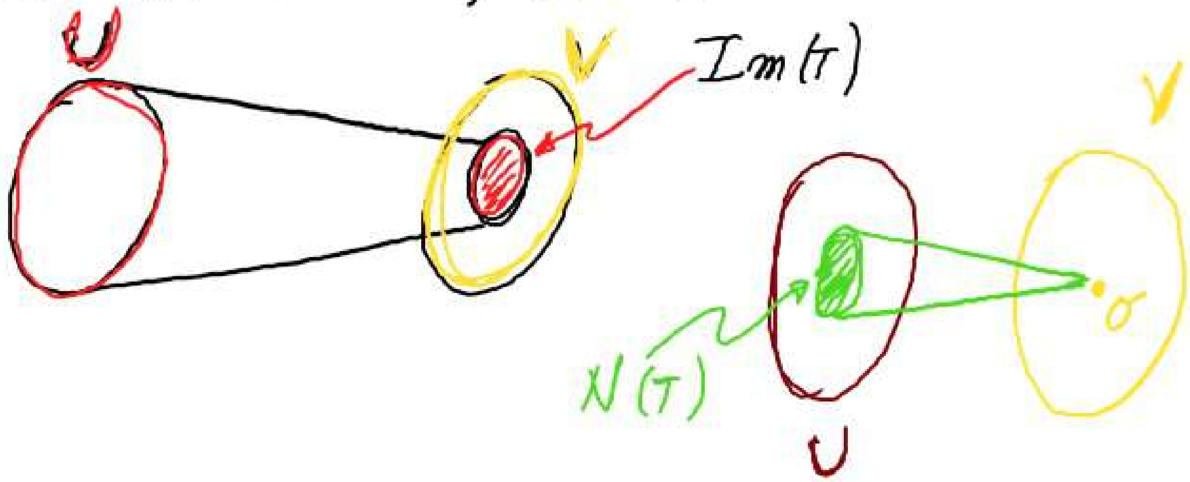
Recordar: Sea $T: U \to V$ t.l

a) $N(T) = \{ u \in U / Tu = 0 \}$

b) $Im(T) = \{ v \in V / \exists\, u, Tu = v \}$



Proposición. $N(T)$ e $Im(T)$ son subespacios vectoriales.

dem: $N(T)$

a) $0_U \in N(T)$ porque $T 0_U = 0_V$

$\longrightarrow N(T) \neq \phi$

b) Sean $u, v \in N(T)$

Queremos probar que: $\alpha u + \beta v \in N(T)$

$\alpha u + \beta v \in N(T) \longleftrightarrow T(\alpha u + \beta v) = 0$

$T(\alpha u + \beta v) = \alpha Tu + \beta Tv = \alpha . 0 + \beta 0 = 0$

Im(T)

a) $T(0_U) = 0_V \Rightarrow 0_V \in Im(T) \Rightarrow Im(T) \neq \emptyset$

b) Sean $u, v \in Im(T)$. Queremos probar que

$\alpha u + \beta v \in Im(T) \leftrightarrow \exists y / Ty = \alpha u + \beta v$

Como $u \in Im(T) \Rightarrow \exists x_1 / Tx_1 = u$

Como $v \in Im(T) \Rightarrow \exists x_2 / Tx_2 = v$

Considero $\alpha x_1 + \beta x_2$

$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha u + \beta v$

$\Rightarrow \alpha u + \beta v \in Im(T)$.

Observación: Sea $T: U \to V$ t. l

Supongamos que $A = \{v_1, .., v_n\} \xrightarrow{g} U$

Si $y \in Im(T) \Rightarrow \exists x \in U / Tx = y$

Como $\left.\begin{array}{l} A \xrightarrow{g} U \\ x \in U \end{array}\right\} \to x = \alpha_1 v_1 + \cdots + \alpha_n v_n$

$\Rightarrow y = Tx = T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 T v_1 + \cdots + \alpha_n T v_n$

$\boxed{\text{Entonces } \{Tv_1, .., Tv_n\} \xrightarrow{g} Im(T)}$

Ejemplos:

1) Sea $T: \mathbb{R}^3 \to \mathbb{R}^4 \,/\, T(a,b,c) = (a, b+a, a+c, 2c)$

$N(T)$: $\begin{cases} a = 0 \\ a+b = 0 \\ a+c = 0 \\ 2c = 0 \end{cases} \longrightarrow a = 0 = b = c$

$\Rightarrow T(T) = \{(0,0,0)\}$

$Im(T)$ $A = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = (1,1,1,0)$

$T(0,1,0) = (0,1,0,0)$

$T(0,0,1) = (0,0,1,2)$

$\{(1,1,1,0), (0,1,0,0), (0,0,1,2)\} \xrightarrow{g} Im(T)$

$(a, b+a, a+c, 2c) = a(1,1,1,0)$

$+ b(0,1,0,0) + c(0,0,1,2)$

2) Sea $T: \mathbb{P}_2(x) \to \mathbb{P}_2(x)$ tal que

$T(p) = p'$

$T(ax^2+bx+c) = 2ax+b$

$\underline{N(T)}$ $\quad 2ax+b = 0 \Rightarrow \begin{matrix} 2a=0 \to a=0 \\ b=0 \end{matrix}$

$N(T)$ están los polinomios constantes.

$\{1\} \xrightarrow{b} N(T)$

$\underline{Im(T)}$ $\quad A = \{x^2, x, 1\}$ $\quad \boxed{\dim \mathbb{P}_2(x) = 3}$

$Tx^2 = 2x \quad Tx = 1 \quad T.1 = 0$

$\{2x, 1\} \xrightarrow{g} Im(T)$

$2ax+b = a(\underline{2x}) + b.\underline{1}$

3) Sea $T: M_{3\times3}(\mathbb{R}) \to M_{3\times3}(\mathbb{R}) \,/\, TA = A^t + A$

$$T\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2a_1 & b_1+a_2 & c_1+a_3 \\ b_1+a_2 & 2b_2 & c_2+b_3 \\ a_3+c_1 & b_3+c_2 & 2c_3 \end{pmatrix}$$

5

$$N(T) \to \begin{pmatrix} 2a_1 & b_1+a_2 & c_1+a_3 \\ b_1+a_2 & 2b_2 & c_2+b_3 \\ a_3+c_1 & b_3+c_2 & 2c_3 \end{pmatrix} = 0$$

$2a_1=0 \to a_1=0 \qquad b_1+a_2=0 \to b_1=-a_2$

$2b_2=0 \to b_2=0 \qquad c_1+a_3=0 \qquad c_1=-a_3$

$2\ c_3=0 \to c_3=0 \qquad c_2+b_3=0 \qquad c_2=-b_3$

$$\begin{pmatrix} 0 & a_2 & a_3 \\ -a_2 & 0 & -c_2 \\ -a_3 & c_2 & 0 \end{pmatrix} = a_2\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} +$$

$$a_3\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow \dim N(T) \leq 3$. En este caso

$\dim N(T) = 3$

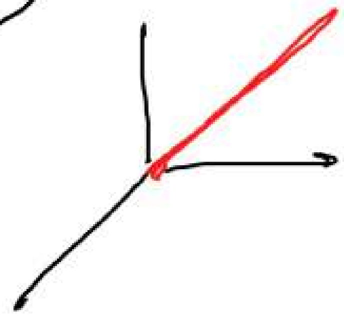
$$Im(T) \Rightarrow a_1\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b_2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} + b_1\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$c_2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + a_2\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_3\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + b_3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$c_1\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

③ $T: \mathbb{R}^3 \to \mathbb{R}^3 \,/\, T(a,b,c) = (a+b,\, b+c,\, a-c)$

$$N(T) \to \begin{cases} a+b=0 \\ b+c=0 \\ a-c=0 \end{cases} \to \begin{matrix} a=-b \\ c=-b \end{matrix}$$



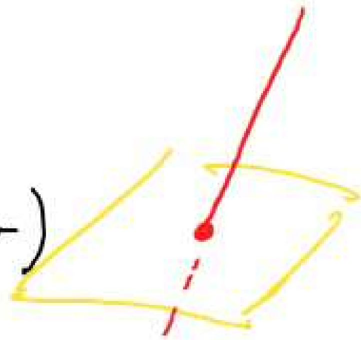
$$(-b, b, -b) = b(-1, 1, -1)$$

$$Im(T) \quad (a+b,\, b+c,\, a-c) =$$

$$= a(1,0,1) + b(1,1,0) + c(0,1,-1)$$

$$\{(1,0,1),\ (1,1,0),\ (0,1,-1)\}$$

$$\{(1,0,1),\ (1,1,0)\} \xrightarrow{\text{base}} Im(T)$$



¿ $N(T) \subset Im(T)$?

$$(-1,1,-1) = \alpha(1,0,1) + \beta(1,1,0)$$

$$\begin{cases} \alpha+\beta = -1 \\ \beta = 1 \\ \alpha = -1 \end{cases} \quad S.I \Rightarrow \underline{N(T) \not\subset Im(T)}$$