Who controls the controller? A dynamical model of corruption

Abstract

The aim of this paper is to give at least a partial answer to the question made in the title. Several works analyze the evolution of the corruption in different societies. Most of such papers show the necessity of several controls displayed by a central authority to deterrence the expansion of the corruption. However there is not much literature that addresses the issue of who controls the controller. This article aims to approach an answer to this question. Indeed, as it is well known. in democratic societies an important role should be played by citizens. We show that politically active citizens can prevent the spread of corruption.

1 Introduction

In February 2014, the European Union published its first ever anti-corruption report. Over 41 pages, it concluded that bribery, tax evasion, cronyism, embezzlement, political fraud, and the like, cost the European economy 120 billion euros at year, just short of the EU annual budget. Corruption is costly, but it deprives citizens of more than money. There is a lot of empirical and theoretical evidence showing that high and rising corruption increases income inequality and poverty.

In this paper we conclude that citizens are key in the fight against corruption, because in a democratic country they have the possibility to exert pressure demanding the government to combat this scourge,

There is a profuse economic literature related with the topic of administrative and political corruption. Pioneering works in the area are [Rose-Ackerman, S. (a)] and [Rose Ackerman, S. (b)]. A basic insight that emerges from many studies is the self-enforcing nature of corruption: in an environment where corruption is the norm, corruption tends to persist and to be imitated, see for instance [Lui, Francis T. (a))] [Lui, Francis T. (b)]; [Sah, R.]. In recent works the evolution of the corruption in a given society is modelled using the evolutionary game theory. Even when initially individuals choice their strategies independently, after some time, they compare the obtained payoffs and copy the apparently more profitable strategy. Under this evolutionary approach and under given social conditions, corruption, can become a dominant strategy. See for instance [Accinelli, E.; Carrera, E. (a)], [Accinelli, E.;Carrera,E. (b)]. In much of this literature the conditions under which the public officials are willing to be corrupted are analysed. These officials must ensure compliance with the law, payment of taxes by citizens, compliance with rules aimed at preventing pollution and annoying sounds, etc.. But often, officials themselves are willing to accept bribes from citizens who do not want to be punished for breaching the rules of coexistence. See for instance [Accinelli, E.; Carrera, E.; Policardo, L.]. The increasing of official corruption, in turn, creates incentives for the development of the corrupt behaviour and in this way the society as a whole becomes corrupt. The question about how to avoid the evolution of corruption is not easy to be answered however of the great importance.

On the other hand, in many specialized papers it is considered that the central authority, the government and or central agencies, should play an important role to deterrence and to control the evolution of corruption. The government is considered as a benevolent planner trying to maximize the social welfare. But many times, individuals who are members of these central agencies (political elites) can benefit by the evolution of corruption among officials. In such cases these agents act maximizing their own selfish interest rather than benevolent agents maximizing the social welfare. Models of this behavior are considered for instance in [Becker, G.], and [Grossman, G. M. and Helpman, E], but the question that remains unanswered is: who and how controls the controller?

An interesting discussion on this point is introduced in [Hurwicz, L.][1]. The cited work, takes up the question raised by the Roman author Juvenal, which suggests that there is no way to control the behavior of

[1] This reference is brought to bear with the aim of showing that the question that inspires our work has been the object of human concern since at least classical antiquity. Obviously the social paradigms of that time were different from the current ones.

the wives. To keeping them under guard is not a solution, because guardians are neither reliable. For similar considerations a finite succession of guardians of guards does not seem be a solution. So, Juvenals suggests that the problem to control wives has not solution, because there is not way to control the guardians. The aim of this paper is to give at least a partial answer to this question without recourse to an endless succession of guardians of different levels. In our paper, the role of wives is played by officials, the set of guardians of first level corresponds to the government and citizens are the second level of guardians. We argue that an infinite cycle of guardians is not necessary to control the officials when citizens are voters of an elective government.

We do not give an exact definition of corruption[2], moreover we consider only one of its forms of expression, the abuse of the officials, more interested in their own profits, rather than fulfilling their duties. We will show that the persistence of this behavior in a democratic country depends on the degree of intolerance of citizens with respect to this behavior. To measure this degree of intolerance we introduce an index. This index is function of the percentage of corrupt officials existing in each time, and is a measure of the probabilities that a government will be re-elected. We focuses on the dynamics of corruption, and we analyze how certain patterns of behavior may evolve, and give the conditions under which the stationary points become stable. In particular, we want to show how — if the parameters of the model are exogenous — a sudden change in the evolution of corruption might occur as consequence of changes in the intolerance index.

The rest of the work is organized as follows: In the next section we introduce a formal model of a process that involves, official citizens and government. To analyse the evolution of the corruption we consider a normal form game with three players. In section (3) we consider the corruption as a self reinforcement mechanism. In section (4) we consider a dynamical system to explain the evolution of the corruption in a society. In section (5) we analyse the rs between dynamical and Nash equilibria. The analysis of the stability of the equilibria are given in section (6). In section (7) we analyse with some detail the role of the index of intolerance of corruption by citizens. In the last section we present some conclusions.

2 The model

Consider an economy or society, where the central authority is elected by universal suffrage of citizens. By central authority or national government, we understand the president and his political sector. The members of the government can be reelected or not after each period of government through universal suffrage. The president and members of his political sector, in turn, appoint public officials who may or may not be renewed by the new government. These officials are in charge of carrying out the legal and administrative management of the government and serve directly to the citizens when they require to carry out this type of formalities before the central authority These officials must choose between two different behaviors namely, properly fulfilling its role or, when her participation is required by a citizen, he fulfils his duty as long as the citizen pay for it a certain amount of money.

We call an honest or non-corrupt official the one that chooses to unconditionally fulfill its functions, otherwise we call the official a dishonest or corrupt official. Sometimes, a dishonest official is colluded with a member of the central authority and both take advantage for this behavior. Several examples of this kind of collusion are considered in [Thompsom D.] and [Lessig, L.].

In general, corruption can be defined as the misuse of public power for private benefit. For instance, government official collect bribes for providing permits, licences, passage through costumers, or avoiding the entrance to competitors in a given market. Such behavior may give room to an increase of the dishonest behavior in the whole society.

[2]Different ways of defining corruption and its limitations are considered in [Jain, A. K.])

Following [Shleifer, A. and Vishny, R.], we define the governmental corruption as the complicity of the government (the political elite) with officials that sell government property for personal gain.

But even when some members of the government can be attracted to acting in collusion with dishonest officials, it is necessary to consider that the government is interested in being re-elected for the next period, and they know that this happens only if citizens are satisfied with the performance of the government. Citizens will judge the performance of the central authority through the work of officials who deal directly with them. We assume as mandatory, that at the end of every period each citizen must vote to re-elect or not the government. Citizens prefer a non corrupt government, but they do not have complete information about the behavior of the government. They know this information only in an indirect way, and only if they have taking contact with some official.

If the current government is not re-elected then a new government takes the place of the former.

We summarize the activity of the government saying that it must choose between to follow a corrupt behavior or a non-corrupt behavior, meaning to act in complicity with corrupt officials, or alternatively, punishing them.

The model can be formalized as a normal form game with three different populations. The sets of pure strategies are as follows:

1. Officials must choose between two pure strategies: to be corrupt or not, respectively symbolized by O_c and O_{nc}, so that we have $\Gamma_O = \{O_c, O_{nc}\}$.

2. The central authority or government must choose in the set of pure strategies $\Gamma_G = \{G_c, G_{nc}\}$. A corrupt policy (meaning to collude with corrupt officials) is symbolized by G_c while an honest or non-corrupt policy is denoted by G_{nc}. This represents to the behavior of the political elites.

3. Citizens must choose between to re-elect the government or not. We assume that every citizen prefers a non-corrupt government to a corrupt one, however they do not have perfect information about the governmental corruption. They perceive the corruption only through the behavior of the official ones with which they are related. So, even when citizens prefer a non corrupt government to a corrupt one, some of them can vote by mistake, for the re-election of a corrupt government or vote against the re-election of a non-corrupt one. This possibilities will be considered when we define the index of intolerance to corruption.

The payoffs for officials and government are represented in the following two tables. Table 1 corresponds to the case in which the government was reelected, while Table 2 corresponds to the case in which the government was not reelected. The difference is in the profits obtained by the government, if a corrupt government was reelected, then he obtain the additional profit V_{G_c} and in the case of being non-corrupt (and reelected) the amount of this profit is V_{Gnc}. The government does not obtain this profits if it is not reelected.

	G/O	G_c	G_{nc}
$C_R \rightarrow$	O_c	$W + M_c - M_g,\ M_g - W + V_{G_c}$	$W + M_c - M,\ M - W - e + V_{G_{nc}}$
	O_{nc}	$W - M'_g,\ M'_g - W + V_{G_c}$	$W,\ -W + V_{G_{nc}}$

(1)

	G/O	G_c	G_{nc}
$C_{NR} \rightarrow$	O_c	$W + M_c - M_g,\ M_g - W$	$W + M_c - M,\ M - W - e$
	O_{nc}	$W - M'_g,\ M'_g - W$	$W,\ -W$

Where:

- By W we symbolize the wage of the officials which is paid by the government.

- M is the fine imposed by an honest government to a dishonest official.
- M_c corresponds to the bribe that a dishonest official takes from a citizen when his participation is required.
- θ is the percentage over the bribe that a dishonest official may pay to a corrupt government to keep his position. $M_g = \theta M_c$ is the amount that the dishonest official must pay to his partner in the government.
- M'_g is the amount that an honest official must pay to a dishonest government to keep his position. or because they do not want to be punished for breaching the rules of coexistence.
- e is the cost associated with the capture of a corrupt official. We assume that this cost It is a measure of the governmental efficiency in fight against corruption
- V_{G_c} and $V_{G_{nc}}$ correspond respectively to the value that a corrupt government and a non-corrupt government assign to be re-elected for the next period.

Government must choose between two possible behaviors or pure strategies: to be corrupt, or to be honest (or non-corrupt). Analogously, officials must choose between being corrupt and being honest. Citizens must choose between to re-elect or not the current government.

3 Corruption as a self-reinforcing mechanism

The von Neumann-Morgenstern utility theorem shows that, under certain axioms of rational behavior, a decision-maker faced with risky outcomes of different choices will behave as if he is maximizing the expected values of some function (the von Neumann-Morgenstern utility function) defined over the potential outcomes at some specified point in the future. We will follow this point of view to describe the behavior of the agents involved in our model. We assume that the values of the utility function associated with each choice, are the potential profits in each state of the world.

The total payoff of a dishonest government corresponds to

$$\Pi(G_c)(t) = N_c(t)\theta M_c + N_{nc}(t)M'_g - NW + R_{G_c} \tag{2}$$

and the total payoff of a honest government corresponds to

$$\Pi(G_{nc})(t) = -NW + (M - e)N_c(t) + R_{G_{nc}} \tag{3}$$

where

- $N_c(t)$ is the quantity of corrupt officials in time t, $N_{nc}(t)$ the quantity of honest officials in time t, and $N = N_c + N_{nc}$. N is fixed, but the distribution of officials can change along the time.
- R_{G_c} and $R_{G_{nc}}$ are the expected profits by governments in case of re-election, i.e., $R_{G_c} = V_{G_c} q_{G_c}$ and analogously for a non-corrupt government $R_{G_{nc}} = V_{G_{nc}} q_{G_{nc}}$, where q_{G_c} and $q_{G_{nc}}$ are respectively, the probabilities that a corrupt and a non-corrupt government get re-elected.

The expected profit of a dishonest official is given by

$$E(O_c) = (W + M_c - \theta M_c)P(G_c) + (W + M_c - M)P(G_{nc}) \tag{4}$$

The expected profit of an honest official is given by

$$E(O_{nc}) = (W - M'_g)P(G_c) + WP(G_{nc}). \tag{5}$$

Let $P(G_c)$ tbe the probability that the government follows a corrupt policy, as we shall see later, this probability is determined endogenously. Note that $P(G_{nc}) = 1 - P(G_c)$ is the probability that the government follows a non-corrupt policy.

Given that we assume a rational behavior of the different agents involved, it follows that, the quantity of dishonest official increases if and only if $E(O_c) > E(O_{nc})$ i.e., if and only if:

$$(W + M_c - \theta M_c')P(G_c) + (W + M_c - M)(1 - P(G_c)) > (W - M_g')P(G_c) + WP(G_{nc}). \tag{6}$$

After some algebra we obtain the following statements: $E(O_c) > E(O_{nc})$ if and only if:

$$P(G_c) > \frac{M - M_c}{M - \theta M_c + M_g'} \tag{7}$$

and $\Pi(G_c) > \Pi(G_{nc})$ if and only if

$$N_c > \frac{(R_{G_{nc}} - R_{G_c}) - NM_g'}{\theta M_c - M_g' - M + e}. \tag{8}$$

The next proposition summarizes these facts

Proposition 1 *Officials prefer to choose a dishonest behavior if and only if the government corruption is large enough, and reciprocally a high number of corrupt officials encourage governmental corruption.*

Remark 1 *Note that if the fines are relatively low relative to what a corrupt officer can obtain as an illegal payment for his services, i.e: if* $1 \leq \frac{M_c}{M}$ *then, even in the case where the government is non-corrupt, a corrupt conduct is more profitabe to officials. This situation would raise the number of corrupt officials and consequently, the government would prefer over time to be corrupt. More precisely, this will happen as soon as the inequality (8) is verified.*

A general conclusion can be obtained from proposition (1): corruption corrupts. More explicitly, this proposition says that corruption is a self-reinforcing mechanism. The question now is how to break down this process. The answer is in the degree of intolerance of citizens.

Definition 1 (The Index of Intolerance to Corruption) *Let* $q_{G_{nc}}$ *be the probability that a corrupt government is re-elected given that the percentage of corrupt officials is* n_c *and let* q_{G_c} *be the probability that a non-corrupt government is re-elected. We define the index of intolerance to corruption by the difference:*

$$D_{it} = q_{G_{nc}} - q_{G_c}. \tag{9}$$

This index captures the social sensibility to the corruption. Note that:

$$\begin{aligned} R_{G_{nc}} - R_{G_c} &= V_{G_{nc}} q_{G_{nc}} - V_{G_c} q_{G_c} = \\ &= \left(V_{G_{nc}} - V_{G_c}\right) q_{G_{nc}} - V_{G_c}\left(q_{G_c} - q_{G_{nc}}\right) = \\ &= \left(V_{G_{nc}} - V_{G_c}\right) q_{G_{nc}} + V_{G_c} D_{it}. \end{aligned} \tag{10}$$

Because corruption is wilfully hidden, it is not easy to measure directly [Seligson, M.]. There have been many attempts to solve this problem but they have all came up with limitations, see for instance [Campbell, S. V.] and [Mauro, P.]. However, we consider that citizens perceive the degree of corruption of government through the services that officials provide. Consequently, the indignation that corrupt services cause among citizens can help to stop corruption. If citizens are sufficiently intolerant with the bad services provided by corrupt officials, then, according to (8), it becomes more unlikely that there are enough corrupt officials so that governments prefer to be corrupt, so that the government loses incentives to tolerate or to allow corruption. Insofar as that the degree of tolerance of citizens for the services of corrupt officials decreases or, equivalently, insofar the degree of intolerance for corrupt services increases, the government prefers to punish corrupt officials.

However, note that the strategy "to be corrupt" can be a dominant strategy for the government if its efficiency to capture corrupt officials is low, or equivalently the cost to catch the corrupt officials is high, i.e., if $e > M - M_g = M - \theta M_c$[3], or if the degree of intolerance D_{it} of citizens to corruption is not high enough. The cost to catch the corrupt is higher in those countries where the effectiveness of the legal system is low. We are in presence of a negative cycle were, an inefficient legal system becomes a cause and a consequence of corruption. In this cases to revert the process big changes in the parameters of the models are necessary, but this change are exogenous to the model.

Let us analyse the social evolution of corruption by means of the replicator dynamics.

4 The evolution of corruption

To explain the social evolution of corruption, we shall follows an evolutionary approach. This approach is based on the fact that strategies that makes a person do better than others will be retained, while strategies that lead to failure will be abandoned. The success of a strategy is measured by its relative frequency in the population at any given time. Strategies change over time as a function of their relative success in an environment that is made up of other players that keep changing their own strategies adaptively.

Initially people decide their strategies independently. We assume that individuals in every time try to improve his welfare and that they follows a myopic behavior. In addition we consider that officials do not know with absolute accuracy the likelihood that the government act corruptly, neither the government knows exactly the percentage of corrupt officials.

Periodically they compare the obtained returns and after some time, some of them updated their strategic choices, switching for the, apparently, most profitable strategies. So, in each period, the percentage of individuals that follows a given strategy increases if the payoff of such strategy is greater than the average payoff obtained by the population. Along the time, more profitable strategies become the most widely used. In addition we consider that, depending on the prevailing social conditions in each period, the strategy that offers the best return can change.

The dynamical summarizing these facts is the replicator dynamics, see [Weibull, W. J.].

According with this considerations we e assume that, the players of our model try, in each period, to improve his welfare however however they follow a kind of myopic behavior, because officials do not know, with absolute accuracy, the likelihood that the government act corruptly, neither the government knows exactly the percentage of corrupt officials, existing in each period.

Let $N_c(t) + N_{nc}(t) = N$, be the total amount of officials. $N_c(t)$ is the number of corrupt official in time t and $N_{nc}(t)$ the amount of not corrupt officials in time t. The amount of officials following one or another strategy may change, but, we assume that the total amount of officials is constant and equal to N.

We denote by $n_i(t) = \frac{N_i(t)}{N}$ the percentage of corrupt officials following the strategy $i \in \{O_c, O_{nc}\}$. By $n(t) = (n_c(t), n_{nc}(t))$ we symbolize the distribution of the officials over the set of pure strategies, in each time t, by $g(t) = (g_c(t), g_{nc}(t))$ a mixed strategy of the government in time t.

Thus the replicator dynamics becomes the dynamical side of our model. Ar the end of every period, players compare the obtained payoff, and even when they do not choice its strategies following an absolutely

[3]We assume that this cost is funded by sanctions that a non-corrupt government obtains from fines to corrupt officials. Certainly if this cost exceeds this amount, the government will have to appeal to other sources to perform this task. This point is not considered in this work.

rational behavior, the most successful strategy ends up by prevail in the society. According with this approach, the growth rate of corrupt officials is given by the following differential equation.

$$\dot{n}_c = n_c[E(O_c|g, n) - \bar{E}]$$

where $\bar{E} = n_c E(O_c|n, g) + n_{nc} E(O_{nc}|n, g)$ is the expected payoff of the officials and $E(O_c|n, g)$ and $E(O_{nc}|n, g)$ denote, respectively, the expected value of a corrupt behavior and a non-corrupt behavior by an official given a distribution g over the government behavior and a distribution n of the officials over its available pure strategies. Analogously for the percentage of individuals following the honest behavior, where for all t, $n_{nc}(t) = 1 - n_c(t)$. After some algebra we obtain the equivalent dynamical system:

$$\begin{aligned} \dot{n}_c &= n_c(1 - n_c)(E(O_c|n, g) - E(O_{nc}|n, g)). \\ \dot{n}_{nc} &= -\dot{n}_c. \end{aligned} \tag{11}$$

By $\dot{n}_i$ we represent the derivative with respect to the time of the percentage of official following the strategy i. All these variables are time depending, but to simplify we do not write the variable t.

To measure the evolution of the governmental corruption we introduce g_c as an index[4] measuring the percentage of corrupt acts committed in public offices regarding the total of acts performed in these government agencies in each time t.

To endogenize the probability of a government being corrupt we will consider the index g_c that represents the percentage of corrupt acts made by a government on the total acts of government performed. Then, in a similar way, we obtain that the evolution of the government policy can be represented by the following dynamical system.

$$\begin{aligned} \dot{g}_c &= g_c(1 - g_c)(\Pi(G_c|n, D_{it}) - \Pi(G_{nc}|n, D_{it})). \\ \dot{g}_{nc} &= -\dot{g}_c. \end{aligned} \tag{12}$$

where $\dot{g}_i = \frac{dg_i}{dt}$ represents the derivative with respect to time of the probability $g_i(t)$ that the government follows strategy i.

Note that the expected value of each possible behavior of a government depends on the current distribution of the official and on the degree of intolerance of citizens to the corruption of the officials.

This dynamical system with four equations can be summarized in the following system with only two differential equations:

$$\begin{aligned} \dot{n}_c &= n_c(1 - n_c)(E(O_c|n, g) - E(O_{nc}|n, g)) \\ \dot{g}_c &= g_c(1 - g_c)(\Pi(G_c|n, D_{it}) - \Pi(G_{nc}|n, D_{it})) \end{aligned} \tag{13}$$

Using equalities (2), (3), (4) and (5), after some algebra we obtain:

$$\begin{aligned} \dot{n}_c &= n_c(1 - n_c)\left[(M_c(1 - \theta) - M_c + M + M'_g)P(G_c) + M_c - M\right] \\ \dot{g}_c &= g_c(1 - g_c)\left[Nn_c\left(\theta M_c - M'_g - M + e\right) + NM'_g + R_{G_c} - R_{G_{nc}}\right] \end{aligned} \tag{14}$$

To simplify the writing we can consider

$$\begin{aligned} A &= -\theta M_c + M + M'_g, & B &= M_c - M \\ A' &= N(\theta M_c - M'_g - M + e), & B' &= NM'_g + R_{G_c} - R_{G_{nc}} \end{aligned} \tag{15}$$

[4]The dominant approach to measuring corruption has been Transparency International's Corruption Perceptions Index (CPI). The CPI captures information about the administrative and political aspects of corruption.

then the dynamical system (14) takes de form:

$$\dot{n}_c = n_c(1-n_c)\,(Ag_c + B)$$
$$\dot{g}_c = g_c(1-g_c)\,(A'n_c + B')\,. \tag{16}$$

5 Dynamic equilibrium and Nash equilibria

To advance in the analysis let us consider the index of intolerance as a given characteristic of the society. In section (7) we shall leave aside this restriction on the index of intolerance and we shall consider this index as a function of the relative proportion of corrupt officials.

The dynamical system (13) has five equilibria or steady states. If A and A' are not equal to zero, then the point (n_c^T, g_c^T) is an steady state, where

$$\bar{n}_c^T = -\frac{B'}{A'} = \frac{(R_{G_{nc}} - R_{G_c}) - M'_g}{N(\theta M_c - M'_g - M + e)} \quad \text{and} \quad \bar{g}_c^T = -\frac{B}{A} = \frac{M - M_c}{M - \theta M_c + M'_g}$$

Note that, in our framework, this equilibrium makes sense if and only if

$$0 \leq \frac{(R_{G_{nc}} - R_{G_c}) - NM'_g}{N(\theta M_c - M'_g - M + e)} \leq 1 \;\text{ and }\; 0 \leq \bar{g}_c^T = \frac{M - M_c}{M - \theta M_c + M'_g} \leq 1$$

are verified.
Under these conditions this steady state is at the same time a mixed Nash equilibria for the game where

$$E(O_c) = E(O_{nc}) \text{ and } E(G_c) = E(G_{nc})$$

where $E(G_c) = \Pi(G_c)$ is the governmental expected payoff corresponding to follow a corrupt behavior or strategy, and $E(G_{nc}) = \Pi(G_{nc})$ is the governmental expected payoff that corresponds to follows an honest, or non-corrupt, strategy.

The dynamical system has four other equilibria:

$$n^1 = (1,0), \quad g^1 = (1,0) \to (n_c^1, g_c^1) = (1,1)$$

$$n^2 = (1,0), \quad g^2 = (0,1) \to (n_c^2, g_c^2) = (1,0)$$

$$n^3 = (0,1), \quad g^3 = (1,0) \to (n_c^3, g_c^3) = (0,1)$$

$$n^4 = (0,1), \quad g^4 = (0,1) \to (n_c^4, g_c^4) = (0,0)$$

These four dynamic equilibria correspond to pure strategies of the game. Depending on the value of parameters, these points may or may not correspond to Nash equilibria for the subgame played by officials and government (1).

- Case 1 is NE $\Leftrightarrow E(O_c) \geq E(O_{nc})$ and $E(G_c) \geq E(G_{nc})$
- Case 2 is NE $\Leftrightarrow E(O_c) \geq E(O_{nc})$ and $E(G_c) \leq E(G_{nc})$
- Case 3 is NE $\Leftrightarrow E(O_c) \leq E(O_{nc})$ and $E(G_c) \geq E(G_{nc})$
- Case 4 is NE $\Leftrightarrow E(O_c) \leq E(O_{nc})$ and $E(G_c) \leq E(G_{nc})$

6 Stability of equilibria

The Hartman-Grobman theorem states that the orbit structure of a dynamical system in a neighbourhood of a hyperbolic equilibrium point is topologically equivalent to the orbit structure of the linearized dynamical system.

Assuming that A and A' are nonzero, then the point $(n_c^T, g_c^T) = (-\frac{B'}{A'}, -\frac{B}{A})$ is an steady state for the dynamical system. The linearization at this point is given by the matrix:

$$J\left(-\frac{B'}{A'}, -\frac{B}{A}\right) = \begin{bmatrix} 0 & -\frac{A'B}{A^2}(B+A) \\ -\frac{AB'}{A'^2}(B'+A') & 0 \end{bmatrix}$$

The eigenvalues of this matrix are:

$$\lambda = \pm\sqrt{\frac{B'B}{AA'}(B'+A')(B+A)}.$$

Thus, if $\frac{B'B}{AA'}(B'+A')(B+A) > 0$ then this point is a saddle point for the dynamics. In other cases the Hartman-Grobman theorem is not conclusive, because the matrix J has eigenvalues with zero real parts, meaning that the point is not hyperbolic.

For the case 1 above, i.e., the 'bad' equilibrium $(n_c^B, g_c^B) = (1,1)$ corresponding to a fully corrupt society where all officials are corrupt and government always acts in a corrupt way, the matrix corresponding to the linearization is:

$$J(1,1) = \begin{bmatrix} -(A+B) & 0 \\ 0 & -(A'+B') \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -(A+B)$ and $\lambda_2 = -(A'+B')$.

The case 2 above $(n_c^2, g_c^2) = (1,0)$ corresponds to a situation where all officials are corrupt but the government always acts is a honest way. The linearization is

$$J(1,0) = \begin{bmatrix} -B & A' \\ 0 & A'+B' \end{bmatrix}$$

The eigenvalues are $\lambda_1 = B$ and $\lambda_2 = -(A'+B')$.

The case 3 above $(n_c^3, g_c^3) = (0,1)$ corresponds to a situation where the government always acts in a corrupt way, but officials prefer to be honest. The linearization is

$$J(0,1) = \begin{bmatrix} A+B & A' \\ 0 & -B' \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -(A+B)$ and $\lambda_2 = B'$.

For the case 4 above i.e., the 'good' equilibrium without corruption, $(n_c^4, g_c^4) = (0,0)$, we obtain the linearization:

$$J(0,0) = \begin{bmatrix} B & 0 \\ 0 & B' \end{bmatrix}$$

This matrix has two real eigenvalues $\lambda_1 = B$ and $\lambda_2 = B'$ and then this equilibrium is asymptotically stable if and only in $B < 0$ and $B' < 0$.

6.1 The transition paths: Possible cases

For each time t we say that the pair $(n_c(t), g_c(t))$ defines the state of corruption of the society in time t. Thus, given the dynamical system (16) and an initial condition in time $t = t_0$ (i.e., an initial state of corruption), $(n_c(t_0), g_c(t_0)) = (n_{c_0}, g_{c_0})$, we say that $\xi(\cdot, (n_{c_0}, g_{c_0})) \to \Re^2$ is a solution of the dynamical system with such initial condition if and only if $\xi(t, (n_{c_0}, g_{c_0}))$ verifies the system (16), and $\xi(t_0, (n_{c_0}, g_{c_0})) = (n_{c_0}, g_{c_0})$. It can be proved that once an initial condition is fixed, there is an unique solution for the dynamical system. Also, it can be proved that the function $\xi(t, \cdot) : \Re^2 \to \Re^2$ is continuous. i.e., the solution of the dynamical system (16) is continuous with respect to initial conditions, see for instance [Hirsh, M.; Smale S.; Devanay, R.].

Definition 2 (The trajectory of corruption) *Given the dynamical system (16) and an initial condition in time $t = t_0$, we define the trajectory of the corruption, as the set $\Gamma \subset \Re^2$ given by:*

$$\Gamma = \{(n_c(t), g_c(t)) = \xi(t, (n_c(t_0), g_c(t_0))), \ \forall\, t \geq t_0\}.$$

Note that each trajectory defines a set of possible future states of corruption, i.e., for each initial condition, there is only one set of possible future states. So, the corruption in a given society, once the initial condition is fixed, evolves along a trajectory.

Definition 3 (The transition path*) Given the dynamical system (16) and an initial condition the set of possible states for all $t > t_0$ will be called the transition path.*

This transition path is given by the set of possible states of corruption, from a fixed initial time $t = t_0$ until the system rests in a dynamical equilibrium.

To analyze the possible evolution of the corruption, i.e, the possible transition paths, in a given society, let us begin considering the cases where the Hartman-Grobman theorem is conclusive.

(1) Let $B < 0, \ \ B' < 0$. In this case the good equilibrium $(n_c^4, g_c^4)) = (0, 0)$ is always asymptotically stable. When $A > -B, \ A' > -B'$ then $A + B > 0$ and $A' + B' > 0$. From these conditions the following inequalities are verified: $A > 0, A' > 0, \ \ 0 < -\frac{B}{A} < 1$, and $0 < -\frac{B'}{A'} < 1$, implying the existence of a mixed equilibrium in the interior of the unit square. In this case we also have that $\frac{B'B}{AA'}(B'+A')(B+A) > 0$, so the Hartman-Grobman theorem can be applied to the mixed equilibrium. In this case the 'bad' equilibrium and the 'good' equilibrium without corruption are asymptotically stable, and the mixed equilibrium is a saddle point. See Figure (1) for the general picture of the dynamics in this case.

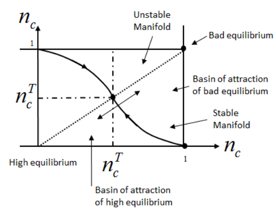

Figure 1: The dynamics of the system, with the basin of attraction of the equilibria and the mixed Nash equilibrium.

This is a good example of ongoing spontaneous coordination. Note that this case corresponds to a social situation where:

1. The amount M of the fine imposed by a non-corrupt government to a corrupt official is relatively high, meaning that it is greater than the bribe M_c the official takes from citizens, i.e., $M > M_c$.
2. The inequality $(1-\theta)M_c > -M'_g$ is verified. Recall that $(1-\theta)M_c$ is the net amount that a corrupt official retains of the bribe M_c (that he imposes to citizens), after the payment that he must to do to a dishonest government, and M'_g is the amount that an honest official must pay to a dishonest government to keep his place.
3. The government is inefficient to catch corrupts officials, or equivalently, e is relatively high (relatively high costs to combat corruption).
4. The index of intolerance is high enough, and the government is interested in being re-elected meaning that the inequality $V_{G_c}D_{it} > NM'_g + (V_{G_{nc}} - V_{G_c})q_{G_{nc}}$ is verified.

In this case corruption can be regarded as a social trap. If the initial distributions of officers and government actions correspond with a point in the basin of attraction of bad equilibrium, then officials and government have incentives to act in a corrupt way. Thus, the general levels of corruption will increase, and corruption becomes a self-enforcing mechanism over time.

However, this situation can change if the degree of intolerance of citizens increases. If the government believes that this change in intolerance can take place then (depending also on the value that the government assigns to be re-elected), it may result in a change in the basin of attractions of the good and bad equilibria, making the path remains from that time within the basin of attraction of the equilibrium without corruption. This possibility is supported in the following fact:

Remark 2 *The basin of attraction of the bad equilibrium* $(n_c^B, g_c^B) = (1,1)$ *decreases when the degree of intolerance increases.*

Thus, the index of intolerance of citizens with respect to corruption, if high enough, and if the government is interested in being re-elected can play an important role at the time to control the controller acting as a servomechanism correcting the evolution of corruption. It acts as a barrier stopping corruption, since, under several circumstances, it can reverse a process of growing corruption. The higher it is, the more difficult it gets that corruptions grows and develops within the government. In Figure (**??**) we plot some trajectories of the system that exemplify the previous remark. For the same initial conditions with different model parameters, corresponding to an increase in the degree of intolerance, we see that initial conditions originally in the basin of attraction of the bad equilibrium are instead converging to the good equilibrium. This illustrates the shrinking of the basin of attraction of the bad equilibrium as the degree of intolerance grows.

(2) Assuming that $B < 0,\ B' < 0,\ \ A > -B$ and $A' < -B'$ it follows that $(A+B) > 0,\ (A'+B') < 0$ then there is not a mixed Nash equilibrium because either $-\frac{B'}{A'} > 1$ or $-\frac{B'}{A'} < 0$. The bad equilibrium $(n_c^1, g_c^1) = (1,1)$ is a saddle point, as well as the equilibrium $(n_c^2, g_c^2) = (1,0)$, and the equilibrium $(n_c^3, g_c^3) = (0,1)$ is a repulsor. In this case there is a unique asymptotically stable dynamic equilibrium and this is the Nash equilibrium without corruption, i.e., $(n_c^4, g_c^4) = (0,0)$. See Figure (2).

(3) Assuming that $B < 0,\ B' < 0,\ A < -B$ and $A' < -B'$ it follows that $(A+B) < 0,\ (A'+B') < 0$ then the bad equilibrium is a repulsor, there is no mixed equilibrium, and there is a unique equilibrium that is asymptotically stable, that is the 'good' equilibrium $(n_c^4, g_c^4) = (0,0)$, with all interior initial conditions being attracted to this point. The equilibria $(n_c^2, g_c^2) = (1,0)$ and $(n_c^3, g_c^3) = (0,1)$ are saddle points.

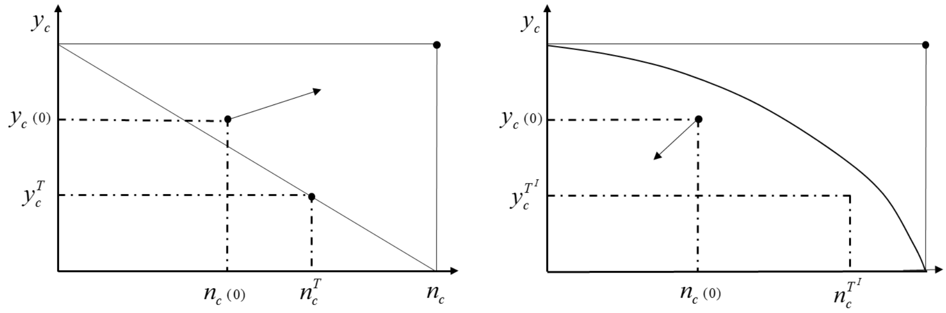

Figure 2: Some trajectories of the system for the same initial conditions with different parameters. Left-hand side: lower degree of intolerance. Right-hand side: higher degree of intolerance.

(4) Assuming that $B < 0,\ B' < 0,\ A < -B$ and $A' > -B'$ it follows that $(A + B) < 0,\ (A' + B') > 0$. Then, there is no mixed equilibrium and the 'bad' equilibrium $(n_c^1, g_c^1) = (1, 1)$ is a saddle point as well as the equilibrium $(n_c^3, g_c^3) = (0, 1)$, and the equilibrium $(n_c^2, g_c^2) = (1, 0)$ is a repulsor. The only asymptotically stable equilibrium is the 'good' equilibrium $(n_c^4; g_c^4) = (0, 0)$, with all interior initial conditions being attracted to this point.

Note that the cases (2), (3) and (4) are mathematically similar, but from a social point of view they are very different, because in our model, if $0 < \theta < 1$ ¡ 1 the condition $A + B = M'_g + (1 - \theta)M_c$ can not be negative for any possible values of M'_g and M_c given that these are, in each possible case, non negative numbers, so that cases (3) and (4) don't occur. As we shall see there are other two apparently paradoxical cases. See cases (5) and (7) below.

Remark 3 *It is possible to consider the case where $\theta > 1$ but, this could happen if for example, a corrupt government was charging corrupt officials more than bribes they overcharged citizens. In this case the government retains a percentage of the salaries of officials, or equivalently officials are paying the government a portion of their salaries to keep their jobs..*

In case (2), the most probably of the last three considered, there are no mixed equilibria. The assumptions are describing a political situation corresponding with a governments applying heavy fines on corrupt officials at the same time the index of intolerance is relatively high or the governmental elite has a high interest in to be re-elected. The government is highly efficient in fighting corruption, i.e; low values of e. In this cases $E(O_{nc}) > E(O_c)$ and $E(G_{nc}) > E(G_c)$ and the evolution is to a society without corruption.

(5) Assuming that $B > 0,\ B' > 0$ and $A < -B, A' < B'$ then the inequalities $(A + B) < 0,\ (A' + B) < 0$ hold. This is a seemingly paradoxical situation, where both the 'good' non-corruption equilibrium $(n_c^4, g_c^4) = (0, 0)$ and the 'bad' corrupt equilibrium $(n_c^1, g_c^1) = (1, 1)$ are repulsors and the mixed Nash equilibrium is a saddle point. The equilibria $(n_c^2, g_c^2) = (1, 0)$ and $(n_c^3, g_c^3) = (0, 1)$ are attractors, i.e., government prefers to be honest but officials prefer to be corrupt, or reciprocally, government prefers to be corrupt but officials prefer to be honest. Which one of these two situations occurs is initial conditions dependent. Figures (3) and (4) are illustrative of these possibilities. See remark(3).

The following two cases are mathematically very different from the previous one, but with social interpretations that are similar to each one of the possibilities in the previous case.

(6) Assuming that $B > 0,\ B' > 0$ and $A > -B, A' < -B'$ then $A + B > 0$ and $A' + B' < 0$. There is no mixed Nash equilibrium and the equilibrium $(n_c^2, g_c^2) = (1, 0)$ is the only equilibrium point that is

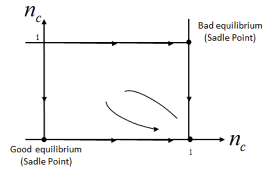

Figure 3: Different trajectories of the system for case (5).

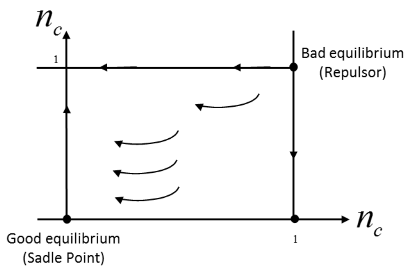

Figure 4: Different trajectories of the system for case (5).

asymptotically stable, and all initial conditions in the interior of the unit square are attracted to this equilibrium. This case is similar to the previous one when the initial conditions were in the basin of attraction of the equilibrium $(n_c^2, g_c^2) = (1, 0)$.

(7) Assuming that $B > 0$, $B' > 0$ and $A < -B$, $A' > -B'$ then $A + B < 0$ and $A' + B' > 0$. There is no mixed Nash equilibrium and the equilibrium $(n_c^3, g_c^3) = (0, 1)$ is the only equilibrium point that is asymptotically stable, and all initial conditions in the interior of the unit square are attracted to this equilibrium. This case is similar to the previous one when the initial conditions were in the basin of attraction of the equilibrium $(n_c^3, g_c^3) = (0, 1)$.

In the case (6) society is evolving to an equilibrium where officials prefer to be corrupt, even with an honest government. Our assumptions imply that governmental fines to punish corrupt behavior are relatively low: note that $B > 0 \Leftrightarrow M_c > M$. Also we have that $A + B > 0$ as we have seen before. These conditions make that to follow a corrupt behavior is a dominant strategy for officials.

The government prefers to be honest if and only if $E(G_{nc}) > E(G_c)$. In our case, the intolerance index is relatively low, so the government has relatively high chances of being re-elected. Also, we have that $N\theta M_c + R_{G_c} < R_{G_{nc}} + NM'_g - Ne$. So, with time, the governmental elite diminishes its own corruption, since this solution is preferable to colluding with corrupt officials.

Summarizing it, we can say that this case corresponds to a socio-political situation where government has a relatively high interest in being re-elected and then prevents the spreading corruption. However, because of being focused on re-election, and because of low inefficiency, for instance because of low fines imposed on corrupt officials, government is unable to diminish corruption, even though it would prefer to be able to do it.

The apparently paradoxical case (7) where the society could be evolving to an equilibrium where government is corrupt but officials prefer to be honest can not take place at least that $\theta > 1$ (because as in case (4) the condition $A + B < 0$ is necessary). See remark (3).

(8) Assuming that $B > 0$, $B' > 0$ and $A > -B$, $A' > -B'$ then $A + B > 0$ and $A' + B' > 0$. In this case

the 'good' equilibrium $(n_c^4, g_c^4) = (0,0)$ is a repulsor and the corner equilibria $(n_c^3, g_c^3) = (0,1)$ and $(n_c^2, g_c^2) = (1,0)$ are saddle points. The only equilibrium point that is asymptotically stable is the "bad" equilibrium $(n_c^1, g_c^1) = (1,1)$, with all interior initial conditions being attracted to this point. See Figure (**??**).

In case (8) society is evolving towards full corruption both on the governmental level and on the officials' level, due to general inefficacy of government, low fines to punish corrupt officials, high costs to capture corrupt officials and because of low intolerance index .A corrupt government would have developed ways to protect the corrupts. This extreme situation is characteristic of dictatorships where the dictator confuses his own interests with national interests. It becomes a cause of several social and economic ills.

6.2 Some particular cases

1. Note that in cases where $A = 0$ or $A' = 0$, the system becomes the usual logistic equation system and we can obtain an analytic solution.

   If $A = A' = 0$ we obtain the analytical solution of the dynamical system.

$$n_c(t) = \frac{n_c(0)e^{Bt}}{(1 - n_c(0)) + n_c(0)e^{Bt}}, \text{ and } g_c(t) = \frac{g_c(0)e^{B't}}{(1 - g(0)) + g_c(0)e^{B't}}.$$

   The evolution depends on the signs of B and B'.

   Recall that $B = M_c - M$ and $B' = NM'_g - D_{it}$ then:

   - $B < 0$ if and only the value M of the fine is high enough, i.e.; in and only if $M > M_c$ and $B' < 0$ if and only if the intolerance index D_{it} is high enough, i.e, if and $D_{it} > NM'_g$ then $g_c(t) \to 0$, and $n_c(t) \to 0$, when $t \to \infty$. If this inequalities are verified then, independently of the initial conditions the society is evolving to a situation where officials and government prefer to follows an honest behavior. In this case $\xi(t, n_{c_0}, g_{c_0}) \to (0,0)$, when $t \to \infty$.
   - If $B > 0$ and $B' > 0$ it follows that $g_c(t) \to 1$, and $n_c(t) \to 1$ when $t \to \infty$. Independently of the initial conditions the economy is evolving to an economy where officials and government prefer to follows a corrupt behavior, $\xi(t, n_{c_0}, g_{c_0}) \to (1,1)$, when $t \to \infty$.
   - If $B > 0$ and $B' < 0$ then $g_c(t) \to 1$, and $n_c(t) \to 0$, so $\xi(t, n_{c_0}, g_{c_0}) \to (0,1)$, when $t \to \infty$.
   - If $B < 0$ and $B' > 0$ then $g_c(t) \to 0$, and $n_c(t) \to 1$, so $\xi(t, n_{c_0}, g_{c_0}) \to (1,0)$, when $t \to \infty$.
   - If $B < 0$ and $B' < 0$ the "good" equilibrium $(n_c^4, g_c^4) = (0,0)$ is asymptotically stable.
   - If$B > 0$ and $B' > 0$ the "bad" equilibrium $(n_c^3, g_c^3) = (0,1)$ is asymptotically stable.
   - Note that the cases where $B < 0$, $B' > 0$ and $B > 0, B' < 0$ the behavior of officials and the government end up being antagonistic

   In all cases, the basin of attraction of the asymptotically stable equilibria is the whole interior of the unit square.

   In the degenerate cases when $B = 0$ or $B' = 0$, we have that, respectively, $n_c(t)$ or $g_c(t)$ is constant.

2. The remaining cases, where B, B' and $(A + B)$ or $(A' + B')$ are equal to zero, should be studied directly from the dynamical system (16) because the Hartman-Grobman theorem is no longer valid.

For instance, if $(A+B)=(A'+B')=0$ then the dynamical system (16) takes the form:

$$\dot{n}_c = n_c(1-n_c)(1-g_c)B$$

$$\dot{g}_c = g_c(1-g_c)(1-n_c)B'$$

(a) The sign of the derivatives $\dot{n}_c$ and $\dot{g}_c$ in the rectangle $[0,1]\times[0,1]$ depends only in the sign of B and B'. In the case where $B<0$ and $B'<0$, the point $(0,0)$, the 'good' equilibrium is the only steady state of this dynamical system that is asymptotically stable, and its basin is the whole interior of the unit square. Recall that $B<0$ if and only if $M>M_c$. This means that fines imposed by the government to corrupt officials are greater than bribes required by corrupt officials to citizens. $B'<0$ means that the index of intolerance to corruption of citizens is relatively high, and so the expected value of a corrupt government to be re-elected is low. Under these conditions the economy is evolving to an equilibrium without corruption.

(b) The contrary happens when $B>0$ and $B'>0$. In this case the society is evolving towards an economy where corruption prevails, where both officials and government prefer to be corrupt. In this case only if the index of intolerance increases enough the course of this evolution could change.

(c) In the situation where $B>0$ and $B'<0$, meaning that fines to corrupt officials are not high enough, but that citizens are relatively intolerant, then the system will evolve towards the equilibrium $(n_c^2, g_c^2)=(1,0)$, meaning that the government will evolve to a non-corrupt behavior while officers will be corrupt. In the remainder case where $B<0$ and $B'>0$, then high fines will cause officers to be honest, while on the other hand, low intolerance, meaning that citizens don't care about having a corrupt government, will cause the government to be corrupt. In this case the systems evolves towards the equilibrium $(n_c^3, g_c^3)=(0,1)$.

In the next section we show that the index of intolerance to corruption plays a central role to stop or reverse a process of growing corruption.

6.3 Complex eigenvalues

Let as consider now the case where $\frac{B'B}{AA'}(B'+A')(B+A)<0$. Note that in this case the Hartman-Grobman's theorem is not applicable, because the eigenvalues of the mixed equilibrium are purely imaginary numbers.

Let us consider the case where $B<0$, $B'>0$ and $A>-B$, $A'<-B'$. These inequalities imply that the bad equilibrium $(n_c^1, g_c^1)=(1,1)$ and the high equilibrium $(n_c^4, g_c^4)=(0,0)$ are saddle points. The corner equilibria $(n_c^2, g_c^2)=(1,0)$ and $(n_c^3, g_c^3)=(0,1)$ are also saddle points. These inequalities imply the existence of a mixed Nash equilibrium in the interior of the unit square, and by the previous formula, the eigenvalues of its linearization are purely imaginary numbers. It corresponds to cycles of growth and decline of corruption. Recall that in this case there are low costs to capture corrupt officials, resulting in high efficiency, and there are high fines to punish corrupt officials, but the intolerance index is low. This interplay between these quantities results in the appearance of periodic orbits, as shown in Figure (**??**). The mixed equilibrium is a focus. The rationale behind this situation is the following. The low index of intolerance causes an increase in government corruption, which in turn causes more officials to prefer to be corrupt. Facing an increasingly bigger number of corrupt officials, government decides to be less corrupt, taking advantage of low costs to capture corrupt officials and high fines, which in turn cause a disincentive for officials to become corrupt, thus increasing the number of honest officials. Hence, the overall levels of corruption in society have declined to the original levels so that the cycle restarts again. A similar situation with the appearance of periodic orbits occurs if $B > 0$, $B' < 0$ and $A < -B$, $A' > -B'$ but, this last case has not social meaning in our model (see the considerations given for cases (3) (4) and (6)).

Periodic orbits appear naturally if the intolerance index is a function of the percentage of corrupt agents. This index increases when the number of corrupt officials grows, and decreases as does the percentage of corrupt officials. The corrupt political elite feels the pressure of a high index of intolerance, possibly reducing its expected value in this case of re-election, because the probability of being re-elected is reduced. As a result, the government corruption is reduced, and government will seek to punish corrupt officials more severely. But by reducing the amount of corrupt officials, the index of intolerance decreases, and therefore the pressure on the government declines, again permitting an increase in governmental corruption and allowing for an increase in the number of corrupt officials, thus restarting the cycle of corruption. We discuss this possibility in more detail in the next section, in which we consider a variable index of intolerance, namely, depending on the relative number of corrupt officials.

7 Stability of equilibria and the index of intolerance

Let us change our evolutionary dynamics of corruption by considering instead that the index of intolerance changes with time. More precisely, we will consider that time dependence of the index is implicit, with the index evolving through time only as the relative proportion of corrupt officials changes. Assuming that the probability that a corrupt government is re-elected is a decreasing and a differentiable function of the percentage of corrupt officials, i.e., $\frac{dq_{G_c}(n_c)}{dn_c} \leq 0$, then the degree of intolerance $D_{it}(n_c) = q_{G_{nc}}(n_c) - q_{G_c}(n_c)$ increases when the percentage of corrupt officials increases.

Note that if in a given time t_0 the inequality $n_c(t_0) > \bar{n}_c^T$ holds, then the inequality $E(G_c) > E(G_{nc})$ holds. Then the government prefers to follows a corrupt policy until a certain time $t \geq t_1 > t_0$ when this inequality reverses. This time is finite if and only if there exists a time t_1 such that the probability that a corrupt government is re-elected decreases enough, so that the inequality

$$V_{G_{nc}} q_{G_{nc}}(t) - V_{G_c} q_{G_c}(t) > \frac{n_c(t)}{N} \left(\theta M_c - M_g' - M + e\right) + M_g'$$

is verified. See equations (2) and (3). Note that this process could be cyclic.
In the particular case when $V_{G_{nc}} = V_{G_c} = V_G$ the processes of growing corruption is broken only if

$$D_{it}(n_c(t)) > \frac{1}{V_G} \left[\frac{n_c(t)}{N} \left(\theta M_c - M_g' - M + e\right) + M_g'\right] \tag{17}$$

See equation (9).

Given that the parameters of the model are fixed, only with an index of intolerance of corruption that is high enough can reverse a process of self reinforcement of corruption.

Assuming that a citizen votes for re-election of non corrupt government with probability $\frac{1}{2}$, so an honest government will be re-elected if at least one half of the citizen votes for re-election.

If we symbolize the probability that a non corrupt government be re-elected by $q_{G_{nc}}$ we have that:

$$q_{G_{nc}} = \sum_{j \geq \frac{H}{2}}^{H} \binom{H}{j} \left(\frac{1}{2}\right)^H$$

where H is the total amount of citizens.

On the other hand, we consider that the probability $K(n_c)$ that a citizen votes for a corrupt government depends inversely on the amount of corrupt officials. When $K(n_c) = \frac{1}{2}(1 - n_c)$ so, the probability that a corrupt government be re-elected is

$$q_{G_c}(n_c) = \sum_{j \geq \frac{H}{2}}^{H} \binom{H}{j} (K(n_c))^j \, (1 - K(n_c))^{(H-j)} \, .$$

Note that this degree of intolerance verifies all our conditions in the beginning of this section. In particular, since $\frac{dq_{G_c}(n_c)}{dn_c} \leq 0$ we have that $\frac{dD_{it}(n_c)}{dn_c} \geq 0$ for all $0 \leq n_c \leq 1$. Also observe that $D_{it}(n_c) \geq 0$.

$$\begin{aligned} \dot{n}_c &= n_c(1 - n_c) \, (Ag_c + B) \\ \dot{g}_c &= g_c(1 - g_c) \, (A' n_c + B') \\ D_{it}(n_c) &= q_{G_{nc}} - q_{G_c}(n_c). \end{aligned} \tag{18}$$

Recall that $B' = NM'_g + R_{G_c} - R_{G_{nc}}$. So, if we assume that $V_{G_{nc}} = V_{G_c} = V_G$, i.e., the value government assigns to re-election is the same in all circumstances, then,

$$B'(n_c) = NM'_g - V_G D_{it}(n_c).$$

Consequently $\frac{dB'(n_c)}{dn_c} = -\frac{dD_{it}(n_c)}{dn_c} \leq 0$.

Equivalently the dynamical system of the corruption takes the form:

$$\begin{aligned} \dot{n}_c &= n_c(1 - n_c) \, (Ag_c + B) \\ \dot{g}_c &= g_c(1 - g_c) \left(A' n_c + NM'_g - V_G D_{it}(n_c) \right) \end{aligned} \tag{19}$$

Considering that in $t = t_0$ we have that $g_c(t_0) = g_{c_0} > 0$. Note that $\dot{g}_c < 0$ if and only if $A' n_c(t) + NM'_g < V_G D_{it}(n_c(t))$ (see section (3)) or equivalently if and only if

$$\frac{1}{V_G} \left[N(\theta M_c - M'_g - M + e) NM'_g \right] < D_{it}(n_c(t)),$$

see (15). So the percentage of the corrupt acts of government decrease, if and only if the index of intolerance to corruption is high enough. And, $\dot{n}_c < 0$ if and only if $g_c(t) < \frac{-B}{A}$. So, it is possible that a big enough intolerance's index can make that the corruption decrease. However the economy could be in a cyclical process of evolution of corruption, because $D_{it}(n_c)$ decreases when n_c decreases.

Substituting $D_{it}(n_c)$ by its value in terms of n_c we obtain the system:

$$\begin{aligned} \dot{n}_c &= n_c(1 - n_c) \, (Ag_c + B) \\ \dot{g}_c &= g_c(1 - g_c) \left[A' n_c + NM'_g - \tfrac{V_G}{2^H} \textstyle\sum_{j \geq \frac{H}{2}}^{H} \binom{H}{j} \left(1 - (1 - n_c)^j \, (1 + n_c) \right)^{(H-j)} \right]. \end{aligned} \tag{20}$$

Simplifying the notation we can write

$$\begin{aligned} \dot{n}_c &= n_c(1 - n_c) \, (Ag_c + B) \\ \dot{g}_c &= g_c(1 - g_c) \, (A' n_c + K + P_H(n_c)) \end{aligned} \tag{21}$$

where $K = NM'_g - \frac{V_G}{2^H} \sum_{j \geq \frac{H}{2}}^{H} \binom{H}{j}$ and $P_H(n_c)$ is a polynomial of degree H with coefficients $a_j = \binom{H}{j}$.

Note that, since $\frac{d}{dn_c} P_H(n_c) > 0$, one of the possible effects of the intolerance index in the evolution of the society is the shrink of the attraction basin of the bad equilibrium. Under similar considerations as those in case (1) in section (6.1) there will be a mixed fixed point (n_c^*, g_c^*) such that $n_c^T = n_c^*$ and $g_c^* = g_c^T + \epsilon(n_c)$ where $\frac{d}{dn_c}\epsilon > 0$.

Thus, under these assumptions more strong initial conditions will be required to start a process of increasing corruption, and this is a good news.

8 Some additional comments on the index of intolerance

To finish we introduce some additional considerations on the importance of the intolerance index. Certainly corruption can be considered as a social trap [Rothstein, B.]. Under several circumstances the bad equilibrium is asymptotically stable. In this case, if the initial distribution of corrupt officials and governments corrupt acts are in the basin of attraction of this equilibrium, neither official nor the government

have incentives to act in a non corrupt way. It is in this sense that we consider the corruption as a self reinforcing mechanism. Corrupt actions by a party encourage corrupt actions by the other. If everybody is corrupt, nobody wants to be honest. To be corrupt is the rational way, because under these initial conditions, the expected value of this behavior is higher than the expected value of the non corrupt behavior. Under this prospect, corruption looks like a sticky problem that can not be changed for internal agents. This grim prospect is analysed in several works. See for instance. [Rothstein, B.] and [Kornai, I].

However there is a more encouraging prospect. There are examples of success in deterrence of corruption, for instance the cases of Singapore and Honk Kong, see [Root, H.]. If the government has some interest in being re-elect, and the degree of intolerance of citizens increases enough, the above considered situation can be reverted. The basin of attraction of the bad equilibrium shrinks and the evolution of corruption can be reversed. This possibility shows the leading role that intolerance index can play in the fight against corruption. On the other hand note that the evolution of this index can be favoured if citizens have any other access to the knowledge of the corrupt government actions. The press in general and modern media can play an important role in this regard. It would not be mathematically complicated to add in the index this case, but we would need to do some additional analyses to consider the stability of the stationary points (we will consider this possibility in future researches). However, in principle, we believe that the social conclusions would be similar to those already obtained:

The degree of intolerance of citizens to corruption, plays an important role to deterrence the corruption.

9 Conclusions

With increasing frequency, in a process that seems to grow indefinitely, politicians and governments of all countries and across the whole of the political and ideological spectrum are involved in corruption.

It is possible to deterrence this process? To give an answer to this question was the main issue of this paper. We considered a dynamical system explaining the evolution of corruption. We do not deepen in the role of the imitative behavior, but it is possible to do this. The main conclusions would not change but the model could be mathematically more complicated. In cite [Accinelli, E.;Carrera,E. (b)] a model with imitation is considered. In our model we consider only that at the end of every period individuals compare their respective payoffs, and they choose according with their expectations, however, according with the evolutionary models, we assume that the most profitable behavior end by prevail.

Our main conclusion is that corruption corrupts, and that corruption is a self-reinforcing mechanism. This is a straightforward conclusion of proposition (1). However, fortunately, we have obtained a positive answer for such main question. This self-reinforcing mechanism can be weakened or broken by the public degree of intolerance to corruption. Even in situations where corruption tends to expand in government circles, if the intolerance index becomes large enough, it can help to stop or even to reverse this regressive process.

In the case, when the degree of intolerance is relatively low and the political elite in the government has little interest in being re-elected and much interest in gain immediate benefits, an external push looks like a necessary condition. This is the most worrying situation, because it escapes any self-monitoring mechanism, and there is no way to control the controller.

Our answer supports an optimist point of view: the degree of intolerance of corruption can play an important role to make government fulfil the role that society has assigned it, even when some of its members are attracted by the individual benefits that corruption offers.

We believe that to raise the level of intolerance of citizens against corruption is equivalent to creating an antidote against corruption[5]. It is therefore a very important task that should be carried out by democratic governments when they are willing to play the role that society assigns, or at least that is proclaimed by politicians from Plato to our days the safeguarding the interests of society and the welfare of the citizens.

Certainly economics, sociology, such as physics or astronomy, need empirical verification to validate their theories. However in this paper we only intend to show that an adequate combination of mathematics and game theory can help us to model and give a formal framework to social processes. It is possible to perform numerical analysis to obtain more accurate conclusions from our model, as well as necessary is, to obtain statistical data and examples that confirm or deny it. Nevertheless this will be the product of future works, in the present article we only intend to present a formal model that shows the evolution of the corruption and give a possible answer to a question that has always preoccupied humanity: Who controls the controllers?

In future works we will consider other aspects that can influence the performance of the index of intolerance.

[5]Recent events in South Korea, where citizens unanimously react to the corrupt practices of Prime Minister Park Geun-hye show that, if the Index of Intolerance of citizens to Corruption is high enough, it is possible to stop the growth of corruption. See http://www.abc.net.au/news/2016-11-15/south-korea-park-geun-hye-hopes-political-crisis-be-contained/8024978

References

[Accinelli, E.; Carrera, E.; Policardo, L.] "On the dynamics and effects of corruption on environmental protection", a chapter in 'Modelling, Dynamics, Optimization and Bioeconomics I', editors: Alberto A. Pinto and David Zilberman, Series: Springer Proceedings in Mathematics & Statistics, Vol. 73, Chapter 3, pp. 23-41, (2014).

[Accinelli, E.; Carrera, E. (a)] "Corruption, Inequality, and Income Taxation" in 'Dynamics, Games and Science', eds. Jean-Pierre Bourguignon, Rolf Jeltsch, Alberto Adrego Pinto and Marcelo Viana, published by Springer-Verlag in CIM Series in Mathematical Sciences, vol. 1, chapter 1, pp. 1-16, Springer-Verlag, (2015).

[Accinelli, E.;Carrera,E. (b)] "Corruption driven by imitative behavior", *Economics Letters*, Vol. 117, Issue 1, pp. 84-87, (2012).

[Becker, G.] "A theory of competition among pressure groups for political influence", *The Quarterly Journal of Economics* 98, no. 3, pp. 371-400, (1983).

[Campbell, S. V.] "Perception is Not Reality: The FCPA, Brazil, and the Mismeasurement of Corruption", *Minnesota Journal of International Law*, Vol. 22, No. 1, p. 247-281, (2013).

[Grossman, G. M. and Helpman, E] "Protection for sale", *The American Economic Review*, Vol. 84, No. 4, pp. 833-850, September 1994.

[Huberts, L. W.] "Expert views on corruption around the world", Amsterdan: Vrij University, (1996).

[Hurwicz, L.] "But who will guard the guardians?" Prize Lecture 2007. Available at http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2007/hurwicz_lecture.pdf.

[Jain, A. K.] "Corruption: A review', *Journal of Economics Surveys*, Vol. 15, No. 1, pp. 1-51 (2001).

[Kornai, I] "Hidden in an Envelope. Gratitude Payments to Medical Doctors in Hungary" Budapest Collegium Budapest Institute for Advanced Studies, (2000).

[Kruger, A.] "The political economy of the rent seeking society", *American Economic Review*, Vol. 64, issue 3, pp. 291-303, (1974).

[Hirsh, M.; Smale S.; Devanay, R.] "Differential Equations, Dynamical Systems, and an Introduction to Chaos, Elsevier Ltd, Oxford, 3rd edition, (2012).

[Lessig, L.] "Republic Lost: How Money Corrupts Congress - and a Plan to Stop It", ISBN-13: 978-0446576437, Twelve, (2011).

[Lui, Francis T. (a))] "An Equilibrium Queuing Model of Bribery", *Journal of Political Economy* Vol. 93, No. 4, August 1985, pp. 760-781.

[Lui, Francis T. (b)] "A Dynamic Model of Corruption Deterrence", *Journal of Public Economics* Volume 31, Issue 2, pp 215-236, (November 1986).

[Mauro, P.] "Corruption and Growth", *The Quarterly Journal of Economics*, Vol. 110, No. 3, pp. 681-712 (August 1995).

[Root, H.] "Small countries, Big Lessons: Governance and the Rise of East Asia", Hong Kong, Oxford University Press (1996).

[Rose-Ackerman, S. (a)] "The Economics of Corruption", *Journal of Public Economics* Vol. 4, Issue 2, pp. 187-203, (February 1975).

[Rose Ackerman, S. (b)] "Corruption. A Study in Political Economy", Academic Press, New York, (1978).

[Rothstein, B.] "Social traps and the Problem of Trust", Cambridge University Press, (2005).

[Sah, R.] "Persistence and Pervasiveness of Corruption", in Conference on Political Economy: Theory and Policy Implications, Washington D. C., World Bank, (1987).

[Shleifer, A. and Vishny, R.] "Corruption", *The Quarterly Journal of Economics*, Vol. 108, No. 3, pp. 599-617, (1993).

[Seligson, M.] "The Impact of Corruption on Regime Legitimacy: A Comparative Study of Four Latin American Countries", *Journal of Politics*, Vol. 64, Issue 2, pp. 408-433, (2002).

[Thompsom D.] "Ethics in Congress: From Individual to Institutional Corruption", Brookings Institution Press, (1995).

[Weibull, W. J.] "Evolutionary Game Theory", The MIT Press, (1995).