Solving problems with sign-changing coefficients: T-coercivity and beyond

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In electromagnetic theory, the effective response of specifically designed materials is modeled by strictly negative coefficients: these are the so-called negative materials. Transmission problems with discontinuous, sign-changing coefficients then occur in the presence of negative materials surrounded by classical materials. Establishing Fredholmness is well-understood thanks to the T-coercivity approach, which was first introduced for these transmission problems by Bonnet-Ben Dhia, Ciarlet and Zwölf [1].
The T-coercivity approach relies on the design of an operator T to realize the inf-sup condition and, in principle, it can be applied to any variational formulation in Hilbert spaces. If the operator is carefully chosen, it can provide useful insight for a "natural" definition of the approximation of the exact problem. The derivation of the discrete inf-sup condition then becomes elementary, at least when one considers conforming methods, that is when the discrete finite element spaces are subspaces of the exact Hilbert spaces : both the exact and the approximate problems are considered, analysed and solved at once.
For problems with sign-changing coefficients, the shape of the interface separating the two materials must be taken into account to solve the problems numerically: specific meshing rules have to be implemented to design the meshes. And, when one is solving the full Maxwell system, the operator T at hand does not lead to a "natural" discrete counterpart.
To overcome these limitations, another method has been recently developed by Ciarlet, Lassounon and Rihani [3], which relies on the use of an optimal control reformulation of the problem. On the one hand, the resulting numerical method converges without any restriction on the mesh and, on the other hand, it can be used to solve the full Maxwell system [2]. This method is inspired by the smooth extension method, originally introduced by Fabrèges, Gouarin and Maury [4].

References

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