SRB-like Measures for C^0 Dynamics

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June 21th., 2011

Abstract

For any continuous map $f\colon M \to M$ on a compact manifold M, we define the SRB-like probabilities as a generalization of Sinai-Ruelle-Bowen (i.e. physical) measures. We prove that f has SRB-like measures, even if SRB measures do not exist. We prove that the definition of SRB-like measures is optimal, provided that the purpose of the researcher is to describe the asymptotic statistics for Lebesgue almost every initial state. We prove that any isolated measure in the set $\mathcal{O}$ of SRB-like measures is SRB. Finally we conclude that if $\mathcal{O}$ is finite or countable infinite, then there exist (up to countable many) SRB measures such that the union of their basins cover M Lebesgue a.e.

1   Introduction

Let $f\colon M \to M$ be a continuous map in a compact, finite-dimensional manifold M. Let m be a Lebesgue measure normalized so that $m(M) = 1$, and not necessarily f-invariant. We denote $\mathcal{P}$ the set of all Borel probability measures in M, provided with the weak* topology, and a metric structure inducing this topology.

For any point $x \in M$ we denote $p\omega(x)$ to the set of all the Borel probabilities in M that are the limits in the weak* topology of the convergent subsequences of the following sequence

$$\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right\}_{n \in \mathbb{N}} \tag{1.1}$$

where δ_y is the Dirac delta probability measure supported in $y \in M$. We call the probabilities of the sequence (1.1) *empiric probabilities* of the orbit of x. We call $p\omega(x)$ the *limit set* in $\mathcal{P}$ corresponding to $x \in M$.

It is classic in Ergodic Theory the following definition:

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Definition 1.1 A probability measure μ is *physical* or *SRB* (Sinai-Ruelle-Bowen), if $\{\mu\} = pw(x)$ for a set $A(\mu)$ of points $x \in M$ that has positive Lebesgue measure. The set $A(\mu)$ is called *basin of attraction* of μ.

In this paper, as in [V98] and Chapter 11 of [BDV05], we agree to name such a probability μ an SRB measure (and also physical as in [Y02]). This preference is based in three reasons, which are also our motivations:

1. Our scenario includes *all the continuous systems.* Most (namely C^0 generic) continuous f are not differentiable. So, no Lyapunov exponents necessarily exist, to be able to assume some kind of hyperbolicity. Thus, we can not assume the existence of an unstable foliation with differentiable leaves. Therefore, we aim to study those systems for which the SRB measures usually defined in the literature (related with an unstable foliation $\mathcal{F}$), do not exist. We recall a popularly required property for μ: the conditional measures μ_x of μ, along the local leaves $\mathcal{F}_x$ of a hyperbolic unstable foliation $\mathcal{F}$, are absolute continuous respect to the internal Lebesgue measures of those leaves. But this latter assumption needs the existence of such a regular foliation $\mathcal{F}$. It is well known that the ergodic theory based on this absolute continuity condition does not work for generic C^1 systems (that are not $C^{1+\alpha}$), see [RY80, BH98, AB07]. So, it does not work for most C^0-systems.

2. In the modern Differentiable Ergodic Theory, for $C^{1+\alpha}$-systems that have some hyperbolic behavior, one of the ultimate purposes of searching measures with absolute continuity properties respect to Lebesgue is to find probabilities that satisfy Definition 1.1. Therefore, if the system is not $C^{1+\alpha}$, or is not hyperbolic-like, but nevertheless exists some probability μ describing the asymptotic behavior of the sequence (1.1) for a Lebesgue-positive set of initial states (i.e. μ satisfies Definition 1.1), then one of the initial purposes of research of Sinai, Ruelle and Bowen in [B71, BR75, R76, S72], is also achieved. Therefore, it makes sense (principally for C^0-systems) to call μ an SRB measure, if it satisfies Definition 1.1.

3. The SRB-like property of some invariant measures which describe (modulus ε for all $\varepsilon > 0$) the behavior of the sequence (1) for n large enough and for a Lebesgue-positive set of initial states can be also achieved considering the *observable measures* that we introduce in Definition 1.2, instead of restricting to those in Definition 1.1. This new setting will describe the statistics defined by the sequence (1.1) of empiric probabilities for Lebesgue almost all initial state (see Theorem 1.5). This is particularly interesting in the cases in which SRB-measures do not exist (for instance [K04] and some of the examples in Section 5 of this paper.) So, in the sequel, we use the words physical and SRB as synonymous, and we apply them only to the probability measures that satisfy Definition 1.1. To generalize this notion, we will call *observable or SRB-like or*

physical-like, to those measures introduced in Definition 1.2. After this agreement all SRB measure are SRB-like but not conversely (we provide Examples in Section 5).

One of the major problems of the Ergodic Theory of Dynamical Systems, is to find SRB measures. They are widely studied occupying a relevant interest for those systems that are $C^{1+\alpha}$ and show some kind of hyperbolicity ([PS82], [PS04], [V98], [BDV05]). One of the reasons for searching those measures, is that they describe the asymptotic behavior of the sequence (1.1) for a Lebesgue-positive set of initial states, namely, for a set of spatial conditions that is not negligible from the viewpoint of an observer. One observes, through the SRB measures, the statistics of the orbits through experiments that measure the time-mean of the future evolution of the system, with Lebesgue almost all initial states. But it is unknown if most differentiable systems exhibit SRB measures ([P99]). It seems to be true that most C^0-systems *do not exhibit SRB* measures, because for them, there is evidence that Lebesgue almost all initial states define non convergent sequences (1.1) of empiric probabilities [AA10]. In [K98], Keller considers an SRB-like property of a measure, even if the sequence (1.1) is not convergent. In fact, he takes those measures μ that belong to the set $pw(x)$ for a Lebesgue-positive set of initial states $x \in M$, regardless if $pw(x)$ coincides or not with $\{\mu\}$. Precisely, Keller considers those measures μ for which dist$(\mu, pw(x)) = 0$ for a Lebesgue positive set of points $x \in M$. But, as he also remarks in his definition, that kind of weak-SRB measures may not exist. We introduce now the following notion, which generalizes the notion of observability of Keller, and the notion of SRB measures in Definition 1.1:

Definition 1.2 A probability measure $\mu \in \mathcal{P}$ is *observable* or *SRB-like* or *physical-like* if for all $\varepsilon > 0$ the set $A_\varepsilon(\mu) = \{x \in M : \text{dist}(pw(x), \mu) < \varepsilon\}$ has positive Lebesgue measure. The set $A_\varepsilon(\mu)$ is called basin of ε-attraction of μ. We denote with $\mathcal{O}$ the set of all observable measures.

It is immediate from Definitions 1.1 and 1.2, that every SRB measure is observable. But not every observable measure is SRB (we provide examples in Section 4). It is standard to check that any observable measure is f-invariant. (In fact, if $\mathcal{P}_f \subset \mathcal{P}$ denotes the weak*-compact set of f-invariant probabilities, since $pw(x) \in \mathcal{P}_f$ for all x, we conclude that $\mu \in \overline{\mathcal{P}_f} = \mathcal{P}_f$ for all $\mu \in \mathcal{O}$.)

For the experimenter, the observable measures as defined in 1.2 should have the same relevance as the SRB measures defined in 1.1. In fact, the basin of ε-attraction $A_\varepsilon(\mu)$ has positive Lebesgue measure *for all $\varepsilon > 0$.* The ε-approximation lays in the space $\mathcal{P}$ of probabilities, but it can be easily translated (through the functional operator induced by the probability μ in the space $C^0(M, \mathbb{R})$) to an ε-approximation (in time-mean) towards an "attractor" in the ambient manifold M. Precisely, if μ is

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observable and $x \in A_\epsilon(\mu)$ then, with a frequency that is near 1, the iterates $f^n(x)$, for certain values of n large enough, will ε-approach the support of μ. Note that also for an SRB measure μ this ε-approximation to the support of μ holds in the ambient manifold M with $\varepsilon \neq 0$. Namely, assuming that there exists an SRB measure μ, the empiric probability (defined in (1.1) for Lebesgue almost all orbit in the basin of μ) approximates, but in general differs from μ, after any time $n \geq 1$ of experimentation which is as large as wanted but finite. If the experimenter aims to observe the orbits during a time n large enough, but always finite, Definition 1.2 of observability ensures him a 2ε-approximation to the "attractor", for any given $\varepsilon > 0$, while Definition 1.1 of physical measures ensures him an ε-approximation. As none of them guarantees a null error, and both of them guarantee an error smaller than $\epsilon > 0$ for arbitrarily small values of $\epsilon > 0$ (if the observation time is large enough), the practical meanings of both definitions are similar.

Statement of the results

Main Theorem 1.3 (Existence of observable measures)
For every continuous map f, the space $\mathcal{O}$ of all observable measures for f is nonempty and weak*-compact.

We prove this theorem in Section 3. It says that Definition 1.2 is weak enough to ensure the existence of observable measures for any continuous f. But, if considering the set $\mathcal{P}_f$ of all the invariant measures, one would obtain also the existence of probabilities that describe completely the limit set $pw(x)$ for a Lebesgue-positive set of points $x \in M$ (if so, for all points in M). Nevertheless, that would be less economic. In fact, along Section 5, we exhibit paradigmatic systems for which most invariant measures are not observable. Also we show that observable measures (as well as SRB measures defined in 1.1) are not necessarily ergodic. The ergodic measures, or a subset of them, may be not suitable respect to a non-invariant Lebesgue measure describing the probabilistic distribution of the initial states in M. In fact, there exist examples (we will provide one in Section 5), for which the set of points $x \in M$ such that $p\omega(x)$ is an ergodic probability has zero Lebesgue measure.

In Definition 1.1, we called basin of attraction $A(\mu)$ of an SRB-measure μ to the set $A(\mu) = \{x \in X : p\omega(x) = \{\mu\}\}$. Inspired in that definition we introduce the following:

Definition 1.4 We call *basin of attraction $A(\mathcal{K})$* of any nonempty weak* compact subset $\mathcal{K}$ of probabilities, to

$$A(\mathcal{K}) := \{x \in M : \ p\omega(x) \subset \mathcal{K}\}.$$

We are interested in those sets $\mathcal{K} \subset \mathcal{P}$ having basin $A(\mathcal{K})$ with positive Lebesgue measure. We are also interested in not adding unnecessary probabilities to the set $\mathcal{K}$. The following result states that the optimal choice, under those interests, is a nonempty compact subset of the observable measures defined in 1.2.

Main Theorem 1.5 (Full optimal attraction of $\mathcal{O}$)

The set $\mathcal{O}$ of all observable measures for f is the minimal weak* compact subset of $\mathcal{P}$ whose basin of attraction has total Lebesgue measure. In other words, $\mathcal{O}$ is minimally weak* compact containing, for Lebesgue almost all initial state, the limits of the convergent subsequences of (1.1).

We prove this theorem in Section 3. Finally, let us state the relations between the cardinality of $\mathcal{O}$ and the existence of SRB measures according with Definition 1.1.

Theorem 1.6 (Finite set of observable measures) *$\mathcal{O}$ is finite if and only if there exist finitely many SRB measures such that the union of their basins of attraction cover M Lebesgue a.e. In this case $\mathcal{O}$ is the set of SRB measures.*

We prove this theorem in Section 4.

Theorem 1.7 (Countable set of observable measures) *If $\mathcal{O}$ is countably infinite, then there exist countably infinitely many SRB measures such that their basins of attraction cover M Lebesgue a.e. In this case $\mathcal{O}$ is the weak*-closure of the set of SRB measures.*

We prove this theorem in Section 4.

For systems preserving the Lebesgue measure the main question is their ergodicity, and most results of this work translate, for those systems, as equivalent conditions to be ergodic. The proof of the following result is standard after Theorem 1.5:

Remark 1.8 (Observability and ergodicity.) *If f preserves the Lebesgue measure m then the following assertions are equivalent:*

1. *f is ergodic respect to m.*
2. *There exists a unique observable measure μ for f.*
3. *There exists a unique SRB measure ν for f attracting Lebesgue a.e.*
   Moreover, if the assertions above are satisfied, then $m = \mu = \nu$

The ergodicity of most maps that preserve the Lebesgue measure is also an open question. ([PS04], [BMVW03]). Due to Remark 1.8 this open question is equivalent to the *unique observability.*

2 The convex-like property of $p\omega(x)$.

For each $x \in M$ we have defined the nonempty compact set $p\omega(x) \subset \mathcal{P}_f$ composed by the limits of all the convergent subsequences of the empiric probabilities in Equality (1.1). For further uses we state the following property for the $p\omega$-limit sets:

Theorem 2.1 (Convex-like property.) *For every point $x \in M$:*
1. If $\mu, \nu \in p\omega(x)$ then for each real number $0 \le \lambda \le 1$ there exists a measure $\mu_\lambda \in p\omega(x)$ such that $\operatorname{dist}(\mu_\lambda, \mu) = \lambda \operatorname{dist}(\nu, \mu)$.
2. $p\omega(x)$ either has a single element or non-countable infinitely many.

Proof:

The statement 2 is an immediate consequence of 1. To prove 1 it is enough to exhibit, in the case $\mu \ne \nu$, a convergent subsequence of (1.1) whose limit μ_λ satisfies 1. It is an easy exercise to observe that the existence of such convergent sequence follows (just taking $\varepsilon = 1/n$) from the following lemma 2.2. $\square$

Lemma 2.2 *For fixed $x \in M$ and for all $n \ge 1$ denote $\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$. Assume that there exist two weak*-convergent subsequences $\mu_{m_j} \to \mu$, $\mu_{n_j} \to \nu$. Then, for all $\varepsilon > 0$ and all $K > 0$ there exists a natural number $h = h(\varepsilon, K) > K$ such that $|\operatorname{dist}(\mu_h, \mu) - \lambda \operatorname{dist}(\nu, \mu)| \le \varepsilon$.*

Proof: First let us choose m_j and then n_j such that

$$m_j > K; \quad \frac{1}{m_j} < \frac{\varepsilon}{4}; \quad \operatorname{dist}(\mu, \mu_{m_j}) < \frac{\varepsilon}{4}; \quad n_j > m_j; \quad \operatorname{dist}(\nu, \mu_{n_j}) < \frac{\varepsilon}{4}.$$

We will consider the following distance in $\mathcal{P}$:

$$\operatorname{dist}(\rho, \delta) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int g_i \, d\rho - \int g_i \, d\delta \right|$$

for any $\rho, \delta \in \mathcal{P}$, where $\{g_i\}_{i \in \mathbb{N}}$ is a countable dense subset of $C^0(M, [0, 1])$. Note from the sequence (1.1) that $|\int g \, d\mu_n - \int g \, d\mu_{n+1}| \le (1/n)\|g\|$ for all $g \in C(M, [0, 1])$ and all $n \ge 1$. Then in particular for $n = m_j + k$, we obtain

$$\operatorname{dist}(\mu_{m_j+k}, \mu_{m_j+k+1}) \le \frac{1}{m_j} < \frac{\varepsilon}{4} \quad \text{for all } k \ge 0 \tag{2.1}$$

Now let us choose a natural number $0 \le k \le n_j - m_j$ such that

$$\left| \operatorname{dist}(\mu_{m_j}, \mu_{m_j+k}) - \lambda \operatorname{dist}(\mu_{m_j}, \mu_{n_j}) \right| < \varepsilon/4 \quad \text{for the given } \lambda \in [0, 1]$$

Such k does exist because inequality (2.1) is verified for all $k \ge 0$ and moreover if $k = 0$ then $\operatorname{dist}(\mu_{m_j}, \mu_{m_j+k}) = 0$ and if $k = n_j - m_j$ then $\operatorname{dist}(\mu_{m_j}, \mu_{m_j+k}) = \operatorname{dist}(\mu_{m_j}, \mu_{n_j})$.

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Now renaming $h = m_j + k$, applying the triangular property and tying together the inequalities above, we deduce:

$$\left| \operatorname{dist}(\mu_h, \mu) - \lambda \operatorname{dist}(\nu, \mu) \right| \leq \left| \operatorname{dist}(\mu_h, \mu) - \operatorname{dist}(\mu_h, \mu_{m_j}) \right|$$

$$+ \left| \operatorname{dist}(\mu_h, \mu_{m_j}) - \lambda \operatorname{dist}(\mu_{m_j}, \mu_{n_j}) \right| + \lambda \left| \operatorname{dist}(\mu_{m_j}, \mu_{n_j}) - \operatorname{dist}(\mu_{m_j}, \nu) \right|$$

$$+ \lambda \left| \operatorname{dist}(\mu_{m_j}, \nu) - \operatorname{dist}(\mu, \nu) \right| < \varepsilon$$

□

3   Proof of Theorems 1.3 and 1.5.

From the beginning we have fixed a metric in the space $\mathcal{P}$ of all Borel probability measures in M, inducing its weak* topology structure. We denote as $B_\varepsilon(\mu)$ the open ball in $\mathcal{P}$, with such a metric, centered in $\mu \in \mathcal{P}$ and with radius $\varepsilon > 0$.

Proof: (*of Theorem* 1.3.) Let us prove that $\mathcal{O}$ is compact. The complement $\mathcal{O}^c$ of $\mathcal{O}$ in $\mathcal{P}$ is the set of all probability measures μ (not necessarily f-invariant) such that for some $\varepsilon = \varepsilon(\mu) > 0$ the set $\{x \in M : p\omega(x) \cap B_\varepsilon(\mu) \neq \emptyset\}$ has zero Lebesgue measure. Therefore $\mathcal{O}^c$ is open in $\mathcal{P}$, and $\mathcal{O}$ is a closed subspace of $\mathcal{P}$. As $\mathcal{P}$ is compact we deduce that $\mathcal{O}$ is compact as wanted.

We now prove that $\mathcal{O}$ is not empty. By contradiction, assume that $\mathcal{O}^c = \mathcal{P}$. Then for every $\mu \in \mathcal{P}$ there exists some $\varepsilon = \varepsilon(\mu) > 0$ such that the set $A = \{x \in M : p\omega(x) \subset (B_\varepsilon(\mu))^c\}$ has total Lebesgue probability. As $\mathcal{P}$ is compact, let us consider a finite covering of $\mathcal{P}$ with such open balls $B_\varepsilon(\mu)$, say $B_1, B_2, \ldots B_k$, and their respective sets $A_1, A_2, \ldots A_k$ defined as above. As $m(A_i) = 1$ for all $i = 1, 2, \ldots, k$ we have that the intersection $B = \cap_{i=1}^k A_i$ is not empty. By construction, for all $x \in B$ the $p\omega$-limit of x is contained in the complement of B_i for all $i = 1, 2 \ldots, k$, and so it would not be contained in $\mathcal{P}$, that is the contradiction ending the proof.   □

Proof: (*of Theorem* 1.5.) Recall Definition 1.4 of the basin of attraction $A(\mathcal{K})$ of any weak*-compact and nonempty set $\mathcal{K}$ of probabilities. We must prove the following two assertions:

1. $m(A(\mathcal{O})) = 1$, where m is the Lebesgue measure.
2. $\mathcal{O}$ is minimal among all the compact sets $\mathcal{K} \subset \mathcal{P}$ with such a property.

Define the following family $\aleph$ of sets of probabilities:

$$\aleph = \{\mathcal{K} \subset \mathcal{P} : \mathcal{K} \text{ is compact and } m(A(\mathcal{K})) = 1\}.$$

Therefore $\aleph$ is composed by all the weak* compact sets $\mathcal{K}$ of probabilities such that $p\omega(x) \subset \mathcal{K}$ for Lebesgue almost every point $x \in M$. The family $\aleph$ is not empty since it contains the set $\mathcal{P}_f$ of all the invariant probabilities. So, to prove Theorem 1.5, we must prove that $\mathcal{O} \in \aleph$ and $\mathcal{O} = \bigcap_{\mathcal{K} \in \aleph} \mathcal{K}$.

Let us first prove that $\mathcal{O} \subset \mathcal{K}$ for all $\mathcal{K} \in \aleph$. This is equivalent to prove that if $\mathcal{K} \in \aleph$ and if $\mu \notin \mathcal{K}$ then $\mu \notin \mathcal{O}$.

If $\mu \notin \mathcal{K}$ take $\varepsilon = \text{dist}\,(\mu, \mathcal{K}) > 0$. For all $x \in A(\mathcal{K})$ the set $p\omega(x) \subset \mathcal{K}$ is disjoint from the ball $B_\varepsilon(\mu)$. But almost all Lebesgue point is in $A(\mathcal{K})$, because $\mathcal{K} \in \aleph$. Therefore $p\omega(x) \cap B_\varepsilon(\mu) = \emptyset$ Lebesgue a.e. This last assertion, combined with Definition 1.2 and the compactness of the set $p\omega(x)$ imply that $\mu \notin \mathcal{O}$, as wanted.

Now let us prove that $m(A(\mathcal{O})) = 1$. After Theorem 1.3 the set $\mathcal{O}$ is compact and nonempty. So, for any $\mu \notin \mathcal{O}$ the distance $\text{dist}\,(\mu, \mathcal{O})$ is positive. Observe that the complement $\mathcal{O}^c$ of $\mathcal{O}$ in $\mathcal{P}$ can be written as the increasing union of compacts sets $\mathcal{K}_n$ (not in the family $\aleph$) as follows:

$$\mathcal{O}^c = \bigcup_{n=1}^{\infty} \mathcal{K}_n, \qquad \mathcal{K}_n = \{\mu \in \mathcal{P} : \text{dist}\,(\mu, \mathcal{O}) \geq 1/n\} \subset \mathcal{K}_{n+1} \tag{3.1}$$

Let us consider the sequence $A'_n = A'(\mathcal{K}_n)$ of sets in M, where $A'(\mathcal{K})$ is defined as follows:

$$A'(\mathcal{K}) := \{x \in M : p\omega(x) \cap \mathcal{K} \neq \emptyset\}. \tag{3.2}$$

Denote $A'_\infty = A'(\mathcal{O}^c)$. We deduce from (3.1) and (3.2) that:

$$A'_\infty = \bigcup_{n=1}^{\infty} A'_n, \quad m(A'_n) \to m(A'_\infty) = m(A'(\mathcal{O}^c)).$$

To end the proof is now enough to show that $m(A'_n) = 0$ for all $n \in \mathbb{N}$.

In fact, $A'_n = A'(\mathcal{K}_n)$ and $\mathcal{K}_n$ is compact and contained in $\mathcal{O}^c$. By Definition 1.2 there exists a finite covering of $\mathcal{K}_n$ with open balls $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ such that

$$m(A'(\mathcal{B}_i)) = 0 \quad \text{for all } i = 1, 2, \ldots, k \tag{3.3}$$

By (3.2) the finite collection of sets $A'(\mathcal{B}_i);\ i = 1, 2, \ldots, k$ cover A'_n and therefore (3.3) implies $m(A'_n) = 0$ ending the proof. $\square$

4 Proof of Theorems 1.6 and 1.7

To prove Theorems 1.6 and 1.7 we need the following Lemma:

Lemma 4.1 *If an observable or SRB-like measure μ is isolated in the set $\mathcal{O}$ of all observable measures, then it is an SRB measure.*

Proof: Recall that we denote $\mathcal{B}_\varepsilon(\mu)$ the open ball in $\mathcal{P}$ centered in $\mu \in \mathcal{P}$ and with radius $\varepsilon > 0$. Since μ is isolated in $\mathcal{O}$, there exists $\varepsilon_0 > 0$ such that the set $\overline{\mathcal{B}_{\varepsilon_0}(\mu)} \setminus \{\mu\}$ is disjoint from $\mathcal{O}$. After Definition 1.2, $m(A) > 0$, where $A := A_{\varepsilon_0}(\mu) = \{x \in M : \text{dist}(p\omega(x), \mu) < \varepsilon_0\}$.

After Definition 1.1, to prove that μ is SRB it is enough to prove that for m-almost all $x \in A$ the limit set $p\omega(x)$ of the sequence (1.1) of empiric probabilities, is $\{\mu\}$. In fact, fix and arbitrary $0 < \varepsilon < \varepsilon_0$. The compact set $\overline{\mathcal{B}_{\varepsilon_0}(\mu)} \setminus \mathcal{B}_\varepsilon(\mu)$ is disjoint from $\mathcal{O}$, then it can be covered with a finite number of open balls $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ such that $m(A_i) = 0$ for all $i = 1, \ldots, k$, where $A_i := \{x \in M : p\omega(x) \cap \mathcal{B}_i \neq \emptyset\}$. Thus, for m-a.e. $x \in A$ the limit set $p\omega(x)$ intersects $\mathcal{B}_\varepsilon(\mu)$ but it does not intersect $\mathcal{B}_{\varepsilon_0}(\mu) \setminus \mathcal{B}_\varepsilon(\mu)$. After Theorem 2.1, we conclude that $p\omega(x) \subset \mathcal{B}_\varepsilon(\mu)$ for Lebesgue almost all $x \in A$. Taking the values $\varepsilon_n = 1/n$, for all $n \geq 1$, we deduce that $p\omega(x) = \{\mu\}$ for $m-$ a.e. $x \in A$, as wanted. $\square$

Proof: (*of Theorem* 1.6.) Denote SRB to the (a priori maybe empty) set of all SRB measures, according with Definition 1.1. It is immediate, after Definition 1.2, that SRB $\subset \mathcal{O}$. If $\mathcal{O}$ is finite, then all its measures are isolated, and after Lemma 4.1, they are all SRB measures. Therefore SRB $= \mathcal{O}$ is finite. Applying Theorem 1.5 which states the full attraction property of $\mathcal{O}$, it is obtained that $m(A(\text{SRB})) = 1$, where $A(\text{SRB}) = \bigcup_{\mu \in \text{SRB}} A(\mu)$, being $A(\mu)$ the basin of attraction of the SRB measure μ. Therefore, we conclude that, if $\mathcal{O}$ is finite, there exist a finite number of SRB measures such that the union of their basins cover Lebesgue almost all $x \in M$, as wanted. Now, let us prove the converse statement. Assume that SRB is finite and the union of the basins of attraction of all the measures in SRB cover Lebesgue almost all $x \in M$. After the minimality property of $\mathcal{O}$ stated in Theorem 1.5, $\mathcal{O} \subset$ SRB. On the other hand, we have SRB $\subset \mathcal{O}$. We deduce that $\mathcal{O} = $ SRB, and thus $\mathcal{O}$ is finite, as wanted. $\square$

To prove Theorem 1.7 we need the following Lemma (which in fact holds in any compact metric space $\mathcal{P}$).

Lemma 4.2 *If the compact subset $\mathcal{O} \subset \mathcal{P}$ is countably infinite, then the subset $\mathcal{S}$ of its isolated points is not empty, countably infinite and $\overline{\mathcal{S}} = \mathcal{O}$. Therefore, $\text{dist}(\nu, \mathcal{O}) = \text{dist}(\nu, \mathcal{S})$ for all $\nu \in \mathcal{P}$.*

Proof: The set $\mathcal{O} \subset \mathcal{P}$ is not empty and compact, after Theorem 1.3. Assume by contradiction that $\mathcal{S}$ is empty. Then $\mathcal{O}$ is perfect, i.e. all measure of $\mathcal{O}$ is an accumulation point. The set $\mathcal{P}$ of all the Borel probabilities in M is a Polish space, since it is metric and compact. As nonempty perfect sets in a Polish space always have the cardinality of the continuum [K95], we deduce that $\mathcal{O}$ can not be countably infinite, contradicting the hypothesis.

Even more, the argument above also shows that if $\mathcal{O}$ is countable infinite, then it does not contain nonempty perfect subsets.

Let us prove now that the subset $\mathcal{S}$ of isolated measures of $\mathcal{O}$ is countably infinite. Assume by contradiction that $\mathcal{S}$ is finite. Then $\mathcal{O} \setminus \mathcal{S}$ is nonempty and compact, and by construction has not isolated points. Therefore it is a nonempty perfect set, contradicting the assertion proved above.

It is left to prove that $\text{dist}(\nu, \mathcal{O}) = \text{dist}(\nu, \mathcal{S})$ for all $\nu \in \mathcal{P}$. This assertion, if proved, implies in particular that $\text{dist}(\mu, \mathcal{S}) = 0$ for all $\mu \in \mathcal{O}$, and therefore, recalling that $\mathcal{O}$ is compact, it implies $\overline{\mathcal{S}} = \mathcal{O}$.

To prove that $\text{dist}(\nu, \mathcal{O}) = \text{dist}(\nu, \mathcal{S})$ for all $\nu \in \mathcal{P}$, first fix ν and take $\mu \in \mathcal{O}$ such that $\text{dist}(\nu, \mathcal{O}) = \text{dist}(\nu, \mu)$. Such a probability μ exists because $\mathcal{O}$ is compact. If $\mu \in \mathcal{S}$, then the equality in the assertion is obtained trivially. If $\mu \in \mathcal{O} \setminus \mathcal{S}$, fix any $\varepsilon > 0$ and denote $\mathcal{B}_\varepsilon(\mu)$ to the ball of center μ and radius ε. Take $\mu' \in \mathcal{S} \bigcap \mathcal{B}_\varepsilon(\mu)$. Such μ' exists because, if not, the nonempty set $\mathcal{B}_\varepsilon(\mu) \cap \mathcal{O}$ would be perfect, contradicting the above proved assertion. Therefore, $\text{dist}(\nu, \mathcal{S}) \leq \text{dist}(\nu, \mu') \leq \text{dist}(\nu, \mu) + \text{dist}(\mu, \mu') = \text{dist}(\nu, \mathcal{O}) + \text{dist}(\mu, \mu')$. So, $\text{dist}(\nu, \mathcal{S}) < \text{dist}(\nu, \mathcal{O}) + \varepsilon$. As this inequality holds for all $\varepsilon > 0$, we conclude that $\text{dist}(\nu, \mathcal{S}) \leq \text{dist}(\nu, \mathcal{O})$. The opposite inequality is immediate, since $\mathcal{S} \subset \mathcal{O}$. $\quad\square$

Proof: (*of Theorem 1.7.*) Denote $\mathcal{S}$ to the set of isolated measures in $\mathcal{O}$. After Lemma 4.2, $\mathcal{S}$ is countably infinite. Thus, applying Lemma 4.1, μ is SRB for all $\mu \in \mathcal{S}$. Then, there exist countably infinitely many SRB measures (those in $\mathcal{S}$ and possibly some others in $\mathcal{O} \setminus \mathcal{S}$). Denote SRB to the set of all SRB measures. After Lemma 4.2 $\mathcal{O} = \overline{\mathcal{S}} \subset \overline{\text{SRB}} \subset \mathcal{O}$. So $\overline{\text{SRB}} = \mathcal{O}$. It is only left to prove that the union of the basins of attractions $A(\mu_i)$, for all $\mu_i \in \text{SRB}$ covers Lebesgue almost all M. Denote m to the Lebesgue measure. Applying Theorem 1.5: $p\omega(x) \subset \mathcal{O} \quad m-$a.e. $x \in M$. Together with Theorem 2.1 and with the hypothesis of countability of $\mathcal{O}$, this last assertion implies that for $m-$ a.e. $x \in M$ the set $p\omega(x)$ has a unique element $\{\mu_x\} \subset \mathcal{O}$. Then:

$$p\omega(x) = \{\mu_x\} \subset \mathcal{O} \quad m - \text{a.e. } x \in M. \tag{4.1}$$

We write $\mathcal{O} = \{\mu_i : i = 1, \ldots, n\}$, where $\mu_i \neq \mu_j$ if $i \neq j$. Denote $A = \bigcup_{i \in \mathbb{N}} A(\mu_i)$, where $A(\mu_i) := \{x \in M : \mu_x = \mu_i\}$. Assertion (4.1) can be written as $m(A) = 1$. In addition, $A(\mu_i) \cap A(\mu_j) = \emptyset$ if $\mu_i \neq \mu_j$. So $1 = \sum_{i=1}^{+\infty} m(A(\mu_i))$. After Definition 1.1: $\text{SRB} = \{\mu_i \in \mathcal{O} : m(A(\mu_i)) > 0\}$. We conclude that $\sum_{\mu_i \in \text{SRB}} m(A(\mu_i)) = \sum_{i=1}^{+\infty} m(A(\mu_i)) = 1$, as wanted. $\quad\square$

5 Examples

Example 5.1 For any transitive $C^{1+\alpha}$ Anosov diffeomorphism the unique SRB measure μ is the unique observable measure. But there are also infinitely many other ergodic and non ergodic invariant probabilities, that are not observable (for instance those supported on the periodic orbits).

Example 5.2 In [HY95] it is studied the class of diffeomorphisms f in the two-torus obtained from an Anosov when the unstable eigenvalue of df at a fixed point x_0 is weakened to be 1, maintaining its stable eigenvalue strictly smaller than 1, and the

uniform hyperbolicity outside a neighborhood of x_0. It is proved that f has a single SRB measure, which is the Dirac delta δ_{x_0} supported on x_0, and that its basin has total Lebesgue measure. Therefore, this is the single observable measure for f, it is ergodic and there are infinitely many other ergodic and non ergodic invariant measures that are not observable.

Example 5.3 The diffeomorphism $f\colon [0,1]^2 \to [0,1]^2$; $f(x,y) = (x/2, y)$ has $\mathcal{O}$ as the set of Dirac delta measures $\delta_{(0,y)}$ for all $y \in [0,1]$. In this case $\mathcal{O}$ coincides with the set of all ergodic invariant measures for f. Note that, for instance, the one-dimensional Lebesgue measure on the interval $[0] \times [0,1]$ is invariant and not observable, and that there are not SRB-measures as defined in 1.1. This example also shows that the set $\mathcal{O}$ of observable measures is not necessarily closed on convex combinations.

Example 5.4 The maps exhibiting infinitely many simultaneous hyperbolic sinks $\{x_i\}_{i \in \mathbb{N}}$, constructed from Newhouse's theorem ([N74]) has a space $\mathcal{O}$ of observable measures which contains δ_{x_i} for all $i \in \mathbb{N}$, which, moreover, are physical measures and isolated in $\mathcal{O}$. Also the maps exhibiting infinitely many Hénon-like attractors, constructed by Colli in [C98], has a space of observable measures that contains countably infinitely many isolated probabilities (the SRB measures supported on the Hénon-like attractors).

Example 5.5 The following example due to Bowen (cited for instance in [Y02]), shows that, even for a C^∞ system, the space of observable measures may be formed by the limit set of the non convergent sequence (1.1) for Lebesgue almost all initial state. In fact, consider a diffeomorphism f in a ball of $\mathbb{R}^2$ with two hyperbolic saddle points A and B such that the unstable global manifold $W^u(A) \setminus \{A\}$ is an embedded arc that coincides with the stable global manifold $W^s(B) \setminus \{B\}$, and conversely, the embedded arc $W^u(B) \setminus \{B\} = W^s(A) \setminus \{A\}$. Let us take f such that there exists a source $C \in U$ where U is the open ball with boundary $W^u(A) \cup W^u(B)$. One can design f such that for all $x \in U$ the α-limit is $\{C\}$ and the ω-limit contains $\{A, B\}$. If the eigenvalues of the derivative of f at A and B are adequately chosen, then the empiric sequence (1.1) for all $x \in U \setminus \{C\}$ is not convergent, has at least one subsequence convergent to the Dirac delta δ_A, and has another subsequence convergent to δ_B.

After Theorem 2.1, for each $x \in U \setminus \{C\}$ there are non countably many probability measures which are the weak* limits of the convergent subsequences of (1.1). All these measures are, then, observable. In addition, as they are invariant under f, due to Poincaré Recurrence Theorem, all of them are supported on $\{A\} \cup \{B\}$. So, after Theorem 2.1, all the convex combinations of δ_A and δ_B are observable and conversely. Therefore, the set $\mathcal{O}$ of observable measures for f coincides with the set of convex combinations of δ_A and δ_B and no physical measure exists. This example also shows that the observable measures are not necessarily ergodic.

In the Bowen example f described above, the eigenvalues of df at the saddles A and B, can be adequately designed, also, to achieve that the sequence (1.1), for all $x \in U \setminus \{C\}$, is convergent to a single measure $\mu = (\lambda)\delta_A + (1 - \lambda)\delta_B$, with a fixed constant $0 < \lambda < 1$. So, μ is physical according with definition 1.1, and is the unique observable measure. This proves that physical measures are not necessarily ergodic, and moreover, that the set of observable measures does not depend continuously on the map.

Example 5.6 Consider a partially hyperbolic C^1 diffeomorphism f, as defined in Section 11.2.3 of [BDV05], where it is proved that f has not SRB-measures, according to Definition 1.1. Nevertheless, in 11.2.3 of [BDV05], it is proved that f possesses a probability measure μ that is a Gibbs-u-state, namely, μ has conditional measures μ_x respect to an unstable foliation $\mathcal{F}$ that are absolute continuous respect to the internal Lebesgue measures of the leaves $\mathcal{F}_x$. Even more, Theorem 11.16 of [BDV05] states that, for a Lebesgue-positive set A of initial states x, the sequence (1.1) of empiric probabilities, converges to an ergodic component of μ. Therefore, combining this result with Theorem 1.5 of this paper, all the ergodic components of μ are observable or SRB-like, and their closure is the set $\mathcal{O}$ for $f|_A$. Applying Lemma 11.13 of [BDV05], all those observable measures are Gibbs-u-states. Moreover, after Theorems 1.6 and 1.7, and since in this example there does not exist SRB measures, the set $\mathcal{O}$ is non countably infinite.

Acknowledgements

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