ON THE COMPLETENESS OF THE SET OF RADIAL TREFFTZ FUNCTIONS USED BY THE BKM IN THE SOLUTION OF VISCOELASTICITY PROBLEMS

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Governing equation of viscoelasticity problems

$$c_P^2(t,\cdot) * \nabla\nabla\cdot\mathbf{u}(x,\cdot) - c_S^2(t,\cdot) * \nabla\times\nabla\times\mathbf{u}(x,\cdot) + \mathbf{b}(x,t) = \ddot{\mathbf{u}}(x,t) \qquad (1)$$

Where '*' represents the viscoelastic operator:

$$f(t,\cdot) * g(\cdot) = \int_{\tau_0}^{t} f(t,\tau)\frac{\partial g}{\partial \tau}(\tau)\,\mathrm{d}\tau \qquad (2)$$

$$c_P^2(t,\cdot)* = \frac{1-\nu}{(1+\nu)(1-2\nu)\rho} R_E(t,\cdot)* \qquad c_S^2(t,\cdot)* = \frac{1}{2(1+\nu)\rho} R_E(t,\cdot)* \qquad (3)$$

R_E : Relaxation function of the material.

Governing equation of viscoelasticity problems

Applying the Fourier transform, we obtain the governing equation for harmonic problems. For **b**=0 it is:

$$\bar{c}_P^2(\omega)\nabla\nabla\cdot\bar{\mathbf{u}}(x,\omega)-\bar{c}_S^2(\omega)\nabla\times\nabla\times\bar{\mathbf{u}}(x,\omega)+\omega^2\bar{\mathbf{u}}(x,\omega)=0 \qquad (1)$$

$$\bar{c}_P^2(\omega)=\frac{1-\nu}{(1+\nu)(1-2\nu)\rho}E^*(\omega) \qquad \bar{c}_S^2(\omega)=\frac{1}{2(1+\nu)\rho}E^*(\omega) \qquad (2)$$

$E^*(\omega)$: Complex Modulus.

We can consider Kelvin or Boltzmann fractional models.

In this work we use:

$$E^*(\omega)=(1+2i\beta)E \qquad (3)$$

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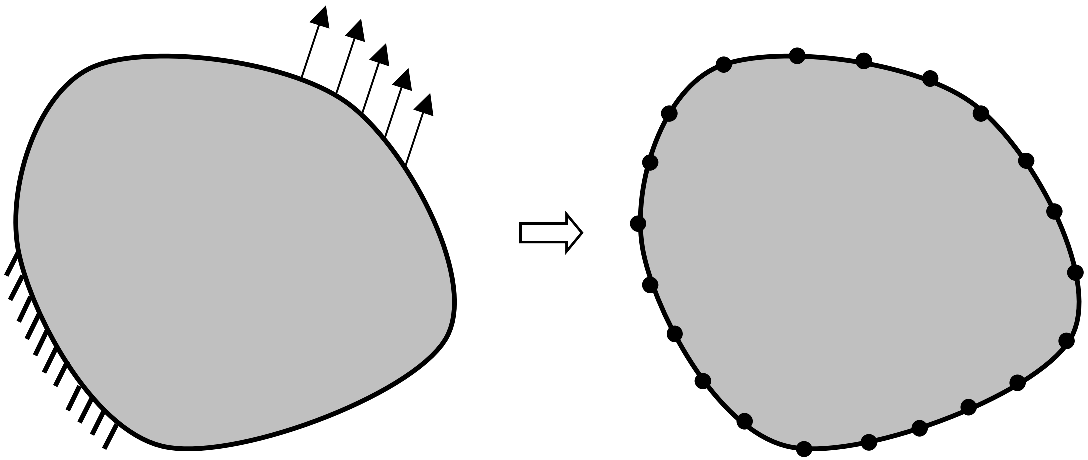
Boundary Knot Method



For each node in the boundary we define a radial function: $\hat{\mathbf{u}}(x, x_j, \omega)$

For N nodes in the boundary, the numerical approximation to the solution is:

$$\mathbf{u}_N(x,\omega) = \sum_{j=1}^{N} \hat{\mathbf{u}}(x, x_j, \omega)\boldsymbol{\alpha}_j$$

<u>The radial function is smooth and satisfies the governing equation</u>

Boundary Knot Method

Example: Helmholtz problem

$$c^2 \Delta u + \omega^2 u = 0 \qquad (1)$$

Radial Trefftz solution:

$$\hat{u}(x, x_j, \omega) = J_0(k r_j) \quad \text{con} \quad k = \sqrt{c / \omega}\ , \quad \text{y} \quad r_j = \left\| x - x_j \right\| \qquad (2)$$

$$u_N(x, \omega) = \sum_{j=1}^{N} J_0(k r_j) \alpha_j$$

Boundary conditions by collocation:

$$\left. \begin{aligned} u_N(x_i, \omega) &= \overline{u}(x_i, \omega), \quad \forall x_i \in \partial\Omega_u \\ p_N(x_i, \omega) &= \overline{p}(x_i, \omega), \quad \forall x_i \in \partial\Omega_p \end{aligned} \right\} \quad \sum_{j=1}^{N} \mathbf{K}_{ij} \alpha_j = \mathbf{b}_i \quad \Leftrightarrow \quad \mathbf{K}\boldsymbol{\alpha} = \mathbf{b} \qquad (3)$$

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Radial Trefftz function

Canelas and Sensale (2010):

$$\hat{\mathbf{u}}_{\ell k} = \frac{1}{4\pi\rho\bar{c}_S^2}\left[\psi\delta_{\ell k} - \chi\frac{\partial r}{\partial x_\ell}\frac{\partial r}{\partial x_k}\right] \quad (1)$$

$$\begin{aligned}\hat{\mathbf{p}}_{\ell k} = \frac{1}{4\pi}\left(\frac{\partial\psi}{\partial r} - \frac{\chi}{r}\right)\left(\delta_{\ell k}\frac{\partial r}{\partial\mathbf{n}} + \mathbf{n}_\ell\frac{\partial r}{\partial x_k}\right) - \frac{2\chi}{r}\frac{\partial r}{\partial x_\ell}\left(\mathbf{n}_k - 2\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial\mathbf{n}}\right) \\ -2\frac{\partial\chi}{\partial r}\frac{\partial r}{\partial x_\ell}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial\mathbf{n}} + \mathbf{n}_k\left(\frac{\bar{c}_P^2}{\bar{c}_S^2} - 2\right)\left(\frac{\partial\psi}{\partial r} - \frac{\partial\chi}{\partial r} - \frac{2\chi}{r}\right)\frac{\partial r}{\partial x_\ell}\end{aligned} \quad (2)$$

$$\psi(r) = r^{-3}\left[k_P r\cos(k_P r) - \sin(k_P r) - k_S r\cos(k_S r) + \sin(k_S r) - k_S^2 r^2\sin(k_S r)\right] \quad (3)$$

$$\begin{aligned}\chi(r) = r^{-3}\big[3k_P r\cos(k_P r) - 3\sin(k_P r) + k_P^2 r^2\sin(k_P r) \\ -3k_S r\cos(k_S r) + 3\sin(k_S r) - k_S^2 r^2\sin(k_S r)\big]\end{aligned} \quad (4)$$

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Objective

- Improve the Boundary Knot Method for viscoelasticity problems presented by Canelas and Sensale (2010) by modifying the radial Trefftz basis.

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New Radial Trefftz function

Using the Cauchy-Kovalevski-Somigliana solution:

$$\bar{c}_S^2(\omega)\nabla\nabla\cdot\mathbf{g}(x,\omega)-\bar{c}_P^2(\omega)\nabla\times\nabla\times\mathbf{g}(x,\omega)+\omega^2\mathbf{g}(x,\omega)=\mathbf{u}(x,\omega) \qquad (1)$$

$$(\bar{c}_S^2(\omega)\nabla\cdot\nabla+\omega^2)(\bar{c}_P^2(\omega)\nabla\cdot\nabla+\omega^2)\mathbf{g}(x,\omega)=0 \qquad (2)$$

$$\mathbf{g}_N(x,\omega)=\sum_{j=1}^{N}\boldsymbol{\varphi}_S(x,x_j,\omega)\ \boldsymbol{\alpha}_j+\sum_{j=1}^{N}\boldsymbol{\varphi}_P(x,x_j,\omega)\ \boldsymbol{\beta}_j \qquad (3)$$

with: $\boldsymbol{\varphi}_S(r,\omega)=J_0(k_S r,\omega)\ \mathbf{I},\quad \boldsymbol{\varphi}_P(r,\omega)=J_0(k_P r,\omega)\ \mathbf{I}$

$$\mathbf{u}_N(x,\omega)=\sum_{j=1}^{N}\hat{\mathbf{u}}_S(x,x_j,\omega)\ \boldsymbol{\alpha}_j+\sum_{j=1}^{N}\hat{\mathbf{u}}_P(x,x_j,\omega)\ \boldsymbol{\beta}_j \qquad (4)$$

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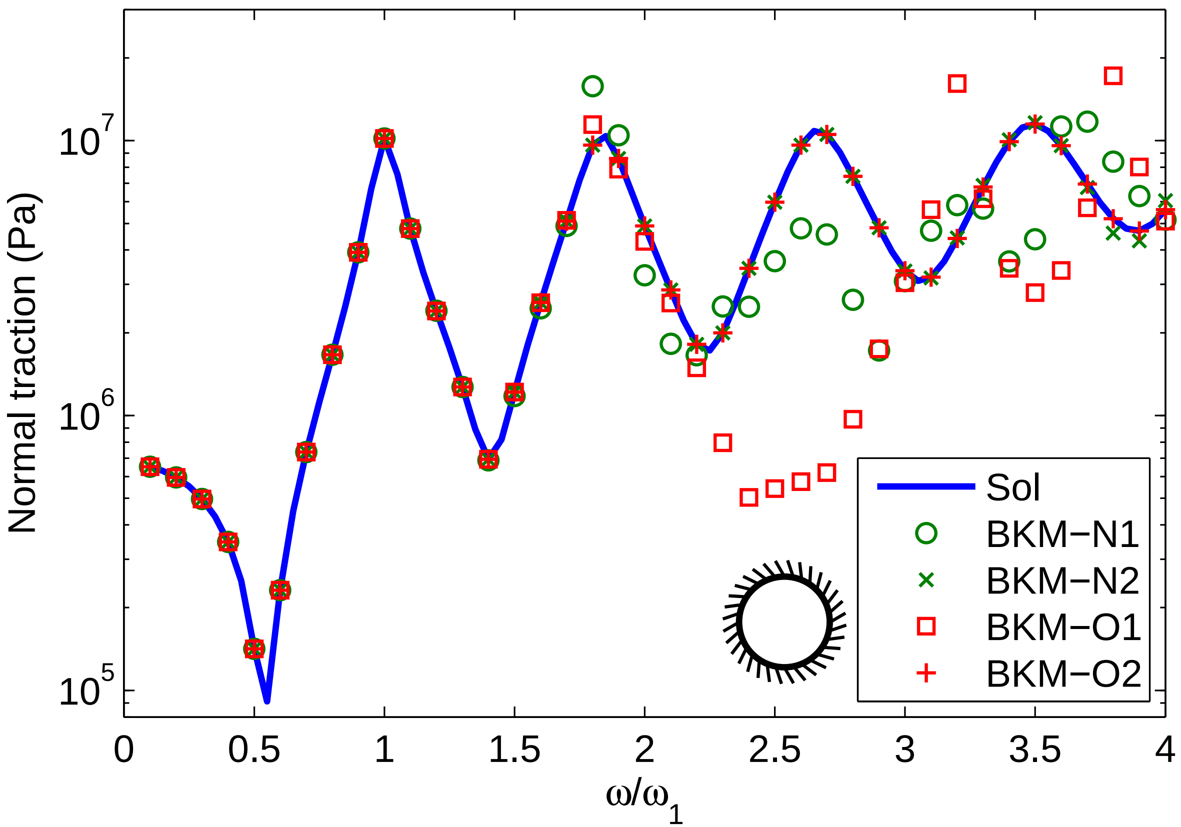
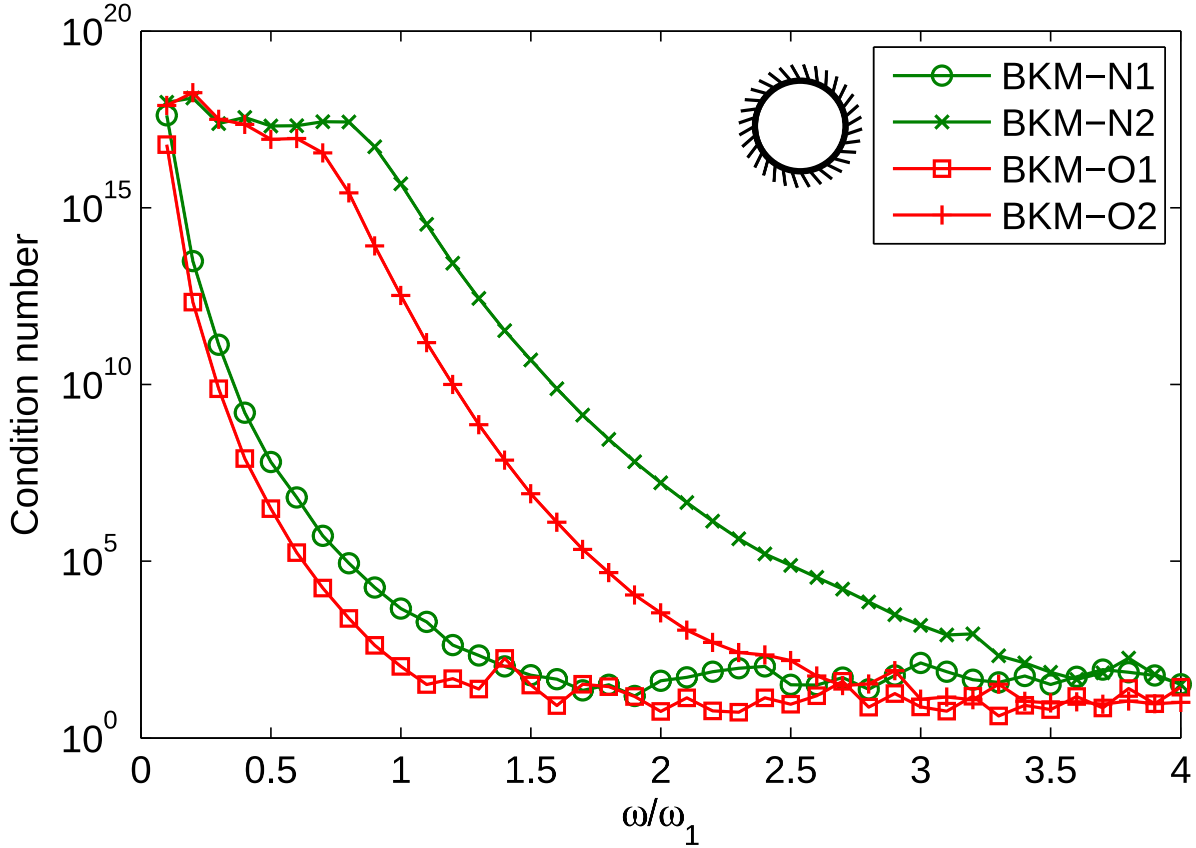
Circle

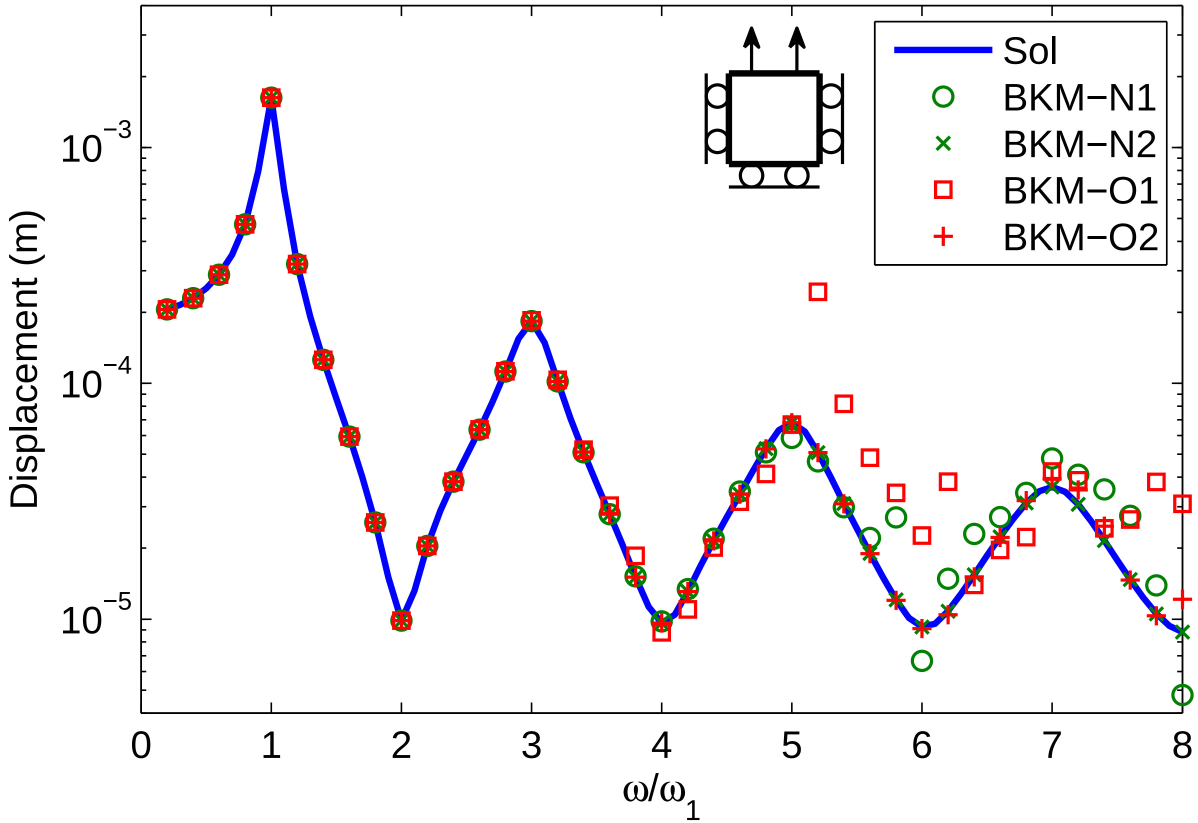


Normal traction (Pa)
10^7
10^6
10^5
0
0.5
1
1.5
2
2.5
3
3.5
4
ω/ω_1
Sol
BKM−N1
BKM−N2
BKM−O1
BKM−O2

Circle: condition number

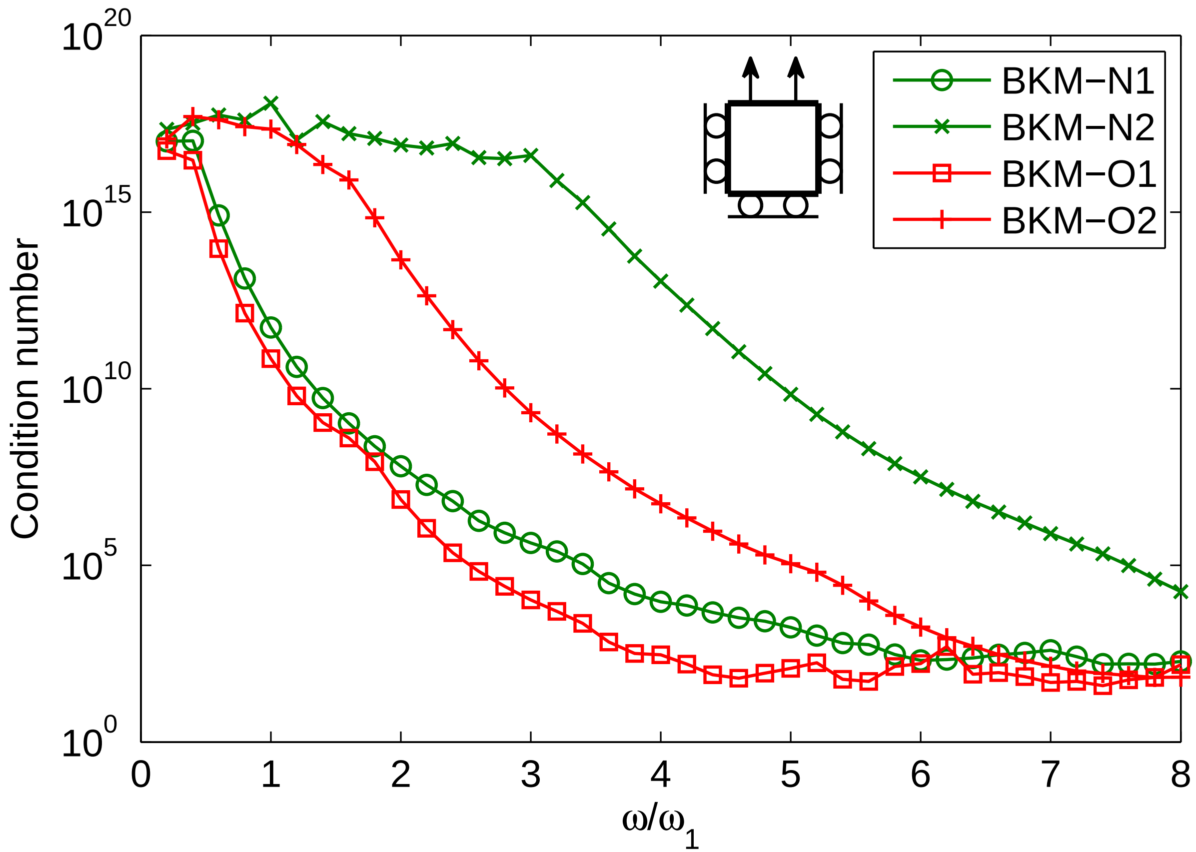


Square



Sol
BKM−N1
BKM−N2
BKM−O1
BKM−O2
Displacement (m)
10−3
10−4
10−5
0
1
2
3
4
5
6
7
8
ω/ω1

Scuare: condition number



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Conclusions

- A new radial Trefftz function for the BKM representation of the solution was obtained by means of the Cauchy-Kovalevski-Somigliana solution of the displacements.
- The new basis can represent exactly any solution obtained by the BKM using the old basis.
- It was observed that the new basis leads to worse conditioned linear systems. Thus, for practical applications the old basis presented in (Canelas and Sensale, 2010) is recommended.
- The ill-conditioning should be addressed using other techniques:
  - Extended precision.
  - Regularization techniques.