# Simultaneous Continuation of Infinitely Many Sinks Near a Quadratic Homoclinic Tangency.

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### Abstract

We prove that the $C^3$ diffeomorphisms on surfaces, exhibiting infinitely many sinks near the generic unfolding of a quadratic homoclinic tangency of a dissipative saddle, can be perturbed along an infinite dimensional manifold of $C^3$ diffeomorphisms such that infinitely many sinks persist simultaneously. On the other hand, if they are perturbed along one-parameter families that unfold generically the quadratic tangencies, then at most a finite number of those sinks have continuation.

## 1 Introduction and statement of the main results.

Let $M$ be a two-dimensional $C^\infty$ compact and connected riemannian manifold, and let $\mathrm{Diff}^3(M)$ be the infinite dimensional manifold of all $C^3$-diffeomorphisms $f \colon M \mapsto M$.

Let $f_0 \in \mathrm{Diff}^3(M)$ having a saddle fixed point $P_0$. We denote $\lambda_0 < 1 < \sigma_0$ the eigenvalues of $Df_0(P_0)$.

We consider diffeomorphisms that are dissipative in a saddle point, i.e. $\lambda_0\sigma_0 < 1$. We also assume that the diffeomorphism $f_0$ exhibits at $q_0$ a quadratic homoclinic tangency (see [PT 1993]) of the saddle point $P_0$, recalling the following definition:

DEFINITION 1.1 We say that the homoclinic tangency at $q_0$ of the periodic saddle point $P_0$ is quadratic if there exists a $C^2$ local chart in a neighborhood of $q_0$ such that the stable arc of $P_0$ which contains the tangency point $q_0$ has equation $y = 0$, and the unstable arc has equation $y = \overline{\beta}\, x^2$ with $\overline{\beta} \neq 0$.

Take a one-parameter family $\{\widetilde{f}_t\}_{t \in I} \subset \mathrm{Diff}^3(M)$ through the given map $\widetilde{f}_0 = f_0$, such that the quadratic homoclinic tangency unfolds generically into two transversal homoclinic intersections for $t > 0$.

The Newhouse-Robinson Theorem ([N 1974], [R 1983]) asserts that, as near as wanted from $\widetilde{f}_0$ in the one-parameter family $\{\widetilde{f}_t\}_{t \in I}$, there exists an interval $I_0$ and a dense set $J_0 \subset I_0$ of values of the parameter such that for all $t \in J_0$, $\widetilde{f}_t$ exhibits infinitely many simultaneous sinks.

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We will prove that for values $t$ in a dense set $J \subset J_0$, the map $\widetilde{f}_t$ is bifurcating: in fact, our Theorem 2 asserts that at most a finite number of certain sequence of infinitely many sinks of $\widetilde{f}_t$ can simultaneously persist when we perturb $\widetilde{f}_t$ along certain one-parameter families in $\mathrm{Diff}^3(M)$. Nevertheless, in Theorem 1 we prove that the bifurcation of infinite many simultaneous sinks has infinite dimension in $\mathrm{Diff}^3(M)$.

Now let us define the kind of perturbations of each diffeomorphism and the kind of persistence of each sink which we will consider all along this paper:

DEFINITION 1.2 Let us suppose that $g_0 \in \mathrm{Diff}^3(M)$ exhibits a sink $s_0$. Consider $g_1 \in \mathrm{Diff}^3(M)$, isotopic to $g_0$. We say that $g_1$ exhibits the continuation $s_1 = s(g_1)$ of the sink $s_0$, if there is a differentiable isotopy $\{g_t\}_{t \in \mathbb{R}} \subset \mathrm{Diff}^3(M)$ such that for all $t \in [0,1]$ there exists a sink $s_t = s(g_t)$ of $g_t$ and the transformation $t \in [0,1] \mapsto s_t \in M$ is of $C^1$ class.

We are now ready to state the main result of this paper:

THEOREM 1 *Let $M$ be a $C^\infty$ two dimensional compact connected riemannian manifold. Let $f_0 \in \mathrm{Diff}^3(M)$ exhibiting a quadratic homoclinic tangency of the saddle point $P_0$. Assume that the saddle is dissipative, i.e. its eigenvalues $\lambda_0 < 1 < \sigma_0$ verify $\lambda_0 \sigma_0 < 1$.*

*Then, given an arbitrarily small neighborhood $\mathcal{N}$ of $f_0$ in $\mathrm{Diff}^3(M)$ there exists a $C^1$ arc-connected infinite-dimensional local submanifold $\mathcal{M} \subset \mathcal{N}$ such that:*

*(a) Every $g \in \mathcal{M}$ exhibits infinitely many simultaneous sinks $s_i(g)_{i \in \mathbb{N}}$.*

*(b) Each sink $s_i(g)$ is the continuation of the respective sink $s_i(g_0)$, for any pair of diffeomorphisms $g_0, g \in \mathcal{M}$.*

Note that the given diffeomorphism $f_0$ does not necessarily belong to $\mathcal{M}$. We prove theorem 1 through sections 2 to 8.

REMARK 1.3 *Notation*

Let $f \in \mathrm{Diff}^3(M)$ have a horseshoe $\Lambda \subset M$, as defined in [PT 1993] Chapter II, Section 3. As $\Lambda$ is an hyperbolic set, there exist constants $C > 0$, $\widetilde{\lambda} < 1$ and $\widetilde{\sigma} > 1$ and a splitting $T_p M = E^u \oplus E^s$ for all $p \in \Lambda$, such that $\|Df^n(v)\| \geq C\widetilde{\sigma}^n \|v\|$ $\forall v \in E^u$ and $\|Df^n(v)\| \leq C^{-1}\widetilde{\lambda}^n \|v\|$ $\forall v \in E^s$. Besides the horseshoe $\Lambda$ is the maximal invariant set in an open neighborhood $U$ of itself. In Section 3 we will add some other restriction to $U$.

We assume that $f$ exhibits at the point $q_1 \in U$ a quadratic homoclinic tangency of the invariant manifolds of a periodic saddle point $P_1 \in \Lambda$. We choose $q_1$ such that $f^{-1}(q_1) \notin U$. It is not restrictive to consider $q_1 \in W^s(P_1)$ such that for all $n \geq 1$, $f^n(q_1)$ belongs to $U$ and then call $N_1 > 1$ to an integer number such that $f^{-N_1}(q_1) \in U$ belongs to the local stable manifold of $P_1$. We will take a small neighborhood $V$ of $q_1$ such that $V \subset U$, $f^{-N_1}(V) \subset U$, and $f^{-i}(V)$ and $\Lambda$ are pairwise disjoint for $i = 0, \ldots N_1$. We also assume that $f^{-1}(V) \cap U = \emptyset$. We will work with small perturbations of $f$, such that the former properties of $V$ persist.

As shown in [PT 1993], Chapter II Section 3 and Appendix 1, it is possible to construct invariant stable and unstable local foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ in a neighborhood of $\Lambda$. We will denote $W^s_{\mathrm{loc}}$ and $W^u_{\mathrm{loc}}$ the respective leaves of the foliations. These foliations are $C^{1+\varepsilon}$, meaning in particular that the tangent directions to the leaves are $C^{1+\varepsilon}$.

DEFINITION 1.4 The line of tangencies $L(f)$ is the set of points in a small neighborhood $V$ of $q_1$ where the leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ are tangent.

$\Omega$
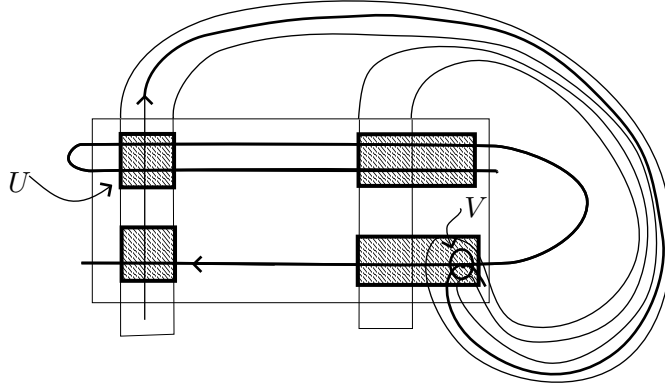
$U$

$q$

$\sigma$

$\lambda$

$f^n$

$f^N$

$V$

Figure 1: Horseshoe

REMARK 1.5 Since the tangent directions are $C^{1+\varepsilon}$, the tangencies on $L(f)$ are also quadratic, and $L(f)$ is a differentiable curve (see [PT 1993] Chapter V, Section 1). It persists and depends continuously on $f$.

DEFINITION 1.6 The stable (unstable) Cantor set $K^s$ (resp. $K^u$) is the intersection with the line of tangencies $L(f)$ of the local leaves through the points $P \in \Lambda$ of the stable foliation $\mathcal{F}^s$ (resp. the $f^{N_1}$ iterates of the local leaves of $\mathcal{F}^u$ passing through the points $P \in \Lambda$).

DEFINITION 1.7 We say that a one-parameter family $\{\widetilde{f}_t\}_{-\varepsilon \leq t \leq +\varepsilon} \subset \mathrm{Diff}^3(M)$ passing through a diffeomorphism $f_0$, unfolds generically the quadratic tangencies of the horseshoe $\Lambda(f_0)$, if there exists a velocity $v > 0$ such that

$$\left| \frac{d\mu_{p,q}(\widetilde{f}_t)}{dt} \right| \geq v > 0 \ \ \forall p, q \in \Lambda, \ \forall t \in (-\varepsilon, +\varepsilon)$$

where $\mu_{p,q}(\widetilde{f}_t)$ is the distance along the line of tangencies $L(\widetilde{f}_t)$ between $p^s = W^s_{\mathrm{loc}}(p) \bigcap L(\widetilde{f}_t) \in K^s$ and $q^u \in f_t^{N_1}(W^u_{\mathrm{loc}}(q)) \bigcap L(\widetilde{f}_t) \in K^u$ of any two points $p, q \in \Lambda(\widetilde{f}_t)$.

THEOREM 2 *In the hypothesis of Theorem 1, given a one-parameter family $\{\widetilde{f}_t\}_{-\varepsilon \leq t \leq \varepsilon} \subset Diff^3(M)$ which generically unfolds the quadratic homoclinic tangency at $q_0$ exhibited by $f_0$, there exist an open real interval $I \subset (-\varepsilon, \varepsilon)$ and a dense set $J \subset I$ of the parameter values such that if $f_\infty \in \{\widetilde{f}_t\}_{t \in J}$, then:*

(A) *$f_\infty$ exhibits infinitely many sinks $s_i(f_\infty)_{i \geq 1} \in V$ with periods $p_i(\to +\infty)$, and there exists $0 < \rho < 1$ such that the eigenvalues of $df_\infty^{p_i}(s_i)$ have modulus smaller than $\rho$ for all $i \geq 1$.*

(B) *There exists a local $C^1$ infinite-dimensional, arc-connected manifold $\mathcal{M} \subset \mathrm{Diff}^3(M)$, such that:*

    *1. $f_\infty \in \mathcal{M}$*

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2. If $g \in \mathcal{M}$ then $g$ exhibits the continuation $s_i(g) \in V$ of the infinitely many sinks $s_i(f_\infty)$.

(C) Any one-parameter family $\{g_\mu\}_{-\varepsilon \leq \mu \leq +\varepsilon}$ of $C^3$ diffeomorphisms passing through $g_0 = f_\infty$ and unfolding generically the quadratic tangencies on $L(g_0)$, exhibits for $\mu \neq 0$ at most a finite number of simultaneous continuations $s_i(g_\mu)$ of the sinks $s_i(g_0)$ constructed in part (A).

Remark to thesis (A): The sinks $s_i(\widetilde{f_t})$ for $t \in J$ are not necessarily the continuation of the sinks $s_i(f_\infty)$, at least not for infinitely many values of $i \geq 1$.

We prove Theorem 2 in Section 7.

An interesting open problem is the prevalence of infinite sinks. A conjecture of Palis ([P 2000]) asserts that there exists a dense set of $C^r$ diffeomorphisms with a finite number of attractors with a total Lebesgue measure attracting basins. In dimension two the main obstruction to this conjecture is that the phenomenon of the coexistence of infinite simultaneous sinks occurs for a whole open set in $\mathrm{Diff}^r(M)$. We observe that our Theorem 2 does not solve the problem, since infinite sinks could appear from other homoclinic tangencies.

To prove Theorem 1, inspired in the Newhouse-Robinson Theorem, we construct a one-parameter family $\{\widetilde{f_t}\}_{t \in I}$ perturbing in an adequate way the diffeomorphism $f_0$. This perturbation is constructed so that there exists a nested sequence of intervals of values of the parameter such that in the $i+1$-interval there exists a sink $s_{i+1}$ and the $i$ sinks constructed in the former intervals still persist. In the intersection of all these intervals we obtain a parameter $t_\infty$ in which there exist infinitely many sinks.

This construction is possible because $f_0$ has a homoclinic tangency and perturbing $f_0$, a horseshoe is created. Newhouse remarked the persistence of homoclinic tangencies of saddle points of a horseshoe whose unstable and stable Cantor sets $K^u$ and $K^s$ along the line of tangencies have large thickness. Since near a homoclinic tangency there exists a sink (see the Yorke-Alligood theorem, ([YA 1983])), it is possible to reason inductively in order to construct the nested sequence of intervals.

Our purpose to prove Theorems 1 and 2 in this paper, goes beyond the construction of Newhouse: we shall be able, besides, to perturb the primary family of diffeomorphisms in the functional space $\mathrm{Diff}^3(M)$, considering what we call "secondary diffeomorphisms", along a properly defined manifold $\mathcal{M} \subset \mathrm{Diff}^3(M)$, in such a way that the infinite sinks, constructed for the diffeomorphism in the primary family, persist simultaneously.

Taking a nearby family in an adequate infinite dimensional set of $\mathrm{Diff}^3(M)$, we will prove that the values of the parameter where the tangencies and the sinks are produced, are near those of the original family, and then the sinks continue, obtaining in this way the manifold $\mathcal{M}$.

To prove part A of Theorem 2, with a suitable change of coordinates, the diffeomorphisms of the family are near the functions of the classical quadratic family. For certain functions of the quadratic family, the sink has eigenvalues as contractive as wanted. This property is maintained for the diffeomorphisms of the original family.

Part B will be proved perturbing the diffeomorphism in such a way that the sinks, which are far from a bifurcation after part A, persist.

To prove part C we will show that any perturbation of the diffeomorphism generically unfolding the quadratic tangencies allows to persist only a finite number of the sinks, because the range of the values of the parameter for which the sinks persist decreases monotonically to 0.

The main tools that will allow us to make such proofs are Propositions 4.3 and 5.7 of this paper. This last guarantees the existence of uniform sized manifolds of codimension one in some

infinite dimensional subset $\mathcal{N}_1 \subset \mathrm{Diff}^3(M)$, along which all sinks persist simultaneously.

As far as we prove our theorems, we construct the manifold $\mathcal{M}$ having infinite dimension and also infinite codimension. We do not assert that the manifold $\mathcal{M}$ that we construct is maximal verifying the conditions (a) and (b) of the thesis of Theorem 1. Nevertheless, if such a maximal manifold exists, it must have at least codimension one, as a consequence of the part (C) of the thesis in Theorem 2, which we prove at the end of the paper.

We also answer to other open question: Can the infinitely many simultaneous sinks exhibited by a diffeomorphism $g_0$ constructed as in Newhouse-Robinson Theorem simultaneously continue in an open set? In fact, we prove that the answer is negative, provided that $g_0$ is a diffeomorphism constructed as in Theorem 2.

As a consequence of the proof of part (C) of Theorem 2, it is immediate the following last result:

COROLLARY 1.8 *To continue infinitely many sinks* (from those in $\{s_i(f_\infty)\}$) *of a diffeomorphism $f_\infty$ constructed as in Theorem 2 it is necessary to move along their respective stable local leaves all the points in the unstable Cantor set $K^u$ of the line of tangencies $L(f_\infty)$ of $f_\infty$, that are in the parabolic unstable arcs of the accumulation points of the sequence of sinks.*

This last result is the main reason why we restricted our constructions (to prove the theorems of this paper) to an infinite codimension manifold of diffeomorphisms, obtained from $f_\infty$ perturbing only inside $V$. In that way we can control easily the unstable Cantor set $K^u$ while the stable Cantor set remains fixed.

Finally, we pose the following open question. Let $f_\infty$ and $\mathcal{M}$ verifying parts (A), (B)1 and (B)2 of Theorem 2. Has $\mathcal{M}$ necessarily infinite codimension?

# 2 Persistence of tangencies.

We recall the definition of line of tangencies (see Definition 1.4) and stable and unstable Cantor sets (see Definition 1.6).

DEFINITION 2.1 Given a Cantor set $K \subset \mathbb{R}$, the thickness at $u \in K$ in the boundary of a gap $U$ is defined as $\tau(K, u) = \frac{l(C)}{l(U)}$ where $C$ is the bridge of $K$ at $u$ (see [PT 1993]). The thickness of $K$, denoted $\tau(K)$ is the infimum of the $\tau(K, u)$ over $u$.

DEFINITION 2.2 *Two Cantor sets, $K_1$, $K_2 \subset \mathbb{R}$ have large thickness if $\tau(K_1)\tau(K_2) > 1$.*

DEFINITION 2.3 *The horseshoe $\Lambda$ verifies the large thickness condition if $\tau(K^s)\tau(K^u) > 1$ where $K^u$ and $K^s$ are defined in 1.6.*

The importance of this definition resides in the following lemma:

LEMMA 2.4 *Let $K_1$, $K_2 \subset \mathbb{R}$ two Cantor sets with large thickness. Then, one of the following three alternatives occurs: $K_1$ is contained in a gap of $K_2$; $K_2$ is contained in a gap of $K_1$; $K_1 \bigcap K_2 \neq \emptyset$.*

For a proof, see [PT 1993]. We will apply the lemma to the stable and unstable Cantor sets on the line of tangencies: the third alternative assures the persistence of tangencies.

Now, we will define strongly dissipative horseshoes. Let us consider unstable and stable foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ defined in a neighborhood $U$ of a horseshoe $\Lambda$. Let us take nonzero $C^1$ vector fields $X^u$ and $X^s$, tangent to the leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$, and let us define the functions $\lambda: U \to \mathbb{R}$ and $\sigma: U \to \mathbb{R}$ as:

$$df(X^u(x)) = \sigma(x) X^u(f(x))$$
$$df(X^s(x)) = \lambda(x) X^s(f(x))$$

Redefining $X^u$ and $X^s$ if necessary, we obtain that $\sigma > 1$ and $\lambda < 1$ for every point of $U$ if $U$ is a small enough neighborhood of $\Lambda$.

DEFINITION 2.5 We say that a horseshoe is strongly dissipative if for every $x \in U$, $\lambda(x)\sigma^2(x) < 1$

THEOREM 2.6 (**Newhouse-Robinson**) *Let $\widetilde{f}_t$ be a monoparametric family which generically unfolds a quadratic homoclinic tangency $q_0$ exhibited at $t = 0$ of a fixed dissipative saddle point $P_0$ (i.e. the eigenvalues of $df_0(P_0)$ are $\lambda_0 < 1 < \sigma_0$ and $\lambda_0\sigma_0 < 1$.)*

*Then, given $\varepsilon > 0$, there exists an interval $I \subset (0, \varepsilon)$ of values of the parameter and an open set $V$ such that:*

**(i)** *For every $t \in I$ the diffeomorphism $\widetilde{f}_t$ exhibits a horseshoe $\Lambda$ which verifies the condition of large thickness as in Definition 2.3 and it is strongly dissipative (i.e. $\lambda(x)\sigma^2(x) < 1$, $\forall x \in \Lambda$).*

**(ii)** *For every value $t$ of the parameter in a dense set in $I$, there exists a saddle point $P \in \Lambda$ which exhibits an homoclinic tangency $q \in V$.*

**(iii)** *For every value $\tau$ of the parameter in a dense set in $I$ there exists infinite simultaneous sinks in $V$.*

*Proof:* See [N 1974] and [R 1983] and the lemma below.

The horseshoes created by unfolding tangencies have an important property:

LEMMA 2.7 *The horseshoe created by the unfolding of a homoclinic quadratic tangency of a dissipative saddle point can be taken strongly dissipative just taking the number of iterates large enough.*

*Proof:* It is a consequence of the scaling in Section 4, Chapter III, of [PT 1993]. The horseshoe is diffeomorphically conjugated to a horseshoe near a map of the quadratic family which is infinite contractive ($\lambda = 0$) along its stable foliation. $\square$

To prove our first main result in Theorem 1 it is enough to join the statements of Theorem 2.6 with the following:

THEOREM 2.8 *Let $M$ be a $C^\infty$ two dimensional compact connected riemannian manifold. Let $f_1 \in \mathrm{Diff}^3(M)$ exhibiting a strongly dissipative horseshoe $\Lambda \subset M$. Let $P_1 \in \Lambda$ be a saddle periodic point with a quadratic homoclinic tangency at $q_1$.*

*Assume that the stable and unstable Cantor sets of $\Lambda$ along the line of tangencies $L$ in a neighborhood $V$ of $q_1$ verify the condition of large thickness defined in 2.3.*

*Then, given an arbitrarily small neighborhood $\mathcal{N}$ of $f_1$ in $\mathrm{Diff}^3(M)$, there exists a $C^1$ arc-connected infinite-dimensional manifold $\mathcal{M} \in \mathcal{N}$ such that:*

**(a)** *Every $g \in \mathcal{M}$ exhibits infinitely many simultaneous sinks $\{s_i(g)\}_{i \in \mathbb{N}} \in V$*

**(b)** *Each sink $s_i(g)$ is the continuation of the respective sink $s_i(g_0)$ for any pair of diffeomorphisms $g_0, g \in \mathcal{M}$.*

The proof of this Theorem is in Section 6.

# 3   Local coordinates.

We remark some known facts on the existence of a regular coordinate system in a neighborhood $U$ of the horseshoe $\Lambda$ that trivialize its local stable and unstable foliations.

REMARK 3.1 *Regularity of the local invariant foliations.*
Given the horseshoe $\Lambda$ in a two dimensional manifold $M$, and given a sufficiently small neighborhood $U \subset M$ of $\Lambda$, there exist the stable local foliation $\mathcal{F}^s$ and the unstable local foliation $\mathcal{F}^u$ that are invariant while their iterates remain in $U$ (see Appendix 1 in [PT 1993]; see also [M 1973]). Moreover, if $f \in \mathrm{Diff}^2(M)$, then both invariant local foliations are of $C^1$-class (see [PT 1993] Chapter II Section 3 and also Appendix 1.) Then the stable leaves are $C^2$ and the tangent space of the stable leaf through a point $P \in U$ depends $C^1$ on the point $P$. In particular the concavity of each leaf depends continuously on the point $P$.

Besides, if $f \in \mathrm{Diff}^3(M)$ and $\lambda\sigma^2 < 1$ then the local stable foliation is of $C^3$-class while the unstable foliation is not necessarily more then $C^{1+\varepsilon}$. In fact, the $C^3$ differentiability of the stable foliation follows after its $r-$normality (see [HPS 1977]): arguing as in [PT 1993] Appendix 1, and working in the space $L(M) = \{(x, L) : x \in M \text{ and } L \text{ is a 1-dimensional linear subspace of } T_x M\}$, it follows that the local stable foliation is $C^r$ with $r$ such that $\dfrac{\sigma\lambda^{-1}}{\sigma^r} > 1$ for all $x$, or, equivalently, $r < 1 + \dfrac{-\log\lambda}{\log\sigma}$. The $C^3$ regularity of $\mathcal{F}^s$ follows recalling that we assumed that $\lambda\sigma^2 < 1$.

We will mainly work with $C^3$ diffeomorphisms, so the stable foliation will be $C^3$, and the unstable foliation, will be $C^{1+\varepsilon}$.

REMARK 3.2 *Local coordinate system in the neighborhood $U$ of the horseshoe.*
As a consequence of Remark 3.1, if $f \in \mathrm{Diff}^3(M)$ and $\lambda\sigma^2 < 1$, we can take $C^1$ local coordinates $(x, y)$ of the two-dimensional manifold in the neighborhood $U$ containing the horseshoe $\Lambda$, such that the local stable leaves of $\Lambda$ are horizontal lines $y = $ constant and its local unstable leaves are vertical lines $x = $ constant.

We get $f(x, y) = (\xi(x), \eta(y))$ with $\xi$ of $C^1$ class and $\eta$ of $C^3$ class.

Also, given any $C^3$ map $H : U \mapsto U$ its computation in the local coordinates $H(x, y) = (H_1, H_2)$ will be of $C^1-$class and besides the second and third order partial derivatives of $H_2$ respect to $y$ exist and are continuous.

Such a regular coordinate system in $U$ exists for any map $g$ in a neighborhood $\mathcal{N} \subset \mathrm{Diff}^3(M)$ of the given map $f_1$: in fact, the hypothesis of existence of the hyperbolic horseshoe $\Lambda(g)$ verifying $\lambda\sigma^2 < 1$, is persistent under small perturbations of $f_1$.

Besides, the local unstable and stable foliations and their tangent spaces depend continuously on $g \in \mathcal{N}$. Therefore the local coordinate system in $U$ chosen as above for each $g \in \mathcal{N} \subset \mathrm{Diff}^3(M)$, depends continuously on $g$.

### 3.3 *The local coordinates computation of the map.*

Take $f_1 \in \mathrm{Diff}^3(M)$ exhibiting a horseshoe $\Lambda$ as in the hypothesis of Theorem 2.8 and a small neighborhood $U$ of $\Lambda$ in $M$.

We will perturb $f_1$ with diffeomorphisms $\xi$, i.e. $f = \xi \circ f_1$ such that $\xi = \mathrm{Id}$ in a neighborhood of $\Lambda$, so that the horseshoe remains the same. Later, we will choose an adequate sequence of periodic saddle points $P_i \in \Lambda$ and use the following notation:

REMARK 3.4 *Notation:*

$P_i = P_i(f) \in \Lambda(f)$ is a saddle point. Let us denote $\{f_i\}_{i\geq 1}$ a sequence of diffeomorphisms $f_i \in \mathrm{Diff}^3(M)$ along a one-parameter family from $f_1$, such that $f_i$ exhibits a homoclinic quadratic tangency at $q_i$ of the saddle $P_i$. Each of the points $q_i$ shall be chosen in a certain horizontal arc $A_i^s \subset \{y = y(q_i)\} \subset V$ of the stable manifold of the saddle $P_i$, but not necessarily in the local connected component $y = y(P_i)$ through $P_i$. The tangency points $q_i$, for all $i \geq 1$, shall be chosen in the line of quadratic tangencies $L = L(f_i)$ contained in the small open set $V \subset U$ defined in remark 1.3.

The horizontal arc $A_i^s \ni q$ is chosen small enough such that $f_i^n(A_i^s) \subset U \, \forall n \geq 0$. Then we choose $n_i$ large enough so that $f_i^{n_i}(A_i^s) \subset \{y = y(P_i)\}$. Let us take a height $h$ of a vertical segment $I_h$ such that $f_i^j(A_i^s \times I_h) \subset U$ for $j = 0, 1, \ldots, n_i$. Now we choose a region $D \subset f_i^{n_i}(A_i^s \times I_h)$ which projects in a fundamental domain on $W_{\mathrm{loc}}^u(P_i)$. Finally, we take $V_i = f_i^{-n_i}(D)$, (let us observe that $n_i$ can be taken as large as wanted) where we will rescale the coordinates in the next Section.

Consider any $f_i$ as above. We will argue as in [PT 1993] Chapter III, Section 4:

Taking $n_i$ large enough, and observing that in that case the number of iterates near $P_i$ can be taken as large as wanted, we have that the length contraction $\lambda_i^{(n_i)}(A_i^s)$ of the horizontal compact arc $A_i^s$ of stable manifold of $P_i$ in $U$, when applied $f_i^{n_i}$, and the expansion $\sigma_i^{(n_i)}(A_i^u)$ of a (vertical) compact arc $A_i^u$ of its unstable manifold in $U$, verify $\lambda_i^{(n_i)}(A_i^s) < \widetilde{\lambda}^{n_i} < 1$ and $\sigma_i^{(n_i)}(A_i^u) > \widetilde{\sigma}^{n_i} > 1$ (see Remark 1.3).

We have for all $(x, y) \in V_i$

$$f_i(x, y) \quad = \quad (\xi(x), \eta(y)) \tag{3.1}$$

$$\mathrm{dist}^s(f_i(P_i), (\xi(x), \eta(y))) = \lambda_i(x)\mathrm{dist}^s(P_i, (x, y))$$

$$\mathrm{dist}^u(f_i(q_i), (\xi(x), \eta(y))) = \sigma_i(y)\mathrm{dist}^u(q_i, (x, y))$$

where $\mathrm{dist}^s$ and $\mathrm{dist}^u$ can be taken as the distances along compact arcs of stable and unstable manifolds of $\Lambda$ as follows: $\mathrm{dist}^u(q_i, (x, y)) = |y - y_{q_i}|$; $\mathrm{dist}^s(q_i, (x, y)) = |x - x_{q_i}|$ and $\mathrm{dist}^s(P_i, q_i)$ can be taken for instance as the length of the compact stable arc between $P_i$ and the homoclinic tangency $q_i$ (we remark that this arc is not necessarily contained in $U$, see Figure 2).

If the point $(x, y)$ is such that $f_i^j(x, y) \in U$ for all $j = 0, 1, \ldots, n_i$ and if $n_i$ is large enough:

$$\sigma_i^{(n_i)}(y) = \left( \prod_{j=0}^{n_i-1} \sigma_i(f_i^j(y)) \right) > \widetilde{\sigma}^{n_i}, \quad \lambda_i^{(n_i)}(x) = \left( \prod_{j=0}^{n_i-1} \lambda_i(f_i^j(x)) \right) < \widetilde{\lambda}^{n_i} \tag{3.2}$$

In the last equations $f_i^j(y)$ denotes the ordinate of the point $f_i^j(x, y)$, which depends only on $y$. Similarly $f_i^j(x)$ denotes its abscise, which depends only on $x$.
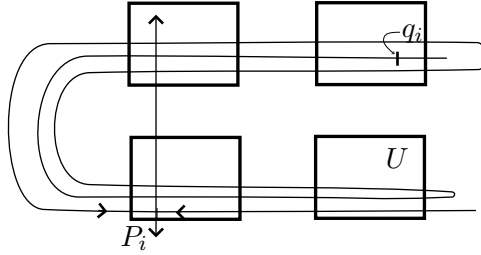
Figure 2: Stable distance.

### 3.5 *The local coordinates computation near the quadratic homoclinic tangencies.*

Let us study now the behavior of the map $f$ and its computation in the linearizing coordinates near the homoclinic tangency. Let us consider $f_i$ as in 3.4, and the points $r_i = (x(P_i), a_i)$ and $q_i = (b_i, c_i) \in V$ in the homoclinic orbit of the saddle $P_i$, where the quadratic tangency is exhibited such that $q_i = f_i^{N_i}(r_i)$ for some integer number $N_i \geq 1$, where $N_i$ is defined in the same way as $N_1$ in remark 1.3, but referred to the point $P_i$ .

Take $f \neq f_i$ in $\mathcal{N}$, and consider for such $f$ the points $r_i(f) = (x(P_i(f)), a_i(f))$ and $q_i(f) = (b_i(f), c_i(f))$ as in figure 3, being the "remaining points" of the tangency that the map $f_i$ exhibited. These points $r_i(f)$ and $q_i(f)$ are defined as follows:

First, we denote $r_i(f_i) = r_i$, $q_i(f_i) = q_i$. The point $q_i$ belongs to a compact arc $A_i^s$ of the stable manifold of the saddle $P_i$, with equation $y = c_i$ when $f = f_i$. It is not necessarily in the local stable manifold $y = y_{P_i}$ of the saddle $P_i$.

Second, if $f \neq f_i$ in $\mathcal{N}$ , then the homoclinic tangency may disappear, but we still have the continuation $A_i^s(f)$ of the compact stable arc $A_i^s$, with equation $y = c_i(f)$, and a new line of tangencies $L(f) \subset V$. We first define the point $r_i(f)$ belonging to the connected component of the local unstable leaf $x = x_{P_i}$ of $P_i$ in $U$, and being such that $f^{N_i}(r_i) \in K^u(f)$ is in the line of tangencies $L(f) \subset V$.

Afterwards, we take the coordinates of the point $f^{N_i}(r_i(f)) = (b_i(f), \nu_i(f)) = (b_i(f), \mu_i(f) + c_i(f))$. Its ordinate $\nu_i(f)$ is the height of the parabolic arc in the compact piece of unstable manifold of $P_i$ that made the tangency for the diffeomorphism $f_i$. The height $\nu_i(f)$ is the sum of two terms: the "relative height" $\mu_i(f)$ respect to the stable arc $A_i^s(f)$ with which that parabolic arc made the tangency (i.e. $\mu_i(f_i) = 0$), and the ordinate $c_i(f)$ of the arc $A_i^s(f)$. Finally we define the point $q_i(f) = (b_i(f), c_i(f)) \in A_i^s(f)$ as the projection of $f^{N_i}(r_i(f))$ along the vertical direction on the stable leaf $A_i^s(f)$, see Figure 3.

We remark that $r_i, q_i, a_i, b_i, c_i$ and $\mu_i$ depend continuously on $f \in \mathcal{N}$.

We compute the equations of the transformation $f^{N_i}$ which goes from a small neighborhood of $r_i = (x(P_i), a_i)$ to the neighborhood $V$ of $q_i = (b_i, c_i)$ in the coordinates $(x, y)$. We take $x^* = x - x(P_i)$, $y^* = y - a_i$, and after [PT 1993]:

$$f^{N_i} \colon (x(P_i) + x^*, a_i + y^*) \mapsto (b_i, c_i) + (H_1(\mu_i, x^*, y^*), H_2(\mu_i, x^*, y^*))$$

Compute now the Taylor expansion of $H_1$ and $H_2$ in a neighborhood of $(\mu_i, x^*, y^*) = (0, 0, 0)$, that is in a spacial neighborhood of the point $r_i$ and a neighborhood of the diffeomorphism $f_i$ in a one-parameter family $\{f_{i,\mu_i}\} \subset \mathrm{Diff}^2(M)$ such that $f_{i,0} = f_i$ for $\mu_i = 0$:
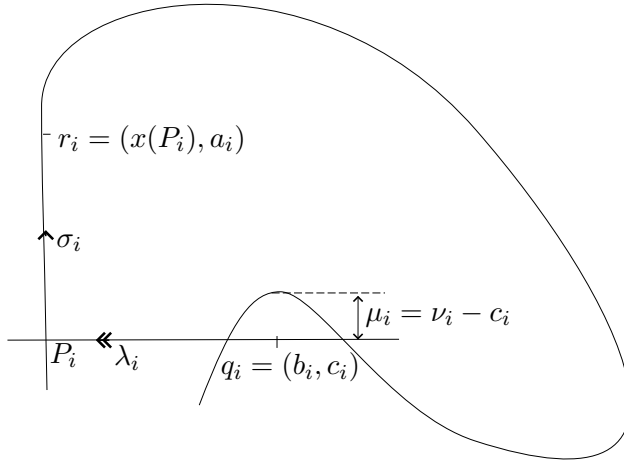
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Figure 3: Unfolding the tangency

$$
\begin{aligned}
H_1(\mu, x^*, y^*) &= \alpha_i y^* + \widehat{H}_1(\mu_i, x^*, y^*) \\
H_2(\mu, x^*, y^*) &= \beta_i y^{*2} + \mu_i + \gamma_i x^* + \widehat{H}_2(\mu_i, x^*, y^*)
\end{aligned}
\tag{3.3}
$$

$$
\text{where } \alpha_i = \frac{\partial H_1(0,0,0)}{\partial y^*}, \qquad \beta_i = \frac{\partial^2 H_2(0,0,0)}{\partial y^{*2}}, \qquad \gamma_i = \frac{\partial H_2(0,0,0)}{\partial x^*}
$$

REMARK 3.6 Observe that the construction in the subsection 3.5 is applicable for $f \in \mathrm{Diff}^2(M)$. Note that $\alpha_i, \beta_i, \gamma_i$ depend continuously on $f \in \mathrm{Diff}^2(M)$: in fact, the numbers $\alpha_i$ and $\gamma_i$ are first order derivatives of the $C^1$- functions $H_1$ and $H_2$ which depend continuously on the given $f \in \mathcal{N}$. And $\beta_i$ is a second order derivative along the stable foliation, which is of $C^2$ class, and depends continuously on $f$, due to the regularity of the chosen local coordinates and its continuous dependence on $f$, as observed in Remark 3.2.

LEMMA 3.7 $\alpha_i(f) \neq 0$, $\beta_i(f) \neq 0$ and $\gamma_i(f) \neq 0$ for all $f$ near enough $f_1$ in the $C^2$ topology.

*Proof:* The result is due to the quadratic hypothesis. In fact, $\partial H_2(0,0,0)/\partial y^* = 0$ due to the tangency at the point $f^{N_i}(r_i)$. That is why we have chosen $r_i$ such that $f^{N_i}(r_i)$ belongs to the line of tangencies $L_f \subset V$. As $f_i$ is a diffeomorphism, the derivative $Df_i(0,0,0) \neq 0$ and so its determinant is not null, i.e. $\alpha_i \gamma_i \neq 0$.

Besides, the tangency is quadratic and therefore Definition 1.1 holds. Consider now the change of coordinates from the $C^2$-system given in Definition 1.1 to the coordinates leading to Equations (3.3). We get the relation $\overline{\beta_i} \neq 0$ (in Definition 1.1) if and only if $\beta_i/\alpha_i^2 \neq 0$ in Equations (3.3). Then $\beta_i \neq 0$ as wanted. $\square$

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LEMMA 3.8 *If $\mathcal{N}$ is small enough, then for all $i$ there exists a real constant $K_i > 0$ such that for $f \in \mathcal{N}$ the coefficients $\alpha_i, \beta_i, \gamma_i$ in equations 3.3 verify:*

$$\frac{1}{K_i} \le |\alpha_i|, \ |\beta_i|, \ |\gamma_i| \le K_i$$

*Proof:*

It is not restrictive to suppose a bounded small open set $\mathcal{N} \subset \text{Diff}^3(M)$, so for some $\varepsilon_0 > 0$:

$$||f - f_1||_{C^3} \le \varepsilon_0 \quad \forall f \in \mathcal{N} \tag{3.4}$$

We will prove that there exists a positive lower bound of $|\beta_i|$ for all $f \in \mathcal{N}$. The proof of the existence of the upper bound has a similar argument.

By contradiction, suppose that there exists a sequence of diffeomorphisms $g_j \in \mathcal{N} \subset \text{Diff}^3(M)$ such that

$$|\beta_i(g_j)| \le 1/j \ \ \forall j \ge 1 \tag{3.5}$$

The sequence $g_j$ of diffeomorphisms in $\mathcal{N}$ is $C^3$-bounded due to condition (3.4). By the Arzela-Ascoli Theorem there exists a subsequence, which we still call $g_j$, convergent in the $C^2$ topology to a map $g_0 \in \text{Diff}^2(M)$. For this map $g_0$ the number $\beta_i(g_0)$ in Equations (3.3) is still defined and different from zero, due to Lemma 3.7.

As remarked in 3.6, the real number $\beta_i(g)$ depends continuously on $g \in \text{Diff}^2(M)$. Therefore we get:

$$\lim_{j \to \infty} g_j = g_0 \ \Rightarrow \ |\beta_i(g_j)| \to |\beta_i(g_0)| \ne 0$$

Therefore the sequence of real numbers $|\beta_i(g_j)|$ is bounded away from zero, contradicting the inequality (3.5). $\square$

# 4 Approximation to the one-dimensional quadratic family.

We continue arguing as in [PT 1993] Chapter III, Section 4:

Consider $f \in \mathcal{N} \subset \text{Diff}^3(M)$ as in Section 3, and for fixed $i \ge 1$ take the periodic saddle point $P_i \in \Lambda$ and the coordinate system $(x, y)$ defined in Remark 3.2 in the neighborhood $U$ of $\Lambda$.

Take the point $r_i = (x(P_i), a_i)$ in the local unstable vertical arc through $P_i$, and the point $q_i = (b_i, c_i)$ in the horizontal leaf $y = c_i$ contained in the global stable manifold of $P_i$ in $U$, as defined in Section 3 and Figure 3.

We recall equations (3.1) and will consider a change of coordinates in the small open rectangle $V_i \subset U$ near $q_i$ defined in 3.4.

The following change of variables, and also the reparametrization on the value of $\mu_i$, are defined in [PT 1993] Chapter III, Section 4, near a quadratic homoclinic tangency. We have made some minor adaptation to our context, in which the coordinate system $(x, y)$ in the neighborhood $U$ of the horseshoe $\Lambda$ is independent of the saddle point $P_i \in \Lambda$ with which we work. Therefore $P_i$ does not have necessarily coordinates $(0, 0)$. We write:

$$
\begin{aligned}
\widehat{\mu}_i &= \left( \mu_i [\sigma_i^{(n_i)}(y)]^2 + \text{dist}^s(q_i, P_i) \gamma_i [\lambda_i^{(n_i)}(x)][\sigma_i^{(n_i)}(y)]^2 - (a_i - y(P_i))[\sigma_i^{(n_i)}(y)] \right) \beta_i \\
\widehat{x} &= (x - b_i)[\sigma_i^{(n_i)}(y)]\beta_i \alpha_i^{-1} \\
\widehat{y} &= ((y - c_i)[\sigma_i^{(n_i)}(y)]^2 - (a_i - y(P_i))[\sigma_i^{(n_i)}(y)])\beta_i
\end{aligned}
\tag{4.6}
$$

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where the definition of the coefficients $\sigma_i^{(n_i)}(y)$ and $\lambda_i^{(n_i)}(x)$ are in Equations (3.2), and $x(Q)$ and $y(Q)$ denote respectively the abscissa and ordinate of $Q$. We recall that if $n_i$ is large enough then:

$$\sigma_i^{(n_i)}(y) > \widetilde{\sigma}^{n_i}, \qquad \lambda_i^{(n_i)}(x) < \widetilde{\lambda}^{n_i}$$

where $\widetilde{\lambda} < 1$ and $\widetilde{\sigma} > 1$ are the exponential contractive and expansive rates of the hyperbolic set $\Lambda$. The Inverse Function theorem allows us to assert that the former equations define invertible $C^1$ change of coordinates. We recall that at each point, $\lambda\sigma^2 < 1$.

For later use we write the following equations, obtained from 4.6:

$$
\begin{aligned}
\mu_i &= \beta_i^{-1}\widehat{\mu}_i[\sigma_i^{(n_i)}(y)]^{-2} - \operatorname{dist}^s(q_i, P_i)\gamma_i[\lambda_i^{(n_i)}(x)] + (a_i - y(P_i))[\sigma_i^{(n_i)}(y)]^{-1} \\
x &= b_i + \alpha_i\beta_i^{-1}\widehat{x}[\sigma_i^{(n_i)}(y)]^{-1} \\
y &= c_i + (a_i - y(P_i))[\sigma_i^{(n_i)}(y)]^{-1} + \beta_i^{-1}\widehat{y}[\sigma_i^{(n_i)}(y)]^{-2}
\end{aligned}
\tag{4.7}
$$

Given a point $(\widehat{x}, \widehat{y})$ in the new system of coordinates, we apply $f^{n_i+N_i}$ (with $\mu$ constant), using Equations 3.1 for the first $n$ iterates of $f$, and Equations 3.3 for the last $N_i$ iterates. The detailed computations are explicit in [PT 1993] Chapter III, Section 4. We get

$$
\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} \xrightarrow{f^{n_i+N_i}} \begin{pmatrix} F_1(\widehat{x}, \widehat{y}, \widehat{\mu}_i, n_i) \\ F_2(\widehat{x}, \widehat{y}, \widehat{\mu}_i, n_i) \end{pmatrix}
$$

The value of $\widehat{\mu}_i$ is obtained computing $(x, y)$ through the last two equations of 4.7 and then substituting in the first equation 4.6. We note from the first equation 4.6 that being $\mu_i$ constant, the value of $\widehat{\mu}_i$ changes when applying $f^{n_i+N_i}$ because it depends on $(x, y)$ which changes when applying the map.

For the next lemma, we consider in $D = [-1, 1]^3$ the 2-dimensional manifold $\mathcal{S}$ of points $(\widehat{x}, \widehat{y}, \widehat{\mu}_i)$ implicitly defined by equations 4.6 with a fixed value $\mu_i$. It can be written as $\widehat{\mu}_i = g(\widehat{x}, \widehat{y})$. Let us observe that for $n_i$ large enough, $\mathcal{S}$ approaches to a horizontal surface:

LEMMA 4.1  $\frac{\partial \widehat{\mu}_i}{\partial \widehat{x}}$ and $\frac{\partial \widehat{\mu}_i}{\partial \widehat{y}}$ converge uniformly to 0 for $n_i \to \infty$ and $(\widehat{x}, \widehat{y}, \widehat{\mu}_i) \in [-1, 1]^3$.

*Proof*

$$
\left| \frac{\partial \widehat{\mu}_i}{\partial \widehat{y}} \right| = \left| \frac{\partial \widehat{\mu}_i}{\partial x} \cdot \frac{\partial x}{\partial \widehat{y}} + \frac{\partial \widehat{\mu}_i}{\partial y} \cdot \frac{\partial y}{\partial \widehat{y}} \right|
$$

Computing:

$$
\frac{\partial y}{\partial \widehat{y}} = \frac{\beta_i^{-1}[\sigma_i^{(n_i)}]^{-2}}{1 + \left( \frac{a_i - y(P_i)}{[\sigma_i^{(n_i)}]^2} + \frac{2\beta_i^{-1}\widehat{y}}{[\sigma_i^{(n_i)}]^3} \right) \sum_j \sigma_{i,j}^{(n_i-1)}\sigma_{i,j}'}
$$

where $\sigma_{i,j}^{(n_i-1)}$ is a notation for the product $\sigma_i^{(n_i)}$ (see equation (3.2)) where we take out the $j$-th factor, and $\sigma_{i,j}'$ is the notation for the derivative of the omitted factor. It follows that there exists $k_i$ such that if $n_i$ is large:

$$
\left| \frac{\partial y}{\partial \widehat{y}} \right| \leq k_i[\sigma_i^{(n_i)}]^{-2}
$$

Similarly,

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$$\frac{\partial x}{\partial \widehat{y}} = -\frac{\alpha_i \beta_i^{-1} \widehat{x}}{[\sigma_i^{(n_i)}]^2} \sum_j \sigma_{i,j}^{(n_i-1)} \sigma'_{i,j} \frac{\partial y}{\partial \widehat{y}}$$

$$\frac{\partial \widehat{\mu}}{\partial x} = \operatorname{dist}^s(q_i, P_i) \gamma_i \left( \sum_j \lambda_{i,j}^{(n_i-1)} \lambda'i, j \right) [\sigma_i^{(n_i)}(y)]^2 \beta_i$$

$$\frac{\partial \widehat{\mu}}{\partial y} = \left( 2 \left( \mu_i + \operatorname{dist}^s(q_i, P_i) \gamma_i [\lambda_i^{(n_i)}(x)] \right) [\sigma_i^{(n_i)}(y)] - (a_i - y(P_i)) \right) \beta_i \sum_j \sigma_{i,j}^{(n_i-1)} \sigma'_{i,j}$$

Moreover, from the definition of strongly dissipative horseshoe and the first equality of (4.7), it follows that there exists $k_i$ large enough such that $|\mu_i| \leq k_i(\sigma_i^{(n_i)})^{-2}$. Therefore, increasing $k_i$ if necessary,

$$\left| \frac{\partial \widehat{\mu}}{\partial y} \right| \leq k_i n_i [\sigma_i^{(n_i)}(y)]$$

We will take numbers $\lambda^*$, $\lambda^+$, $\mu^*$, $\mu^+$ with $\lambda^* < \lambda(x) < \lambda^+ < 1 < \sigma^* < \sigma(y) < \sigma^+ \ \forall\,(x,y) \in V_i$ such that $\lambda^+ \sigma^+ < 1$.

$$\left| \frac{\partial \widehat{\mu}_i}{\partial \widehat{y}} \right| \leq k_i (n_i^2 (\lambda^+ \sigma^+)^{n_i} (\sigma^*)^{-2n_i} + n_i \sigma^{*-n_i}) \longrightarrow_{n_i \to \infty} 0$$

uniformly in $(\widehat{x}, \widehat{y}, \widehat{\mu}_i) \in [-1,1]^3$ as wanted. Analogously it is proved for $\frac{\partial \widehat{\mu}_i}{\partial \widehat{x}}$. $\quad\square$

We conclude that taking $n_i \to \infty$, $f^{n_i+N_i}|_{V_{n_i}}$ converges in the $C^1$ topology, uniformly to the asymptotic map:

$$\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} \mapsto \begin{pmatrix} \widehat{y} \\ \widehat{y}^2 + \widehat{\mu} \end{pmatrix} \tag{4.8}$$

REMARK 4.2 Note that the family defined by Equation 4.8 is the one-dimensional quadratic family with parameter $\widehat{\mu}$. It is standard to verify that this quadratic family exhibits a fixed point which is a sink for the parameter values $\widehat{\mu} \in \left(-\frac{3}{4}, \frac{1}{4}\right)$ and that its basin of attraction includes all points $(\widehat{x}, \widehat{y})$ with $\widehat{y}$ in the interval $(-1/4, 1/4)$.

Even more, if $\widehat{\mu} < -3/4$ of if $\widehat{\mu} > 1/4$, the fixed point in the one-dimensional quadratic map does not exist or it is not a sink.

Observe that for the one-dimensional quadratic family, the sink has two eigenvalues: one is always zero, along the horizontal lines $y = $constant, because it has infinite contraction transforming the horizontal line onto one single point. The other eigenvalue is the slope at the sink of the parabola $\widehat{y} \mapsto \widehat{y}^2 + \widehat{\mu}$.

We note that for $\widehat{\mu} = -3/4$ the sink has an eigenvalue equal to $-1$ and the quadratic unidimensional family exhibits there a period doubling bifurcation. On the other hand, if $\widehat{\mu} = 1/4$ the sink has eigenvalue equal to 1, and the family has a saddle node bifurcation. For $\widehat{\mu} \in (-3/4, 1/4)$ the slope of the parabola at the sink, (being less than 1 in absolute value), is continuous and monotone with $\widehat{\mu}$. Therefore, given any $0 < \rho < 1$, there exist numbers $-3/4 < 2k^-(\rho) < 0 < 2k^+(\rho) < 1/4$ such that if $\widehat{\mu} \in (2k^-(\rho), 2k^+(\rho))$ then the sink has both eigenvalues smaller than $\rho$ in absolute value.

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After the changes of coordinates and the reparametrization given in Equations (4.6), $f^{n_i+N_i}$ converges uniformly to the quadratic family when $n_i \to +\infty$. The speed of convergence depends on the values of the hyperbolic expansive rates $\sigma(f), \lambda(f)$ in the horseshoe exhibited by $f$ and also of the values of $\alpha_i(f), \beta_i(f), \gamma_i(f), \|\widehat{H}_1(f)\|_{C^0}, \|\widehat{H}_2(f)\|_{C^0}$, defined in equations (3.3). Due to Lemma 3.8, these are uniformly bounded for all $f \in \mathcal{N}$.
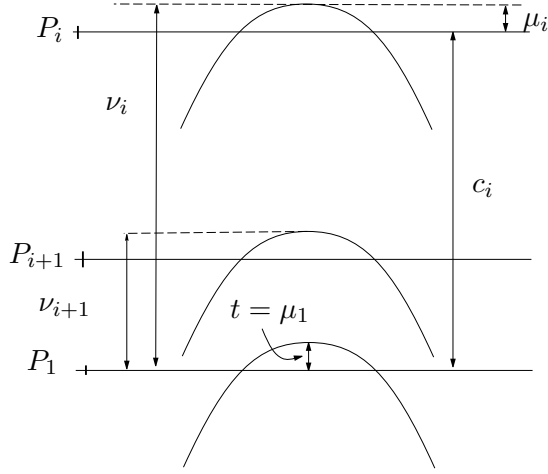


Figure 4: Construction of sinks.

We recall the notation of subsection 3.5 and Figures 3 and 4. We choose a large enough natural number $n_i$ and the small rectangle $V_i \subset V$, defined in Remark 3.4, to apply Equations (4.6) which lead asymptotically to the one-dimensional quadratic family, and apply the results in Remark (4.2). We now resume in Proposition 4.3 all the conclusions obtained in this section.

PROPOSITION 4.3 **(A)** *If for some real number $\widehat{\mu} \in (-3/8,\ 1/8)$ and for some $n_i$ large enough is verified*

$$
\begin{aligned}
\nu_i(f) &= c_i(f) + \mu_i(f) = \\
&= c_i(f) + (a_i(f) - y(P_i)) \cdot [\sigma_i(f)^{(n_i)}(\nu_i(f))]^{-1} + \\
&\quad + \widehat{\mu} \cdot \frac{[\sigma_i(f)^{(n_i)}(\nu_i(f))]^{-2}}{\beta_i(f)} - \mathrm{dist}^s(q_i, P_i) \cdot \gamma_i(f) \cdot [\lambda_i(f)^{(n_i)}(b_i(f))]
\end{aligned}
\tag{4.9}
$$

*then $f$ exhibits a sink $s_i = s_i(f)$ in the given open set $V$.*

**(B)** *If $f, g \in \mathcal{N}$ are arc-connected in $\mathcal{N}$ by a one-parameter family $\{\widetilde{f}_t\}_t$ that verifies the equality (4.9) for some $C^1$ real function*

$$\widehat{\mu}(\widetilde{f}_t) \in (-3/8,\ 1/8)\ \ \forall t$$

*then the sink $s_i(g)$ is the continuation of the sink $s_i(f)$.*

**(C)** *If $f$ and $g$ verify equality (4.9) for some $\widehat{\mu}(f) \in (-3/8,\ 1/8)$ and some $\widehat{\mu}(g) < -1$ or $\widehat{\mu}(g) > 1$, then there does not exist the continuation of the sink $s_i(f)$ for such $g$.*

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**(D)** *Given $0 < \rho < 1$ there exist constants $-3/8 < k^-(\rho) < 0 < k^+(\rho) < 1/8$ such that if for some real number $\widehat{\mu} \in (k^-(\rho), k^+(\rho))$ and for some $n_i$ large enough is is verified equation (4.9) then $f$ exhibits a sink $s_i = s_i(f)$ in the open set $V$ with both eigenvalues smaller than $\rho$.*

*Proof:* Recall that $\nu_i(f) = c_i(f) + \mu_i(f)$ and take into account (4.7). The real number $\widehat{\mu}$ in equation (4.9) is the parameter $\widehat{\mu}_i$ in the first equality of (4.7) for $x = b_i(f)$ and $y = \nu_i(f)$. As $\widehat{\mu} \in (-3/8, 1/8)$ and $n_i$ is large enough the reparametrized map $f^{n_i+N_i}$ is uniformly near the quadratic family and due to the remark 4.2, it exhibits a sink.

Part B is a consequence of the Implicit Function Theorem applied to (4.7).

Parts C and D follow after the remark 4.2. $\square$

# 5   Uniform sized continuation of the sinks.

Given $f_1 \in \mathrm{Diff}^3(M)$ verifying the hypothesis of Theorem 2.8 let us consider the neighborhood $\mathcal{N} \subset \mathrm{Diff}^3(M)$ of $f_1$ as in Section 3.

Take a small neighborhood $U$ of the horseshoe $\Lambda$ and the coordinate system as in Section 3. In this Section we shall assume the hypothesis of the strong dissipative horseshoe $\lambda\sigma^2 < 1$ so the local coordinate system that trivializes the foliation of $\Lambda$ is of $C^3$ class.

Consider the homoclinic tangency point $q_1 = (b_1, 0) \in U$ of the saddle $P_1 = (0,0) \in \Lambda$ as in the hypothesis of Theorem 2.8, and the line of tangencies $L_1 = L(f_1) \ni q_1$ in $V$.

Let us define the following $C^1$ manifold $\mathcal{N}_1 \subset \mathcal{N} \subset \mathrm{Diff}^3(M)$, which has infinite dimension and codimension in $\mathrm{Diff}^3(M)$, contains $f_1$, and will be considered our universe where $f_1$ shall be perturbed.

For any $\delta > 0$ small enough, (to be fixed later) we define:

$$\mathcal{N}_1 = \{f \in \mathcal{N} : f = \xi \circ f_1\} \tag{5.10}$$

where $\xi \in \mathrm{Diff}^3(M)$, $\|\xi - id\|_{C^3} < \delta$ and besides:

$$
\begin{aligned}
&\xi(p) = p \; \forall p \notin V \\
&\exists k = k(\xi) \in \mathbb{R} \text{ such that } \forall\, (x,y) \in L_1 : \\
&\pi_2(\xi(x,y)) = y + k, \quad (D(\pi_2 \circ \xi)(x,y)) \cdot (1,0) = 0
\end{aligned}
\tag{5.11}
$$

Here we denote $\pi_2$ to the horizontal projection $\pi_2(a,b) = b$, which is of $C^3$ class, due to the choice of the coordinate system in Remark 3.2, under the assumption of the strong dissipative hypothesis $\lambda\sigma^2 < 1$.

We observe that $\xi$ is isotopic with the identity map, so $\mathcal{N}_1$ is an arc connected manifold.

LEMMA 5.1 *For $K$ small enough the following set*

$$\mathcal{M}_K = \{f = \xi \circ f_1 \in \mathcal{N}_1 : k(\xi) = K\}$$

*is a $C^1$ submanifold of codimension one in $\mathcal{N}_1$.*

*Proof:*

The map $k : \xi \mapsto k(\xi)$, defined for all $\xi \in \mathrm{Diff}^3(M)$ which verify the conditions (5.11), is the second coordinate of the vector obtained by the evaluation of $\xi - Id$ at $q_1 = (b_1, 0) \in L_1 =$

$L(f_1)$. Therefore, the map $k$ is a $C^1$ real function defined in the set of diffeomorphims $\xi$ verifying conditions (5.11). Besides, its Fréchet derivative respect to $\xi$ is the evaluation $\xi \mapsto \xi(q_1)$ which is a not null linear transformation on $\xi$ in the tangent space of $\mathcal{N}_1$. Therefore the value $K$ is a regular value of the real function $k$, and so the equation $k(\xi) = K$ for $\xi$ such that $\xi \circ f_1 \in \mathcal{N}_1$ defines a $C^1$ submanifold $\mathcal{M}_K \subset \mathcal{N}$ of codimension one in $\mathcal{N}_1$, as wanted. $\square$

The conditions (5.10) and (5.11) mean that we are perturbing $f_1$ only in the neighborhood $V$ near the line of tangencies $L_1 = L(f_1)$ in such a way that we apply a vertical translation of amplitude $k(\xi)$ to $L_1$ to obtain the new line of tangencies $L(f)$ for $f = \xi \circ f_1$, and a horizontal deformation.

In particular we neither perturb the horseshoe $\Lambda$, nor the diffeomorphism in a neighborhood of $\Lambda$. Therefore, the local stable and unstable manifolds of $\Lambda$ are the same, and the system of local coordinates in $U$, as defined in Section 3, does not change when perturbing $f$.

For each $i \geq 1$ we choose any sequence of periodic saddle points $P_i \in \Lambda$ as in the subsection 3.3. Let us suppose a one-parameter family of diffeomorphisms in $\mathcal{N}_1$ having a sequence of diffeomorphism $f_i \in \mathcal{N}_1$ which exhibits a homoclinic tangency at $q_i$ of the saddle $P_i$. We use the notation of subsection 3.4.

We are working along the restricted space $\mathcal{N}_1$ of diffeomorphisms that coincide with $f_1$ in a neighborhood of the horseshoe $\Lambda$. Therefore, the values $\lambda_i(x)$, $\sigma_i(y)$ in Equation (3.1) and of $\lambda_i^{(n)}(x)$ and $\sigma_i^{(n)}(y)$ in Equations (3.2) and (4.6), are the same for all $f \in \mathcal{N}_1$.

When passing from $f_1$ to $f \in \mathcal{N}_1$, the horizontal local stable foliation remains fixed, and we apply a transformation $\xi$ preserving the horizontal direction in the points of the line of tangencies $L_1$ to obtain $L(f)$. Then, the point $q_i = q_i(f) = (b_i(f), c_i)$ remains in the same horizontal line ($c_i$ is fixed) and the point $f^{N_i}(r_i) = (b_i(f), c_i + \mu_i(f)) \in L(f)$ moves from $f_1^{N_i}(r_i) = (b_i(f_1), c_i + \mu_i(f_1)) \in L(f_1)$ a vertical distance $k = k(\xi) = k(f \circ f_1^{-1})$, and slides horizontally preserving its quality of being a point in the line of tangencies. Therefore, the numbers $a_i$, $c_i$ and the point $r_i$, defined in subsection 3.5 and Figure 3, remain the same for all $f \in \mathcal{N}_1$ and $\mu_i(f) - \mu_i(f_1) = k(\xi) = k(f \circ f_1^{-1})$. In particular if $f = f_i = \xi_i \circ f_1$ such that $\mu_i(f_i) = 0$, we obtain $-\mu_i(f_1) = k(\xi_i)$ and therefore $\mu_i(f) + k(\xi_i) = k(\xi) \forall f \in \mathcal{N}_1 \forall i \geq 1$.

We conclude the following:

REMARK 5.2 The points $P_i$ and $r_i$, the real numbers $a_i$ and $c_i$, and the functions $\sigma_i^{(n)}$ and $\lambda_i^{(n)}$ do not depend on the diffeomorphism $f \in \mathcal{N}_1$.

REMARK 5.3 $\mu_1(f) = k(\xi) = k(f \circ f_1^{-1}) \forall f \in \mathcal{N}_1$, and for $f_i = \xi_i \circ f_1 \in \mathcal{N}_1$ such that $\mu_i(f_i) = 0$ we obtain

$$\begin{aligned} k(\xi) &= k(\xi_i) + \mu_i(f) \forall f \in \mathcal{N}_1 \\ \nu_i(f) &= c_i + \mu_i(f) = c_i + k(f \circ f_1^{-1}) - k(f_i \circ f_1^{-1}) \forall f \in \mathcal{N}_1 \end{aligned} \qquad (5.12)$$

We recall that (5.11) assumes $||\xi - id||_{C^3} < \delta$, so we obtain the following:

LEMMA 5.4 *Given $\varepsilon > 0$ there exists $\delta > 0$ such that if the manifold $\mathcal{N}_1 = \mathcal{N}_1(\delta)$ is constructed fulfilling Equations (5.10) and (5.11), then the number $\beta_i(f)$ defined by the equations (3.3), verifies the following inequalities for all $f \in \mathcal{N}_1$ and all $i \geq 1$:*

$$(1 - \varepsilon)|\beta_i(f_1)| < |\beta_i(f)| < (1 + \varepsilon)|\beta_i(f_1)|$$

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*Proof:* We have $f = \xi \circ f_1$, with $\xi(p) = p \; \forall p \notin V$, and $V$ the neighborhood of the line of tangencies defined in Remark 1.3.

Recall that for all $i$ the point $r_i$, and its first $N_i - 1$ forward iterates, do not lay in $V$, and $f_1^{N_i}(r_i) \in V$. And this also holds for all the points $p$ in a small open neighborhood of $r_i$. As $f(p) = f_1(p) \; \forall p \notin V$, we deduce $f^{N_i}(p) = f \circ f^{N_i - 1}(p) = \xi \circ f_1^{N_i}(p)$ for all the points $p$ in a small open neighborhood of $r_i$.

We recall the definition in equalities (3.3): the number $\beta_i(f)$ is the second order partial derivative respect to $y$ of the $C^3$ transformation $\pi_2 \circ f^{N_i} = \pi_2 \circ \xi \circ f_1^{N_i}$ at the point $r_i$.

We will work in a new $C^2$ system of coordinates (to be able to apply the chain rule), such that the lines $y$ constant coincide with the stable foliation in $U$ and we take a non invariant foliation as $x$ constant. In these new coordinates the value of $\beta_i(f)$ is the same as in the former system. The first derivative respect to $x$ of $\pi_2 \circ \xi$ at $f_1^{N_1}(r_i) \in L(f)$ is null due to our assumption that $D(\pi_2 \circ \xi)(1, 0) = 0$ in the line of tangencies $L(f)$. On the other hand, its first derivative respect to $y$ is in $(1 - \delta, 1 + \delta)$ due to $||\xi - id||_{C^3} < \delta$.

Now, denoting $(u, v) = f_1^{N_i}(x, y)$:

$$\frac{\partial(\pi_2 f^{N_i})}{\partial y} = \frac{\partial(\pi_2 \xi)}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial(\pi_2 \xi)}{\partial v} \frac{\partial v}{\partial y}$$

and then, (we omit the points at which we evaluate the partial derivatives) using that $\frac{\partial(\pi_2 \xi)}{\partial u} = 0$, $\frac{\partial v}{\partial y} = 0$ it follows:

$$\beta_i(f) = \frac{\partial^2(\pi_2 f^{N_i})}{\partial y^2} = \frac{\partial^2(\pi_2 \xi)}{\partial u^2} \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial(\pi_2 \xi)}{\partial v} \frac{\partial^2 v}{\partial y^2}$$

The first term is bounded by $\delta \alpha_i^2(f_1)$, which can be taken smaller than $\beta_i(f_1)\varepsilon/2$ taking $\delta < \inf_i\{\frac{\varepsilon \beta_i}{2\alpha_i^2}\}$. We note that $\frac{\beta_i}{\alpha_i^2}$ is the concavity of the unstable parabolic arcs which are uniformly bounded away from 0 due to the quadratic hypothesis. The second term belongs to $((1 - \delta)\beta_i(f_1), (1 + \delta)\beta_i(f_1))$. Then, taking $\delta < \varepsilon/2$, $\beta_i(f)$ belongs to the interval $((1 - \varepsilon)\beta_i(f_1), (1 + \varepsilon)\beta_i(f_1))$. $\square$

LEMMA 5.5 *For each $i \geq 1$ there exist real constants $m_i$, $K_i > 0$ and $\nu_i^{(0)}$ (that are independent of $f \in \mathcal{N}_1$) such that if $n_i > m_i$ then the implicit function $\nu_i(f)(\widehat{\mu}) = G_f(\widehat{\mu})$ defined by the equation (4.9) in Lemma 4.3, verifies:*

$$|G_f(0) - \nu_i^{(0)}| \leq K_i(\max_{(x,y) \in \overline{V}_i}\{\lambda(x)\})^{n_i}$$

$$\frac{(\max_{(x,y) \in \overline{V}_i}\{\sigma(y)\})^{-2n_i}}{(1 + \varepsilon)|\beta_i(f_1)|} < |G_f'(\widehat{\mu})| \approx \frac{[\sigma_i^{(n_i)}(\nu_i)]^{-2}}{|\beta_i(f)|} < \frac{\widetilde{\sigma}^{-2n_i}}{(1 - \varepsilon)|\beta_i(f_1)|} \; \forall \widehat{\mu} \in (-2, 2)$$

*Moreover the constant number $\nu_i^{(0)}$ is the sum of the following two terms, each independent of $f \in \mathcal{N}_1$:*

$$\nu_i^{(0)} = c_i + (a_i - y(P_i)) \cdot [\sigma_i^{(n_i)}(\nu_i^{(0)})]^{-1} \tag{5.13}$$

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*Proof:* See figure 5. The equality (4.9) in Lemma 4.3 which defines the implicit function $\nu_i(\widehat{\mu}) = G_f(\widehat{\mu})$ has now the following expression, due to Remark 5.2:

$$
\begin{aligned}
\nu_i \;=\; & c_i + (a_i - y(P_i)) \cdot [\sigma_i^{(n_i)}(\nu_i)]^{-1} + \\
& + \widehat{\mu} \cdot \frac{[\sigma_i^{(n_i)}(\nu_i)]^{-2}}{\beta_i(f)} - \mathrm{dist}^s(q_i, P_i) \cdot \gamma_i(f) \cdot [\lambda_i^{(n_i)}(b_i(f))]
\end{aligned}
\tag{5.14}
$$

If $\widehat{\mu} = 0$ the equation (5.14) depends on $f \in \mathcal{N}_1$ only because its last term does, and defines $G_f(0)$.

On the other hand, the equation (5.13), which is independent on $f \in \mathcal{N}_1$, defines $\nu_i^{(0)}$.

Subtracting (5.13) and (5.14) with $\widehat{\mu} = 0$ and applying the Lagrange Theorem we obtain:

$$
|G_f(0) - \nu_i^{(0)}| = \frac{|\mathrm{dist}^s(q_i, P_i)\gamma_i(f)[\lambda_i^{(n_i)}(b_i(f))]|}{1 + (a_i - y(P_i))[\sigma_i^{(n_i)}]^{-2}[\sum_j \sigma_{i,j}^{(n_i-1)}(Y)]\sigma_{i,j}'(Y)}
$$

where $Y$ is an intermediate value between $\nu_i(0)$ and $G_f(0)$. Using inequality (3.2) and arguing as in Lemmas 3.8 and 4.1 we obtain:

$$
|G_f(0) - \nu_i^{(0)}| \leq k_i[\max_{(x,y)\in\overline{V}_i} \lambda(x)]^{n_i}
$$

We now compute its derivarive $G_f'$ respect to $\widehat{\mu}$ in any point where $G_f$ is defined:

$$
G_f' = \frac{\left[\sigma_i^{(n_i)}\right]^{-2}}{\beta_i(f) \cdot \left(1 + (a_i - y(P_i)) \cdot \left[\sigma_i^{(n_i)}\right]^{-2} \sum_j \left[\sigma_{i,j}^{(n_i-1)}\right]\sigma_{i,j}'\right) + 2\widehat{\mu} \cdot \left[\sigma_i^{(n_i)}\right]^{-3} \sum_j \left[\sigma_{i,j}^{(n_i-1)}\right]\sigma_{i,j}'}
$$

Arguing as in Lemmas 3.8 and 4.1 we conclude that it is uniformly bounded for $f \in \mathcal{N}_1$.

Finally, using Lemma 5.4, the inequalities (3.2) and recalling that $n_i$ is large enough, we deduce the bounds of $|G_f'(\widehat{\mu})|$ in the thesis. $\square$

Combining the results in Proposition 4.3 and Lemma 5.5 we obtain the following:

LEMMA 5.6 *For each $i \geq 1$ and for each sufficiently large $n_i$ there exist constants $\nu_i^- < \nu_i^0 < \nu_i^+$, independent of $f \in \mathcal{N}_1$, such that:*

$$
|\nu_i^+ - \nu_i^-| = \frac{1}{8(1+\varepsilon)\beta_i(f_1)}(\max_{y\in\overline{V}_i}\{\sigma(y)\})^{-2n_i}
$$

$$
\nu_i^{(0)} = c_i + (a_i - y(P_i))[\sigma_i^{(n_i)}(\nu_i(0)]^{-1}
$$

*and, if*

$$
f \in \mathcal{N}_1 : \quad \nu_i(f) \in (\nu_i^-, \nu_i^+)
\tag{5.15}
$$

*then $f$ exhibits a sink $s_i(f)$. Even more, if $f, g \in \mathcal{N}_1$ are arc connected in $\mathcal{N}_1$ and all the connecting arc verifies the condition (5.15), then the sink $s_i(g)$ is the continuation of the sink $s_i(f)$.*

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*Proof:* See figure 5. Consider the $C^1$ real function $\nu_i(f) = G_f(\widehat{\mu})$ of real variable $\widehat{\mu}$ defined as in Lemma 5.5. It is strictly monotone because its first derivative is never zero. Applying the Lagrange Theorem to $G_f$, and the lower bound of its derivative given in Lemma 5.5, we deduce that the images by $G_f$ of the intervals $(-3/8, 0]$ and $[0, 1/8)$ are two intervals of length:

$$|G_f(1/8) - G_f(0)| = \frac{|G_f'(\widehat{\mu}_f^{(1)})|}{8} > \frac{1}{8(1+\varepsilon)\beta_i(f_1)}(\max_{y \in \overline{V}_i}\{\sigma(y)\})^{-2n_i} = C_i > 0 \qquad (5.16)$$

$$|G_f(0) - G_f(-3/8)| = \frac{3|G_f'(\widehat{\mu}_f^{(2)})|}{8} > \frac{3}{8(1+\varepsilon)\beta_i(f_1)}(\max_{y \in \overline{V}_i}\{\sigma(y)\})^{-2n_i} = 3C_i > 0$$

where $C_i$ is a constant independent of $f \in \mathcal{N}_1$, but depending on $n_i$.

On the other hand, due to Lemma 5.5, there exists a real number $\nu_i^{(0)}$, independent of $f \in \mathcal{N}_1$ such that

$$|\nu_i^{(0)} - G_f(0)| \leq K_i[\max_{(x,y) \in \overline{V}_i} \lambda(x)]^{n_i}, \quad G_f(0) \in G_f((-3/8, 1/8))$$

As the horseshoe is strongly dissipative, $\lambda^{n_i}(x) \ll \sigma^{-2n_i}(y)$ if $n_i$ is large enough, for all points of the rectangle $\overline{V}_i$, in particular, for the point $(x, y)$ where $\max \lambda(x)$ and $\max \sigma(y)$ are obtained. So we can assume that $|\nu_i^{(0)} - G_f(0)| < C_i/2$ and then $[\nu_i^{(0)} - C_i/2, \nu_i^{(0)} + C_i/2] \subset G_f((-3/8, 1/8))$.

We define $\nu_i^+ = \nu_i^{(0)} + C_i/2$ and $\nu_i^- = \nu_i^{(0)} - C_i/2$. Both values are independent of $f \in \mathcal{N}_1$ and included in the image by $G_f$ of the interval $(-3/8, 1/8)$.

We use the definition of the constant $C_i$ in the Equality (5.16) to get the exact value of $|\nu_i^+ - \nu_i^-|$.

The real function $G_f(\widehat{\mu})$ is strictly monotone. Therefore, given $\nu_i(f) \in (\nu_i^-, \nu_i^+)$ there exists a single real replacement $1/8$) such that $\nu_i(f) = G_f(\widehat{\mu})$. Therefore, the hypothesis of Proposition 4.3 is fulfilled, and so its thesis about the existence and continuation of the sink $s_i$ is verified. $\square$
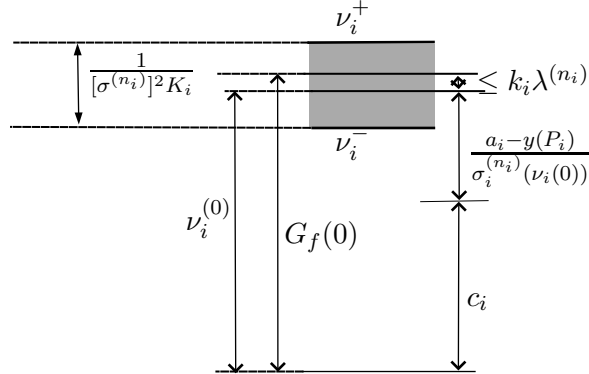


Figure 5: Determination of parameters for the sinks

PROPOSITION 5.7 *The infinite dimensional arc-connected manifold $\mathcal{M}_K \subset \mathcal{N}_1$ in Lemma 5.1, (which has codimension one in $\mathcal{N}_1$) verifies, for each $i \geq 1$, the following properties:*

*(a) If $f \in \mathcal{M}_K$ with the constant $K = k(f_i \circ f_1^{-1})$, then $f$ exhibits a homoclinic tangency at the point $q_i \in V$ of the saddle $P_i \in \Lambda$.*

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*(b) If $f \in \mathcal{M}_K$ with the constant $K \in k(f_i \circ f_1^{-1}) - c_i + (\nu_i^-, \nu_i^+)$, where $\nu_i^- < \nu_i^+$ are defined as in Lemma 5.6, then $f$ exhibits a sink $s_i(f) \in V$. Even more, if $f, g \in \mathcal{M}_K$ then they are isotopic and the sink $s_i(g)$ is the continuation of the sink $s_i(f)$.*

*Proof:*

As proved in Lemma 5.1, the manifolds $\mathcal{M}_K$ with $K$ constant are codimension one submanifolds of $\mathcal{N}_1$.

First choose, for each $i \geq 1$, a fixed $f_i \in \mathcal{N}_1$ such that $f_i$ exhibits a homoclinic quadratic tangency at the point $q_i \in V$. Due to the definition of $\mu_i(f)$ in subsection 3.5, such $f_i$ verifies $\mu_i(f_i) = 0$. Recall Equalities (5.12) and note that for all $f \in \mathcal{N}_1$: $\mu_i(f) = 0$ if and only if $f$ exhibits a homoclinic tangency at the point $q_i$. This condition is fulfilled if and only if $k(f \circ f_1^{-1}) = k(f_i \circ f_1^{-1})$, which proves part a).

To prove part b) argue similarly, using Equalities (5.12) with $\nu_i(f) \in (\nu_i^-, \nu_i^+)$, and applying Lemma 5.6. $\square$

# 6 Proof of Theorem 2.8.

For the given diffeomorphism $f_1$ as in the hypothesis of Theorem 2.8, we shall work along the infinite dimensional manifold $\mathcal{N}_1 \ni f_1$ of $C^3$ diffeomorphisms, defined in Section 5, by conditions (5.10) and (5.11).

Let us construct, as in Newhouse-Robinson theorem (see [N 1974] and [R 1983]), a sequence of sinks $s_i$ which are produced along a monoparametric family of diffeomorphisms (which we will call *primary family*), which generically unfolds a sequence of homoclinic quadratic tangencies $q_i$. By induction, each sink $s_{i+1}$ shall be produced while the $i$ sinks that were previously generated, still survive. The key to get this result is the persistence of tangencies of the Theorem of Newhouse in [N 1974]. For a seek of completeness we reproduce here the details of the inductive proof of Newhouse-Robinson Theorem to obtain infinitely many simultaneous sinks. We improve the argument, adding the conclusions of our previous sections, to obtain also the simultaneous continuation of the infinitely many sinks.

DEFINITION 6.1 Given a small enough real number $\delta > 0$ we fix a one-parameter family $\{\widetilde{f}_t\}_{t \in (-\varepsilon, \varepsilon)} \in \mathcal{N}_1$, called the *primary family*, such that:

$\widetilde{f}_t = \xi_t \circ f_1$ where $\xi_t$ verifies the conditions (5.11) and besides:

$\|\xi_t - \text{id}\|_{C^3} < \delta \forall t \in (-\varepsilon, \varepsilon)$

$\xi_0 = Id$, and thus $\widetilde{f}_0 = f_1$ and $k(\xi_0) = 0$

$k(\xi_t) = t$

We call "secondary family" $g_t$ to any other one-parameter family $\{g_t\}_{t \in (-\varepsilon, \varepsilon)} \in \mathcal{N}_1$ such that $k(g_t \circ f_1^{-1}) = k(\widetilde{f}_t \circ f_1^{-1}) = t$ for all $t \in (-\varepsilon, \varepsilon)$.

We observe that the primary family is transversal to the manifolds $\mathcal{M}_K$ defined in Lemma 5.1, for all $K \in (-\varepsilon, +\varepsilon)$, and that it unfolds generically any homoclinic tangency $q_i$ produced in the line of tangencies $L_{\widetilde{f}_t}$, in particular the given homoclinic tangency $q_1 \in L_{f_1}$. After the density of tangencies (see [N 1974]) the hypothesis of large thickness of the stable and unstable Cantor sets $K^s$ and $K^u$ of $\Lambda$ along the line of tangencies $L_f$, assumed in Theorem 2.8, implies the following:

REMARK 6.2 *If $\varepsilon > 0$ is small enough then there exists a dense set of parameter values $t \in (-\varepsilon, \varepsilon)$ in the primary family, such that the diffeomorphism $\widetilde{f}_t$ exhibits a homoclinic tangency in some point $q$ in the line of tangencies $L_{\widetilde{f}_t}$ of some saddle periodic point $P \in \Lambda$.*

We take the system of coordinates in the open neighborhood $U \supset \Lambda$, as defined in Section 3 and such that the saddle $P_1 = (0,0)$. We shall choose the tangency $q_1 \in U$ such that $q_1 = (b_1, 0)$ (i.e. in the connected local stable leaf of $P_1$), and then the small neighborhood $q_1 \in V \subset U$ (see Remark 1.3). We use the notation of Section 5:

We have $t = 0$, $k(\xi_0) = k(id) = 0$ and $\mu_1(f_1) = 0$, $\nu_1(f_1) = c_1 + \mu_1(f_1) = 0$ and there is a tangency at the point $q_1 = (b_1, 0) \in L_1 = L(f_1)$.

Applying Lemma 5.6 we choose $n_1$ sufficiently large so the fixed numbers $\nu_1^-$ and $\nu_1^+$ verify $|\nu_1^{\pm}| < \varepsilon$. Applying Proposition 5.7, if $t = k(\widetilde{f}_t \circ f_1^{-1}) \in (\nu_1^-, \nu_1^+)$ then $\widetilde{f}_t$ exhibits a sink $s_1(\widetilde{f}_t)$ and for all secondary diffeomorphism $g \in \mathcal{N}_1$ such that $k(g \circ f_1^{-1}) = t$ there exists the continuation $s_1(g)$ of the sink $s_1(\widetilde{f}_t)$.

We now argue by induction in the number of simultaneous sinks exhibited by $\widetilde{f}_t$ in the primary family:

Let us suppose that there exist parameter values $-\varepsilon < t_i^- < t_i^+ < \varepsilon$ such that for all $t \in (t_i^-, t_i^+)$ the diffeomorphism $\widetilde{f}_t$ of the primary family exhibits $i$ sinks $s_1(\widetilde{f}_t), s_2(\widetilde{f}_t), \ldots, s_i(\widetilde{f}_t) \in V$, and any secondary diffeomorphism $g_t$ exhibits the continuations $s_1(g_t), s_2(g_t), \ldots, s_i(g_t) \in V$ of those $i$ sinks.

We shall construct an interval with non void interior $[t_{i+1}^-, t_{i+1}]^+ \in (t_i^-, t_i^+)$ such that if $t \in (t_{i+1}^-, t_{i+1}^+)$ then the diffeomorphisms $\widetilde{f}_t$ of the primary family exhibit a new sink $s_{i+1}(\widetilde{f}_t) \in V$, and besides all secondary diffeomorphisms $g_t$ with $t \in (t_{i+1}^-, t_{i+1}^+)$ exhibit the continuation $s_{i+1}(g_t)$ of $s_{i+1}(\widetilde{f}_t)$.

After the density of tangencies (see Remark 6.2), there exists a parameter value

$$t_{i+1} \in \left( t_i^- + \frac{t_i^+ - t_i^-}{4}, t_i^- + \frac{3(t_i^+ - t_i^-)}{4} \right) \tag{6.17}$$

such that some periodic saddle point $P_{i+1}$ in the horseshoe $\Lambda$, has a homoclinic tangency at $q_{i+1} = (b_{i+1}, c_{i+1}) \in L(\widetilde{f}_{t_{i+1}})$.

Unfolding the tangency of $f_{i+1} = \widetilde{f}_{t_{i+1}}$ when moving along the primary family, we will create a new sink $s_{i+1}$ in such a way that $t$ is still in the interval $(t_i^-, t_i^+)$ where the $i$ previous sinks still persist.

To construct the new sink $s_{i+1}$ and a parameter interval inside $(t_i^-, t_i^+)$ in which this new sink is exhibited, we argue as follows:

If $t = t_{i+1}$ verifies condition (6.17), there exists a homoclinic tangency at $q_{i+1} = (b_{i+1}, c_{i+1})$ of a saddle $P_{i+1} \in \Lambda$. Therefore the height of the parabolic unstable arc of $P_{i+1}$ is $\mu_{i+1}(f_{i+1}) = 0$, $\nu_{i+1} = c_{i+1}$ (see Figure 4). On the other hand, by Definition 6.1 of the primary family, we have: $t_{i+1} = k(f_{i+1} \circ f_1^{-1})$.

Applying Lemma 5.6, we shall find the parameter values $t_{i+1}^- < t_{i+1}^+$ and the diffeomorphisms $\widetilde{f}_t$ in the primary family such that $\nu_{i+1}(\widetilde{f}_t) \in [\nu_{i+1}^-, \nu_{i+1}^+]$ if $t \in [t_{i+1}^-, t_{i+1}^+] \subset (t_i^-, t_i^+)$. Also from Lemma 5.6, we can choose a sufficiently large $n_{i+1}$ so that $|\nu_{i+1}^{\pm} - c_{i+1}| < (t_i^+ - t_i^-)/8$.

After equalities (5.12), and recalling that $t = k(\widetilde{f}_t \circ f_1^{-1})$ we obtain: $\nu_{i+1}(\widetilde{f}_t) - c_{i+1} = t - t_{i+1}$. So, if $t \in t_{i+1} - c_{i+1} + (\nu_{i+1}^-, \nu_{i+1}^+)$, then $\nu_{i+1}(\widetilde{f}_t) \in [\nu_{i+1}^-, \nu_{i+1}^+]$, and applying Lemma 5.6, the map $\widetilde{f}_t$ will exhibit a new sink $s_{i+1}(\widetilde{f}_t)$.

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Applying Proposition 5.7, for any secondary family $\{g_t\}_t$, if $t = k(g \circ f_1^{-1}) \in t_{i+1} - c_{i+1} + [\nu_{i+1}^-, \nu_{i+1}^+]$ then the map $g_t$ will exhibit the continuation $s_{i+1}(g_t)$ of the sink $s_{i+1}(\widetilde{f}_t)$.

We then define

$$t_{i+1}^\pm = t_{i+1} - c_{i+1} + \nu_{i+1}^\pm \in (t_{i+1} - (t_i^+ - t_i^-)/8, t_{i+1} + (t_i^+ - t_i^-)/8)$$

From condition (6.17), the equality above implies $[t_{i+1}^-, t_{i+1}^+] \subset (t_i^-, t_i^+)$ as wanted. We conclude that for $t \in (t_{i+1}^-, t_{i+1}^+)$ the diffeomorphisms $\widetilde{f}_t$ of the primary family exhibit the sink $s_{i+1}(\widetilde{f}_t)$ and the secondary diffeomorphisms $g_t$ exhibit the continuation $s_{i+1}(g_t)$ of that sink.

We observe that we can make this new sink $s_{i+1}$ of arbitrary period $n_{i+1} + N_{i+1}$, provided that it shall be large enough, because in Lemma 5.6 we can arbitrarily choose the natural number $n_{i+1}$, from a minimum value. If we choose $n_{i+1}$ such that $n_{i+1} + N_{i+1}$ is not a multiple of the periods of the previous $i$ sinks $s_1, s_2, \ldots, s_i$, then the sink $s_{i+1}$ shall be necessarily a new one.

Finally, taking $t_\infty = \bigcap_{i=1}^\infty [t_i^-, t_i^+]$, as we constructed each compact interval in the interior of the previous one, the real value $t_\infty$ is in the interior of all intervals, and thus, by construction, there exists the sink $\{s_i(g)\}$ for all $i \geq 1$ and for all diffeomorphism $g \in \mathcal{M}_{t_\infty}$, being each sink $s_i(g)$ the continuation of the respective sink $s_i(\widetilde{f}_{t_\infty})$ exhibited by the diffeomorphism $\widetilde{f}_{t_\infty}$ in the primary family. This ends the proof of Theorem 2.8. □

# 7 Conclusion of the main results.

*End of the Proof of Theorem* 1:

Due to Newhouse-Robinson Theorem 2.6, we find a diffeomorphism $f_1 \in \mathcal{N}$, arbitrarily near the given $f_0$, such that $f_1$ has a horseshoe $\Lambda$, which is strongly dissipative and fulfils the condition of large thickness, and besides there is an homoclinic quadratic tangency $q_1$ of a saddle $P_1 \in \Lambda$. These last assertions are the hypothesis of Theorem 2.8 which we have already proved in the last section, ending the proof of Theorem 1. □

*Proof of part A) of Theorem* 2 (see Figure 3): The given one-parameter family $\{\widetilde{f}_t\}_{t \in (-\varepsilon, \varepsilon)}$ generically unfolds the quadratic tangency at $q_0$ of a saddle point $P_0$. The generic unfolding is defined by the condition. $v = d\mu_0(\widetilde{f}_t)/dt \neq 0$ for all $t \in (-\varepsilon, \varepsilon)$, where $\mu_0(\widetilde{f}_t)$ is the height of the parabolic arc in the unstable leaf of the saddle $P_0$ respect to the local stable arc of $P_0$, to which it is tangent when $t = 0$ (i.e. $\mu_0(\widetilde{f}_t)|_{t=0} = 0$).

After the Theorem of Newhouse-Robinson (revisited in Theorem 2.6), there exists an interval $I \subset (-\varepsilon, \varepsilon)$ such that for all $t \in I$ there is a horseshoe $\Lambda$ fulfilling the large thickness condition (see Definition 2.3) which is strongly dissipative (see Lemma 2.7). Even more, Newhouse-Robinson Theorem asserts that there exist a dense set $H \subset I$ of parameter values, such that $\widetilde{f}_t$ exhibits some homoclinic tangency, for all $t \in H$. Let us choose some $t_1 \in H$, such that $\widetilde{f}_{t_1} = f_1$ exhibits such homoclinic tangency at a point $q_1 \in V$ (see remark 1.3) of a saddle $P_1 \in \Lambda$.

Consider for this $f_1$ a small neighborhood $\mathcal{N}$ such that, for all $f \in \mathcal{N}$ (in particular for all $\widetilde{f}_t$ with $t$ near $t_1$), there exist the real numbers $a_1(f), b_1(f), c_1(f), \alpha_1(f), \beta_1(f), \gamma_1(f), \nu_1(f), \mu_1(f)$ defined in Section 3, Figure 3 and Equations (3.3).

Observe that $|d\mu_1(\widetilde{f}_t)/dt| \geq |v|/2 \neq 0$. Suppose $d\mu_1(\widetilde{f}_t)/dt > 0$, then $\mu_1(\widetilde{f}_t)$ is a strictly increasing diffeomorphic function which is zero for $t = t_1$, and there exists $\delta_1 > 0$ such that $\mu_1(\widetilde{f}_t) \in (-\delta_1, \delta_1) \Rightarrow t \in I$.

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We will repeat the well known argument of Newhouse, improving it to get the eigenvalues of the sinks as small as wanted:

First, we shall construct some interval $(t_1^-, t_1^+) \subset I$ of the parameter values $t$ for which $\widetilde{f}_t$ exhibits a sink in $V$, whose eigenvalues have modulus smaller than the given number $0 < \rho < 1$.

Consider, for each $\widetilde{f}_t$, the first equation (4.7) giving $\mu_1(\widetilde{f}_t)$ diffeomorphically as a function of the new parameter $\widehat{\mu}$, for each fixed $n = n_1 \geq 1$. If $n_1$ is large enough, there exist $-\delta_1 < \mu_1^- < \mu_1^+ < \delta_1$ such that if $\mu_1(\widetilde{f}_t) \in (\mu_1^-, \mu_1^+)$ then $\widehat{\mu} \in (k(\rho)^-, k(\rho)^+)$, defined in Proposition 4.3. Therefore, the thesis of this proposition implies that $\widetilde{f}_t$ has a sink in $V$ whose eigenvalues have modulus smaller than $\rho$. Considering that the real function $\mu_1(\widetilde{f}_t)$ depends diffeomorphically on $t$, the preimage by $\mu_1(f_t)$ of the interval $(\mu_1^-, \mu_1^+)$ is an interval $(t_1^-, t_1^+) \subset I$. By construction, if $t \in (t_1^-, t_1^+)$ then $\widetilde{f}_t$ exhibits a sink $s_1$ in $V$ whose eigenvalues have modulus smaller than $\rho$.

Now, by induction, suppose that there is an open interval $(t_i^-, t_i^+) \subset I$ such that, if $t \in (t_i^-, t_i^+)$, then $\widetilde{f}_t$ exhibits $i$ simultaneous different orbits of the sinks $s_1, s_2, \ldots, s_i$ in $V$, whose eigenvalues have all modulus smaller than $\rho$. As the set $H \subset I$, where the homoclinic tangencies are produced, is dense in $I$, we can choose $t_{i+1} \in (t_i^- + (1/4)(t_i^+ - t_i^-), \; t_i^- + (3/4)(t_i^+ - t_i^-))$ such that $\widetilde{f}_{t_i}$ exhibits a homoclinic point $q_{i+1} \in V$ of a saddle $P_{i+1}$. As above, the function $\mu_{i+1}(f_t)$ is an increasing diffeomorphism from the interval $(t_i^-, t_i^+)$, to an interval of the real variable $\mu_{i+1}(\widetilde{f}_t)$, such that $\mu_{i+1}(\widetilde{f}_{t_{i+1}}) = 0$. Therefore, there exists $\delta_{n+1} > 0$ such that, if $\mu_{i+1}(\widetilde{f}_t) \in (-\delta_{n+1}, \delta_{n+1})$, then $|t - t_{i+1}| < 1/8(t_i^+ - t_i^-)$.

Arguing as in the first step, if $n_{i+1}$ is large enough there exist $-\delta_{n+1} < \mu_{n+1}^- < \mu_{n+1}^+ < \delta_{n+1}$ such that if $\mu_{n+1}(\widetilde{f}_t) \in (\mu_{n+1}^-, \mu_{n+1}^+)$ then $\widehat{\mu} \in (k(\rho)^-, k(\rho)^+)$, defined in Proposition 4.3. Therefore, the thesis of this proposition implies that $\widetilde{f}_t$ has a sink $s_{i+1}$ in $V$ whose eigenvalues have modulus smaller than $\rho$. This new sink is different from the $i$ sinks that were previously constructed, provided one can choose any integer number $n_{i+1}$ large enough, so one can get the period of the new sink larger and not a multiple, of the periods of the $i$ sinks that were previously constructed.

Considering that the real function $\mu_{i+1}(\widetilde{f}_t)$ depends diffeomorphically on $t$, the preimage by $\mu_{i+1}(f_t)$ of the interval $(\mu_{i+1}^-, \mu_{i+1}^+)$ is an interval

$$[t_{i+1}^-, t_{i+1}^+] \in t_{i+1} + [-(t_i^+ - t_i^-)/8, (t_i^+ . - t_i^-)/8] \subset (t_i^-, t_i^+)$$

By construction, if $t \in (t_{i+1}^-, t_{i+1}^+)$ then $\widetilde{f}_t$ exhibits simultaneously $i+1$ sinks in $V$, whose eigenvalues have modulus smaller than $\rho$.

Finally define $g_0 = \widetilde{f}_{t_\infty}$ where $t_\infty \in \bigcap_{i=1}^{\infty}[t_i^-, t_i^+]$. By construction, the set $J$ of such values where the inifinitely many sinks exist is dense in $I$. $\square$

REMARK 7.1 We observe that in in the proof of part (A) of Theorem 2, the construction of the map $g_0 = f_{t_\infty}$, which exhibits infinitely many sinks in $V$, allows us to choose, for each $i \geq 1$, any integer $n_i$ provided it is large enough. So, we can obtain the same thesis if, besides, we ask $n_i \geq m_i$, where $m_i \to +\infty$ is any previously specified sequence of integer numbers.

*Proof of part B) of Theorem* 2: We will show that there exists a sequence $m_i \to +\infty$ such that, if $g_0$ is constructed as in the proof of part (A) and besides verifying $n_i > m_i$ for all $i \geq 1$, then the thesis (B) of Theorem 2 holds for this $g_0$.

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Choose $\delta > 0$ small enough (to be fixed at the end of the proof) and define the following manifold $\mathcal{N}_1 \subset \mathrm{Diff}^3(M)$, which is $\delta - C^3$ near $g_0$:

$$\mathcal{N}_1 = \{g \in \mathrm{Diff}^3(M) : g = \xi \circ g_0\}$$

where $\xi \in \mathrm{Diff}^3(M)$ is such that $\|\xi - \mathrm{id}\|_{C^3} < \delta$ and besides it verifies conditions (5.11), replacing $g_0$ instead of $f_1$, in a small fixed neighborhood $V$ of the line of tangencies $L_0 = L(g_0)$ (instead of the line of tangencies $L_1 = L(f_1)$).

As in equalities (5.12), we now have

$$k(g \circ g_0^{-1}) = \mu_i(g) - \mu_i(g_0) = \nu_i(g) - \nu_i(g_0) \ \ \forall\, i \geq 1$$

Consider the set $\mathcal{M}$, as follows:

$$\mathcal{M} = \{g \in \mathcal{N}_1 : \ k(g \circ g_0^{-1}) = 0\}$$

As seen in Lemma 5.1, $\mathcal{M}$ is an infinite dimensional, arc connected manifold, with codimension one in $\mathcal{N}_1$. By construction:

$$g \in \mathcal{M} \ \ \Rightarrow \ \ \mu_i(g) = \mu_i(g_0), \ \ \nu_i(g) = \nu_i(g_0) \ \ \forall\, i \geq 1 \tag{7.18}$$

Let us prove that, if $g \in \mathcal{M}$, then the infinitely many sinks $s_i(g_0)$ have continuation sinks $s_i(g)$.

We apply, for $g$ and $g_0$, the respective changes of variables and parameter given by Equations (4.6) and (4.7). We recall from the proof of part (A) of Theorem 2, that $\mu_i(g_0)$ was constructed such that:

$$\widehat{\mu}(g_0) \in (k^-(\rho), k^+(\rho))$$

where

$$-\frac{3}{8} < k^-(\rho) < 0 < k^+(\rho) < \frac{1}{8}$$

are the numbers defined in Proposition 4.3.

By contradiction, if $g \in \mathcal{M} \subset \mathcal{N}_1$ did not have the continuation of the sink $s_i(g_0)$, then, due to Proposition 4.3 we obtain $\widehat{\mu}(g) \notin (-3/8, 1/8)$. In fact, we note that $g_0$ and $g$ are not isotopic by any one-parameter family of diffeomorhphisms $\{g_t\}_{0 \leq t \leq 1}$ such that $\widehat{\mu}(g_t) \in (-3/8, 1/8) \ \forall\, t \in [0, 1]$. So considering in particular some one-parameter family in $\mathcal{M}$, there would exist an interval $[t_0, t_1]$ such that $\widehat{\mu}(g_{t_0}) \in \{k^-(\rho), k^+(\rho)\}$, $\widehat{\mu}(g_{t_1}) \in \{-3/8, 1/8\}$ and $\forall\, t \in [t_0, t_1]$, $\widehat{\mu}(g(t)) \neq 0$.

Therefore:

$$\left| \frac{\widehat{\mu}(g_{t_0}) - \widehat{\mu}(g_{t_1})}{\widehat{\mu}(g_{t_1})} \right| \geq \frac{\min\{|1/8 - k^+(\rho)|, |k^-(\rho) - 3/8|\}}{3/8} = \eta(\rho) = \eta > 0$$

$$0 < \frac{\widehat{\mu}(g_{t_0})}{\widehat{\mu}(g_{t_1})} \notin (1 - \eta, 1 + \eta) \tag{7.19}$$

From equation (7.18), we observe that $\mu_i(g_{t_1}) = \mu_i(g_{t_0})$. Then, taking into account (4.7) in the points $y = \nu_i(g_{t_1}) = \nu_i(g_{t_0})$, $x = b_i(g_{t_1})$ or $x = b_i(g_{t_0})$ and subtracting:

$$0 = \frac{\widehat{\mu}(g_{t_1})}{\beta_i(g_{t_1})}(\sigma_i^{(n_i)})^{-2} - \frac{\widehat{\mu}(g_{t_0})}{\beta_i(g_{t_0})}(\sigma_i^{(n_i)})^{-2} + R_i(g_{t_1}, g_{t_0}) \tag{7.20}$$

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where $R_i(g_{t_1}, g_{t_0})$ is obtained as the difference of the two terms (one computed for $g_{t_1}$ and the other for $g_{t_0}$) in Equations (4.7), that have the factor $\lambda_i^{(n_i)}(x)$. By the strong dissipative condition we have $\lambda_i^{(n_i)} \ll (\sigma_i^{(n_i)})^{-2}$ if $n_i$ is large enough. Besides, the other terms or coefficients depending continuously on $g$ in the Equation (4.7), are upper and lower bounded from zero, due to Lemma 3.8. Therefore given $0 < \varepsilon$ there exists $m_i$ large enough such that if $n_i \geq m_i$ then

$$|R_i(g_{t_1}, g_{t_0})| \leq k_i \lambda_i^{(n_i)} \leq \frac{\varepsilon}{8 \sup_{g \in \mathcal{N}} |\beta_i(g)|} \; (\sigma_i^{n_i})^{-2} \leq \frac{\varepsilon |\widehat{\mu}(g_{t_1})|}{|\beta_i(g_{t_1})|} \; (\sigma_i^{n_i})^{-2}$$

Substituting in 7.20 we obtain:

$$\frac{1 - \varepsilon}{1 + \varepsilon} \leq \frac{1}{1 + \varepsilon} \left| \frac{\beta_i(g_{t_1})}{\beta_i(g_{t_0})} \right| \leq \left| \frac{\widehat{\mu}(g_{t_1})}{\widehat{\mu}(g_{t_0})} \right| \leq \frac{1}{1 - \varepsilon} \left| \frac{\beta_i(g_{t_1})}{\beta_i(g_{t_0})} \right| \leq \frac{1 + \varepsilon}{1 - \varepsilon} \tag{7.21}$$

In the last inequalities we have used Lemma 5.4. Take $\varepsilon > 0$ such that

$$\left( \frac{1 - \varepsilon}{1 + \varepsilon}, \frac{1 + \varepsilon}{1 - \varepsilon} \right) \subset (1 - \eta, 1 + \eta)$$

and then fix $\delta = \delta(\varepsilon)$ as in Lemma 5.4 to define $\mathcal{N}_1$. Therefore (7.21) implies

$$\left| \frac{\widehat{\mu}(g_{t_1})}{\widehat{\mu}(g_{t_0})} \right| \in (1 - \eta, 1 + \eta)$$

contradicting (7.19). □

*Proof of part C) of Theorem 2:*

Consider $g_0$ constructed as in the proof of the part (A) of Theorem 2. The given one-parameter family $\{g_t\}_{t \in (-\varepsilon, \varepsilon)}$ is not necessarily in the space $\mathcal{N}_1$ defined in that proof, but nevertheless it unfolds generically the tangencies along $L(g_t)$, i.e.:

$$\left| \frac{d\mu_i(t)}{dt} \right| \geq v > 0 \;\; \forall i \geq 1, \;\; \forall t \in (-\varepsilon, \varepsilon)$$

where $\mu_i$ is defined in subsection 3.5. We have:

$$|\mu_i(g_t) - \mu_i(g_0)| = |(\nu_i(g_t) - c_i(g_t)) - (\nu_i(g_0) - c_i(g_0))| \geq v|t| \;\; \forall \, i \geq 1$$

Consider the implicit function $\nu_i(f) \in G_f^{(i)}(\widehat{\mu})$ verifying equation (4.9) for any $f \in \mathcal{N}$. We can not apply directly the thesis of Lemma 5.5 because it is valid only if $f \in \mathcal{N}_1$ and our diffeomorphism $g_t$ does not necessarily belong to the manifold $\mathcal{N}_1$. Nevertheless we use equation (4.9), Lemma 3.8 and similar arguments of those in the proof of Lemma 5.5 to obtain the following bounds for all $f \in \mathcal{N}$ and for all $n_i$ large enough:

$$|G_f^{(i)}(0) - c_i(f)| = \left| \frac{a_i(f) - y(P_i(f))}{\sigma_i^{(n_i)}(G_f^{(i)}(0))} - \text{dist}^s(q_i(f).P_i(f))\gamma_i(f)\lambda_i(f)^{(n_i)}(b_i(f)) \right| < \frac{1}{i} \tag{7.22}$$

$$\left| \frac{dG_f^{(i)}(\widehat{\mu})}{d\widehat{\mu}} \right| \leq \frac{2}{K_i(\sigma_i^{(n_i)}(G_f^{(i)}(\mu)))^2} \tag{7.23}$$

In particular we apply (7.22) to $f = g_t$ and $f = g_0$ to obtain

$$|(G_{g_t}^{(i)}(0) - c_i(g_t)) - (G_{g_0}^{(i)}(0) - c_i(g_0))|_{i \to +\infty} \to 0 \tag{7.24}$$

Suppose that $g_t$ exhibits infinitely many sinks $s_i(g_t)$ that are continuations from those of $g_0$. Applying part (C) of Proposition 4.3 we deduce that there exists $\widehat{\mu} \in (-1, 1)$ such that

$$\nu_i(g_t) = G_{g_t}(\widehat{\mu}) \tag{7.25}$$

Combining (7.25) with (7.23) applied to $f = g_t$ and using the Lagrange Theorem:

$$\nu_i(g_t) \in G_{g_t}^{(i)}(0) + \frac{[\sigma_i^{(n_i)}(g_t)(Y_i(t))]^{-2}}{K_i} (-2, 2)$$

But $\sigma_i^{(n_i)}(g_t)(Y_i(t)) > \widetilde{\sigma}^{n_i}$ with $\widetilde{\sigma} > 1$ and $n_i \to \infty$ as fast as needed.

Recalling (7.24):

$$0 < v, \quad 0 \le v|t| \le |(\nu_i(g_t) - c_i(g_t)) - (\nu_i(g_0) - c_i(g_0))|_{i \to +\infty} \to 0$$

and then $|t| = 0$. $\square$

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