i^l]_<$$

joining α_i and ω_i . So, Γ separates in $\overline{\mathbb{D}}$ α_{i-1} from ω_{i-1} . This implies that there exist two positive integers j, k such that

$$[b_{i-1}^{-j}]_> \cap [b_i^k]_< \neq \emptyset$$

(note that $\cup_{j>0} [b_{i-1}^{-j}]_>$ is a connected set whose closure contains α_{i-1} and ω_{i-1}). So, we can find a sequence of bricks $(b_m)_{0 \leq m \leq p}$ such that $b_0 = b_{i-1}^{-j}$, $b_p = b_i^k$ and $f(b_m) \cap b_{m+1} \neq \emptyset$ if $m \in \{0, \dots, p-1\}$. We will suppose that this sequence is of minimal length, that is:

$$f(b_m) \cap b_{m'} \neq \emptyset \Rightarrow m' = m + 1(*).$$

We define for all $1 \leq m \leq p-1$

$$X_m = b_m \setminus O.$$

We also define

$$X_0 = b_0 \setminus (O - \{f^{-k_{i-1}-j+1}(z_{i-1})\})$$

and

$$X_p = b_p \setminus (O - \{f^{k_i+k-1}(z_i)\})$$

(we recall from Lemma 2.9 that $f^{-l_{i-1}-j+1}(z_{i-1})$ is the only point of the orbit of z_{i-1} which lies in b_0 , and that $f^{l_i+k-1}(z_i)$ is the only point of the orbit of z_i which lies in b_p). As every $x \in O$ belongs to the interior of a brick, we know that

$$f(X_m) \cap X_{m+1} \neq \emptyset$$

if $m \in \{0, \dots, p-1\}$.

For each $m \in \{0, \dots, p-1\}$, we take $x_m \in X_m$ such that $f(x_m) \in X_{m+1}$. We take an arc $\gamma_0 \subset X_0$ from $f^{-k_{i-1}-j+1}(z_{i-1})$ to x_0 , and an arc $\gamma_p \subset X_p$ from $f(x_{p-1})$ to $f^{k_i+k-1}(z_i)$. For each $m \in \{1, \dots, p-1\}$ we take an arc $\gamma_m \subset X_m$ joining $f(x_{m-1})$ and x_m . As the interiors of the sets $\{X_m\}$ are pairwise disjoint, the arcs $\{\gamma_m\}$ can only meet in their extremities. However, condition (*) implies that the points $\{x_m\}$ (and thus the points $\{f(x_m)\}$) are all different. Indeed, if $x_m = x_{m'}$, then $f(x_m) \in X_{m'+1}$, and so $f(b_m) \cap b_{m'+1} \neq \emptyset$. It follows by (*) that $m = m'$. On the other hand, if $f(x_m) = x_{m'}$, we obtain that $f(b_m) \cap b_{m'} \neq \emptyset$, and so $m' = m+1$. This means that the arcs $\{\gamma_m\}$ are pairwise disjoint (some of them maybe reduced to a point).

It follows that we can thicken these arcs $\{\gamma_m\}$ into free, open and pairwise disjoint disks $\{D_m\}$, such that $\gamma_m \subset D_m$, and such that $D_m \cap O = \emptyset$.

We are done by Lemma 6.5. □

Lemma 6.6. *Let f realize a minimal n -gon P such that $i(P) < 0$. If $\delta_i = 0$ for some $i \in \mathbb{Z}/n\mathbb{Z}$, then either*

1. *there exists $g \in \text{Homeo}^+(\mathbb{D})$ realizing an $n-1$ -gone P' such that $i(P') = i(P)$ and $\text{Fix}(g) = \text{Fix}(f)$,*
2. *$\text{Fix}(f) \neq \emptyset$.*

Proof. By Lemma 6.4, there exists $g \in \text{Homeo}^+(\mathbb{D})$ such that :

1. $\text{Fix}(g) = \text{Fix}(f)$;
2. $g = f$ on the orbits of the points z_j , $j \in \mathbb{Z}/n\mathbb{Z}$, $j \notin \{i-1, i\}$,
3. there exists $z \in \mathbb{D}$ such that $\lim_{k \rightarrow -\infty} g^k(z) = \alpha_{i-1}$ and $\lim_{k \rightarrow +\infty} g^k(z) = \omega_i$.

If the lines $(\Delta_j)_{j \in \mathbb{Z}/n\mathbb{Z} \setminus \{i, i-1\}}$ and the straight (oriented) line Δ_* from α_{i-1} to ω_i bound a polygon P' , then P' is an $n-1$ -gon, $i(P') = i(P)$, and g realizes P' . Otherwise, the line Δ_* must coincide with some already existing Δ_j , $j \in \mathbb{Z}/n\mathbb{Z}$. By minimality of P , the only possibility is $\Delta_* = \Delta_{i+2}$. Besides, as $i(P) < 0$ the orientations of these lines cannot coincide. We conclude that P is a pentagone and $i(P) = -1$. We can construct as before a free perturbation g of f such that $\lim_{k \rightarrow +\infty} g^k(z_{i-1}) = \omega_i = \alpha_{i+2}$, $\lim_{k \rightarrow -\infty} g^k(z_{i-1}) = \alpha_{i-1} = \omega_{i+2}$, $g = f$ on the orbits of the points z_j , $j \in \mathbb{Z}/5\mathbb{Z}$, $j \notin \{i-1, i\}$. We define $\mathcal{L} = ((\alpha'_j, \omega'_j))_{j \in \mathbb{Z}/4\mathbb{Z}}$, where $(\alpha'_0, \omega'_0) = (\alpha_{i-1}, \omega_i)$, and for all $j \in \{1, 2, 3\}$, $(\alpha'_j, \omega'_j) = (\alpha_{i+j}, \omega_{i+j})$. Then, \mathcal{L} is a (degenerate) hyperbolic cycle of links, and g realizes \mathcal{L} . We are now done by Theorem 1.1. □

By applying the previous lemma inductively, if $\text{Fix}(f) \neq \emptyset$, then there exists $g \in \text{Homeo}^+(\mathbb{D})$ such that $\text{Fix}(g) = \text{Fix}(f)$ and g realizes a minimal n -gon P such that $i(P) < 0$, and $\delta_i = 1$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

This next lemma finishes the proof of Corollary 1.2:

Lemma 6.7. *If f realizes a minimal n -gon P such that $i(P) < 0$, and $\delta_i = 1$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, then $\text{Fix}(f) \neq \emptyset$.*

Proof. If $\delta_i = 1$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, then the points in ℓ satisfy the hyperbolic order property. We are now done by Theorem 1.1. □

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