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SRB MEASURES OF CERTAIN ALMOST HYPERBOLIC DIFFEOMORPHISMS WITH A TANGENCY

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Abstract. We study topological and ergodic properties of some almost hyperbolic diffeomorphisms on two dimensional manifolds. Under generic conditions, diffeomorphisms obtained from Anosov by an isotopy pushing together the stable and unstable manifolds to be tangent at a fixed point, are conjugate to Anosov. For a finite codimension subset at the boundary of Anosov there exist a SRB measure and an unique ergodic attractor.

1. Introduction.

We consider some C^3 diffeomorphisms at the boundary of Anosov, inspired in the examples of Lewowicz [18]. We first prove that they are topologically conjugate to Anosov (with a conjugation that is not necessarily Hölder nor absolutely continuous). Second, we prove that such systems exhibit only one ergodic attractor, as in Palis'conjecture [29], although their stable and unstable foliations are not C^1 transversal and there is not a uniform separation between positive and negative Lyapounov exponents. Also, they are examples with non-zero Lyapounov exponents for Lebesgue almost all regular points, that have a SRB measure as in Viana's conjecture [38].

Let us consider a continuous map $f : M \mapsto M$ on a compact manifold M and μ an f-invariant probability measure on M. We call *basin of attraction* $Y(\mu)$ of μ to the set of points $S \in M$ such that the averages of Dirac measures along the forward orbit of S converge to μ in the weak* topology. An *ergodic attractor*, if it exists, is an f-invariant set $A \subset M$ that is the support of an ergodic probability μ, that we call SRB measure, whose basin of attraction has positive Lebesgue measure.

Sinai, Ruelle and Bowen ([37], [5], [3], [35]), prove the existence and finitude of ergodic attractors for uniformly hyperbolic systems. A crucial ingredient in their constructions of ergodic attractors consists in proving the existence of a Gibbs probability measure for f, that is a f-invariant probability for which conditional measures along (strong) unstable manifolds are absolutely continuous with respect to the Lebesgue measure.

Uniformly hyperbolic systems have a Gibbs measure. In [33] it is proved that if a system has a Gibbs measure μ and if the Lyapounov exponents are non-zero

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almost everywhere, then there are (countable many) ergodic attractors. So, for non-uniformly hyperbolic systems, the construction of Gibbs measures is a key step to get ergodic attractors.

In [29] Palis proposes a route of research toward a general theory of dynamical systems, looking for an answer to the following open questions: Do most systems have ergodic attractors? Does Lebesgue almost all point belong to the basin of attraction of an ergodic attractor? Palis conjectures that a dense class of systems has finitely many ergodic attractors and that their basins of attractions have full Lebesgue measure.

We focus our attention on Palis' conjecture. One should try to extend the class of dynamical systems for which an ergodic attractor is known to exist. Also, one should try to understand how ergodic attractors persist or disappear when the dynamical system is perturbed. Substantial progress in the study of the stability of ergodic properties can be found in the work of Mañé ([24], [25]) and also in [12].

For one-dimensional maps the existence of SRB measures is known for some non-hyperbolic maps ([15], [7], [22]). However, the existence of SRB measures in a general non-hyperbolic setting in higher dimensions, remain mostly unknown. In [38] Viana includes the conjecture which states that smooth maps with only non-zero Lyapounov exponents for Lebesgue almost all points, admit SRB measures. Progress in the knowledge of classes of systems with some kind of non-uniform hyperbolicity or singularities is found in [31], [8], [1], [9], [6] and [2].

The diffeomorphisms we study in this paper are non uniformly hyperbolic examples in dimension two. We work with C^3 diffeomorphisms at the boundary of Anosov, that have stable and unstable manifolds of a fixed point, tangent at that point. This is achieved pushing by an isotopy the stable and unstable eigenvalues at the fixed point of an Anosov map in dimension two, to join in a double 1 (or in a double -1), with non-diagonalizable derivative.

Both positive and negative Lyapounov exponents become zero at the fixed point (and in a dense set of points). We still have a continuous invariant splitting of the tangent bundle (in two one-dimensional sub-bundles), except in the non-hyperbolic fixed point where stable and unstable directions collapse in the single eigendirection. We assume that the unstable and stable cone fields still exist outside the fixed point, but in a non-uniform hyperbolic sense: at all points except at the fixed point, the cone fields close to transversal directions when iterating the map, but the angles between stable and unstable directions are not uniformly bounded away from 0. This is because at the fixed point the cone fields have a common direction to which they close (non exponentially) when iterating the map.

The existence of the unstable and stable cone fields is equivalent to the existence of an appropriate indefinite quadratic form in the tangent bundle, as was introduced by Lewowicz in [17]. All along this work we use this characterization with quadratic forms of almost hyperbolic maps.

We first prove some topological results in the part 1 of Theorem 1, based in the arguments of [18] and [19]: *generically, such diffeomorphisms are conjugate to Anosov.* The conjugation is only C^0. As a consequence the stable and unstable foliations of the Anosov diffeomorphisms still persist, but are only C^0 foliations (and not necessarily Hölder continuous). The existence of stable and unstable cone fields produces C^1-leaves of such foliations.

In the part 2 of Theorem 1, under some additional codimension-one hypothesis (one coefficient of the Taylor development up to order three is zero) we prove that *there exist a Gibbs measure that is also a SRB measure μ, with non-zero Lyapounov* *exponents μ almost everywhere and Lebesgue almost everywhere, and that its basin of attraction has full Lebesgue measure.* Consequently, there exists a unique ergodic attractor.

To obtain the Gibbs measure, we apply Sinai's construction [37] for uniformly hyperbolic systems: first take the forward iterates of a small rectangle R (using the local product structure of stable and unstable foliations). Then define the measures induced by f^n from the volume of R. Finally choose a weak* limit of the average of these measures. If the volume distortion is bounded when applying f^n, this limit invariant measure has chance to have absolutely continuous conditional measures on unstable manifolds. In section 3 we estimate a bound of the volume distortion for our examples. In [32] Pesin and Sinai develop a similar construction for partially hyperbolic systems. Instead of considering the whole volume of a rectangle R, they iterate a small unstable disk U, take its riemannian measure restricted to unstable elements, and estimate a bound for unstable elements in backward iterates. We do not have (a priori) a distortion bound of unstable lenghts for the diffeomorphisms that we study in this work. Usual tools to obtain this bound are the Lipschitz or Hölder continuity of the invariant foliations, that fail in our examples.

In some examples at the boundary of Anosov diffeomorphisms, the construction in [32] still works. In [8] Carvalho weakens a stable subspace of a fixed point of a n-dimensional Anosov diffeomorphism, but maintaining strong the unstable space. She bounds distortion of backward iterates of unstable volume elements, to conclude that there exists a SRB measure. In [9] a heteroclinic intersection of an Anosov diffeomorphism is perturbed to obtain a cubic heteroclinic tangency. Also the unstable distortion is bounded and the SRB measure persists. On the other hand, in [14] the authors weaken the unstable direction of a fixed point of a two-dimensional Anosov diffeomorphism, maintaining strong the stable direction. They prove that the sum of unstable lengths of backward iterates is not bounded, that it does not exist a SRB measure with positive Lyapounov exponents, and that the non hyperbolic fixed point is the unique ergodic attractor. In [13] both unstable and stable eigenvalues of an Anosov diffeomorphism in dimension two are weakened together to a double one, in such a way that the derivative at the fixed point is the identity. The author proves that under some conditions there exists a SRB measure with positive Lyapounov exponentes, and under the complementary conditions, the unique ergodic attractor is the non hyperbolic fixed point.

In this paper we weaken together the stable and unstable directions, to have a double eigenvalue equal to one, with non-diagonalizable derivative. Some technical difficulties arise. First, the angle between stable and unstable foliations accumulates in zero, so there is not uniform transversality. Second, the weak invariant manifolds are not C^2 (although they are C^1), so we are faced to study the tangency between stable and unstable manifolds with other tools than the usual geometrical approach. Third, the unstable foliation is not necessarily Hölder continuous, so we could not compare lengths of unstable arcs of nearby points, to get a bound of the unstable lenght distortion. Fourth, the Lyapounov exponents become zero in a dense set of points.

To avoid the irregularity of the invariant foliations we approximate the unstable local arcs with the leaves of other regular (of C^3 class) but non-invariant foliation Φ. We construct an invariant measure μ as in [37]. Then we apply the arguments of [34] to locally decompose the measure along the partition that is generated by the non-invariant foliation Φ. We use Ledrappier and Young characterization of measures ([23]) and Brin-Katok formula for the metric entropy ([4]) to conclude that μ is a Gibbs measure. We then apply a theorem of Lewowicz, Lima de Sá and Markarian ([21] and [27]) to show that Lyapounov exponents are non-zero μ-almost everywhere. To construct the ergodic attractor we use the Pugh and Schub arguments ([33]).

In Section 3 we show that the area distortion is bounded, under an additional codimension-one hypothesis: we assume that certain coefficient of second order in the Taylor development of the diffeomorphism around the non-hyperbolic fixed point, is null. This hypothesis is verified in all the examples of almost hyperbolic diffeomorphisms studied in [18]. It could be removed, (and also the whole Section 3), if instead we had, by other means, that the distortion of area when iterating the diffeomorphism is bounded (for instance if the diffeomorphism is area preserving).

The diffeomorphisms we study are of two classes: first, the linear part at the fixed point is of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \neq 0$, or second, it is of the form $\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$ with $a \neq 0$.

Let us give some examples, taken from [18]: for $t \in [0,1]$ consider the family given by

$$F_t(x,y) = \left(2x + y - \frac{t}{2\pi}[\sin(2\pi x)\cos^2(\pi y)], \quad x + y - \frac{t}{2\pi}[\sin(2\pi x)\cos^2(\pi y)]\right)$$

in the two dimensional torus $[0,1] \times [0,1]$. For $t = 1$, we have a map in the first class. An example in the second class is the following:

$$F_t(x,y) = \left(-2x - y + \frac{t}{2\pi}[\sin(2\pi x)\cos^2(\pi y)], \quad -x - y + \frac{t}{2\pi}[\sin(2\pi x)\cos^2(\pi y)]\right)$$

1.1. **Statements of the results.** To state the main theorem we begin by giving some definitions (see 1.3 of [30]). Let M be a compact riemannian C^∞ manifold, of finite dimension, and $f : M \mapsto M$ be a C^r diffeomorphism, with $r \geq 1$.

Definition 1.1. The point $S \in M$ is *regular for the diffeomorphism* f if there exist real numbers

$$\chi_1(S) > \chi_2(S) > \ldots > \chi_m(S)$$

(called Lyapounov exponents) and a decomposition

$$T_S M = E_1(S) \oplus E_2(S) \oplus \ldots \oplus E_m(S)$$

such that

$$\lim_{i \to \pm\infty} \frac{1}{i} \log \|Df^i(S)\mathbf{v}\|$$

exists and is equal to $\chi_j(S)$ for $\mathbf{0} \neq \mathbf{v} \in E_j(S)$ and $1 \leq j \leq m$.

The theorem of Oseledec asserts that for any f-invariant measure μ, the set of regular points has μ measure 1.

Definition 1.2. The *Pesin region* Σ is the set of regular points whose Lyapounov exponents are not null.

Definition 1.3. The set

$$W^{uu}(S) = \left\{ S^* \in M : \limsup_{i \to \infty} \frac{\log \text{dist}\,(f^{-i}(S), f^{-i}(S^*))}{i} < 0 \right\},$$

is called *the strong unstable manifold of* f *at* S. For S a regular point $W^{uu}(S)$ is an immersed submanifold of M (possibly reduced to a point) tangent at S to $E^{uu}(S) = \oplus_{\chi_i > 0} E_i(S)$. (see [11], see also [30]).

Definition 1.4. We say that an f-invariant probability μ is a *Gibbs measure* if its continuous conditional measures along strong unstable manifolds are absolutely continuous with respect to Lebesgue measure.

Definition 1.5. An *ergodic attractor for* f (see [33]) is a f-invariant set $A \subset M$ with a f-invariant Borel probability μ (called SRB measure) on A such that for some set $Y \subset M$ with positive Lebesgue measure (called *basin of attraction*) : **(i)** $\lim_{i\to\infty} d(f^i(S), A) = 0$ for $S \in Y$, **(ii)** μ is f-ergodic, **(iii)** Lebesgue a.e. point $S \in Y$ is generic respect μ, that is, $\lim_{i\to\infty} \frac{1}{i}\sum_{j=0}^{i-1} \delta_{f^j(S)} = \mu$ in the weak* topology, where δ_Q is the Dirac measure concentrated on Q.

Definition 1.6. We say that a map $f : M \mapsto M$ with an invariant probability measure μ is *Bernoulli* if it is equivalent to a Bernoulli shift.

To state our result we need some other definitions and results.

Definition 1.7. A function $B : TM \mapsto \mathbb{R}$ is a quadratic form if $B_P = B|_{T_PM}$ is a quadratic form on the vector space T_PM for each $P \in M$.

Definition 1.8. A quadratic form B is non degenerate if for each $P \in M$, B_P is non-degenerate; B is positive definite ($B > 0$) if $B_P(\mathbf{v}) > 0$ for every $\mathbf{v} \in T_PM$, $\mathbf{v} \neq \mathbf{0}$ and every $P \in M$; B is semipositive definite ($B \geq 0$) if $B_P(\mathbf{v}) \geq 0$ for every $\mathbf{v} \in T_PM$ and every P in M; B is indefinite if for every $P \in M$ there exist $\mathbf{v}$ and $\mathbf{w} \in T_PM$ such that $B_P(\mathbf{v}) > 0$ and $B_P(\mathbf{w}) < 0$.

If f is a diffeomorphism on M, and B is a quadratic form on TM, we will denote $f^{\#}(B)$ the quadratic form defined by $f^{\#}(B)_P(\mathbf{v}) = B_{f(P)}(Df(P)(\mathbf{v}))$, $P \in M$, $\mathbf{v} \in T_PM$. Also we denote $\Delta_{f,B}$ or simply Δ to $f^{\#}B - B$.

Theorem 1.9 (Lewowicz)**.** *Let $F : M \mapsto M$ be a C^r diffeomorphism, $r \geq 1$. Then F is Anosov if and only if there exists a continuous non-degenerate indefinite quadratic form $B : TM \mapsto \mathbb{R}$ such that $\Delta_{F,B} > 0$.*

Proof. See [18]. □

Remark 1.10. Due to the density of C^k functions in the set of C^0 functions, it is not restriction, in the former theorem, to write "C^k" instead "continuous".

The non-degenerate indefinite quadratic form B implies the existence of two cone fields (namely, $\mathcal{U} = \{(P, \mathbf{v}) : B_P(\mathbf{v}) \geq 0\}$ and $\mathcal{S} = \{(P, \mathbf{v}) : B_P(\mathbf{v}) \leq 0\}$. The condition $\Delta_{F,B} \geq 0$ implies that $\mathcal{U}$ is forward invariant and $\mathcal{S}$ is backward invariant. The main idea in the proof of the Theorem 1.9 is to show that when $\Delta_{F,B} > 0$ the invariant cone fields close, while the vectors in $\mathcal{U}$ grow, and the vectors in $\mathcal{S}$ contract, with exponential rate uniformly bounded away from zero.

Definition 1.11. Let $f : M \mapsto M$ be a C^r diffeomorphism, $r \geq 1$. We say that f is *almost hyperbolic* if there exists a continuous non-degenerate indefinite quadratic form $B : TM \mapsto \mathbb{R}$ such that $\Delta_{F,B} > 0$ except in a finite invariant subset M_0 of M.

Observe that if f is almost hyperbolic, then $\Delta_{F,B} \geq 0$ on M_0.

To prove the existence of ergodic attractors (and SBR measures) of an almost hyperbolic diffeomorphism, it is enough to construct a Gibbs measure μ such that $\mu(M_0) = 0$, and then apply the following theorems:

Theorem 1.12 (Lewowicz, Lima de Sá, Markarian)**.** *Let $f : M \mapsto M$ be a C^r diffeomorphism, $r \geq 1$. Let μ any f-invariant probability measure, and B a non-degenerate indefinite quadratic form such that $\Delta_{f,B} > 0$ μ almost everywhere. Then the Lyapounov exponents for f are non zero μ almost everywhere.*

Proof. See Lemma 2 of the appendix of [27]. (See also [21]). □

Theorem 1.13 (Pugh-Shub)**.** *If there exists a Gibbs measure μ such that the Lyapounov exponents are non-zero μ almost everywhere, then there exist (at most countably many) ergodic attractors.*

Proof. See [33]. □

We now state our result. Let M be a C^∞ connected compact two dimensional manifold. Let $F : M \mapsto M$ be an order preserving Anosov diffeomorphism. Therefore, M is homeomorphic to a two dimensional torus ([10] and [28]). Let P_0 be a fixed point, let B be a C^3 quadratic form as in the theorem 1.9. Our first goal is to obtain a good local chart in a neighborhood N_1 of P_0. We integrate, in a neighborhood of P_0, the two directions such that $B = 0$, obtaining two C^3 local foliations. We construct a C^3 local chart h_1 around P_0, so that $h_1(P_0) = (0,0)$, that trivializes the two foliations and that $B_{(x,y)}(u,v)$ has locally the expression $(1 + 2dx + 2ey + \text{h.o.t.})uv$. Let us transform F to $f \in C^3$ by an isotopy such that $Df(0,0) = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}$ where λ is equal to 1 or -1. Let us denote the Taylor development of f around $(0,0)$ as $f(x,y) = (\lambda x + ay + a_{10}x^2 + 2a_{11}xy + a_{12}y^2 + b_{10}x^3 + 3b_{11}x^2y + 3b_{12}xy^2 + b_{13}y^3 + \text{h.o.t.}, \lambda y + a_{20}x^2 + 2a_{21}xy + a_{22}y^2 + b_{20}x^3 + 3b_{21}x^2y + 3b_{22}xy^2 + b_{23}y^3 + \text{h.o.t.})$.

We note that $\Delta_{f,B}$ can not be positive definite at the fixed point P_0. At this point, and along the direction $[(u,v)]$, the quadratic form $\Delta_{f,B}$ is $a\lambda v^2$, so it is null along the eigendirection $[(1,0)]$.

Theorem 1.
1. *If $\Delta_{f,B} > 0$ except at the fixed point P_0, $a \neq 0$ and $b_{20} \neq 0$ then f is conjugate to an Anosov diffeomorphism.*
2. *If moreover $\lambda = -1$, or $a_{10} = 0$ and $\lambda = 1$, then there exists a unique ergodic attractor whose basin of attraction has Lebesgue-measure 1 and the corresponding SRB measure μ is a Gibbs measure. Besides, μ is Bernoulli, and the Pesin region has μ-measure 1 and contains Lebesgue-almost all regular points.*

2. Topological properties.

We will prove Part 1 of Theorem 1. We need the following definition:

Definition 2.1. A C^0 homeomorphism f on a compact riemannian manifold M is *expansive* if there exists a constant $\alpha > 0$ (called expansivity constant) such that, if x and y are in M and if $\text{dist}(f^n(x), f^n(y)) \leq \alpha$ for all integer n, then $x = y$.

In [19] are topologically classified the expansive homeomorphisms on compact connected two-dimensional manifolds. In particular, it is proved the following theorem:

Theorem 2.2 (Lewowicz)**.** *If a homeomorphism f on the two-dimensional torus is expansive, then it is conjugated to an Anosov diffeomorphism.*

Proof. See Theorem 5.5 of [19] □

As the two-dimensional manifold M is homeomorphic to the two-dimensional torus, to prove Part 1 of Theorem 1 it is enough to prove that f is expansive. We shall do that using a Lyapounov function.

The following ideas were obtained from [18]. In that article Lewowicz studies the topological properties of diffeomorphisms in a n-dimensional compact manifold, introducing a rather more general concept of almost hyperbolicity than that of our definition 1.11. That concept includes several assumptions; the most important of them is the existence of a Lyapounov function. We will construct (Lemma 2.4) such a function. Instead of showing that the other assumptions of [18] are also fulfilled in our case, we found easier to reproduce some parts of his proofs, applied to our simpler particular case.

Now, we need a technical lemma:

Lemma 2.3. *The hypothesis in part 1. of Theorem 1 implies* $\lambda a > 0$, $a_{20} = 0$, $a_{21} = 0$, *and* $0 < 3ab_{20} \geq (a_{10} + d - \lambda d)^2$.

Proof. We will develop the proof for $\lambda = 1$; the proof is similar with small changes if $\lambda = -1$. We first write $\Delta(\mathbf{v})$ in the local chart; if $P = (x, y)$, and $\mathbf{v} = (u, v)$, then such expression is equal to $v^2(a+\text{h.o.t.})+2uv[(a_{21}+aa_{20}+a_{10})x+(da+a_{22}+aa_{21}+a_{11})y+\text{h.o.t.}]+u^2[2a_{20}x+2a_{21}y+(3b_{20}+4a_{10}a_{20}+4da_{20})x^2+2(3b_{21}+2a_{10}a_{21}+2da_{21}+2da_{20}a+2ea_{20}+2a_{20}a_{11})xy+(3b_{22}+4a_{11}a_{21}+4daa_{21}+4ea_{21})y^2+\text{h.o.t.}]$. Taking $u = 0$, we deduce that $a > 0$. If we now take $v = 0$, from the former development and the fact that Δ is positive definite, we obtain that $a_{20} = a_{21} = 0$. The resulting expression is, then, $\Delta(\mathbf{v}) = v^2(a + \text{h.o.t.}) + 2uv[a_{10}x + (ad + a_{22} + a_{11})y + \text{h.o.t.}] + 3u^2(b_{20}x^2 + 2b_{21}xy + b_{22}y^2 + \text{h.o.t.})$; it must be positive definite. Then

$$3a(b_{20}x^2 + 2b_{21}xy + b_{22}y^2) \geq [a_{10}x + (ad + a_{22} + a_{11})y]^2 \tag{1}$$

for all (x, y). If we take $y = 0$ in (1), we obtain $3ab_{20} \geq a_{10}^2$. As $a > 0$ and $b_{20} \neq 0$ we deduce $3ab_{20} > 0$, as wanted. □

At most rescaling the horizontal direction with a positive factor, it is no restriction to consider $a\lambda = 1$, that is $Df(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. After the rescaling, the coefficient d of the quadratic form B, and some coefficients of the Taylor development of f, will change, but the inequalities of Theorem 1 and Lemma 2.3 are homogeneous in the rescaling factor and remain true.

In the proof of Lemma 2.7 we will apply Lemma 2.3 in the case $a = \lambda = 1$ and use that $3b_{20} \geq a_{10}^2$. In Lemmas 2.8 and 3.8 we will apply again Lemma 2.3 in the case $a = \lambda = 1$ and $a_{10} = 0$, and use that $b_{20} > 0$. The conditions $a_{20} = a_{21} = 0$ are used all along this section and the following.

We write the quadratic form

$$\Delta_{(x,y)}(u, v) = (f^{\#}B - B)_{(x,y)}(u, v) = v^2\theta_1(x, y) + 2uv\theta_2(x, y) + u^2\theta_3(x, y)$$

where θ_1, θ_2 and θ_3 are continuous real functions such that $\theta_1(0, 0) = \lambda a = 1$ and $\theta_2(0, 0) = \theta_3(0, 0) = 0$. As $\Delta_{(x,y)}$ is positive definite for all $(x, y) \neq (0, 0)$, we have that

$$\theta_1\theta_3 - \theta_2^2 > 0$$

if $(x, y) \neq (0, 0)$.

Lemma 2.4 (Existence of a Lyapounov function)**.** *In the hypothesis of Part 1 of Theorem 1 there exists a continuous real function V defined in a neighborhood of the diagonal of $M \times M$, that is null in the diagonal, and such that* $\overline{V}(P, Q) = V(f(P), f(Q)) - V(P, Q) > 0$ *for all* $P \neq Q$ *in some neighborhood of the diagonal of* $M \times M$.

Proof. We will define a real function V and then show that it verifies the required conditions. First, let us take a metric in M such that in a neighborhood of P_0, the local chart h_1 be an isometry. Let us consider a finite family of local charts, h_1, $h_2, \dots h_s$ with domains $E_1, \dots, E_s$. Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{E_i\}$ such that in a neighborhood of P_0, $\varphi_1 = 1$. Then, we define a metric in M by $\langle u, v\rangle_P = \sum_{i=i}^{s} \varphi_i(P)\langle Dh_i u, Dh_i v\rangle_{h_i(P)}$ for $u, v \in T_P M$, where $\langle , \rangle_{h_i(P)}$ is the usual scalar product in $\mathbb{R}^2$.

Next, given (P, Q) in a neighborhood of the diagonal of $M \times M$, we call R the middle point between P and Q, that is $R = \exp_P((1/2)\exp_P^{-1}(Q))$, and we define $V(P, Q) = B_R(\exp_R^{-1}(P))$. We shall prove that $\overline{V}(P, Q) = V(f(P), f(Q)) - V(P, Q)$ is greater than 0 if $P \neq Q$ are in some small neighborhood of the diagonal:

Let us see that for $P \neq Q$ far from P_0 (outside a small open neighborhood, say N, of P_0) the property follows from $\Delta = f^{\#}B - B > 0$. Call $H = \min\{\Delta_P(\mathbf{v}),\ P \in M \setminus N,\ \mathbf{v} \in T_P(M),\ \|\mathbf{v}\| = 1\} > 0$. Call L to some Lipschitz constant for the diffeomorphism f. Call $K = \max\{|B_P(\mathbf{v})|,\ P \in M,\ \mathbf{v} \in T_P(M),\ \|\mathbf{v}\| = 1\} > 0$. Given $P \neq Q$ in a neighborhood of the diagonal, call $S = \exp_{f(P)}((1/2)\exp_{f(P)}^{-1}(f(Q)))$, $\mathbf{v} = \exp_R^{-1}(P)$ and $\mathbf{w} = \exp_S^{-1}(f(P))$. Call $\hat{\mathbf{v}} = \exp_P^{-1}(R)$ and $\hat{\mathbf{w}} = \exp_{f(P)}^{-1}(S)$. The norms of $\mathbf{v}$ and $\hat{\mathbf{v}}$ ($\mathbf{w}$ and $\hat{\mathbf{w}}$)are equal to half the distance between P and Q ($f(P)$ and $f(Q)$). Write: $\overline{V}(P, Q) = B_S(\mathbf{w}) - B_R(\mathbf{v}) = B_S(\mathbf{w}) - B_{f(P)}(\hat{\mathbf{w}}) + B_{f(P)}(\hat{\mathbf{w}}) - B_{f(P)}(df_P(\hat{\mathbf{v}})) + \Delta_P(\hat{\mathbf{v}}) + B_P(\hat{\mathbf{v}}) - B_R(\mathbf{v})$ Then:

$$\overline{V}(P, Q) \geq \|\mathbf{v}\|^2 H - \|\mathbf{w}\|^2 \left| B_S(\mathbf{w}/\|\mathbf{w}\|) - B_{f(P)}(\hat{\mathbf{w}}/\|\mathbf{w}\|)\right| - \left| B_{f(P)}(\hat{\mathbf{w}} - df_P(\hat{\mathbf{v}}))\right| -$$

$$-\|\mathbf{v}\|^2 |B_P(\hat{\mathbf{v}}/\|\mathbf{v}\|) - B_R(\mathbf{v}/\|\mathbf{v}\|)|$$

Now use that $\|\mathbf{w}\| \leq L\|\mathbf{v}\|$. Observe that the difference of the continuous quadratic form B in two different nearby points (applied to a unitary vector and to its parallel transport), is as small as wanted if the two points are sufficiently near. Also, taking the linear part df_P of f at P, observe that

$$\left\| \exp_{f(P)}^{-1}(f(Q)) - df_P \exp_P^{-1}(Q) \right\| / \operatorname{dist}(P, Q) \to 0$$

when $\operatorname{dist}(P, Q) \to 0$. So, there exists $\delta > 0$ such that, if $0 < \operatorname{dist}(P, Q) < \delta$ and $P, Q \notin N$, then:

$$\overline{V}(P, Q) \geq \|\mathbf{v}\|^2 H - \|\mathbf{v}\|^2 L^2 H/(6L^2) - \|\mathbf{v}\|^2 \frac{H}{6K} K - \|\mathbf{v}\|^2 H/6 = \operatorname{dist}^2(P, Q) H/8 > 0$$

Now, let as prove that $\overline{V}(P, Q) > 0$ for two different points P and Q in a neighborhood of P_0. We use the following notation: $P = (x - u, y - v)$; $Q = (x + u, y + v)$. It follows $V(P, Q) = \rho(x, y)uv$ where $\rho(x, y)$ is the C^3 real function $\rho(x, y) = 1 + 2dx + 2ey + \text{h.o.t.}$

When applying f we obtain $V(f(P), f(Q)) = \rho(\tilde{x}, \tilde{y})\tilde{u}\tilde{v}$, where $(\tilde{x}, \tilde{y}) = [f(x - u, y - v) + f(x + u, y + v)]/2$ and $(\tilde{u}, \tilde{v}) = [f(x + u, y + v) - f(x - u, y - v)]/2$.

We observe that $\tilde{x}$ and $\tilde{y}$ are C^3 real functions of (x, y, u, v) that stay invariant when changing the signs of u and v. So their odd derivatives respect to (u, v) on $u = 0, v = 0$ are null. Also $\tilde{u}$ and $\tilde{v}$ are C^3 real functions that change sign when u and v do, and so, their even derivatives are null.

We have

$$\overline{V}(P, Q) = \rho(\tilde{x}, \tilde{y})\tilde{u}\tilde{v} - \rho(f(x, y))\tilde{u}\tilde{v} + \rho(f(x, y))(\tilde{u} - u')\tilde{v} +$$

$$+\rho(f(x, y))u'(\tilde{v} - v') + \Delta_{(x,y)}(u, v)$$

where $(u', v') = Df_{(x,y)}(u, v)$.

Taking the Taylor developments on (u,v) around $(0,0)$, with fixed (x,y), up to order two of $\rho(\tilde{x},\tilde{y})-\rho(f(x,y))$, up to order one of $\tilde{u}$ and $\tilde{v}$ and up to order three of $\tilde{u}-u'$ and $\tilde{v}-v'$, we obtain:

$$\overline{V}(P,Q)=v^2[u^2\gamma_1+2uv\gamma_2+v^2\gamma_3]+2uv(u^2\gamma_4)+u^2(u^2\gamma_5)+\Delta_{(x,y)}(u,v)$$

where $\{\gamma_i\}$ is a set of continuous real functions on (x,y,u,v) such that $\gamma_5(0,0,0,0)=\lambda b_{20}>0$.

Therefore:

$$\overline{V}(P,Q)=v^2[\theta_1(x,y)+u^2\gamma_1+2uv\gamma_2+v^2\gamma_3]+2uv[\theta_2(x,y)+u^2\gamma_4]+u^2[\theta_3(x,y)+u^2\gamma_5]$$

As $\theta_1(0,0)=1$, $\theta_2(0,0)=0$ and $\theta_3(0,0)=0$, given any positive real number κ, there exists a neighborhood N of $(0,0,0,0)$ such that $\overline{V}(P,Q)>0$ if $(x,y,u,v)\in N$ and $|u|\leq\kappa|v|\neq 0$.

On the other hand, if $|u|>\kappa|v|$, the value of $\overline{V}(P,Q)$ is positive, for (x,y,u,v) in a small neighborhood of $(0,0,0,0)$, because:

$$(\theta_3+u^2\gamma_5)(\theta_1+u^2\gamma_1+2uv\gamma_2+v^2\gamma_3)-(\theta_2+u^2\gamma_4)^2\geq\theta_3\theta_1-\theta_2^2+u^2\lambda b_{20}/2>0$$

□

We now prove part 1. of Theorem 1.

Proof. Due to Theorem 2.2 it is enough to show that f is expansive. Take $\alpha>0$ such that $\overline{V}(P,Q)>0$ if $0<\text{dist}(P,Q)\leq\alpha$ as in Lemma 2.4. By contradiction assume that there exist two different points P and Q such that $\text{dist}\,(f^n(P),f^n(Q))\leq\alpha$ for all integer n. Suppose $V(P,Q)\geq 0$ (if not, substitute in the following argument f by f^{-1} and V by $-V$). As $\overline{V}(f^n(P),f^n(Q))>0$ for all $n\geq 0$, we have that $V(f^n(P),f^n(Q))$ is strictly increasing with n, and so, it is larger than some $\epsilon>0$ for all $n\geq 1$. As V is continuous and null in the (compact) diagonal of $M\times M$, there exists $\delta>0$ such that $\text{dist}\,(f^n(P),f^n(Q))\geq\delta$ for all $n\geq 1$. Take $K>0$ the minimum value of $\overline{V}$ in two different points whose distance is greater or equal to δ and smaller or equal to α. Thus $V(f^n(P),f^n(Q))\geq(n-1)K$ for all $n\geq 1$ contradicting that V is bounded for all pairs of points whose distance is smaller or equal than α.

This finishes the proof that f is expansive (with α an expansivity constant) and so, f is conjugated to an Anosov diffeomorphism, as wanted.

□

Our next aim is to show that the stable and unstable topological manifolds for f, (defined as the images by the conjugacy of the stable and unstable manifolds of the Anosov diffeomorphism), are indeed C^1 curves. We will also characterize their tangent spaces at all points.

For any point $P\in M$ let

$$S_P=\{\mathbf{v}\in T_PM:\ B(Df^m(P)\mathbf{v})\leq 0\,\forall m\geq 0\}$$
$$U_P=\{\mathbf{v}\in T_PM:\ B(Df^m(P)\mathbf{v})\geq 0\,\forall m\leq 0\}$$

Corollary 2.5. *There exist two continuous stable and unstable invariant foliations for f, whose leaves respectively are:*

$$W^s(P)=\{Q:\ \text{dist}(f^n(P),f^n(Q))\to_{n\to+\infty}0\}$$
$$W^u(P)=\{Q:\ \text{dist}(f^n(P),f^n(Q))\to_{n\to-\infty}0\}$$

defined for all P in M. Each leaf of the foliations is C^1. Besides, $T_PW^s(P)=S_P$, $T_PW^u(P)=U_P$, depend continuously on P, are transversal if $P\neq P_0$ and coincide to $[(1,0)]$ in P_0.

This corollary follows from the conjugation to Anosov and the following lemmas. We took the idea of the proofs from [27], [18] and [21].

Lemma 2.6. *Fixed a small neighborhood E of P_0, there exist constants $a > 0$, $0 < b < 1$, $C_1 > 0$ and $C_2 > 0$ such that:*

1. *If $P \notin E$ and $\mathbf{v} \in T_P(M)$, then $\Delta(\mathbf{v}) \geq aB(\mathbf{v})$.*
2. *If $P \notin E$ and $\mathbf{v} \in S_P$, then $\Delta(\mathbf{v}) \geq -bB(\mathbf{v})$.*
   If $P \notin E$ and $\mathbf{v} \in U_P$, then $\Delta(\mathbf{v}) \geq bB(\mathbf{v})$
3. *If $P \notin E$ and $\mathbf{v} \in S_P$, then $C_1\|\mathbf{v}\|^2 \leq -B(\mathbf{v}) \leq C_2\|\mathbf{v}\|^2$*
   If $P \notin E$ and $\mathbf{v} \in U_P$, then $C_1\|\mathbf{v}\|^2 \leq B(\mathbf{v}) \leq C_2\|\mathbf{v}\|^2$
4. *If $f^{j_i}(P) \notin E$ for $0 = j_0 < j_1 < \ldots < j_i$ and $\mathbf{v} \in S_P$, or if $f^{j_i}(P) \notin E$ for $0 = j_0 > j_1 > \ldots > j_i$ and $\mathbf{v} \in U_P$, then*
$$\|Df^{j_i}(\mathbf{v})\| \leq (C_2/C_1)^{1/2}(1-b)^{i/2}\|\mathbf{v}\|$$

Proof. To prove [1.], take
$$a^{-1} = \max\{B_P(\mathbf{v})/\Delta_P(\mathbf{v}),\ P \in M \setminus E,\ \mathbf{v} \in T_P(M),\ \|\mathbf{v}\| = 1\}$$
The maximum exists and is positive because B and Δ are continuous, Δ is positive definite and B is indefinite. As B and Δ are homogenous on $\|v\|$ the inequality [1.] follows from the definition of a.

To prove the first assertion of [2.], define $K = \max\{-B_P(\mathbf{v})/\Delta_P(\mathbf{v}),\ P \in M \setminus E,\ \mathbf{v} \in T_P(M),\ \|\mathbf{v}\| = 1,\ B_P(\mathbf{v}) \leq 0,\ B_{f(P)}(df_P(\mathbf{v})) \leq 0\}$. The set where the maximum is taken is not empty, because $f^{\#}B$ is indefinite, so there exists a unitary vector $\mathbf{v} \in T_P(M)$, such that $B_{f(P)}(df_P(\mathbf{v})) \leq 0$, and , as $\Delta_P > 0$, we have that such a vector verifies $B_P(\mathbf{v}) < 0$. Besides, in the set where the maximum is taken (in particular if $\mathbf{v}$ is a unitary vector in S_P), $0 < -B_P(\mathbf{v})/\Delta_P(\mathbf{v}) = -B_P(\mathbf{v})/(B_{f(P)}(df_P(\mathbf{v})) - B_P(\mathbf{v})) \geq 1$. So $K \geq 1$, and $-B_P(\mathbf{v}) \leq K\Delta_P(\mathbf{v}) \leq b^{-1}\Delta_P(\mathbf{v})$ for any positive $b < (1/K) \leq 1$.

To prove the second assertion of [2.], use the first assertion applied to f^{-1} instead of f and $-B$ instead of B.

To prove [3.] define $C_1 = \min\{|B_P(\mathbf{v})|,\ P \in M \setminus E,\ \mathbf{v} \in T_P(M),\ \|\mathbf{v}\| = 1,\ B_{f(P)}(df_P(\mathbf{v})) \leq 0 \text{ or } B_{f^{-1}(P)}(df_P^{-1}(\mathbf{v})) \geq 0\}$. The same arguments as before show that $C_1 > 0$, and $|B_P(\mathbf{v})| \geq C_1\|\mathbf{v}\|^2$, if $\mathbf{v} \in S_P \cup U_P$. Analogously define C_2 as the maximum of $|B_P(\mathbf{v})|$ in the same compact set of TM as before, concluding that $|B_P(\mathbf{v})| \leq C_2\|\mathbf{v}\|^2$, if $\mathbf{v} \in S_P \cup U_P$.

To prove [4.] observe that for any point Q in M (also in the neighborhood N) and any vector $\mathbf{u} \in T_Q(M)$, the inequality $B_{f(Q)}Df_Q(\mathbf{u}) \geq B_Q(\mathbf{u})$ follows from $\Delta_Q(\mathbf{u}) \geq 0$. Consider $\mathbf{v} \in S_P$ and apply [2.]: $B(Df^{j_i}(\mathbf{v})) = B(Df^{j_i-1}(\mathbf{v})) + \Delta_{f^{j_i-1}(P)}(Df^{j_i-1}(\mathbf{v})) \geq (1-b)B(Df^{j_i-1}(\mathbf{v})) \geq (1-b)B(Df^{j_{i-1}}(\mathbf{v})) \geq (1-b)^2B(Df^{j_{i-1}-1}(\mathbf{v})) \geq \ldots \geq (1-b)^iB(\mathbf{v})$. Applying [3.] we conclude [4.]. A similar proof stands for $\mathbf{v} \in U_P$.

□

Lemma 2.7. *The subspaces S_P and U_P are one-dimensional, they depend continuously on P and if $P \neq P_0$, $T_PM = S_P \oplus U_P$ while $S_{P_0} = U_{P_0} = [(1,0)]$.*

Proof. At each point P of M, we consider $G_1(T_PM)$, the Grassmanian manifold of the subspaces of dimension 1 in T_PM. When P varies we obtain the manifold $G_1(TM)$; it is a compact manifold.

We fix $P \in M$. For $n \geq 0$, let us choose
$$H_n \in G_1(T_{f^n(P)}M)$$

such that for

$$\mathbf{0} \neq \mathbf{v} \in H_n$$

$B_{f^n(P)}(\mathbf{v}) < 0$. Let us take a convergent subsequence

$$Df^{-n_j}(f^{n_j}(P))H_{n_j} \in G_1(T_P M)$$

to, say, H_∞. Since $\Delta > 0$, for any $m \geq 0$ and

$$\mathbf{0} \neq \mathbf{u} \in Df^{-m}(f^n(P))H_n \in G_1(T_{f^{n-m}(P)}M)$$

we have $B(\mathbf{u}) < 0$. Then with $m \geq 0$ fixed,

$$\lim_{j\to\infty} Df^{m-n_j}(f^{n_j}(P))H_{n_j} = Df^m(P)H_\infty$$

and so, for any $\mathbf{0} \neq \mathbf{v} \in H_\infty$, we deduce that $B(Df^m(P)\mathbf{v}) \leq 0$ for any $m \geq 0$; this proves that S_P contains the one dimensional subspace H_∞. We will prove that $S_P = H_\infty$.

We first claim that for any $P \in M$ there exists a direction $H^* \in T_P M$ such that for $\mathbf{0} \neq \mathbf{v} \in H^*$, $\limsup_{j\to\infty} \Delta(Df^j(\mathbf{v})) > 0$. First, we prove that the property is verified for points that are not in the global stable curve $W^s(P_0)$ of P_0. For such a point P, fixed a small open neighborhood E of P_0, there exists an increasing sequence of natural numbers j_i such that $f^{j_i}(P) \notin E$. (This is a topological characterization of the complement of $W^s(P_0)$, inherited from the Anosov diffeomorphism to which f is conjugated). Take $a > 0$ as in Lemma 2.6. Therefore $\Delta(Df^{j_i}(\mathbf{v})) \geq aB(Df^{j_i}(\mathbf{v})) \geq aB(\mathbf{v})$. Choosing $\mathbf{v} \in T_P M$ such that $B(\mathbf{v}) > 0$, the claim is proved if $P \notin W^s(P_0)$. We are left to prove the same property for a point $P = (x_0, y_0)$ that stay forever in the future in a suitable neighborhood E of $P_0 = (0,0)$. We show the computations with $\lambda = 1$. (When $\lambda = -1$ we should take f^2 instead of f and observe that $\Delta_{f^2,B} = f^\# \Delta_{f,B} + \Delta_{f,B}$, so $\Delta_{f^2,B}(Df^{2j}\mathbf{v}) \to 0$ if and only if $\Delta_{f,B}(Df^j\mathbf{v}) \to 0$). Let us compute the image of the graphic of $y = \alpha x^2$. It is another curve with the same value of the first and second derivatives than $y = \alpha x^2$ at $(0,0)$, but its third derivative is $-12\alpha^2 - 12\alpha a_{10} + 6b_{20}$. Therefore, if α is large enough, the graphic of the invariant local manifolds lies, in a neighborhood E of $(0,0)$, in the region $\{(x,y) \in \mathbb{R}^2;\ -\alpha x^2 \leq y \leq \alpha x^2\}$. (The equalities are only verified for the fixed point $P_0 = (0,0)$). We denote $(x_j, y_j) = f^j(x_0, y_0)$ and choose $\mathbf{v}_0 = (u_0, v_0) \in T_{(x_0,y_0)}M$, such that $u_0 > 0$, $v_0 > 0$. Denoting $\mathbf{v}_j = (u_j, v_j) = Df^j(u_0, v_0) \in T_{(x_j,y_j)}M$, we observe that $u_j > 0, v_j > 0$ for all $j \in I\!N$. (In fact, the derivative of f in E is close to the derivative at $(0,0)$, so $u_{j+1} > (u_j + v_j)/2$ if $u_j > 0$ and $v_j > 0$. Therefore, if u_{j+1} were not positive, for the minimum j, we should have $v_j \leq 0, u_j > 0$, contradicting that $B(\mathbf{v}_j) = (1 + dx_j + ey_j + \text{h.o.t.})u_j v_j \geq B(\mathbf{v}_0) > 0$.) Let us finish now the proof of the claim. By contradiction, let us suppose that $\lim_{j\to\infty} \Delta(\mathbf{v}_j) = 0$. Using that $-\alpha x_j^2 \leq y_j \leq \alpha x_j^2$ we write $\Delta(\mathbf{v}_j) = (v_j + u_j x_j a_{10})^2 + u_j^2 x_j^2(3b_{20} - a_{10}^2) + \text{h.o.t.}$. Due to Lemma 2.3 we have $3b_{20} - a_{10}^2 \geq 0$ and then $\lim v_j = 0$. As $B(\mathbf{v}_j) = (1 + dx_j + ey_j + \text{h.o.t.})u_j v_j$ is increasing and positive, we have that $u_j\ {}_{j\to\infty} \to +\infty$ and $3b_{20}u_j - 3|b_{21}|v_j > 8|a_{22}|v_j\alpha$ for all sufficiently large $j \in I\!N$. We now compute $v_{j+1} - v_j = (3b_{20}u_j + 3b_{21}v_j)x_j^2 + 2a_{22}v_j y_j + \text{h.o.t.} \geq (3b_{20}u_j - 3|b_{21}|v_j)x_j^2/2 - 4|a_{22}|v_j|y_j| \geq 4|a_{22}|v_j(\alpha x_j^2 - |y_j|) \geq 0$. We have the contradiction $v_{j+1} \geq v_j > 0$ and $v_j \to 0$, ending the proof of the claim.

Let us prove now that $S_P = H_\infty$. (We already know that $H_\infty \subset S_P$.) By contradiction, let us suppose that there exists $\mathbf{v}_1 \in S_P$, $\mathbf{v}_1 \notin H_\infty$. Let us take $\mathbf{v}_2 \in H_\infty \subset S_P$ such that $\mathbf{v}_1 + \mathbf{v}_2 \in H^*$. Now use that $\sqrt{\Delta(Df^n(\mathbf{v}_1 + \mathbf{v}_2))} \leq$

$\sqrt{\Delta(Df^n(\mathbf{v}_1))}+\sqrt{\Delta(Df^n(\mathbf{v}_2))}$. The contradiction follows taking $\limsup_{n\to\infty}$ and observing that $\triangle(Df^n(\mathbf{v}))\to 0$ for all $\mathbf{v}\in S_P$ due to the definition of S_P.

In order to prove the continuity of S_P we take $Q_n\to P$ in M and prove that $S_{Q_n}\to S_P$ in $G_1(TM)$. Let us choose any subsequence of S_{Q_n} convergent to some S in $G_1(T_PM)$. For any $m\geq 0$ we have $B(Df^m(Q_n)\mathbf{v}_n)\leq 0$ for $\mathbf{v}_n\in S_{Q_n}$. Taking $n\to+\infty$, as B is continuous, $B(Df^m(P)\mathbf{v})\leq 0$ if $\mathbf{v}\in S$. Therefore $\mathbf{v}\in S_P$, that is $S=S_P$. The same considerations with f^{-1} instead of f show that U_P is a one dimensional continuous field of directions.

Since f increases the values of $B_P(\mathbf{v})$ if $P\neq P_0$ and $\mathbf{v}\neq\mathbf{0}$, then $S_P\bigcap U_P=[\mathbf{0}]$. On the other hand $[(1,0)]\subset S_{P_0}\bigcup U_{P_0}$ and both subspaces are one-dimensional, so they both coincide in P_0 with $[(1,0)]$.

□

We now end the proof of the corollary 2.5.

Proof. We prove it for the stable foliation; the same ideas work for the unstable foliation. Let us take $P\notin W^s_{P_0}$. We can locally integrate the directions S_P, and take any solution through P. We first claim that the lower limit of the lengths of positive iterates of this curve goes to 0. Fix E a small neighborhood of P_0, and take $0<b<1$ and $C=(C_2/C_1)^{1/2}$ as in Lemma 2.6. Defining an increasing sequence $\{j_i\}_i$ such that $f^{j_i}(P)\notin E$, it is verified $\|Df^{j_i}(\mathbf{v})\|\leq C(1-b)^{i/2}\|Df^{j_0}\mathbf{v}\|$ for $\mathbf{v}\in S_P$, $P\notin W^s_{P_0}$. Therefore, the claim is proved. Recalling the conjugation to Anosov, the integral curve has to be on the stable manifold through $P\notin W^s_{P_0}$.

Finally, let us suppose that P is on the stable manifold of P_0. The conjugation to Anosov and the C^0-density of the leaves of the stable foliation which are not in the stable manifold of P_0 imply the local unicity of the curve obtained integrating S_P and that this curve is $W^s(P)$.

□

We are left to prove the second part of Theorem 1. We will see first the case $\lambda=1$ and $a_{10}=0$.

Lemma 2.8. *If $\lambda=1$ and $a_{10}=0$ then there exists a C^3 local chart (ξ,η) defined in a neighborhood of P_0 such that f can be written $(\xi+\eta+\varepsilon(\xi,\eta),\eta+\varepsilon(\xi,\eta))$ where the coefficients of ξ^2 and $\xi\eta$ in the Taylor development of ε are 0, the coefficient of ξ^3 is $b_{20}>0$, and the function $V^*((\xi_1,\eta_1),(\xi_2,\eta_2))=(\eta_2-\eta_1)(\xi_2-\xi_1-\eta_2+\eta_1)$, in a neighborhood of $((0,0),(0,0))$, verifies $\overline{V}^*((\xi_1,\eta_1),(\xi_2,\eta_2))=V^*(f(\xi_1,\eta_1),f(\xi_2,\eta_2))-V^*((\xi_1,\eta_1),(\xi_2,\eta_2))>0$ if $(\xi_1,\eta_1)\neq(\xi_2,\eta_2)$.*

Proof. In N_1 we use the following notation: $f(x,y)=(x+y+\varepsilon_1(x,y),y+\varepsilon_2(x,y))$. Let us take a change of coordinates: $(\xi,\eta)=(x+\varphi(x,y),y+\psi(x,y))=h(x,y)$, with φ and ψ functions, to be chosen, whose first order partial derivatives are equal to zero. We want the diffeomorphism f to be, in the new coordinates, $(\xi+\eta+\varepsilon(\xi,\eta),\eta+\varepsilon(\xi,\eta))$. So, $\varphi(x,y)+\psi(x,y)+\varepsilon(\xi,\eta)=\varepsilon_1(x,y)+\varphi\circ f(x,y)$; $\psi(x,y)+\varepsilon(\xi,\eta)=\varepsilon_2(x,y)+\psi\circ f(x,y)$, obtaining that $\psi(x,y)=(\varepsilon_1-\varepsilon_2-\varphi)\circ f^{-1}(x,y)+\varphi(x,y)$ and $\varepsilon(\xi,\eta)=(\varphi+\varepsilon_2-\varepsilon_1)\circ f^{-1}\circ h^{-1}(\xi,\eta)+(\varepsilon_1-2\varphi)\circ h^{-1}(\xi,\eta)+\varphi\circ f\circ h^{-1}(\xi,\eta)$. Let us take $\varphi(x,y)=d_{10}x^3$, where d_{10} is defined by $6d_{10}=8a_{11}^2+6a_{11}a_{22}+a_{22}^2+(3/b_{20})(b_{10}+b_{21})^2-3b_{10}-6b_{11}-3b_{22}+1$. After some computations, we obtain $\varepsilon(\xi,\eta)=(2a_{11}+a_{22})\eta^2+b_{20}\xi^3+3(b_{10}+b_{21}-b_{20})\xi^2\eta+(1+(3/b_{20})(b_{10}+b_{21}-b_{20})^2+(2a_{11}+a_{22})^2)\xi\eta^2+\beta_{13}\eta^3+\text{h.o.t.}$, where β_{13} is a real number. We are left to prove that the function V^* verifies the thesis of the lemma. Defining $u=\xi_2-\xi_1$, $v=\eta_2-\eta_1$ we can write $\overline{V}^*((\xi_1,\eta_1),(\xi_2,\eta_2))=v^2(1+\text{h.o.t.})+uv[2(2a_{11}+a_{22})\eta_1+\text{h.o.t.}]+u^2[3b_{20}\xi_1^2+(1+(3/b_{20})(b_{10}+b_{21}-b_{20})^2+(2a_{11}+a_{22})^2)\eta_1^2+b_{20}u^2+6(b_{10}+$

$b_{21} - b_{20})\xi_1\eta_1 + 3b_{20}\xi_1 u + 3(b_{10} + b_{21} - b_{20})\eta_1 u + \text{h.o.t.}]$. If $|u| < \alpha|v|$, for certain small positive number α, then V^* is positive. To end the proof of the lemma it is enough to verify that the discriminant is negative if $u \neq 0$. Thus we shall prove that $3b_{20}\xi_1^2 + (1 + (3/b_{20})(b_{10} + b_{21} - b_{20})^2)\eta_1^2 + b_{20}u^2 + 6(b_{10} + b_{21} - b_{20})\xi_1\eta_1 + 3b_{20}\xi_1 u + 3(b_{10} + b_{21} - b_{20})\eta_1 u > 0$, if $u \neq 0$. This is a quadratic form in u, ξ_1, η_1 which is positive definite because $b_{20} > 0$. □

In what follows, we will work with the local charts of the thesis of the former lemma; for convenience we write x instead ξ and y instead η.

3. Distortion estimates.

Let $\mathcal{J}(P)$ be the Jacobian of f at P, i.e. $\mathcal{J}(P)$ is the determinant of $Df(P)$. Our first purpose in this section is to prove that meanwhile the iterates from 0 to n of two points visit a small neighborhood N of the origin, the difference of the Jacobians of f^n at these two points is Hölder dependent on the distance between them, with Hölder constant that does not depend on n; this is the content of the proposition 3.3. Then we use that result to prove a global result: there exists H such that $1/H \leq \prod_{j=1}^{k} \mathcal{J}(f^{-j}(P))/\mathcal{J}(f^{-j}(Q)) \leq H$ for any two points P, Q that maintain close during k iterates, this is the proposition 3.12

Remark 3.1. As f is conjugate to Anosov, it follows that it has a local product structure, i.e., there exists $0 < \gamma$ such that if $0 < \beta < \gamma$ there exists $0 < \alpha = \alpha(\beta)$ verifying that for all P, $Q \in M$ with $\operatorname{dist}(P, Q) \leq \alpha$, $[P, Q] := W^s_\beta(P) \bigcap W^u_\beta(Q)$ contains exactly one point. Here $W^s_\beta(P) = \{Q \in M : \operatorname{dist}(f^n(P), f^n(Q)) \leq \beta \; \forall n \geq 0\} \subset W^s(P)$; similarly $W^u_\beta(Q) \subset W^u(Q)$.

Definition 3.2. A rectangle R is a set in M such that $P, Q \in R$ implies $\emptyset \neq [P, Q] \in R$.

Proposition 3.3 (Local Bounded Area Distortion). *Let N be a sufficiently small rectangle which is a neighborhood of the origin where f has the form of lemma 2.8, we denote $D_1 = f(N) \bigcap f^{-1}(N) \subset N$.*

There exists a positive constant C (that does not depend on P, Q or n) such that, if $P \in M$, $Q \in W^s_\beta(P) \bigcup W^u_\beta(P)$ and $f^i(P)$ and $f^i(Q)$ are in D_1 for $0 \leq i \leq n$, then

$$\left| \log \prod_{i=0}^{n} \frac{\mathcal{J}(f^i(P))}{\mathcal{J}(f^i(Q))} \right| \leq C d^{1/3}$$

where $d = \max\{\operatorname{dist}(P, Q),\ \operatorname{dist}(f^n(P), f^n(Q))\}$.

To prove this proposition we need the following definition and lemmas:

We say that a curve in N is strictly increasing if for two points (x_1, y_1) and (x_2, y_2) in the curve, $x_1 < x_2$ if and only if $y_1 < y_2$. Analogously, we say that the curve is strictly decreasing when $x_1 < x_2$ if and only if $y_1 > y_2$.

Lemma 3.4. *The local unstable manifold of the fixed point $(0, 0)$ is a strictly increasing curve and the local stable manifold is strictly decreasing. The images by f of the horizontal lines $y = y_1$ in a neighborhood of $(0, 0)$ are strictly increasing curves, and the preimages, strictly decreasing curves.*

Proof. The diffeomorphism f has the local form $f(x, y) = (x + y + \varepsilon(x, y), y + \varepsilon(x, y))$. Observe that for $x_1 \neq x_2$,

$$0 < \overline{V}^*((x_1, y), (x_2, y)) = (x_1 - x_2)(\varepsilon(x_1, y) - \varepsilon(x_2, y))$$

Therefore, the image by f of any line $y = y_1$ is strictly increasing. On the other hand, the image by f of the vertical line $x = x_1$ is the line $x - y = x_1$, and so it is also strictly increasing.

As vertical and horizontal lines are transformed by f in strictly increasing curves, we deduce that their preimages are strictly decreasing.

If two different points (x_1, y_1) and (x_2, y_2) belong to the local unstable manifold of $(0,0)$ then

$$V^*_{-n} = V^*(f^{-n}(x_1, y_1), f^{-n}(x_2, y_2)) \to_{n\to\infty} 0$$

By Lemma 2.8 V^*_{-n} is strictly decreasing with n, so $0 < V^*_0 = (y_2 - y_1)(x_2 - x_1 - y_2 + y_1)$. This shows that $y_1 < y_2$ if and only if $x_1 < x_2$ proving that the local unstable manifold of $(0,0)$ is strictly increasing. The same arguments applied to f^{-1} instead of f show that for (x_1, y_1) and (x_2, y_2) in the local stable manifold of $(0,0)$, $0 > V^*_0 = (y_2 - y_1)(x_2 - x_1 - y_2 + y_1)$ and $0 > V^*_1 == (y_2 - y_1 + \varepsilon(x_2, y_2) - \varepsilon(x_1, y_1))(x_2 - x_1)$, so $y_1 > y_2$ if and only if $x_1 < x_2$ proving that the local stable manifold of $(0,0)$ is strictly decreasing. □

We denote $P = (x_0, y_0)$ a point in D_1, and $(x_n, y_n) = f^n(x_0, y_0)$ for n such that $f^i(P) \in N$ for all $i = -1, 0, 1, \dots, n+1$.

Let us consider the stable and unstable local curves of $(0,0)$; they divide N in four open connected components. We call N_j, with $j = 1, 2, 3, 4$ to these connected components. Due to lemma 3.4, the four connected components N_j can be characterized by the following property: In N_1 the abscise x is always positive, in N_2 the ordinate y is always positive, in N_3 the abscise x is always negative, and in N_4 the ordinate y is always negative.

Lemma 3.5. *If $(x_0, y_0) \in N_1 \cup N_3$ then $\{x_i\}_{i=1,2,\dots,n}$ has constant sign with i, and $\{y_i\}_{i=1,2,\dots,n}$ is monotone with i.*

If $(x_0, y_0) \in N_2 \cup N_4$ then $\{y_i\}_{i=1,2,\dots,n}$ has constant sign with i, and $\{x_i\}_{i=1,2,\dots,n}$ is monotone with i.

Proof. By lemma 3.4 $\{x_i\}_i$ and $\{y_i\}_i$ are both monotone with i for points in $W^u_{loc}(0,0)$ or in $W^s_{loc}(0,0)$.

Let us prove that in N_1 the ordinates y_i of iterates of (x_0, y_0) are increasing with i. The horizontal curve $y = y_0$ intersects the boundary of N_1 at a point $(\tilde{x}_0, y_0)$ of the stable or unstable curves of $(0,0)$. We have that $f(\tilde{x}_0, y_0) = (\tilde{x}_1, \tilde{y}_1)$ with $\tilde{y}_1 \geq y_0$. As the image by f of the horizontal curve $y = y_0$ is strictly increasing, then $y_1 > \tilde{y}_1 \geq y_0$ as asserted.

To prove that in N_2 the abscises $\{x_i\}_i$ are increasing, observe that $x_{i+1} - x_i = y_{i+1} > 0$.

The monotony of $\{y_i\}_i$ in N_3 and of $\{x_i\}_i$ in N_4 are proved analogously.

□

In the following statements C denotes a sufficiently large real positive constant that is independent of n or (x_0, y_0).

Lemma 3.6. $\sum_{i=0}^{n-1} |y_i| \leq C$

Proof. Considering the form of f we have:

$$\begin{aligned} x_{i+1} &= x_i + y_i + \varepsilon(x_i, y_i) \\ y_{i+1} &= y_i + \varepsilon(x_i, y_i) \end{aligned}$$

Then $x_{i+1} - x_i = y_{i+1}$ and the sum $\sum_{i=0}^{n-1} y_i$ is telescopic, so it is bounded. The bound of the thesis is obtained from the former bound, because either y_i has constant sign with i or it is monotone with i.

□

Let $k \in \{0, 1, 2, \ldots, n-1\}$ be such that $|y_k| = \min\{|y_i| : i = 0, 1, 2, \ldots, n-1\}$. For each $i \in \{0, 1, 2, \ldots, n-1\}$ denote $r(i)$ the natural number equal to $i+1$ if $0 \le i \le k-1$ and equal to $n-i$ if $k \le i \le n-1$.

Lemma 3.7. 1. $|\sum_{i=0}^{n-1} r(i)\varepsilon(x_i, y_i)| \le C$.

2. $\sum_{i=0}^{n-1} r(i) y_i^2 \le C$ *if* $(x_0, y_0) \in \overline{N}_1 \cup \overline{N}_3$.

Proof. We first prove

$$\left| \sum_{m=0}^{k-1} \sum_{i=m}^{k-1} \varepsilon(x_i, y_i) + \sum_{m=k}^{n-1} \sum_{i=k}^{m} \varepsilon(x_i, y_i) \right| \le C \tag{2}$$

The sum of $\varepsilon(x_i, y_i)$ is telescopic: $\sum_{i=m}^{k-1} \varepsilon(x_i, y_i) = y_k - y_m$ if $0 \le m \le k-1$ and $\sum_{i=k}^{m} \varepsilon(x_i, y_i) = y_{m+1} - y_k$ if $k \le m \le n-1$. On the other hand, the sum of y_i is also telescopic because $y_{i+1} = x_{i+1} - x_i$ so $\sum_{m=0}^{k-1}(-y_m) + \sum_{m=k}^{n-1} y_{m+1} = -x_{k-1} + x_{-1} + x_n - x_k$ is bounded. To prove (2), it is enough to show that $n|y_k|$ is bounded. Let us prove that $n|y_k| \le \sum_{i=0}^{n-1} |y_i|$ and apply the lemma 3.6. If not, as $|y_k|$ is a minimum, we would have that $|y_i| > \sum_{i=0}^{n-1} |y_i|/n$ for all $i = 0, \ldots, n-1$ which is a contradiction.

To prove 1., we observe that $r(i)$ is the number of times that each $\varepsilon(x_i, y_i)$ appears in the sum (2).

To prove 2., let us take C such that $\sum_{i=0}^{n-1} |y_i| \le C$. We assert that $|y_i| \le C/r(i)$ if $(x_0, y_0) \in \overline{N}_1 \cup \overline{N}_3$. If not, there would exist n_0 such that $|y_{n_0}| > C/r(n_0)$. As y_i is monotone, we have that $|y_i| > C/r(n_0)$ for all i between 0 and n_0, (if $n_0 < k$) or between n_0 and $n-1$ (if $n_0 \ge k$). In the first case $\sum_{i=0}^{n_0} |y_i| > C(n_0+1)/r(n_0) = C$ contradicting the choice of C. In the second case we obtain the same contradiction: $\sum_{i=n_0}^{n-1} |y_i| > C(n-n_0)/r(n_0) = C$. Let us prove 2. The sum of $|y_i|$ is bounded, and $r(i)|y_i|$ is also bounded. So the sum of $r(i)y_i^2$ is bounded. □

Due to lemma 2.8 we can use the following notation: $\varepsilon(x, y) = b_{20}x^3 + o_3(x) + yO_2(x) + y^2\alpha(x, y)$, where b_{20} is positive, o_3 is a function of order greater than 3 in x, $O_2(x)$ is a function of order at least 2 in x, and $\alpha(x, y)$ is some C^1 function.

We define $x^{-4/3}O_2(x) = 0$ for $x = 0$ and $x^{-4/3}o_3(x) = 0$ for $x = 0$.

Lemma 3.8. *If* $(x_0, y_0) \in \overline{N}_1 \cup \overline{N}_3$, *then*

$$\sum_{i=0}^{n-1} \left| x_i^{5/3} \right| \le C$$

Proof. As $\varepsilon(x_i, y_i) = y_{i+1} - y_i$ and y_i is monotone then $\varepsilon(x_i, y_i)$ has constant sign. Combining the expression of ε with the results of lemma 3.7, we obtain that $\sum_{i=0}^{n-1} r(i) \left| b_{20}x_i^3 + o_3(x_i) + y_i O_2(x_i) \right|$ is bounded. We will consider

$$\sum_{i=0}^{n-1} \left| b_{20}x_i^{5/3} + x_i^{-4/3}o_3(x_i) + y_i x_i^{-4/3} O_2(x_i) \right|$$

Let us define I as the set of indexes i for which $\left| x_i^{4/3} \right| \ge 1/r(i)$ and J the complementary set in $\{0, 1, 2, \ldots, n-1\}$.

First taking the terms for $i \in I$:

$$\sum_{i \in I} \left| b_{20}x_i^{5/3} + x_i^{-4/3}o_3(x_i) + y_i x_i^{-4/3} O_2(x_i) \right| \le$$

$$\leq \sum_{i\in I} r(i)|x_i|^{4/3}\left|b_{20}x_i^{5/3} + x_i^{-4/3}o_3(x_i) + y_i x_i^{-4/3}O_2(x_i)\right| =$$

$$= \sum_{i\in I} r(i)\left|b_{20}x_i^3 + o_3(x_i) + y_i O_2(x_i)\right| \leq C$$

Second, taking the terms for $i \in J$, and using that $b_{20} > 0$:

$$\sum_{i\in J}\left|b_{20}x_i^{5/3} + x_i^{-4/3}o_3(x_i) + y_i x_i^{-4/3}O_2(x_i)\right| \leq$$

$$\leq \sum_{i\in J} 2b_{20}|x_i|^{5/3} + \sum_{i\in J}|y_i||x_i^{-4/3}O_2(x_i)| \leq \sum_{i\in J} 2b_{20}\left(\frac{1}{r(i)}\right)^{5/4} + C\sum_{i\in J}|y_i|$$

But

$$2\sum_{i\in J} b_{20}\left(\frac{1}{r(i)}\right)^{5/4} \leq 2b_{20}\sum_{i=0}^{k-1}\left(\frac{1}{i+1}\right)^{5/4} + 2b_{20}\sum_{i=k}^{n-1}\left(\frac{1}{n-i}\right)^{5/4} \leq$$

$$\leq 4b_{20}\sum_{j=1}^{\infty} 1/j^{5/4} \leq C$$

Also the sum of $|y_i|$ is bounded, proving that

$$\sum_{i=0}^{n-1}|b_{20}x_i^{5/3} + x_i^{-4/3}o_3(x_i) + y_i x^{-4/3}O_2(x)| \leq C$$

To prove the lemma, see that, due to the fact that $b_{20} > 0$,

$$\left|b_{20}x_i^{5/3}\right| \leq 2|b_{20}x_i^{5/3} + x_i^{-4/3}o_3(x_i)| \leq$$

$$\leq 2\left|b_{20}x_i^{5/3} + x_i^{-4/3}o_3(x_i) + y_i x_i^{-4/3}O_2(x_i)\right| + 2\left|y_i x_i^{-4/3}O_2(x_i)\right|$$

The sum of both terms at right are bounded, so the sum of the term at left also is. □

We denote $P = (x_0, y_0)$, $Q = (\tilde{x}_0, \tilde{y}_0)$ two points in D_1, $(x_n, y_n) = f^n(x_0, y_0)$ and $(\tilde{x}_n, \tilde{y}_n) = f^n(\tilde{x}_0, \tilde{y}_0)$ for n such that $f^i(P) \in N$ and $f^i(Q) \in N$ for all $i = -1, 0, 1, \ldots, n+1$. Denote $\delta = 2\max\{|x_0 - \tilde{x}_0|, |x_n - \tilde{x}_n|\}$.

Lemma 3.9. *If $|\tilde{y}_i - y_i| < |\tilde{x}_i - x_i|$ for $0 \leq i \leq n$, then $|\tilde{x}_i - x_i| \leq 2\delta$, $\tilde{x}_i - x_i$ has constant sign with i for $0 \leq i \leq n$, and $\sum_{i=0}^n |\tilde{y}_i - y_i| \leq 2\delta$.*

Proof. The local expression of f implies that $\tilde{x}_{i+1} - x_{i+1} - y_{i+1} + \tilde{y}_{i+1} = \tilde{x}_i - x_i$. The sign at left is the sign of $\tilde{x}_{i+1} - x_{i+1}$, because $|\tilde{y}_{i+1} - y_{i+1}| < |\tilde{x}_{i+1} - x_{i+1}|$. Thus, the sign of $\tilde{x}_i - x_i$ is constant with i. Let us suppose that $\tilde{x}_0 > x_0$. We have that $\tilde{x}_i > x_i$ for $0 \leq i \leq n$.

Consider $V_i^* = V^*((x_i, y_i), (\tilde{x}_i, \tilde{y}_i)) = (\tilde{y}_i - y_i)(\tilde{x}_i - x_i - \tilde{y}_i + y_i)$. By Lemma 2.8 V_i^* is strictly increasing with i. Therefore, there exists $j_0 \in \{0, 1, \ldots, n\}$ such that $V_i^* > 0$ (so $\tilde{y}_i > y_i$) if $j_0 < i \leq n$; and $V_i^* \leq 0$ (so $\tilde{y}_i \leq y_i$) if $1 \leq i \leq j_0$.

Then $\sum_{i=1}^n |\tilde{y}_i - y_i| = \sum_{i=1}^{j_0}(-\tilde{y}_i + y_i) + \sum_{i=j_0+1}^n(\tilde{y}_i - y_i) = x_{j_0} - \tilde{x}_{j_0} + \tilde{x}_0 - x_0 + \tilde{x}_n - x_n + x_{j_0} - \tilde{x}_{j_0} < \tilde{x}_0 - x_0 + \tilde{x}_n - x_n \leq \delta$ (We used that $\tilde{x}_{j_0} > x_{j_0}$, and the convention $\sum_1^0 = 0$). Now, $\sum_{i=0}^n |\tilde{y}_i - y_i| \leq |\tilde{y}_0 - y_0| + \delta \leq 2\delta$.

To end the proof observe that $0 < \tilde{x}_i - x_i = \tilde{x}_0 - x_0 + \sum_{j=1}^i(\tilde{y}_j - y_j) < |\tilde{x}_0 - x_0| + \delta \leq 2\delta$ □

Lemma 3.10. *If* $(x_0, y_0) \in N$ *then*

$$\sum_{i=0}^{n-1} \left| x_i^{5/3} \right| \le C$$

Proof. The lemma 3.8 states the thesis if $(x_0, y_0) \in \overline{N}_1 \cup \overline{N}_3$. So it is left to prove this lemma when $(x_0, y_0) \in N_2 \cup N_4$.

Take the point $(\overline{x}_0, y_0)$ in the local stable manifold of $(0,0)$, and $(\hat{x}_0, y_0)$ in the local unstable manifold of $(0,0)$. Suppose $\overline{x}_0 < x_0 < \hat{x}_0$. Call $(\overline{x}_i, \overline{y}_i) = f^i(\overline{x}_0, y_0)$, $\quad (\hat{x}_i, \hat{y}_i) = f^i(\hat{x}_0, y_0)$. As $\hat{V}_i^* = V^*((x_i, y_i), (\hat{x}_i, \hat{y}_i)) = (\hat{y}_i - y_i)(\hat{x}_i - x_i + y_i - \hat{y}_i)$ is strictly increasing with i (because of Lemma 2.8), and $\hat{V}_0^* = 0$, we have that $\hat{V}_i^* > 0$ for $1 \le i \le n+1$, so $|\hat{y}_i - y_i| < |\hat{x}_i - x_i|$ for $0 \le i \le n+1$. By Lemma 3.9 $\hat{x}_i - x_i$ has constant sign with i, and analogously for $\overline{x}_i - x_i$. Therefore $\overline{x}_i < x_i < \hat{x}_i$ for $0 \le i \le n-1$. By Lemma 3.8 the sums of $|\overline{x}_i|^{5/3}$ and of $|\hat{x}_i|^{5/3}$ are bounded, so the same holds for the sum of $|x_i|^{5/3}$ as wanted.

□

Proof. (Proposition 3.3):

Let us prove the thesis when the given two different points are in the same stable arc; a similar proof holds when they are in the same unstable arc. The map f is a diffeomorphism, so $\mathcal{J}(P)$ is bounded away from zero.

$$\left| \log \prod_{i=0}^{n-1} \frac{\mathcal{J}(f^i(P))}{\mathcal{J}(f^i(Q))} \right| = \left| \sum_{i=0}^{n-1} \log \mathcal{J}(f^i(P)) - \log \mathcal{J}(f^i(Q)) \right| \le$$

$$\le C \sum_{i=0}^{n-1} |\mathcal{J}(f^i(P)) - \mathcal{J}(f^i(Q))|$$

In the local chart of lemma 2.8, we denote $P = (x_0, y_0) \ne Q = (\tilde{x}_0, \tilde{y}_0)$ and $f^i(P) = (x_i, y_i)$, $f^i(Q) = (\tilde{x}_i, \tilde{y}_i)$. As $f(x,y) = (x + y + \varepsilon(x,y), y + \varepsilon(x,y))$ it follows $\mathcal{J}(P) = 1 + \varepsilon_y(x,y)$, where ε_y denotes the partial derivative of ε respect to y. It is enough to show that

$$\sum_{i=0}^{n-1} |\varepsilon_y(x_i, y_i) - \varepsilon_y(\tilde{x}_i, \tilde{y}_i)| \le C\mathrm{d}^{1/3}$$

We denote $\Delta_i = |\varepsilon_y(x_i, y_i) - \varepsilon_y(\tilde{x}_i, \tilde{y}_i)|$. We have:

$$\Delta_i \le |x_i - \tilde{x}_i| \int_0^1 |\varepsilon_{yx}(x_i + \lambda(\tilde{x}_i - x_i), \tilde{y}_i)|\, d\lambda + |\tilde{y}_i - y_i| \int_0^1 |\varepsilon_{yy}(x_i, y_i + \lambda(\tilde{y}_i - y_i))|\, d\lambda$$

As $\varepsilon_{yx}(0,0) = 0$ and ε are of C^3 class, there exists some constant $C > 0$ such that:

$$\Delta_i \le C\, |\tilde{x}_i - x_i|\, (|\tilde{x}_i| + |x_i| + |\tilde{y}_i|) + C|y_i - \tilde{y}_i|$$

After Lemma 3.6 the sum of $|\tilde{y}_i|$ is bounded. The points, $(x_i.y_i)$ and $(\tilde{x}_i, \tilde{y}_i)$ are in the same local stable arc, and in a sufficiently small neighborhood of the origin. As the stable arcs are C^1 curves, whose tangent subspaces vary continuously, and at the origin the tangent stable subspace is $[(1,0)]$, we have that $|\tilde{y}_i - y_i| < |\tilde{x}_i - x_i|$ and we can apply Lemma 3.9.

We obtain $\sum_{i=0}^{n-1} \Delta_i \le C \sum_{i=0}^{n-1} |\tilde{x}_i - x_i|(|\tilde{x}_i| + |x_i|) + C\delta$ (for a sufficiently large constant $C > 0$). To end the proof it is enough to show that $\sum_{i=0}^{n-1} |\tilde{x}_i - x_i|(|\tilde{x}_i| + |x_i|) \le C\delta^{1/3}$

We write

$$\sum_{i=0}^{n-1} |\tilde{x}_i - x_i|(|\tilde{x}_i| + |x_i|) \leq \sum_{i=0}^{n-1} |\tilde{x}_i - x_i|^{1/3}(|\tilde{x}_i| + |x_i|)^{5/3} \leq (2\delta)^{1/3} \sum_{i=0}^{n-1}(|\tilde{x}_i| + |x_i|)^{5/3}$$

The triangular property and Lemma 3.10 imply that

$$\left(\sum_{i=0}^{n-1}(|x_i| + |\tilde{x}_i|)^{5/3}\right)^{3/5} \leq \left(\sum_{i=0}^{n-1}|x_i|^{5/3}\right)^{3/5} + \left(\sum_{i=0}^{n-1}|\tilde{x}_i|^{5/3}\right)^{3/5} \leq C$$

ending the proof.

□

We will look for global distortion estimates in the hypothesis of Theorem 1, either when λ is 1 or -1. Let us denote $D_r = f^r(N) \bigcap f^{-r}(N) \subset N$, $D_r^c = M \setminus D_r$. We first prove the following lemma:

Lemma 3.11. *There exist $\kappa > 0$ and $0 < \chi < 1$ such that for any $\beta > 0$ smaller than γ of the remark 3.1*

1. *if S_1 and S_2 are in a connected arc W_β^s of stable manifold and if*
$$f^{i_j}(W_\beta^s) \subset D_3^c$$
for $0 \leq i_0 < i_1 < \ldots < i_r$ then $\operatorname{dist}(f^{i_r}(S_1), f^{i_r}(S_2)) \leq \kappa\chi^r$.
2. *if S_3 and S_4 are in a connected arc W_β^u of unstable manifold and if*
$$f^{-i_j}(W_\beta^u) \subset D_3^c$$
for $0 \leq i_0 < i_1 < \ldots < i_r$ then $\operatorname{dist}(f^{-i_r}(S_3), f^{-i_r}(S_4)) \leq \kappa\chi^r$.

Proof. We only prove the first assertion, the second one is proved using the same ideas. The result follows from Lemma 2.6. There we have shown that there exists $0 < \chi = (1-b)^{1/2} < 1$ and $C = (C_2/C_1)^{1/2} > 0$ such that $\|Df^{i_r}(\mathbf{v})\| \leq C\chi^r\|Df^{i_0}(\mathbf{v})\|$ for $\mathbf{0} \neq \mathbf{v} \in S_P$. Then $\text{length} f^{i_r}(W_\beta^s) \leq C\chi^r \text{length} f^{i_0}(W_\beta^s(P)) \leq C\chi^r \text{length} W_\beta^s(f^{i_0}(P))$. The lemma follows defining

$$\kappa = C \max_{P \in M}\{\text{length} W_\beta^s(P)\}$$

□

Let $\beta > 0$ be smaller than γ of the remark 3.1 and also smaller than one half the distance between D_2 and D_1^c. Let α as in remark 3.1 and such that $f^{-1}[P,Q] = [f^{-1}(P), f^{-1}(Q)]$ if $\operatorname{dist}(P,Q) \leq \alpha$.

Proposition 3.12. *In the hypothesis of Theorem 1, there exists a constant H such that for any $P, Q \in M$ and any natural number $k > 0$ with* $\operatorname{dist}(f^{-j}(P), f^{-j}(Q)) \leq \alpha$ *for $j = 0, 1, \ldots, k$ then*

$$\frac{1}{H} \leq \prod_{j=1}^{k} \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(Q))} \leq H$$

Proof. It is enough to prove this proposition when $\lambda = 1$ and $a_{10} = 0$. In the other case, $\lambda = -1$ with any a_{10}, we shall consider f^2 instead of f, reducing the problem to the first case.

We denote $S = [P,Q]$. Let $J = \{1 \leq j \leq k : \; f^{-j}(P) \in D_1, f^{-j}(Q) \in D_1, f^{-j}(S) \in D_1\}$; $K = \{1 \leq j \leq k : \; j \notin J\}$. Observe that

$$\operatorname{dist}(f^{-j}(P), f^{-j}(S)) < \beta$$

and

$$\mathrm{dist}\,(f^{-j}(S), f^{-j}(Q)) < \beta$$

for $j = 0, 1, \ldots, k$. The choice of β implies that for $j \in K$ the stable arc between $f^{-j}(P)$ and $f^{-j}(S)$ and the unstable arc between $f^{-j}(S)$ and $f^{-j}(Q)$ are contained in D_2^c.

$$\left|\log \prod_{j=1}^{k} \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(Q))}\right| \leq \left|\sum_{j\in J} \log \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(Q))}\right| + \left|\sum_{j\in K} \log \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(Q))}\right| \leq$$

$$\leq \left|\sum_{j\in J} \log \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(S))}\right| + \left|\sum_{j\in J} \log \frac{\mathcal{J}(f^{-j}(S))}{\mathcal{J}(f^{-j}(Q))}\right| + \tag{3}$$

$$+ \left|\sum_{j\in K} \log \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(S))}\right| + \left|\sum_{j\in K} \log \frac{\mathcal{J}(f^{-j}(S))}{\mathcal{J}(f^{-j}(Q))}\right| \tag{4}$$

Let us write J as the disjoint union of l (with l minimum) subsets of consecutive naturals, each one corresponding to each passage through D_1, that is $0 = k_0 < j_1 \leq k_1 < j_2 \leq k_2 \ldots < j_m \leq k_m < \ldots < j_l \leq k_l < j_{l+1} = k+1$ such that $J = \cup_{m=1}^{l}\{j \in I\!N : \; j_m \leq j \leq k_m\}$ and $K = \cup_{m=0}^{l}\{j \in I\!N : \; k_m < j < j_{m+1}\}$.

First, consider the iterates corresponding to J. They are within D_1; applying proposition 3.3:

$$\left|\sum_{j\in J} \log \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(S))}\right| = \left|\sum_{m=1}^{l} \sum_{j=j_m}^{k_m} \log \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(S))}\right| \leq C \sum_{m=1}^{l} d_m^{1/3}$$

where

$$d_m \max\{\, \mathrm{dist}\,(f^{-j_m}(P), f^{-j_m}(S)),\;\; \mathrm{dist}\,(f^{-k_m}(P), f^{-k_m}(S))\}$$

After the lemma 3.11

$$\sum_{m=1}^{l} d_m^{1/3} \leq \beta^{1/3} + \sum_{m=1}^{l-1} \left(\kappa\chi^{l-m}\right)^{1/3} \leq \beta^{1/3} + \frac{\kappa^{1/3}\chi^{1/3}}{1-\chi^{1/3}}$$

Then

$$\left|\sum_{j\in J} \log \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(S))}\right| \leq C \tag{5}$$

and, analogously

$$\left|\sum_{j\in J} \log \frac{\mathcal{J}(f^{-j}(S))}{\mathcal{J}(f^{-j}(Q))}\right| \leq C \tag{6}$$

Let us now consider the iterates $i \in K$, these iterates are in D_2^c. Being $\mathcal{J}$ of class C^1, after the lemma 3.11:

$$\left|\sum_{j\in K} \log \frac{\mathcal{J}(f^{-j}(P))}{\mathcal{J}(f^{-j}(S))}\right| \leq C \sum_{j\in K} \mathrm{dist}\,(f^{-j}(P), f^{-j}(S)) \leq C\frac{\kappa}{1-\chi} \tag{7}$$

Analogously,

$$\left| \sum_{j\in K} \log \frac{\mathcal{J}(f^{-j}(S))}{\mathcal{J}(f^{-j}(Q))} \right| \le C \tag{8}$$

The formulas (3), (4), (5), (6), (7) y (8) prove the proposition. □

4. Construction of the measure.

Let N be a rectangle as in 3.3. Let $U^{(0)}$ be a rectangle as in definition 3.2 such that $U^{(0)} \bigcap N = \emptyset$, we also suppose that for any $S, S' \in U^{(0)}$, if $S' \in W^s_\beta(S)$, then $S' \in W^s_\alpha(S)$ with α and β as in the considerations before proposition 3.12. Let $U^{(k)} = f^k(U^{(0)})$. We consider the usual Lebesgue measure ν in M, it is not restriction to suppose $\nu(U^{(0)}) = 1$. We define the sequence ν_k of measures in the borelians of M such that $\nu_k(A) = \nu(f^{-k}(U^{(k)} \bigcap A)) = \nu(U^{(0)} \bigcap f^{-k}(A))$. Then, we define $\mu_n(A) = (1/n)\sum_{k=1}^n \nu_k(A)$. Let μ_{n_j} be a convergent subsequence in the weak* topology, let μ be its limit. We will prove that an ergodic component of this invariant probability measure verifies the thesis of Theorem 1.

For nearby points $S, S_0 \in M$ we denote $S_W = [S_0, S]$. Let us denote $\operatorname{dist}_u(S, S_W)$ the distance between S and S_W measured on unstable manifolds, and $\operatorname{dist}_s(S_0, S_W)$ the distance between S_0 and S_W measured on stable manifolds. For any $S_0 \in M$ and small t, let us denote $R_t(S_0)$ or simply $R(S_0)$ (if there is not confusion) the set $\{S \in M;\ \operatorname{dist}_u(S, S_W) \le t, \operatorname{dist}_s(S_0, S_W) \le t\}$. For fixed S_0 and t, let us denote $W = \{S_W \in W^s_\beta(S_0) :\ \operatorname{dist}(S_0, S_W) \le t\}$.

The unstable foliation in $R(S_0)$ determines a measurable partition on $R(S_0)$ (see [34]) and therefore, the conditional measures on the elements of the partition are well defined.

Let us denote $\partial^u R(S_0)$ the two arcs of boundary of $R(S_0)$ on unstable manifolds; let us denote $\partial^s U^{(k)}$ the arcs of the boundary of $U^{(k)}$ on stable manifolds. We define $A^{(k)} = U^{(k)} \bigcap R(S_0)$, let $B^{(k)}$ be the (possibly empty) union of the connected components of $A^{(k)}$ which intersect $\partial^s U^{(k)}$ or $\partial^u R(S_0)$, finally, let $C^{(k)} = A^{(k)} \backslash B^{(k)}$.

We denote $C^{(k)}_h$ with $1 \le h \le i_k$ each connected component of $C^{(k)}_h$. We consider the arc $W \bigcap C^{(k)}_h$ and denote its extremes as $S^{(k)}_h$ and $S^{(k)}_{h1}$.

Lemma 4.1. *There exists a sequence of measures σ^j en W, $\sigma^j(W) \le 1$ such that for any continuous function $g : R(S_0) \mapsto \mathbb{R}^+$ supported in the interior of $R(S_0)$, it is verified*

$$\int_{R(S_0)} g(S)\, d\mu(S) = \lim_{j\to\infty} \int_W d\sigma^j(S^{(k)}_h) \int_{C^{(k)}_h} \frac{g(S)}{N(S^{(k)}_h)} \prod_{m=1}^{k} \frac{\mathcal{J}(f^{-m}(S^{(k)}_h))}{\mathcal{J}(f^{-m}(S))}\, d\nu(S)$$

where σ^j is concentrated at $S^{(k)}_h$, $1 \le k \le n_j$, $1 \le h \le i_k$, $\mathcal{J}$ is the Jacobian of f, and

$$N(S^{(k)}_h) = \int_{C^{(k)}_h} \prod_{m=1}^{k} \frac{\mathcal{J}(f^{-m}(S^{(k)}_h))}{\mathcal{J}(f^{-m}(S))}\, d\nu(S)$$

Proof. We can write

$$\int_{U^{(k)} \bigcap R(S_0)} g(S)\, d\nu_k(S) = \sum_{h=1}^{i_k} \int_{C^{(k)}_h} g(S)\, d\nu_k(S) + \int_{B^{(k)}} g(S)\, d\nu_k(S) = I^{(k)}_1 + I^{(k)}_2$$

Now, $|I_2^{(k)}| \leq \max(g)\nu_k(B^{(k)}) = C\nu(f^{-k}(B^{(k)}))$. This term converges to 0 when k goes to ∞ because the length of unstable manifolds in $f^{-k}(B^{(k)}) \subset U^{(0)} \bigcap N^c$ goes uniformly to 0 when $k \to \infty$ (cf. lemma 3.11). On the other hand,

$$I_1^{(k)} = \sum_{h=1}^{i_k} \int_{C_h^{(k)}} g(S)\, d\nu_k(S) = \sum_{h=1}^{i_k} \int_{f^{-k}(C_h^{(k)})} g(f^k(S'))\, d\nu(S') =$$

$$= \sum_{h=1}^{i_k} \int_{C_h^{(k)}} \frac{g(S)}{\prod_{m=1}^{k} \mathcal{J}(f^{-m}(S))}\, d\nu(S)$$

where the second equality is due to the definition of ν_k, and the third one to the change of variable $S = f^k(S')$. Therefore,

$$\int_{R(S_0)} g(S)\, d\mu(S) =$$

$$= \lim_{j\to\infty} \frac{1}{n_j} \sum_{k=1}^{n_j} \sum_{h=1}^{i_k} \frac{N(S_h^{(k)})}{\prod_{m=1}^{k} \mathcal{J}(f^{-m}(S_h^{(k)}))} \int_{C_h^{(k)}} \frac{g(S)}{N(S_h^{(k)})} \prod_{m=1}^{k} \frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S))}\, d\nu(S)$$

We define a sequence of measures σ^j on W, (not necessarily probabilities measures) concentrated at $S_h^{(k)}$, with $1 \leq k \leq n_j$, $1 \leq h \leq i_k$, so that if B is a borelian in W, then

$$\sigma^j(B) = \frac{1}{n_j} \sum_{k=1}^{n_j} \sum_{h=1}^{i_k} \frac{N(S_h^{(k)})}{\prod_{m=1}^{k} \mathcal{J}(f^{-m}(S_h^{(k)}))} \delta_h^{(k)}(B)$$

where $\delta_h^{(k)}(B)$ is equal 1 if B contains $S_h^{(k)}$ and 0 in other case. We can write

(9)

$$\int_{R(S_0)} g(S)\, d\mu(S) = \lim_{j\to\infty} \int_W d\sigma^j(S_h^{(k)}) \int_{C_h^{(k)}} \frac{g(S)}{N(S_h^{(k)})} \prod_{m=1}^{k} \frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S))}\, d\nu(S)$$

To prove that $\sigma^j(W) \leq 1$, we observe that

$$1 \geq \mu_{n_j}(R(S_0)) = \frac{1}{n_j} \sum_{k=1}^{n_j} \left[\left(\sum_{h=1}^{i_k} \nu_k(C_h^{(k)}) \right) + \nu_k(B^{(k)}) \right] \geq$$

$$\geq \frac{1}{n_j} \sum_{k=1}^{n_j} \sum_{h=1}^{i_k} \int_{C_h^{(k)}} \frac{d\nu(S)}{\prod_{m=1}^{k} \mathcal{J}(f^{-m}(S))} \geq \frac{1}{n_j} \sum_{k=1}^{n_j} \sum_{h=1}^{i_k} \frac{N(S_h^{(k)})}{\prod_{m=1}^{k} \mathcal{J}(f^{-m}(S_h^{(k)}))} = \sigma^j(W)$$

$\square$

Let us denote $\mathcal{J}^u(P) = \|Df\mathbf{v}\|/\|\mathbf{v}\|$ for $\mathbf{0} \neq \mathbf{v} \in T_P W^u$. Observe that $\mathcal{J}^u(P)$ depends continuously on P_1 due to corollary 2.5. Moreover, $\mathcal{J}^u(P) \to 1$ when $P \to P_0$.

Definition 4.2. We define dynamical ball of p iterates and radius ε centered at S as

$$B_p(S, \varepsilon) = \{Q \in M : \ \operatorname{dist}(f^i(S), f^i(Q)) \leq \varepsilon : \ 0 \leq i \leq p\}$$

Lemma 4.3. *For all $0 < A < 1$ there exist real numbers $C = C(A) > 0$ and $\varepsilon_0 = \varepsilon_0(A) > 0$ such that for all $S_0 \in M$ there exists an increasing sequence of natural numbers $\{p_i\}_{i \in Z^+}$, $p_i = p_i(S_0, A)$ such that*

$$\mu(B_{p_i}(S_0, \varepsilon)) \leq \frac{C}{A^{2(p_i - p_1)} \prod_{m=p_1}^{p_i - 1} \mathcal{J}^u(f^m(S_0))}$$

for all $i > 1$ and all $0 < \varepsilon \leq \varepsilon_0$.

Proof. Given A we determine the value of t defined at the beginning of this section. Let us take $t > 0$ such that for any $S_0 \in M$, if $S \in R_t(S_0)$, then $\operatorname{dist}(S_0, S) < \alpha/2$ (where α is as in the proposition 3.12) and if $S_W = [S_0, S]$, then $S_W \in W^u_\alpha(S)$. Also, given A, we will take $t > 0$ such that $A < \mathcal{J}^u(P)/\mathcal{J}^u(Q) < 1/A$ for all S_0 and for any two points P and Q in $R_t(S_0)$. Let $\varepsilon_0 > 0$ be such that $B_0(S_0, \varepsilon_0)$ is in the interior of $R(S_0)$ for any $S_0 \in M$. Let us observe that for $S_0 \in W^s(P_0)$ the lemma follows immediately because $\mathcal{J}^u(f^m(S_0))$ goes to 1 when m goes to ∞. Then, we fix $S_0 \notin W^s(P_0)$. Let us take $p_i(S_0, A)$, $i = 1, 2, \dots$ so that $R(f^{p_i}(S_0)) \subset N^c$. For i fixed, let $\beta : M \mapsto [0, 1]$ be a continuous bump function supported in $\bigcap_{l=1}^i f^{-p_l+p_1} R(f^{p_l}(S_0))$ such that $\beta(S) = 1$ if $S \in B_{p_i - p_1}(f^{p_1}(S_0), \varepsilon)$. Then

$$\mu(B_{p_i}(S_0, \varepsilon)) \leq \mu(f^{-p_1}(B_{p_i-p_1}(f^{p_1}(S_0), \varepsilon))) =$$

$$= \mu(B_{p_i-p_1}(f^{p_1}(S_0), \varepsilon)) \leq \int_{R(f^{p_1}(S_0))} \beta(S)\, d\mu(S)$$

We now apply the equality (9); only for simplicity in the notation, we will work as if $p_1 = 0$, that is, we will denote W the local stable manifold through $f^{p_1}(S_0)$, σ^j the measure in such manifold, etc, but the reasoning does not depend on this. Then $\mu(B_{p_i}(S_0, \varepsilon)) \leq$

(10)

$$\leq \limsup_{j\to\infty} \int_W d\sigma^j(S_h^{(k)}) \int_{C_h^{(k)}} \frac{\prod_{l=1}^i \chi_{f^{-p_l+p_1}(R(f^{p_l}(S_0)))}(S)}{N(S_h^{(k)})} \prod_{m=1}^k \frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S))}\, d\nu(S)$$

The unstable foliation of f is not C^1, but it is continuous; its leaves are C^1 and the tangent space at P of the unstable leaf through P is $U_P = \{\mathbf{v} \in T_P M : B(Df^m \mathbf{v}) \geq 0\, \forall m \geq 0\}$ (we recall corollary 2.5). For $n \in I\!N$, let us consider the quadratic form $f^{n^\#}B$, and let us integrate the continuous direction such that $f^{n^\#}B = 0$ and that at P_0 does not coincide with U_{P_0}. Thus, we obtain a C^3 foliation Φ, whose generic local leaf will be denoted φ. We choose n (and therefore Φ) so that for all $P \in M$, $T_P\Phi$ and U_P are sufficiently near to obtain $A < \mathcal{J}_\varphi^{(0)}(P)/\mathcal{J}^u(P) < 1/A$, where $\mathcal{J}_\varphi^{(i)}(P) = \|Df\mathbf{v}\|/\|\mathbf{v}\|$ for $\mathbf{0} \neq \mathbf{v} \in Df^i T_{f^{-i}(P)}\Phi$. Let us observe that the angle between $Df^i T_{f^{-i}(P)}\Phi$ and U_P is decreasing with i, so that we can choose n such that $A < \mathcal{J}_\varphi^{(i)}(P)/\mathcal{J}^u(P) < 1/A$ for all $i \geq 0$ and all $P \in M$.

Fixed k and h, we will partitionate $C_h^{(k)}$ so that S and S' in $C_h^{(k)}$ are in the same atom of the partition if either $S' \in \varphi(S)$ or if there exists a finite sequence P_1, $P_2, \dots P_{2r}$ in $W^u_\beta(S_h^{(k)}) \bigcup W^u_\beta(S_{h1}^{(k)})$ so that $P_1 \in \varphi(S)$, $P_2 \in W^s_\beta(P_1), \dots, P_{2i+1} \in \varphi(P_{2i})$, $P_{2i+2} \in W^s_\beta(P_{2i+1}), \dots, S' \in \varphi(P_{2r})$. (The points P_j are not necessarily in $C_h^{(k)}$.) We observe that the leaves of Φ intersect transversally the stable and unstable leaves of f. Let us denote $\mathcal{P}$ this partition, let us denote π a generic atom. As Φ is C^3 it can be C^1 trivialized by a system of a finite number of local charts defined in open sets $E_1, \dots, E_q, \dots, E_s$ covering the manifold M. It follows that we can decompose the Lebesgue area $\nu(B)$ of any borelian set $B \subset C_h^{(k)}$ as $\nu(B) = \int_{C_h^{(k)}/\mathcal{P}} d\rho(\pi) \int_{B\cap\pi} g_q(S) d\nu_\pi(S)$ where ρ is a measure in the quotient space $C_h^{(k)}/\mathcal{P}$ (it depends on $R(S_0)$), ν_π is the riemannian length on π and g_q is a continuous positive density function defined in E_q for some $1 \leq q \leq s$.

The last term of (10) is equal to

$$\limsup_{j\to\infty}\int_W d\sigma^j(S_h^{(k)})\int_{C_h^{(k)}/\mathcal{P}}\frac{I d\rho(\pi)}{N(S_h^{(k)})} \tag{11}$$

where

$$I=\int_\pi\left(\prod_{l=1}^{i}\chi_{f^{-p_l+p_1}(R(f^{p_l}(S_0)))}(S)\right)\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S))}\right)g_q(S)\,d\nu_\pi(S)$$

Now denoting $\pi^{(i)}=\pi\bigcap\bigcap_{l=1}^{i}f^{-p_l+p_1}(R(f^{p_l}(S_0)))$ and changing variables $S'=f^{p_i-p_1}(S)$ it follows

$$I=\int_{f^{p_i-p_1}(\pi^{(i)})}\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m-p_i+p_1}(S'))}\right)\frac{g_q(f^{-p_i+p_1}(S'))d\nu_{f^{p_i-p_1}(\pi)}(S')}{\prod_{m=1}^{p_i-p_1}\mathcal{J}_\varphi^{(p_i-p_1-m)}(f^{-m}(S'))}\leq$$

$$\leq\int_{f^{p_i-p_1}(\pi^{(i)})}\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m-p_i+p_1}(S'))}\right)\frac{g_q(f^{-p_i+p_1}(S'))\,d\nu_{f^{p_i-p_1}(\pi)}(S')}{A^{2(p_i-p_1)}\prod_{m=p_1}^{p_i-1}\mathcal{J}^u(f^m(S_0))} \tag{12}$$

We now claim that there exists a real C independent of $S_0\in M$ (but that may depend on A) such that

$$\int_{f^{p_i-p_1}(\pi^{(i)})}\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m-p_i+p_1}(S'))}\right)g_q(f^{-p_i+p_1}(S'))\,d\nu_{f^{p_i-p_1}(\pi)}(S')\leq$$

$$\leq C\int_\pi\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S))}\right)g_q(S)\,d\nu_\pi(S) \tag{13}$$

The set of functions g_q is bounded by a certain number K, and bounded away from 0 by, say $1/K$. We observe that $\operatorname{dist}(f^{-m}(S),f^{-m}(S_W))\leq\alpha$ for all $m\geq 0$ (by the election of t); similarly, $\operatorname{dist}(f^{-m}(S_W),f^{-m}(S_h^{(k)}))\leq\alpha$ for all $m\leq k$ (by the construction of U_0). Then, writing

$$\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S))}\right)=\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S_W))}\right)\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_W))}{\mathcal{J}(f^{-m}(S))}\right)$$

and applying the proposition 3.12

$$\frac{\displaystyle\int_{f^{p_i-p_1}(\pi^{(i)})}\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m-p_i+p_1}(S'))}\right)g_q(f^{-p_i+p_1}(S'))\,d\nu_{f^{p_i-p_1}(\pi)}(S')}{\displaystyle\int_\pi\left(\prod_{m=1}^{k}\frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S))}\right)g_q(S)\,d\nu_\pi(S)}\leq$$

$$\leq H^4K^2\frac{\displaystyle\int_{f^{p_i-p_1}(\pi^{(i)})}d\nu_{f^{p_i-p_1}(\pi)}(S')}{\displaystyle\int_\pi d\nu_\pi(S)}$$

To prove the claim 13 it is enough to show that there exist positive constants L_1 and L_2 (independent of $f^{p_1}(S_0) \in N^c$ but that depend on t and thus on A) such that

$$\int_\pi d\,\nu_\pi(S) > L_1; \quad \int_{f^{p_i-p_1}(\pi^{(i)})} d\nu_{f^{p_i-p_1}(\pi)}(S') < L_2$$

To prove this we will integrate the continuous field of directions such that $B=0$ and such that at P_0 coincides with U_{P_0}, obtaining the foliation Φ^*. Analogously, we will denote $\tilde{\Phi}$ the foliation obtained after integrating the other field of null directions of B. The unstable foliation and the foliations Φ^*, $\tilde{\Phi}$ and Φ are pairwise transversal in N_0 forming angles bounded away from 0. We project the arcs that form π following the leaves of the foliation Φ^* on a local leaf of $\tilde{\Phi}$. The projections overlap, then L_1 exists.

Similarly, to prove that there exists $L_2 > 0$ we project the arcs that form $f^{p_i-p_1}(\pi^{(i)})$ on a local leaf of Φ^* following the leaves of the foliation $\tilde{\Phi}$, and observe that the projections do not overlap.

Denoting $C = H^4K^2L_2/L_1$ the claim (13) is proved.

Collecting (10), (11), (12) and (13) we have

$$A^{2(p_i-p_1)} \prod_{m=p_1}^{p_i-1} \mathcal{J}^u(f^m(S_0))\mu(B_{p_i}(S_0,\varepsilon)) \leq$$

$$\leq C \limsup_{j\to\infty} \int_W d\sigma^j(S_h^{(k)}) \int_{C_h^{(k)}/\mathcal{P}} d\rho(\pi) \int_\pi \prod_{m=1}^{k} \frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S))} \frac{g_q(S)\,d\nu_\pi(S)}{N(S_h^{(k)})} =$$

$$= C \limsup_{j\to\infty} \int_W d\sigma^j(S_h^{(k)}) \int_{C_h^{(k)}} \prod_{m=1}^{k} \frac{\mathcal{J}(f^{-m}(S_h^{(k)}))}{\mathcal{J}(f^{-m}(S))} \frac{d\nu(S)}{N(S_h^{(k)})} =$$

$$= C \limsup_{j\to\infty} \mu_{n_j}(R(f^{p_1}(S_0))) \leq C$$

$\square$

To continue the proof of the theorem 1, we adapt the proof of the proposition 5.1 in [16]. Recalling the Brin-Katok definition of the entropy (see [4]) we have:

$$h_\mu(f) = \int_M \lim_{\varepsilon\to 0}\left(\limsup_{p\to\infty}\frac{1}{p}\log[\mu(B_p^s(S_0,\varepsilon))]^{-1}\right) d\mu(S_0) \geq$$

$$\geq \int_M \lim_{\varepsilon\to 0}\left(\log A^2 + \limsup_{i\to\infty}\frac{1}{p_i(S_0,A)}\log\prod_{m=p_1(S_0,A)}^{p_i(S_0,A)-1}\mathcal{J}^u(f^m(S_0))\right) d\mu(S_0)$$

For S_0 a regular point; $\mathbf{0} \neq \mathbf{v} \in T_{S_0}(W^u(S_0))$ and p_1 fixed we have:

$$\lim_{p\to\infty}\frac{1}{p}\log\prod_{m=p_1}^{p-1}\mathcal{J}^u(f^m(S_0)) =$$

$$= \lim_{p\to\infty}\frac{1}{p}\left(\log\|Df^p(S_0)\mathbf{v}\| - \log\|Df^{p_1}(S_0)\mathbf{v}\|\right) = \chi^+(S_0)$$

where $\chi^+(S_0)$ is positive or 0. As A can be taken arbitrarily close to 1 and the set of regular points has μ measure equal to 1:

$$h_\mu(f) \geq \int_M \chi^+(S_0)\,d\mu(S_0)$$

After the inequality of Ruelle (see [36]) it follows

$$h_\mu(f) = \int_M \chi^+(S_0)\, d\mu(S_0)$$

We now recall the following theorem:

Theorem 4.4 ([23]). *μ has absolutely continuous conditional measures along strong unstable manifolds if and only if*

$$h_\mu(f) = \int_M \sum_{i:\chi_i(S_0)>0} \chi_i(S_0) \dim E_i(S_0)\, d\mu(S_0)$$

where $h_\mu(f)$ is the metric entropy of f.

Proof. See [23]. □

In fact, it is shown there that the conditional measures on strong unstables manifolds are equivalent to the riemannian measure on $W^{uu}(S_0)$ for μ almost every point S_0.

The former theorem implies that μ has absolutely continuous conditional measures on unstable manifolds. We have not proved that it is necessarily ergodic, so we begin by the following lemma.

Lemma 4.5. $\mu(\{P_0\}) < 1$

Proof. By contradiction, let us suppose $\mu(\{P_0\}) = 1$. Then for n_j large enough and $t > 0$ small, $\mu_{n_j}(R_{t/2}(P_0))$ is as near of 1 as wanted. This implies that $\mu_{n_j}(R_t(P_0) \setminus R_{t/2}(P_0))$ would be as near of 0 as wanted. We denote $B = R_t(P_0) \setminus R_{t/2}(P_0)$. Applying lemma 4.1 and proposition 3.12 we obtain

$$\mu_{n_j}(R_t(P_0)) \leq 2 \int_W d\sigma^j, \qquad \mu_{n_j}(B) \geq \frac{1}{K^2} \int_W \frac{\nu(C_h^{(k)} \cap B)}{\nu(C_h^{(k)})}\, d\sigma^j$$

To obtain the contradiction it is enough to prove that $\nu(C_h^{(k)} \cap B)/\nu(C_h^{(k)})$ is bounded away from zero. In fact, we decompose the Lebesgue measure ν as in the proof of lemma 4.3, along a measurable partition $\mathcal{P}$ of $C_h^{(k)}$, whose atoms π are local leaves of a C^3 foliation ϕ (that is transversal to stable and unstable leaves). We then have:

$$\nu(C_h^{(k)} \cap B) = \int_{C_h^{(k)}/\mathcal{P}} d\rho(\pi) \int_{B\cap\pi} g(S)\, d\nu_\pi(S)$$

$$\nu(C_h^{(k)}) = \int_{C_h^{(k)}/\mathcal{P}} d\rho(\pi) \int_\pi g(S)\, d\nu_\pi(S)$$

where g is a continuous positive function. To end the proof it is enough to show that $\nu_\pi(\pi \cap B)/\nu_\pi(\pi)$ is bounded away from zero. As in the proof of lemma 4.3 we have a positive constant L_1 such that $\nu_\pi(B \cap \pi) > L_1$, because B excludes a neighborhood of P_0, and so the angle between stable and unstable leaves in B is bounded away from zero. We have also a positive constant L_2 such that $\nu_\pi(\pi) < L_2$. (Note that we do not need uniform transversality between stable and unstable leaves to obtain the bound L_2)

□

If $\mu(\{P_0\}) \neq 0$, we define a new measure of any borelian A as

$$\mu(A \setminus \{P_0\})/\mu(\{P_0\}^c)$$

For simplicity we will continue denoting μ to this new measure. It has absolutely continuous conditional measures on unstable manifolds.

The Pesin region Σ has μ-measure 1 after Theorem 1.12. Then, recalling Theorem 1.13 we conclude the existence of a countable union of ergodic attractors, each one corresponding to one ergodic component of μ. After the construction of basin of attraction of the ergodic attractors in [33] (saturation of positive Lebesgue measure sets) and taking into account the density of stable manifolds, it follows that the basin of attraction of each ergodic attractor has total Lebesgue measure (see also subsection 4.3 in [9]). Then, there is a unique ergodic attractor. The same argument proves that the Lyapounov exponents of Lebesgue-almost all regular points are different of zero, because these points are in the strong stable manifolds of points in the attractor. Finally, the theorem 5.10 of [20], asserts that f is Bernoulli, thus ending the proof of the theorem 1.

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