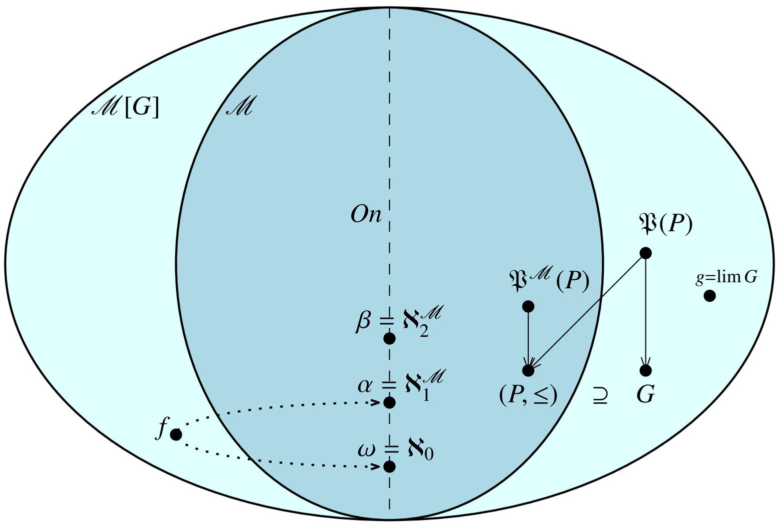
# An axiomatic presentation of the method of forcing

(course notes)





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# Introduction

## The independence of the continuum hypothesis

The method of *forcing* was invented by Paul Cohen [Coh63, Coh64] to prove the relative consistency of the negation of the *Continuum Hypothesis* (CH) with respect to the axioms of set theory. This hypothesis, which was advanced by Cantor in 1878, states that:

> **Continuum Hypothesis:** There is no set whose cardinality is strictly between the cardinality of the set of natural numbers and the cardinality of the real line (i.e. the power of continuum). In symbols:
>
> (CH) $$2^{\aleph_0} = \aleph_1 ,$$
>
> writing $\aleph_1$ the first uncountable cardinal and $2^{\aleph_0}$ the cardinal of $\mathfrak{P}(\omega)$—which is the same as the cardinal of the real line $\mathbb{R}$[1].

For a long time, this problem has been considered as one of the most important problems in mathematics, and Hilbert ranked it first in his famous list of 23 open problems. Eventually, it turned out that CH is independent from the axioms of ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice). That is: if ZFC (and actually: if ZF) is logically consistent, then CH can be neither proved nor disproved from its axioms.

**Gödel's constructible sets** In [Göd38], Gödel solved the first half of the problem, by proving the relative consistency of CH w.r.t. the axioms of ZFC. That is:

*If ZF(C) is consistent, then ZFC + CH is consistent.*

Or, which amounts to the same, that CH cannot be disproved from the axioms of ZFC:

*If ZF(C) is consistent, then ZFC* $\nvdash \neg$*CH.*

For that, Gödel showed that any set-theoretic universe $\mathscr{U}$[2] that fulfills the axioms of ZF contains a sub-universe $L \subseteq \mathscr{U}$, called the *universe of constructible sets* (Chapter 2), that fulfills the axioms of ZF as well as the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH). Recall that the later states that:

[1]Recall that the set $\mathfrak{P}(\omega)$ is equipotent to the real line $\mathbb{R}$ (cf Exercise 1.6 p. 27). Following a longstanding tradition in Logic, we shall (abusively) call *real numbers* the elements of $\mathfrak{P}(\omega)$ in this course.

[2]Here, we use the word 'universe' with its intuitive meaning. Traditionally, this concept is formalized using the notion of model, but in this course, we shall see that universes can be presented as theories as well.

**Generalized Continuum Hypothesis:** There is no cardinality strictly between the cardinality of a given infinite set and the cardinality of its powerset:

$$\text{(GCH)} \qquad (\forall \alpha \in On)\ 2^{\aleph_\alpha} = \aleph_{\alpha+1}\,,$$

writing $\aleph_\alpha$ the infinite cardinal of rank $\alpha \in On$ (and $On$ the class of ordinals).

From this, Gödel deduced the relative consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis w.r.t. the axioms of ZF:

*If ZF is consistent, then ZFC + GCH is consistent,*

a result which is actually stronger than the relative consistency of CH w.r.t. ZFC.

**Cohen's forcing** Gödel's construction is a particular instance of a more general technique in set theory—the technique of *inner models*—, that consists to extract a sub-universe $\mathscr{U}'$ from a given set-theoretic universe $\mathscr{U}$, with the hope that the desired axioms are satisfied in the sub-universe $\mathscr{U}' \subseteq \mathscr{U}$. On the other hand, there was at the time of Gödel no method to extend a given set-theoretic universe $\mathscr{U}$. Defining such a method was hard, since set-theoretic universes are by definition the most constrained of all mathematical structures—remember that by definition, they must fulfill *all* theorems of ordinary mathematics.

To solve the second half of Hilbert's first problem, Cohen precisely introduced such a method—the method of *forcing*—, in which a set-theoretic universe is extended by adding *new subsets* to infinite sets that already exist in the initial universe. Historically, the first example of forcing consisted to add sufficiently many new subsets of $\omega$, known as *Cohen reals*, so that the cardinality of the powerset of $\omega$ (in the extended universe) jumped to at least $\aleph_2$. From this, Cohen deduced the consistency of ¬CH relatively to the axioms of set theory

*If ZF(C) is consistent, then ZFC + ¬CH is consistent,*

thus giving a definitive answer to Hilbert's first problem. (In practice, the method is flexible enough to fix precisely the cardinality of $\mathfrak{P}(\omega)$ to, say, $\aleph_2$, $\aleph_{43}$ or even $\aleph_{\omega+1}$—but not $\aleph_\omega$.)

Since Cohen's achievement (which brought him the Fields medal in 1966), forcing was intensively studied and developed in model theory [Jec02], and it was related to the theory of Boolean-valued models [Bel85]. It quickly brought further independence results in set theory, and proved to be an invaluable tool in the study of large cardinals. Still today, forcing is the only known method to extend a set-theoretic universe.

**The fundamental under-specification of the powerset** The reader might wonder how it is possible to add new subsets to already existing infinite sets, and in particular to the set $\omega$ of natural numbers. Mathematicians are so used to manipulate subsets of $\omega$ (or real numbers) that they often develop the (wrong) intuition that all these subsets are given *a priori*.

But one should remember that the language we use to define subsets of $\omega$ is (intuitively) denumerable, so that the (intuitive) collection formed by all the subsets of $\omega$ that we can individually define is denumerable. The vast majority of the subsets of $\omega$, that actually only exist via Cantor's diagonal argument, are definitely out of reach by linguistic means—at least individually—and their existence is, in some sense, purely formal. The method of forcing precisely exploits this fundamental under-specification of the powerset (of infinite sets) by specifying extra subsets using extra linguistic means that will be presented in Chapter 3.

# The method of forcing: an overview

Basically, forcing is a method to extend—and actually: to *expand*—a given set-theoretic universe into a larger universe by adding new subsets to already existing infinite sets.

**The forcing poset** This method is parameterized by a nonempty poset $(P, \leq)$ taken in the initial universe, which is called the forcing poset and whose elements are called *forcing conditions*, or simply *conditions*. In practice, the forcing poset $(P, \leq)$ is chosen in such a way that its elements represent 'potential approximations' of some 'ideal object' $g$ we would like to create, but that typically does not exist in the initial universe.

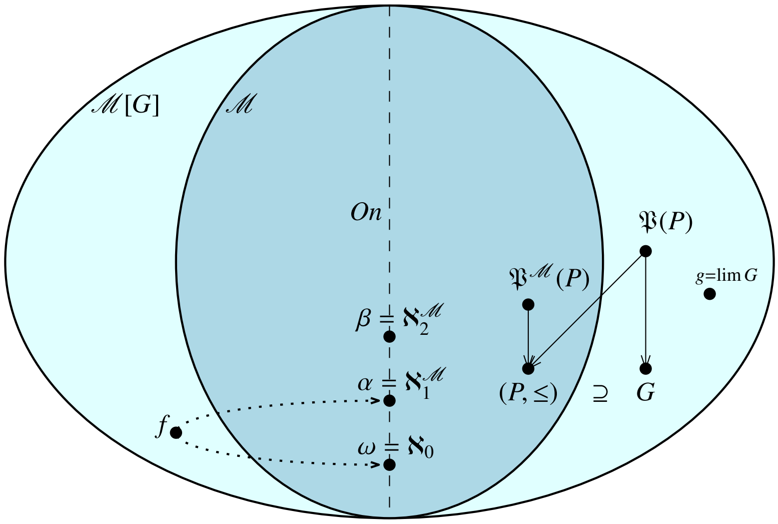
Given two conditions $p, q \in P$, the ordering $p \leq q$ reads '$p$ is stronger than $q$', which intuitively means: '$p$ is a better approximation than $q$', or '$p$ contains more information than $q$'. Since in most cases of forcing, the relation $p \leq q$ is defined as the relation of reverse inclusion $p \supseteq q$, some authors prefer to write it the other way around, i.e. $p \geq q$. (This alternative notation is closer to the spirit of domain theory.) However, from the point of view of Logic, the relation $p \leq q$ may be viewed as a form of implication ('$p$ implies $q$'), which explains why the smaller conditions—that are logically closer to a contradiction—are also the stronger. So that we shall stick to the original notation $p \leq q$ in this course.

**The generic filter** Once the forcing poset $(P, \leq)$ has been chosen in the initial universe, the 'ideal object' $g$ we want to create can be described by a particular subset $G \subseteq P$, called a *generic filter* (Section 3.2.6), whose elements are the actual approximations of the desired ideal object $g$. Alas, such a generic filter $G \subseteq P$ does not exist (in general) in the initial universe, for the same reason that the ideal object $g$ does not exist. (Would the generic filter $G \subseteq P$ exist in the initial universe, then we could directly construct $g$ as some 'limit' of $G$—or as the union $g = \bigcup G$ in the case where ordering is reverse inclusion.)

In this situation, the fundamental theorem of forcing says that we can *enforce* the existence of a generic filter $G \subseteq P$, not in the current universe—this would lead to a contradiction—, but in a larger universe of sets that is generated around the fictitious subset $G \subseteq P$ from the very properties of the forcing poset $(P, \leq)$. The expanded universe that we obtain in this way still fulfils the axioms of ZF, but it now contains a generic filter $G \subseteq P$, from which one can build the desired object $g$ as the limit (or union) of $G$.

**An analogy with Algebra** There is a strong analogy between the method of forcing and the well-known method in Algebra that consists to extend a field $F$ by formally adding a new zero to a given (non constant) polynomial $P \in F[X]$, thus generating the algebraic extension $F[X]/P \supseteq F$. In this analogy, the forcing poset $(P, \leq)$ plays the role of the polynomial whereas the generic filter $G$ (or the ideal object $g$) plays the role of the extra zero. In both cases, the new universe is generated around the old universe from the new ideal object, but its properties are actually determined by the parameter—the poset or polynomial—from which the (fictitious) ideal object is created. However, set-theoretic universes have a much richer structure than fields, which explains why the construction underlying forcing is much more complex.

**Structure of the expanded universe** The expanded universe that is generated from the fictitious subset $G \subseteq P$ (the generic filter) around the initial universe is intuitively 'wider' than the initial universe, since it adds many new subsets to the infinite sets that already exist in the initial universe. (Subsets of finite sets remain unchanged.) The archetypal example of such a new subset is of course the generic filter $G \subseteq P$—that actually generates all new sets. On the other hand, the expanded universe is not 'taller' than the initial universe, since the ordinals are the very same in both universes—they all live in the initial universe.




Surprisingly, the expanded universe may even have fewer cardinals than the initial universe. The reason for this is that new functions—and thus new bijections—may appear in the expanded universe between two already existing sets $X$ and $Y$ (as new subsets of $X \times Y$), so that two distinct cardinals of the initial universe may become in one-to-one correspondence in the expanded universe. In practice, this means that the largest of these two cardinals ceases to be a cardinal from the point of view of the expanded universe—it just becomes one more ordinary ordinal. For instance, the ordinal $\alpha$ that plays the role of $\aleph_1$ (i.e. the first uncountable ordinal) in the initial universe may become denumerable in the expanded universe, due to the creation of a new bijection $f : \omega \xrightarrow{\sim} \alpha$ (as a new subset $f \subseteq \omega \times \alpha$). In this case, the ordinal $\alpha$ becomes one more denumerable ordinal in the expanded universe, while the ordinal $\beta$ formerly playing the role of $\aleph_2$ (in the initial universe) now takes the role of $\aleph_1$ in the expanded universe—provided it did not become denumerable too. In some situations—typically for breaking CH—this phenomenon is undesirable, and we can avoid it by taking a forcing poset $P$ that fulfils the so-called *countable chain condition*, in which case the cardinals are the very same in both universes.

Depending on the choice of the forcing poset, the expanded universe may satisfy some desired properties that are not satisfied in the initial universe. For instance, by taking a suitable poset $P$ of cardinality $\aleph_2$ in the initial universe, we can enforce the existence of an ideal object that is an injection $g : \aleph_2 \to \mathfrak{P}(\omega)$ in the expanded universe (Section 4.3), thus breaking the Continuum Hypothesis. (Note that in this case, the forcing poset must fulfill the countable chain condition to ensure that $\aleph_2$ still plays the role of $\aleph_2$ in the expanded universe.)

# Model-theoretic vs axiomatic approach

**Forcing as a transformation of models** In the literature, the method of forcing is traditionally presented in the language of model theory [Bur77, Bel85, Jec02]. The initial set-theoretic universe is given as a Tarski model $\mathscr{M}$ of ZF, called the *ground model*, and the expanded universe, called a *generic extension of* $\mathscr{M}$ and written $\mathscr{M}[G]$ (where $G$ is the generic filter), is also built as a Tarski model of ZF from the ground model it extends. According to this approach, forcing is thus seen essentially as a model transformation.

The main advantage of the model-theoretic approach is that it puts the accent on the construction of $\mathscr{M}[G]$, and thus justifies the correctness of the method a priori. (Since $\mathscr{M}[G]$ is a model, we know in advance that the theory formed by all formulas it satisfies is consistent.) On the other hand, the complex definition of $\mathscr{M}[G]$ makes that it is sometimes difficult to reason about it. In particular, determining whether a given formula (that is not a consequence of the axioms of ZF) is true in $\mathscr{M}[G]$ often involves complex model-theoretic calculations.

As a matter of fact, the traditional presentation of forcing reflects much more the point of view of the *implementor* than the point of view of the *user*, who is less interested in the details of the construction itself than in the way she can reason in the generic extension. Mathematicians are much more used to axiomatic reasoning than to model-theoretic reasoning, and one of the aims of this course is to present the theory of forcing from an axiomatic point of view.

**Forcing as a transformation of theories** Departing from the usual presentations of forcing in the literature, we shall thus adopt in this course a fully axiomatic approach, by presenting forcing as a method to transform a given *axiomatization* of set theory—that describes some initial universe—into another axiomatization—that describes the corresponding expanded universe—using the language of first-order theories (Section 1.1).

In this approach, the initial universe is described by a first-order theory $\mathscr{T}$, called the *ground theory* (Section 3.1), that may be any standard extension of ZF (Section 1.2.2) whose language contains Skolem symbols $P$ and $(\leq)$ representing the forcing poset—usually accompanied with axioms defining $P$ and $(\leq)$. In the same way, the corresponding expanded universe is described by another first-order theory $\mathscr{T}^*$ (Chapter 3), called the *generic extension of* $\mathscr{T}$ , whose language and axioms are *mechanically derived* from the language and axioms of $\mathscr{T}$. The target theory $\mathscr{T}^*$—another standard extension of ZF that is equiconsistent with $\mathscr{T}$—introduces a constant symbol $G$ representing the canonical generic filter, as well as a class symbol $\mathscr{M}$ that intuitively represents the universe described by $\mathscr{T}$ as a sub-universe in the theory $\mathscr{T}^*$.

The main advantage of the axiomatic approach is that it immediately gives to the user the language and axioms governing the expanded universe, without showing the full construction that justifies these axioms. (Such an approach needs a justification a fortiori, that will be given in Chapter 7.) But from a purely technical point of view, the axiomatic approach has also many advantages. Most notably, it completely removes the need for assuming the existence of *denumerable transitive models of ZF*—an assumption that is stronger than the consistency of ZF. (Although it is not strictly needed, this assumption considerably simplifies the model-theoretic presentation of forcing, as well as it allows us to simplify the construction of the generic extension $\mathscr{M}[G]$ by invoking the Rasiowa-Sikorski Lemma.) Actually, the axiomatic approach allows us to remove any form of model-theoretic reasoning, thus making the equiconsistency proof between the theories $\mathscr{T}$ and $\mathscr{T}^*$ (Chapter 7) purely arithmetical.

However, although this course is *Tarski model free*—we shall develop the theory entirely in the framework of first-order theories—, the concepts, ideas and arguments we shall present here are the very same as in the traditional model-theoretic presentations of forcing. (Readers who already know forcing will find no new idea here, except in some marginal cases.) The only change is in the point of view. Actually, we can even say that the very idea of a *model* is present in every page of this course—but an idea of model that is technically formalized with the notion of first-order theory rather than with Tarski's notion of a model.

**The Axiom of Naming** As we shall see (Section 3.2.7), the key ingredient of the axiomatization of $\mathscr{T}^*$ is a very simple axiom—the *Axiom of Naming*—that contains the seed of the usual model-theoretic construction of generic extensions using $P$-names. (The way we use this axiom to derive the basic properties of forcing is reminiscent from the presentation of forcing given in [Bur77], to which we borrow some ideas and notations.) In practice, the interest of this axiom is that it allows us to unfold only the piece of the underlying model that is needed to derive a particular property—while keeping the rest of the model hidden.

However, the axiomatization of $\mathscr{T}^*$ is powerful enough to allow us to completely unfold the underlying model-theoretic construction based on $P$-names, and to prove in the target theory $\mathscr{T}^*$ (by transfinitely iterating the Axiom of Naming) that every object of the expanded universe is actually represented by a $P$-name in the initial universe. This construction will make clear the connection between our axiomatic approach and the model-theoretic approach, but it will also show that our axiomatization of generic extensions is complete.