# Topological derivatives and a level-set approach for an inverse electromagnetic casting problem

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WCCM 2012

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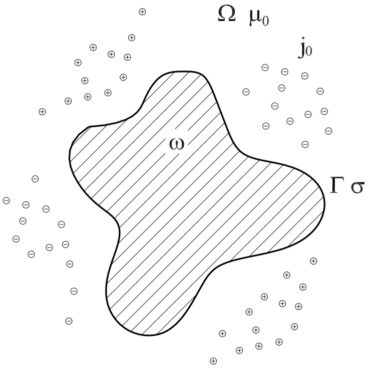
# Electromagnetic Casting Problem




We assume that the electric current frequency is so high that the magnetic field penetrates a negligible distance into the liquid metal (skin effect).

# Electromagnetic Casting Problem

# Magnetic Field Equations

Michel Pierre, Jean R. Roche (1991)

$$\begin{cases} \nabla \times \mathbf{B} = \mu_0 \mathbf{j}_0 & \text{in } \Omega \\ \nabla \cdot \mathbf{B} = 0 & \text{in } \Omega \\ \mathbf{B} \cdot n = 0 & \text{on } \Gamma \\ \|\mathbf{B}\| = O(\|x\|^{-1}) & \text{as } \|x\| \to \infty \text{ in } \Omega \end{cases} \tag{1}$$

$\omega$: domain occupied by the liquid metal.
$\Gamma$: boundary of $\omega$.
$\Omega = \mathbb{R} \setminus \omega$ is the exterior of the liquid metal.
$\mathbf{j}_0 = (0, 0, j_0)$ is the electric current density.
$\mathbf{B} = (B_1, B_2, 0)$ is the magnetic field vector.
$\mu_0$: magnetic permeability of the vacuum.
$n$: outward-pointing unit normal vector of $\Gamma$.

# Equilibrium and constraints

In addition to the field equations we have the equilibrium equation:

$$\frac{1}{2\mu_0}\|\mathbf{B}\|^2 + \sigma\mathcal{C} = p_0 \quad \text{constant on } \Gamma \tag{1}$$

And the volume constraint:

$$\int_\omega \mathrm{d}\Omega = S_0 \tag{2}$$

We also assume that $j_0$ has a compact support in $\Omega$ and satisfies:

$$\int_\Omega j_0 \,\mathrm{d}\Omega = 0 \tag{3}$$

# Magnetic flux function

Then, there exists the magnetic flux funtion $\varphi : \Omega \to \mathbb{R}$ such that $\mathbf{B} = (\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}, 0)$ where $\varphi$ is solution to the state equations:

$$\begin{cases} -\Delta\varphi &= \mu_0 j_0 & \text{in } \Omega \\ \varphi &= 0 & \text{on } \Gamma \\ \varphi(x) &= c + o(1) & \text{as } \|x\| \to \infty \end{cases} \tag{1}$$

$\varphi$ has a unique solution in $W_0^1(\Omega) = \{u : \ \rho u \in L^2(\Omega) \text{ and } \nabla u \in L^2(\Omega)\}$ with $\rho(x) = [\sqrt{1 + \|x\|^2} \log(2 + \|x\|^2)]^{-1}$. $c$ is unique in $\mathbb{R}$.

The equilibrium in terms of the flux function $\varphi$ becomes:

$$\frac{1}{2\mu_0} \left|\frac{\partial \varphi}{\partial n}\right|^2 + \sigma\mathcal{C} = p_0 \quad \text{constant on } \Gamma \tag{2}$$

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# Inverse Problem

- Find the configuration of inductors to have the liquid metal in equilibrium occupying the domain $\omega$.
- Find $j_0$ and $c$ such that $\int_\Omega j_0 \, dx = 0$ and that the solution $\varphi$ of the state equations

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Then:

$$\frac{\partial\varphi}{\partial n} = \varkappa\sqrt{2\mu_0(p_0 - \sigma\mathcal{C})} \quad \text{with} \quad \varkappa = \pm 1\,. \tag{2}$$

Therefore $p_0 \geq \max_\Gamma \sigma\mathcal{C}$. We can show that $\frac{\partial\varphi}{\partial n}$ is zero at some points, therefore:

$$p_0 = \max_\Gamma \sigma\mathcal{C}\,. \tag{3}$$

Calling $\bar{p} = \sqrt{2\mu_0(p_0 - \sigma\mathcal{C})}$ we have:

$$\frac{\partial\varphi}{\partial n} = \varkappa\,\bar{p} \quad \text{on } \Gamma\,, \tag{4}$$

with the sign changes of $\varkappa$ at the zeros of $(p_0 - \sigma\mathcal{C})$, that is at the points of maximum curvature.

# Inverse Problem - Formulation

We can formulate the problem as: find $j_0$ and $c$ such that the system

$$\begin{cases} -\Delta\varphi &= \mu_0 j_0 & \text{in } \Omega\,, \\ \varphi &= 0 & \text{on } \Gamma\,, \\ \dfrac{\partial\varphi}{\partial n} &= \varkappa\,\bar{p} & \text{on } \Gamma\,, \\ \varphi(x) &= c + o(1) & \text{as } \|x\| \to \infty\,, \end{cases} \tag{1}$$

has a solution $\varphi \in W_0^1(\Omega)$.

We known that for a simply connected $\omega$, with $\Gamma$ an analitic Jordan curve, and $p_0 = \max_\Gamma \sigma\mathcal{C}$ (the maximum must be attained at an even number of points) then (Henrot and Pierre 1989):

(i) there exists a solution for the inverse problem,
(ii) the solution is not unique for $j_0$.

# Inverse Problem - Formulation

We propose to minimize the Kohn–Vogelius functional:

$$J(\phi) = \frac{1}{2}\|\phi\|^2_{L^2(\Gamma)} = \frac{1}{2}\int_\Gamma |\phi|^2 \, \mathrm{d}s \,, \tag{1}$$

where the auxiliary function $\phi$ depends implicitly on $j_0$ and $c$ through the solution of the problem:

$$\begin{cases} -\Delta\phi = \mu_0 j_0 & \text{in } \Omega\,, \\ \dfrac{\partial\phi}{\partial n} = \varkappa\,\bar{p} & \text{on } \Gamma\,, \\ \phi(x) = c + o(1) & \text{as } \|x\| \to \infty\,. \end{cases} \tag{2}$$

There is a compatibility condition:

$$\int_\Gamma \varkappa\,\bar{p}\, \mathrm{d}s = 0\,, \tag{3}$$

# Inverse Problem - Formulation

We can eliminate $c$ of the formulation:

$$J(\phi) = \frac{1}{2}\|\phi\|^2_{L^2(\Gamma)} = \frac{1}{2}\int_\Gamma |\phi|^2 \, ds \,, \tag{1}$$

where the auxiliary function $\phi$ depends implicitly on $j_0$ through the solution of the problem:

$$\begin{cases} -\Delta\phi = \mu_0 j_0 & \text{in } \Omega \,, \\ \dfrac{\partial\phi}{\partial n} = \varkappa \bar{p} - d(j_0) & \text{on } \Gamma \,, \\ \displaystyle\int_\Gamma \phi \, ds = 0 \,. \end{cases} \tag{2}$$

With the conditions:

$$\int_\Gamma \varkappa \bar{p} \, ds = 0 \,, \quad d(j_0) = \|\Gamma\|^{-1} \int_\Omega \mu_0 j_0 \, dx \tag{3}$$

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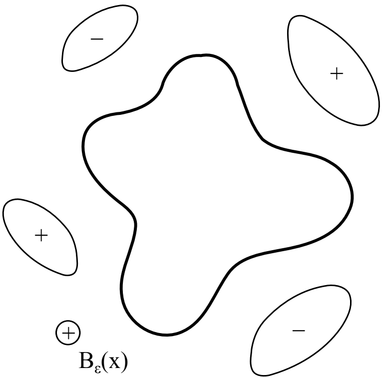
# Computing the topological expansion

We introduce a circular perturbation:



# Computing the topological expansion

The perturbed electric current density is:

$$j_\varepsilon = j_0 + \alpha I \chi_{B_\varepsilon(\hat{x})} \,. \tag{1}$$

Then, for the functional:

$$\psi(\varepsilon) = J(\phi_\varepsilon) = \frac{1}{2}\int_\Gamma |\phi_\varepsilon|^2 \, ds \,, \tag{2}$$

**Theorem**: the topological expansion of $\psi$ is:

$$\psi(\varepsilon) = \psi(0) + (\pi\varepsilon^2) D_T^1\psi + (\pi^2\varepsilon^4) D_T^2\psi \,, \tag{3}$$

# Computing the topological expansion

The topological derivatives are:

$$D_T^1\psi(\hat{x}) = \alpha I \int_\Gamma \phi f \, \mathrm{d}s \,, \tag{1}$$

$$D_T^2\psi(\hat{x}) = \frac{1}{2} I^2 \int_\Gamma f^2 \, \mathrm{d}s \,. \tag{2}$$

Where the function $f$ is the solution to:

$$\begin{cases} -\Delta f &= (\pi\varepsilon^2)^{-1}\chi_{B_\varepsilon(\hat{x})} & \text{in } \Omega \,, \\ \dfrac{\partial f}{\partial n} &= -\|\Gamma\|^{-1} & \text{on } \Gamma \,, \\ \displaystyle\int_\Gamma f \, \mathrm{d}s &= 0 \,. \end{cases} \tag{3}$$

$D_T^1\psi(\hat{x})$ can be computed very efficiently by the adjoint method. $D_T^2\psi(\hat{x})$ cannot be computed by the adjoint method, but it does not depend on the current inductors.

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# Level-set procedure

The region $\Omega^+$ of positive electric current density:

$$\Omega^+ = \{x \in \Omega,\ \psi^+(x) < 0\}\,, \tag{1}$$

Analogously, the region $\Omega^-$ of negative electric current density:

$$\Omega^- = \{x \in \Omega,\ \psi^-(x) < 0\}\,. \tag{2}$$

The region of electric current zero is:

$$\Omega^0 = \Omega - (\Omega^+ \cup \Omega^-)\,. \tag{3}$$

In addition, the 'expected variation' EV of the objective functional produced by the introduction of a small ball of electric current of sign $\alpha$:

$$EV(\hat{x}, \varepsilon, \alpha) = (\pi\varepsilon^2) D_T^1\psi(\hat{x}) + (\pi^2\varepsilon^4) D_T^2\psi(\hat{x})\,. \tag{4}$$

# Sufficient optimality conditions

Sufficient optimality conditions (for the perturbations considered here):

$$EV(\hat{x},\varepsilon,\alpha) > 0\,, \quad \forall \hat{x} \in \Omega^+\,, \text{ and } \alpha = -1\,, \tag{1}$$

$$EV(\hat{x},\varepsilon,\alpha) > 0\,, \quad \forall \hat{x} \in \Omega^-\,, \text{ and } \alpha = +1\,, \tag{2}$$

$$EV(\hat{x},\varepsilon,\alpha) > 0\,, \quad \forall \hat{x} \in \Omega^0\,, \text{ and } \alpha = \pm 1\,. \tag{3}$$

Hence, we define:

$$g^+(x) = \begin{cases} -EV(\hat{x},\varepsilon,-1) & \text{if } \hat{x} \in \Omega^+\,, \\ EV(\hat{x},\varepsilon,+1) & \text{if } \hat{x} \in \Omega^0 \cup \Omega^-\,, \end{cases} \tag{4}$$

$$g^-(x) = \begin{cases} -EV(\hat{x},\varepsilon,+1) & \text{if } \hat{x} \in \Omega^-\,, \\ EV(\hat{x},\varepsilon,-1) & \text{if } \hat{x} \in \Omega^0 \cup \Omega^+\,. \end{cases} \tag{5}$$

The algorithm updates $\psi^+$ and $\psi^-$ iteratively to obtain:

$$\exists\, \tau^+ > 0 \quad \text{s.t.} \quad g^+ = \tau^+ \psi^+\,, \tag{6}$$

$$\exists\, \tau^- > 0 \quad \text{s.t.} \quad g^- = \tau^- \psi^-\,, \tag{7}$$

# Topology optimization procedure

The algorithm is based on the proposed by Amstutz and Andrä (2006).

We can start from $\psi_0^+ = \psi_0^- = 1/\|1\|_{L_2(\Omega)}$ to obtain $\Omega_0 = \Omega$.

1. Compute the topological derivatives and the expected variations $EV(\hat{x}, \varepsilon, +1)$ and $EV(\hat{x}, \varepsilon, -1)$.
2. Compute $g_n^+$ and $g_n^-$.
3. Line Search: find the real value $t_n \in [0, 1]$ such that the following sets minimize the objective functional:

$$\left.\begin{aligned} \psi_{n+1}^+ &= (1 - t_n)\psi_n^+ + t_n g_n^+ \\ \psi_{n+1}^- &= (1 - t_n)\psi_n^- + t_n g_n^- \end{aligned}\right\}, \tag{1}$$

4. Stop or increase $n$ and return to the first step according an optimality criterion.

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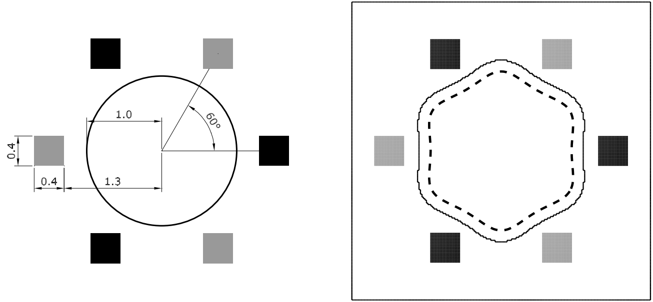
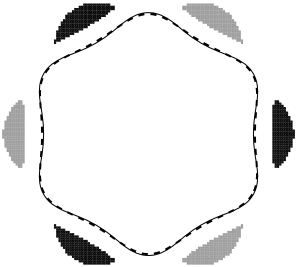
# Example 1

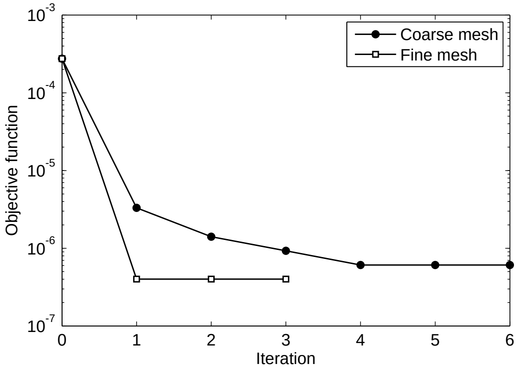



(a) (b)

# Example 1

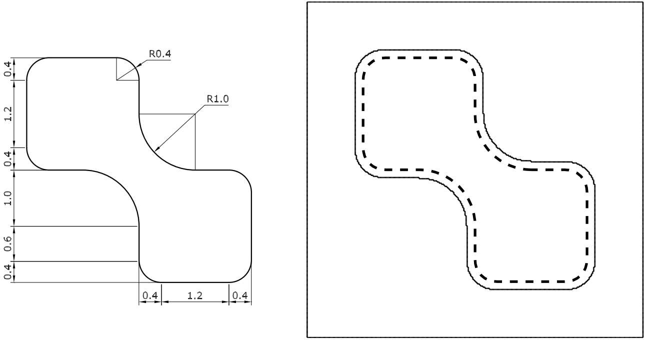


# Example 1



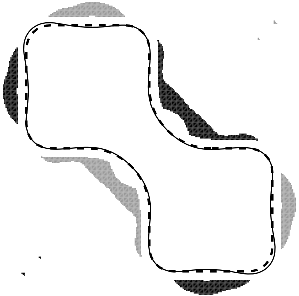
Coarse mesh
Fine mesh
Objective function
Iteration
$10^{-3}$
$10^{-4}$
$10^{-5}$
$10^{-6}$
$10^{-7}$
0
1
2
3
4
5
6

# Example 2

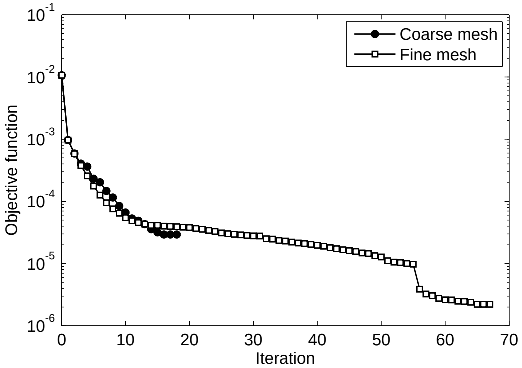



(a) (b)

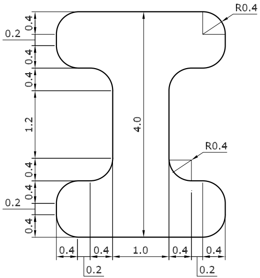
# Example 2



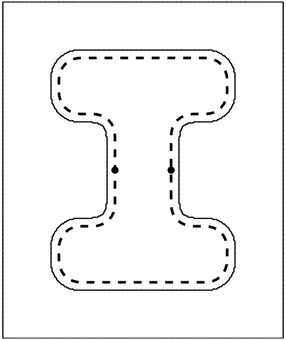
# Example 2


Coarse mesh
Fine mesh
Objective function
Iteration
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
0
10
20
30
40
50
60
70

# Example 3

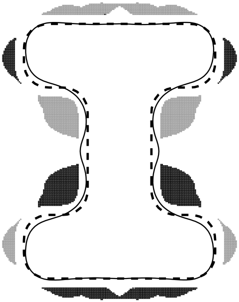



(a)

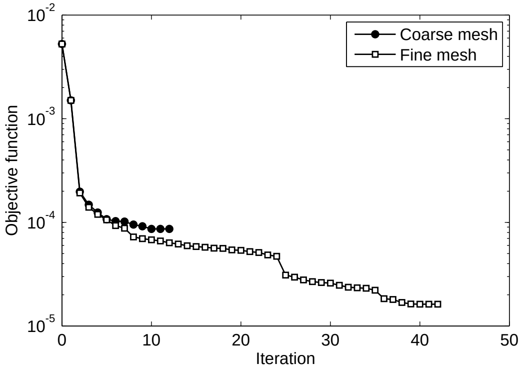


(b)

# Example 3



# Example 3


10-2
10-3
10-4
10-5
Objective function
0
10
20
30
40
50
Iteration
Coarse mesh
Fine mesh

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# Conclusions

- The topological expansion of the Kohn–Vogelius functional was obtained.
- Using this expansion and level-sets, an optimization algorithm was developed.
- Some examples were successfully solved.

We thank the Brazilian councils CAPES, CNPq and FAPERJ, the Uruguayan ANII and CSIC, and the French COFECUB, INRIA and CNRS for the financial support.

Thank you!