Chapter 1

Preliminaries

1.1 First-order theories

In this section we recall some basic definitions and facts about first-order theories. Most proofs are postponed to the end of the chapter, as exercises for the reader.

1.1.1 Definitions

Definition 1.1 (First-order theory) — A *first-order theory* $\mathscr{T}$ is defined by:

- A *first-order language* $\mathscr{L}$, whose terms (notation: t, u, etc.) and formulas (notation: ϕ, ψ, etc.) are constructed from a fixed set of *function symbols* (notation: f, g, h, etc.) and of *predicate symbols* (notation: P, Q, R, etc.) using the following grammar:

Terms	t	$::=$	$x \quad \mid \quad f(t_1, \ldots, t_k)$
Formulas	ϕ, ψ	$::=$	$t_1 = t_2 \quad \mid \quad P(t_1, \ldots, t_k) \quad \mid \quad \top \quad \mid \quad \bot \quad \mid \quad \neg\phi$
			$\mid \quad \phi \Rightarrow \psi \quad \mid \quad \phi \wedge \psi \quad \mid \quad \phi \vee \psi \quad \mid \quad \forall x\, \phi \quad \mid \quad \exists x\, \phi$

  (assuming that each use of a function symbol f or of a predicate symbol P matches its arity). As usual, we call a *constant symbol* any function symbol of arity 0.

- A set of closed formulas of the language $\mathscr{L}$, written $Ax(\mathscr{T})$, whose elements are called the *axioms* of the theory $\mathscr{T}$.

Given a closed formula ϕ of the language of $\mathscr{T}$, we say that ϕ is *derivable in* $\mathscr{T}$—or that ϕ is a *theorem of* $\mathscr{T}$—and write $\mathscr{T} \vdash \phi$ when there are finitely many axioms $\phi_1, \ldots, \phi_n \in Ax(\mathscr{T})$ such that the sequent $\phi_1, \ldots, \phi_n \vdash \phi$ (or the formula $\phi_1 \wedge \cdots \wedge \phi_n \Rightarrow \phi$) is derivable in a deduction system for classical logic. The set of all theorems of $\mathscr{T}$ is written $Th(\mathscr{T})$.

Conventions 1.2 (1) In this course, we only work with first-order theories with equality. In this perspective, we consider the symbol '=' for equality as a logical symbol (as $\neg$, $\wedge$ and $\forall$) that comes with its own deduction rules (cf Fig. 1.1). We use the standard shorthand

$$\exists! x\, \phi(x) \;\equiv\; \exists x\, (\phi(x) \wedge \forall y\, (\phi(y) \Rightarrow y = x))\,.$$

(2) To define the notion of derivability, we may consider any (sequent-based or formula-based) deduction system for classical logic (notation: LK), since all these systems are equivalent in terms of derivability. In this course, we shall implicitly work in Gentzen's Sequent Calculus (with equality), whose rules are recalled in Fig. 1.1.

(3) It is often convenient to mix theories with the sequent notation, writing $\mathscr{T}, \Gamma \vdash \Delta$ (where Γ and Δ are finite lists of formulas) when there is a finite list of axioms $\Gamma_0 \subseteq Ax(\mathscr{T})$ such that the sequent $\Gamma_0, \Gamma \vdash \Delta$ is derivable (in LK). The reader is invited to check that

$$\mathscr{T}, \Gamma \vdash \Delta \qquad \text{iff} \qquad \mathscr{T} \vdash \forall \vec{x} (\wedge\Gamma \Rightarrow \vee\Delta)\,,$$

writing $\wedge\Gamma$ the conjunction of the formulas in Γ, $\vee\Delta$ the disjunction of the formulas in Δ, and $\vec{x}$ the free variables of the formulas occuring in the lists Γ and Δ. Also notice that all the deduction rules of the Sequent Calculus (Fig. 1.1) remain valid if we append the theory $\mathscr{T}$ on the left-hand side of every sequent, both in the premises and in the conclusion of each rule.

Definition 1.3 — Given a first-order theory $\mathscr{T}$, we say that:

- $\mathscr{T}$ *is consistent* when $\mathscr{T} \nvdash \bot$, or—which is equivalent—when there is at least a closed formula ϕ of the language $\mathscr{L}$ such that $\phi \notin Th(\mathscr{T})$.

- $\mathscr{T}$ *is complete* when for every closed formula ϕ of $\mathscr{L}$ we have $\mathscr{T} \vdash \phi$ or $\mathscr{T} \vdash \neg\phi$. Here the 'or' is inclusive, so that by convention, every inconsistent theory is complete.

- $\mathscr{T}$ *is recursively axiomatized* when the set of axioms $Ax(\mathscr{T})$ is recursively enumerable (or semi-recursive). Note that this definition only makes sense when the language of $\mathscr{T}$ is countable, and when it is given with an explicit enumeration of all its symbols.

It is well-known from Gödel's first incompleteness theorem that if a theory $\mathscr{T}$ is at least as expressive as Peano Arithmetic (PA)[1]—which is the case for all the set theories we shall consider in this course—then the above three criteria cannot be fulfilled simultaneously. In practice, all the theories we shall work in are recursively axiomatized, and from Gödel's second incompleteness theorem, we can only hope that they are consistent—thus taking the risk that some of them (if not all) could be instead sadly complete...

However, the fundamental limitations given by Gödel's second incompleteness theorem do not forbid the existence of convincing relative consistency proofs. Formally:

Definition 1.4 (Relative consistency and equiconsistency) — Let $\mathscr{T}$ and $\mathscr{T}'$ be two first-order theories. We say that:

(1) $\mathscr{T}'$ *is consistent relatively to* $\mathscr{T}$ when the consistency of $\mathscr{T}$ implies the consistency of $\mathscr{T}'$, or, which amounts to the same, when $\mathscr{T}' \vdash \bot$ implies $\mathscr{T} \vdash \bot$.

(2) $\mathscr{T}'$ *is equiconsistent with* $\mathscr{T}$ when each of both theories $\mathscr{T}$ and $\mathscr{T}'$ is consistent relatively to the other one, that is, when $\mathscr{T}' \vdash \bot$ iff $\mathscr{T} \vdash \bot$.

[1] Actually, it suffices that $\mathscr{T}$ is at least as expressive as Robinson arithmetic, which is much weaker.

Structural rules

$$\frac{\Gamma \vdash \Delta}{\sigma\Gamma \vdash \Delta}\ (\sigma \text{ perm. of } \Gamma) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \sigma\Delta}\ (\sigma \text{ perm. of } \Delta)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \phi \vdash \Delta} \quad \frac{\Gamma, \phi, \phi \vdash \Delta}{\Gamma, \phi \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \phi, \Delta} \quad \frac{\Gamma \vdash \phi, \phi, \Delta}{\Gamma \vdash \phi, \Delta}$$

Axiom and cut

$$\frac{}{\phi \vdash \phi} \qquad \frac{\Gamma, \phi \vdash \Delta \quad \Gamma' \vdash \phi, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Logical rules

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta \quad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', \phi \Rightarrow \psi \vdash \Delta, \Delta'} \qquad \frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \Rightarrow \psi, \Delta}$$

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta} \quad \frac{\Gamma, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta} \quad \frac{\Gamma \vdash \phi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta} \quad \frac{}{\Gamma \vdash \top, \Delta}$$

$$\frac{\Gamma, \phi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta} \quad \frac{}{\Gamma, \bot \vdash \Delta} \quad \frac{\Gamma \vdash \phi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta} \quad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta}$$

$$\frac{\Gamma, \phi(t) \vdash \Delta}{\Gamma, \forall x\, \phi(x) \vdash \Delta} \qquad \frac{\Gamma \vdash \phi(x), \Delta}{\Gamma \vdash \forall x\, \phi(x), \Delta}\ (x \notin FV(\Gamma,\Delta))$$

$$\frac{\Gamma, \phi(x) \vdash \Delta}{\Gamma, \exists x\, \phi(x) \vdash \Delta}\ (x \notin FV(\Gamma,\Delta)) \qquad \frac{\Gamma \vdash \phi(t), \Delta}{\Gamma \vdash \exists x\, \phi(x), \Delta}$$

Equality rules

$$\frac{}{\phi(t), t = u \vdash \phi(u)} \qquad \frac{}{\vdash t = t}$$

Figure 1.1: Rules of Gentzen's Sequent Calculus (LK)

The main interest of the property of relative consistency is that it can be proved in an ambient theory (or meta-theory) that is much weaker than the two theories under consideration. In this course, we shall only present relative consistency results that are arithmetizable, in the sense that they can be entirely formalized within Peano Arithmetic (PA) or even within a weaker theory such as Primitive Recursive Arithmetic (PRA)[2].

1.1.2 Sub-theories and extensions

Definition 1.5 (Sub-theories and extensions) — Given two first-order theories $\mathscr{T}$ and $\mathscr{T}'$ respectively based on the first-order languages $\mathscr{L}$ and $\mathscr{L}'$, we say that *$\mathscr{T}$ is a sub-theory of $\mathscr{T}'$* (notation: $\mathscr{T} \subseteq \mathscr{T}'$), or that $\mathscr{T}'$ is an *extension* of $\mathscr{T}$ (notation: $\mathscr{T}' \supseteq \mathscr{T}$), when:

(1) $\mathscr{L} \subseteq \mathscr{L}'$ (inclusion of languages);

(2) $Th(\mathscr{T}) \subseteq Th(\mathscr{T}')$ (inclusion of the sets of theorems).

Moreover, we say that $\mathscr{T}$ and $\mathscr{T}'$ are *equivalent* when $\mathscr{T} \subseteq \mathscr{T}'$ and $\mathscr{T}' \subseteq \mathscr{T}$.

Remark 1.6 — The reader is invited to check that the inclusion $Th(\mathscr{T}) \subseteq Th(\mathscr{T})$ can be replaced by the simpler but equivalent inclusion $Ax(\mathscr{T}) \subseteq Th(\mathscr{T}')$ (Exercise 1.1). Also notice that when two theories $\mathscr{T}$ and $\mathscr{T}'$ are equivalent, it does not mean that they have the same axioms; it simply means that every axiom of $\mathscr{T}$ is derivable in $\mathscr{T}'$ and vice-versa.

In what follows, we say that a first-order theory $\mathscr{T}$ is *finite* when the signature of its language is finite as well as the set of its axioms. (But not the set of its theorems.) The fact that any derivation of a theorem in a first-order theory $\mathscr{T}$ involves *finitely many axioms* of $\mathscr{T}$ as well as *finitely many symbols* of the signature of $\mathscr{T}$ implies a central property of Logic, which is the syntactic counterpart of the compactness theorem in model theory:

Proposition 1.7 (Syntactic compactness) — *Let $\mathscr{T}$ be a first-order theory, and ϕ a closed formula of the language of $\mathscr{T}$.*

(1) $\mathscr{T} \vdash \phi$ if and only if there is a finite sub-theory $\mathscr{T}_0 \subseteq \mathscr{T}$ such that $\mathscr{T}_0 \vdash \phi$.

(2) $\mathscr{T}$ is consistent if and only if all its finite sub-theories $\mathscr{T}_0 \subseteq \mathscr{T}$ are consistent.

Proof. (1) We only prove the direct implication, since the converse implication is obvious. For that, let us assume that $\mathscr{T} \vdash \phi$. This means that there are axioms $\phi_1, \ldots, \phi_n \in Ax(\mathscr{T})$ such that the sequent $\phi_1, \ldots, \phi_n \vdash \phi$ has a derivation in LK. Let us now write d such a derivation, and consider the sub-theory $\mathscr{T}_0 \subseteq \mathscr{T}$ whose language is the language induced by the (finitely many) symbols occuring in d and whose axioms are the formulas $\phi_1, \ldots, \phi_n$. Clearly, $\mathscr{T}_0$ is a finite sub-theory of $\mathscr{T}$, and d is a derivation of ϕ in $\mathscr{T}_0$.

(2) Immediately follows from (1). □

[2]Primitive Recursive Arithmetic (PRA, a.k.a. Skolem Arithmetic) is the fragment of Peano Arithmetic (PA) that captures the strength of primitive recursion—but nothing more. Formally, PRA is obtained by enriching the language of PA with function symbols for all primitive recursive functions (that are all definable in PA), while restricting the induction principle to quantifier-free formulas (in the enriched language). PRA is known to be proof-theoretically weaker than PA: its ordinal is ω^ω, whereas the ordinal of PA is $\epsilon_0 = \sup_{n\in\omega}(\omega \Uparrow n)$, where $\omega \Uparrow 0 = 1$ and $\omega \Uparrow (n+1) = \omega^{\omega \Uparrow n}$.

More generally, it is clear that any sub-theory $\mathscr{T}'$ of a given theory $\mathscr{T}$ is consistent relatively to $\mathscr{T}$. On the other hand, *extending* a consistent theory $\mathscr{T}$ may break the property of consistency, and one of the aims of Logic is to determine which extensions are innocuous, and which ones are not. A particular form of extension that is known to be harmless w.r.t. the property of consistency is the notion of *conservative extension*. Formally:

Definition 1.8 (Conservative extension) — An extension $\mathscr{T}' \supseteq \mathscr{T}$ is *conservative* when any formula ϕ of the language of the sub-theory $\mathscr{T}$ that is provable in the extension $\mathscr{T}'$ is actually provable in the sub-theory $\mathscr{T}$ itself, that is: $Th(\mathscr{T}') \cap \mathscr{L} = Th(\mathscr{T})$.

Intuitively, a conservative extension $\mathscr{T}' \supseteq \mathscr{T}$ is a purely linguistic extension of $\mathscr{T}$: it adds new theorems using the new symbols provided by the language of $\mathscr{T}$, but it does not alter the set of theorems expressible in the language of the sub-theory $\mathscr{T}$. We easily check that:

Proposition 1.9 (Equiconsistency by conservativity) — *Any conservative extension $\mathscr{T}'$ of a first-order theory $\mathscr{T}$ is equiconsistent with $\mathscr{T}$.*

Proof. By contraposition: $\mathscr{T} \vdash \bot$ implies $\mathscr{T}' \vdash \bot$ (by extension). Conversely, if $\mathscr{T}' \vdash \bot$, then $\mathscr{T} \vdash \bot$ (by conservativity), since $\bot$ belongs to the language of $\mathscr{T}$. □

1.1.3 Skolemization

A particularly important example of conservative extensions is given by Skolem extensions:

Definition 1.10 (Skolem extension) — Let $\mathscr{T}$ be a first-order theory and $\phi \equiv \phi(\vec{x}, y)$ a formula of the language of $\mathscr{T}$ that only depends on the variables $\vec{x} \equiv x_1, \ldots, x_k$ ($k \geq 0$) and y. We call the *Skolem extension* of the theory $\mathscr{T}$ w.r.t. the formula ϕ the theory $\mathscr{T}'$ defined as follows:

- The language of $\mathscr{T}'$ is the language of $\mathscr{T}$ enriched with a new k-ary function symbol f_ϕ, called the *Skolem symbol* for the formula ϕ.
- The axioms of $\mathscr{T}'$ are the axioms of $\mathscr{T}$, plus the new axiom

  (Skolem's axiom for f_ϕ) $$\forall \vec{x} \forall y\, [\phi(\vec{x}, y) \Rightarrow \phi(\vec{x}, f_\phi(\vec{x}))]\,.$$

The process of extending $\mathscr{T}$ to $\mathscr{T}'$ is called *Skolemization*.

Intuitively, the Skolem function f_ϕ associates to every k-tuple of objects $\vec{x} \equiv x_1, \ldots, x_k$ an object $y = f_\phi(\vec{x})$ such that $\phi(\vec{x}, y)$ when such an object exists, and any object otherwise. In the case where the formula $\forall \vec{x} \exists y\, \phi(\vec{x}, y)$ is derivable in $\mathscr{T}$, Skolem's axiom for the function symbol f_ϕ can be replaced by the simpler but equivalent axiom

(Skolem's axiom for f_ϕ) $$\forall \vec{x}\, \phi(\vec{x}, f_\phi(\vec{x}))\,.$$

Although the process of Skolemization has a strong flavor of the Axiom of Choice, it actually captures nothing of it, since this form of extension is always conservative. (We shall come back to this paradox in Section 1.2.2 and Exercise 1.15.) We present the corresponding result of conservativity only for reference, since we will not use it in this course:

Proposition 1.11 (Conservativity of Skolem extensions) — *Any Skolem extension $\mathscr{T}'$ of a first-order theory $\mathscr{T}$ is a conservative extension of $\mathscr{T}$.*

In what follows, we shall never use Skolem extensions in their most general form, due to the fact that they badly interact with the axiom schemes of set theory (Section 1.2.2 and Exercise 1.15). Instead, we shall only consider two particular forms of Skolem extensions: *Henkin extensions* and *definitional extensions*, that interact much better with the axiom schemes of set theory. Moreover, these particular forms of Skolem extensions have simple syntactic proofs of conservativity, treated in the end of this chapter as exercises.

Henkin extensions Henkin extensions are the particular case of Skolem extensions where there are no parameters $\vec{x}$, that is: when $k = 0$. In this case, the Skolem function symbol f_ϕ becomes a constant symbol c_ϕ, that is called a *Henkin symbol*.

Definition 1.12 (Henkin extension) — Let $\mathscr{T}$ be a first-order theory and $\phi \equiv \phi(y)$ a formula of the language of $\mathscr{T}$ that only depends on the variable y. We call the *Henkin extension* of the theory $\mathscr{T}$ w.r.t. the formula ϕ the theory $\mathscr{T}'$ defined as follows:

- The language of $\mathscr{T}'$ is the language of $\mathscr{T}$ enriched with a new constant symbol c_ϕ.
- The axioms of $\mathscr{T}'$ are the axioms of $\mathscr{T}$, plus the new axiom

  (Henkin's axiom for c_ϕ) $$\forall y\,[\phi(y) \Rightarrow \phi(c_\phi)]\,.$$

Again, the constant c_ϕ intuitively represents any object y such that $\phi(y)$ if there is such an object, and it is unspecified otherwise. In the particular case where $\mathscr{T} \vdash \exists y\,\phi(y)$, the Henkin axiom for the constant symbol c_ϕ can be replaced by the simpler but equivalent axiom

(Henkin's axiom for c_ϕ) $$\phi(c_\phi)\,.$$

Proposition 1.13 (Conservativity of Henkin extensions) — *Any Henkin extension $\mathscr{T}'$ of a first-order theory $\mathscr{T}$ is a conservative extension of $\mathscr{T}$.*

Proof. See Exercise 1.3. □

Definitional extensions Definitional extensions are another particular case of Skolem extensions, that correspond to the situation where, in the theory $\mathscr{T}$, there is (provably) a unique y such that $\phi(\vec{x}, y)$, and this for each k-tuple $\vec{x} \equiv x_1, \dots, x_k$. Formally:

Definition 1.14 (Definitional extension) — Let $\mathscr{T}$ be a first-order theory and $\phi \equiv \phi(\vec{x}, y)$ a formula of the language of $\mathscr{T}$ that only depends on the variables $\vec{x} \equiv x_1, \dots, x_k$ ($k \geq 0$) and y, and such that $\mathscr{T} \vdash \forall\vec{x}\,\exists! y\,\phi(\vec{x}, y)$. We call the *definitional extension* of the theory $\mathscr{T}$ w.r.t. the formula ϕ the theory $\mathscr{T}'$ defined as follows:

- The language of $\mathscr{T}'$ is the language of $\mathscr{T}$ enriched with a new k-ary function symbol f_ϕ.
- The axioms of $\mathscr{T}'$ are the axioms of $\mathscr{T}$, plus the new axiom

  (Definitional axiom for f_ϕ) $$\forall\vec{x}\,\phi(\vec{x}, f_\phi(\vec{x}))\,.$$

As before:

Proposition 1.15 (Conservativity of definitional extensions) *— Any definitional extension of a first-order theory is a conservative extension of that theory.*

Proof. See Exercise 1.4. □

Unlike the most general form of Skolem extensions, we shall see (Section 1.2.2) that definitional extensions interact well with the Separation and Replacement schemes of set theory, in the sense that the defined function symbol f_ϕ can always be used in any instance of these schemes (Prop. 1.25). This important property comes from the fact that we can always eliminate the defined function symbol f_ϕ from any formula in which it appears:

Proposition 1.16 (Elimination of the defined symbol f_ϕ) *— For every formula $\psi(\vec{x})$ of the language of $\mathscr{T}$ (i.e. possibly containing the symbol f_ϕ), there is a formula $\psi_0(\vec{x})$ of the language of $\mathscr{T}$ (i.e. that does not contain f_ϕ) with the same free variables and such that:*

$$\mathscr{T} \vdash \forall \vec{x}(\psi(x) \Leftrightarrow \psi_0(\vec{x}))\,.$$

Proof. See Exercise 1.4 (7). □

1.1.4 The class notation

Given a first-order theory $\mathscr{T}$, we call a *class* of $\mathscr{T}$ any formula $\phi \equiv \phi(x, x_1, \ldots, x_n)$ of $\mathscr{T}$ that is abstracted w.r.t. a particular variable x (usually free in the formula ϕ, but not necessarily). In this course, we shall use the explicit notation

$$\hat{x}.\phi(x, x_1, \ldots, x_n)$$

to denote the class of all objects x such that $\phi(x, x_1, \ldots, x_n)$. In this notation, the variable x (that is bound by the prefix $\hat{x}$) is called the *argument variable*, whereas the other free variables $x_1, \ldots, x_n$ of the formula ϕ are called the *parameters* of the class. Syntactically, the prefix $\hat{x}$ is treated as a binder (as for the prefixes $\forall x$ and $\exists x$) and it is subject to α-conversion.

For instance, if $\phi(E, F)$ is a formula expressing that 'E is a vector space over the field F', the notation $\hat{E}.\phi(E, F)$ denotes the class of vector spaces over the field F (that is parameterized by the field F) whereas the notation $\hat{F}.\phi(E, F)$ denotes the class of all fields over which E is a vector space (that is parameterized by the space E). However, in most situations we shall consider classes *without parameters*, in which case the argument variable (which is the only free variable of the underlying formula) can be left implicit.

The main interest of the class notation is that it allows us to borrow many notations and terminology to set theory—provided we do not mistake classes for sets[3]. For instance, given

[3]Historically, the first attempt (by Frege) to formalize set theory consisted to identify the notion of class (of sets) with the notion of set—a vicious circle that quickly lead to an inconsistency, as shown by Russell. Modern set theories such as Zermelo-Fraenkel set theory or Quine's New Foundations are still based on the idea that certain classes of sets—but not all—may lead to the formation of a set (following the very spirit of set theory), so that it is important to not confuse the two notions, despite their many (intended) similarities.

two classes $C \equiv \hat{x}.\phi(x)$ and $D \equiv \hat{x}.\psi(x)$ (where the formulas $\phi(x)$ and $\psi(x)$ may contain other free variables than x) and a first-order term t, we shall use the notations:

$$\begin{array}{rclrcl} t \in C & \equiv & \phi(t) & C \subseteq D & \equiv & \forall x\,(\phi(x) \Rightarrow \psi(x)) \\ C \cup D & \equiv & \hat{x}.\phi(x) \vee \psi(x) & C^c & \equiv & \hat{x}.\neg\phi(x) \\ C \cap D & \equiv & \hat{x}.\phi(x) \wedge \psi(x) & C \times D & \equiv & \hat{z}.\exists x\,\exists y\,(\phi(x) \wedge \psi(y) \wedge z = (x, y)) \end{array}$$

1.1.5 Relativization

Let $\mathscr{T}$ be a first-order theory and C a class (without parameters) of $\mathscr{T}$. Given a formula ϕ of the language of $\mathscr{T}$, we call the *formula ϕ relativized to C* and write ϕ^C the formula of the same language that is defined by structural induction on ϕ using the equations

$$\begin{array}{rclrcl} \phi^C & \equiv & \phi \qquad \text{(if } \phi \text{ is an atomic formula)} & & & \\ (\neg\phi^C) & \equiv & \neg(\phi^C) & (\phi \Rightarrow \psi)^C & \equiv & \phi^C \Rightarrow \psi^C \\ (\phi \wedge \psi)^C & \equiv & \phi^C \wedge \psi^C & (\phi \vee \psi)^C & \equiv & \phi^C \vee \psi^C \\ (\forall x\,\phi)^C & \equiv & \forall x\,(x \in C \Rightarrow \phi^C) & (\exists x\,\phi)^C & \equiv & \exists x\,(x \in C \wedge \phi^C) \end{array}$$

Strictly speaking, the notation ϕ^C has a meaning only when the function symbols involved in the formula ϕ are compatible with the class C. Formally, we say that a k-ary function symbol f of the language of $\mathscr{T}$ is *compatible* with the class C when

$$\mathscr{T} \ \vdash\ \forall x_1 \cdots \forall x_k\,(x_1 \in C \wedge \cdots \wedge x_k \in C \Rightarrow f(x_1, \ldots, x_k) \in C)\,.$$

In particular, a constant symbol c of the language of $\mathscr{T}$ is compatible with C when $\mathscr{T} \vdash c \in C$. (All predicate symbols are considered to be compatible with C.)

Relativizing a theory The main property of the operation of relativization defined above is that it commutes with all reasoning principles of classical logic, provided the class C is (provably) nonempty, that is, when $\mathscr{T} \vdash \exists x\,(x \in C)$. Formally:

Proposition 1.17 (Logical reasoning relatively to a class) — *Let $\mathscr{T}$ be a first-order theory and C a class of $\mathscr{T}$ (without parameters) that is provably nonempty.*

(1) If ϕ is a closed formula of $\mathscr{T}$ whose symbols are compatible with C and such that ϕ is a classical tautology (i.e. LK $\vdash \phi$), then $\mathscr{T} \vdash \phi^C$.

(2) In particular, if $\phi_1, \ldots, \phi_n$ and ψ are closed formulas of $\mathscr{T}$ whose symbols are compatible with C, and such that ψ is a tautological consequence of $\phi_1, \ldots, \phi_n$ (in the sense of LK), then ψ^C is a consequence of $\phi_1^C, \ldots, \phi_n^C$ in the theory $\mathscr{T}$:

$$\mathscr{T} \vdash \phi_1^C \wedge \cdots \wedge \phi_n^C \Rightarrow \psi^C\,.$$

Proof. (1) See Exercise 1.5 p. 26. Item (2) immediately follows from (1). □

(The reader is invited to check that the above property fails if we relax the hypothesis that the class C is nonempty, or that the formulas $\phi, \phi_1, \ldots, \phi_n, \psi$ are compatible with C.)

From this, we can extend relativization to whole theories as follows:

Definition 1.18 (Relativizing a theory) — Let $\mathscr{T}$ be a first-order theory and C a class of $\mathscr{T}$ (without parameters) that is provably nonempty. We call *the theory $\mathscr{T}$ relativized to C* and write $\mathscr{T}^C$ the first-order theory defined as follows:

- The language of $\mathscr{T}^C$ is the sub-language of the language of $\mathscr{T}$ that is formed by taking all the symbols of the language of $\mathscr{T}$ that are (provably) compatible with the class C. In particular, this language contains the very same predicate symbols as $\mathscr{T}$.

- The axioms of $\mathscr{T}^C$ are all the formulas ϕ of the language of $\mathscr{T}^C$ such that $\mathscr{T} \vdash \phi^C$.

From this definition, we get the following property:

Proposition 1.19 (Characterization of the theorems of $\mathscr{T}^C$) — *Let $\mathscr{T}$ be a first-order theory and C a class of $\mathscr{T}$ (without parameters) that is provably nonempty. Then for every closed formula ϕ of the language of $\mathscr{T}^C$, one has the equivalence:*

$$\mathscr{T}^C \vdash \phi \qquad \textit{iff} \qquad \mathscr{T} \vdash \phi^C .$$

Proof. It is clear that if $\mathscr{T} \vdash \phi^C$, then ϕ is an axiom of $\mathscr{T}^C$ (by definition), so that $\mathscr{T}^C \vdash \phi$. Conversely, let us assume that $\mathscr{T}^C \vdash \phi$. This means that there are axioms $\phi_1, \ldots, \phi_n \in Ax(\mathscr{T}^C)$ such that LK $\vdash \phi_1 \wedge \cdots \wedge \phi_n \Rightarrow \phi$. From Prop. 1.17, we thus have $\mathscr{T} \vdash \phi_1^C \wedge \cdots \wedge \phi_n^C \Rightarrow \phi^C$. But since $\mathscr{T} \vdash \phi_1^C \wedge \cdots \wedge \phi_n^C$ (because $\phi_1, \ldots, \phi_n$ are axioms of $\mathscr{T}^C$), we get $\mathscr{T} \vdash \phi^C$. □

Note that the above proposition expresses that there is no difference between axioms and theorems in the theory $\mathscr{T}^C$, because the set of axioms of $\mathscr{T}^C$ is already closed under logical consequence. (This design is crucial to achieve the equivalence of Prop. 1.19.)

Corollary 1.20 (Equiconsistency of $\mathscr{T}$ and $\mathscr{T}^C$) — *Given a first-order theory $\mathscr{T}$ and a non-empty class C of $\mathscr{T}$ (without parameters), the theories $\mathscr{T}$ and $\mathscr{T}^C$ are equiconsistent.*

Proof. It is clear that $\mathscr{T}^C \vdash \bot$ iff $\mathscr{T} \vdash \bot^C$ (by Prop. 1.19), that is: iff $\mathscr{T} \vdash \bot$. □

Remark 1.21 — If we think of $\mathscr{T}$ as a theory that describes a particular model $\mathscr{M}$, then it is clear that $\mathscr{T}^C$ is nothing but the theory of the sub-structure $\mathscr{M}' = \{a \in \mathscr{M} : \mathscr{M} \models a \in C\} \subseteq \mathscr{M}$. In the framework of first-order theories, the operation of relativization (at the scale of whole theories) is naturally the basis of the notion of inner model.

Definition 1.22 (Inner model) — Let $\mathscr{T}$ and $\mathscr{T}'$ be two first-order theories, and C a nonempty class of $\mathscr{T}$. We say that C is an *inner model* of the theory $\mathscr{T}'$ when $\mathscr{T}' \subseteq \mathscr{T}^C$.

In practice, inner models are useful for proving relative consistency results:

Proposition 1.23 (Relative consistency using an inner model) — *If a first-order theory $\mathscr{T}'$ has an inner model in another theory $\mathscr{T}$, then $\mathscr{T}'$ is consistent relatively to $\mathscr{T}$.*

1.2 Set theories

1.2.1 Zermelo-Fraenkel set theory

The *language of set theory* is the first-order language defined from a single binary predicate symbol $\in$ (membership). Since this language provides no constant/function symbol, its only first-order terms are variables. When working in this language or in any of its extensions, we shall frequently use the following shorthands:

$$\begin{array}{rclrcl} x \subseteq y & \equiv & \forall z\,(z \in x \Rightarrow z \in y) & (\forall x \in y)\,\phi(x) & \equiv & \forall x\,(x \in y \Rightarrow \phi(x)) \\ x \neq y & \equiv & \neg(x = y) & (\exists x \in y)\,\phi(x) & \equiv & \exists x\,(x \in y \wedge \phi(x)) \\ x \notin y & \equiv & \neg(x \in y) & (\exists! x \in y)\,\phi(x) & \equiv & \exists! x\,(x \in y \wedge \phi(x)) \end{array}$$

Formally, *Zermelo-Fraenkel set theory* (ZF) is the first-order theory whose language is the language of set theory (as defined above) and whose axioms are:

Extensionality $\forall a\,\forall b\,[\forall x\,(x \in a \Leftrightarrow x \in b) \;\Rightarrow\; a = b]$

Pairing $\forall a\,\forall b\,\exists c\,\forall x\,[x \in c \;\Leftrightarrow\; x = a \vee x = b]$

Separation $\forall \vec{z}\,\forall a\,\exists b\,\forall\, x[x \in b \;\Leftrightarrow\; x \in a \wedge \phi(x, \vec{z})]$
for every formula $\phi(x, \vec{z})$ of the language of set theory

Union $\forall a\,\exists b\,\forall x\,[x \in b \;\Leftrightarrow\; \exists y\,(y \in a \wedge x \in y)]$

Powerset $\forall a\,\exists b\,\forall x\,[x \in b \;\Leftrightarrow\; x \subseteq a]$

Infinity $\exists a\,[On(a) \;\wedge\; \exists x\,(x \in a) \;\wedge\; (\forall x \in a)\,(\exists y \in a)\,(x \in y)]$

Replacement $\forall \vec{z}\,\forall a\,[(\forall x \in a)\,\exists! y\,\phi(x, y, \vec{z}) \;\Rightarrow\; \exists b\,(\forall x \in a)\,(\exists y \in b)\,\phi(x, y, \vec{z})]$
for every formula $\phi(x, y, \vec{z})$ of the language of set theory

Regularity $\forall a\,[\exists x\,(x \in a) \;\Rightarrow\; (\exists x \in a)\,(\forall y \in x)\,(y \notin a)]$

Recall that Separation (a.k.a. Comprehension) and Replacement are not axioms, but *axiom schemes*, that introduce an axiom for every formula of the language of set theory. In the above statement of the Axiom of Infinity, the formula $On(a)$ (‘a is an ordinal’) is given by

$$On(a) \;\equiv\; a \text{ transitive} \;\wedge\; (a, \in) \text{ strict well ordering}$$

using the shorthands

$$\begin{array}{rcl} a transitive & \equiv & (\forall x \in a)\,(x \subseteq a) \\ (a, R) \text{ strict ordering} & \equiv & (\forall x \in a)\,\neg(x\,R\,x) \;\wedge\; (\forall x, y, z \in a)\,(x\,R\,y \wedge y\,R\,z \Rightarrow x\,R\,z) \\ (a, R) \text{ strict well ordering} & \equiv & (a, R) \text{ strict ordering} \;\wedge \\ & & \forall b\,(b \subseteq a \wedge b \neq \varnothing \;\Rightarrow\; (\exists x_0 \in b)\,(\forall x \in b)\,(x_0\,R\,x \vee x_0 = x)) \end{array}$$

The Axiom of Regularity In this course, we assume that the Axiom of Regularity (also known as the *Axiom of Foundation*) belongs to the ‘official’ axiomatization of ZF, following most authors [Bel85, Jec02]. (Although this axiom is not strictly needed, its presence considerably simplifies the presentation of forcing.) Note that some authors [Kri71, Kri98] consider that ZF does not provide this axiom, which leads to a nonequivalent presentation. However, it can be shown (Exercise 1.12) that both presentations are equiconsistent.

1.2.2 Standard extensions of ZF

In this course, we shall only consider extensions of ZF of the following form:

Definition 1.24 (Standard extension of ZF) — We say that an extension $\mathscr{T} \supseteq \mathrm{ZF}$ is *standard* when the Separation and Replacement axioms (Section 1.2.1) hold in the theory $\mathscr{T}$ for all formulas of the language of $\mathscr{T}$, and not only for the formulas of the (core) language of set theory. Standard extensions of ZF are also called *standard set theories*.

It is clear that all purely axiomatic extensions of ZF (i.e. extensions that do not affect the language) are standard. Typical examples of such extensions are the theories ZFC (= ZF + AC), ZF + ($V = L$) and ZF + (G)CH, which we shall introduce and study in Chapter 2. On the other hand, a typical example of an extension of ZF that is *not* standard is Nelson's *Internal Set Theory* (IST) [Nel77], since its language provides a unary predicate $\mathrm{st}(x)$ ('x is standard') whose use is prohibited in the Separation and Replacement schemes. (This set theory will thus be excluded from our study of forcing, as well as all set theories making similar restrictions.)

An important property of standard set theories is the following:

Proposition 1.25 (Standardness of Henkin/definitional extensions) — *All Henkin and definitional extensions of standard set theories are still standard.*

Proof. Let $\mathscr{T}$ be a standard set theory and $\mathscr{T}' \supseteq \mathscr{T}$ an extension of $\mathscr{T}$ that is either a Henkin extension or a definitional extension. We only treat the case of Separation (the case of Replacement being similar). For that, we consider a formula $\phi(a, x, \vec{z})$ of the language of $\mathscr{T}$, that may contain the extra symbol introduced by the extension $\mathscr{T}' \supset \mathscr{T}$. To prove the separation formula $S_\phi \equiv \forall\vec{z}\,\forall a\,\exists b\,\forall x\,[x \in b \Leftrightarrow x \in a \wedge \phi(a, x, \vec{z})]$ in $\mathscr{T}$, we distinguish two cases:

- $\mathscr{T}'$ is a Henkin extension of $\mathscr{T}$, that introduces the constant symbol c. In this case we write $\phi_0(a, x, \vec{z}, z)$ the formula obtained by replacing in the formula $\phi(a, x, \vec{z})$ every occurrence of the symbol c by an extra free variable z. Since $\phi_0(a, x, \vec{z}, z)$ is a formula of the language of $\mathscr{T}$ and since $\mathscr{T}$ is a standard extension of ZF, the separation formula $S_{\phi_0} \equiv \forall\vec{z}\,\forall z\,\forall a\,\exists b\,\forall x\,[x \in b \Leftrightarrow x \in a \wedge \phi_0(a, x, \vec{z}, z)]$ associated to the formula $\phi_0(a, x, \vec{z}, z)$ is derivable in $\mathscr{T}$, and thus in $\mathscr{T}'$ too (by extension). But the formula S_ϕ immediately follows from the formula S_{ϕ_0} in the theory $\mathscr{T}$, by instanciating z with the constant c.

- $\mathscr{T}'$ is a definitional extension of $\mathscr{T}$, that introduces the function symbol f. In this case, we know by Prop. 1.16 that there is a formula $\phi_0(a, x, \vec{z})$ of the language of $\mathscr{T}$ that is provably equivalent to $\phi(a, x, \vec{z})$ in $\mathscr{T}$. Therefore, the separation formula S_ϕ is provably equivalent (in the theory $\mathscr{T}'$) to the separation formula S_{ϕ_0} associated to the formula $\phi_0(a, x, \vec{z})$. But the latter is provable in $\mathscr{T}'$, since it is already provable in $\mathscr{T}$. □

(This result does not extend to Skolem extensions in general, as shown in Exercise 1.15.)

The above result legitimates the common practice in set theory that consists to enrich the language with various notations that can be used everywhere, including when defining a set by Separation or Replacement. Such notations (that are actually defined constant/function symbols in the sense of Def. 1.14) include $\varnothing$ (the empty set), ω (the set of natural numbers), $\{a, b\}$ (unordered pair), (a, b) (ordered pair), $\mathfrak{P}(a)$ (powerset), $\bigcup a$ (union), $a \cup b$ (binary union), $a \cap b$ (binary intersection), and even the notation $\{x \in a : \phi(x, \vec{z})\}$ (bounded comprehension), that can be seen as a friendly notation for the first-order term $\mathrm{sep}_\phi(a, \vec{z})$, where sep_ϕ is the function symbol defined from the corresponding instance of Separation.

In Chapter 3, we shall restrict our study of forcing to the (standard) set theories with no (proper) function symbol, still allowing constant symbols. (The main reason is that proper function symbols play no essential role in set theory—they are just a convenience—while they add unnecessary complication to the definition of the generic extension.) However, in practice, we shall keep using defined function symbols, considering that these symbols—that do not belong to the official language—can always be eliminated by Prop. 1.16.

1.2.3 The Lévy hierarchy

Let $\mathscr{L}$ be a first-order language that contains the membership predicate $\in$.

Definition 1.26 (Strict Δ_0-formulas) — We say that a formula ϕ (of $\mathscr{L}$) is *strictly* Δ_0 when all quantifications occuring in ϕ are bounded, namely: of the form $(\forall x \in t)\,\psi(t)$ or $(\exists x \in t)\,\psi(t)$ (where t is a first-order term[4]). Formally, strict Δ_0-formulas are generated by the grammar:

Strict Δ_0-Formulas

$$\begin{array}{rcl} \phi, \psi & ::= & t_1 = t_2 \quad | \quad P(t_1, \ldots, t_k) \quad | \quad \top \quad | \quad \bot \\ & | & \neg\phi \quad | \quad \phi \Rightarrow \psi \quad | \quad \phi \wedge \psi \quad | \quad \phi \vee \psi \\ & | & \forall x\,(x \in t \Rightarrow \phi) \quad | \quad \exists x\,(x \in t \wedge \phi) \end{array}$$

Many basic notions of set theory are actually definable in terms of strict Δ_0-formulas as shown in Fig. 1.2 p. 19. This includes most of the vocabulary of elementary set theory (Boolean operations on sets, pairing, Cartesian product, definition and properties of functions, etc.) but also many formulas of ordinal theory: the formula 'α is an ordinal' (cf Exercise 1.9), as well as 'α is a successor ordinal', 'α is a limit ordinal', 'α is a finite ordinal', etc.

On the other hand, many important notions of set theory *cannot* be defined in terms of a strict Δ_0-formula. For instance:

$$Y = \mathfrak{P}(X), \qquad C = B^A, \qquad Cn(\alpha)\ (\alpha \text{ is a cardinal}), \quad \text{etc.} \tag{Π_1}$$

$$X \text{ and } Y \text{ are equipotent}, \qquad X \text{ is denumerable}, \qquad \text{etc.} \tag{Σ_1}$$

The fundamental property of strict Δ_0-formulas is that their meaning is preserved when they are relativized to an arbitrary transitive class. For that, let us recall that a class C of a standard set theory $\mathscr{T}$ is transitive when $\mathscr{T} \vdash (\forall x \in C)(\forall y \in x)\,(y \in C)$.

Proposition 1.27 (Relativizing strict Δ_0-formulas to a transitive class) — *Let $\mathscr{T}$ be a standard set theory and C a transitive class of $\mathscr{T}$. Then for every strict Δ_0-formula $\phi(\vec{x})$ of the language of $\mathscr{T}$ whose constant/function symbols are compatible with C, we have:*

$$\mathscr{T} \vdash (\forall \vec{x} \in C)\,(\phi^C(\vec{x}) \Leftrightarrow \phi(\vec{x}))\,.$$

Proof. The property is proved by (external) induction on the structure of ϕ. The case of atomic formulas is trivial, and the case of formulas constructed by negation, conjunction, disjunction and implication immediately follows from the Induction Hypothesis (IH). We only treat the case where $\phi(\vec{x}) \equiv (\exists x_0 \in t(\vec{x}))\,\phi_0(x_0, \vec{x})$ (the case of universal quantification is dual). Direct

[4] But not a class. Quantifications of the form $(\forall x \in On)\,\psi(x)$ and $(\exists x \in On)\,\psi(x)$ are not bounded.

$$
\begin{aligned}
x \subseteq y &\equiv (\forall z \in x)\,(z \in y)\\
x = \varnothing &\equiv (\forall z \in x)\,(z \notin x)\\
y = \{x\} &\equiv x \in y \;\wedge\; (\forall z \in y)\,(z = x)\\
y = \{x_1, x_2\} &\equiv x_1 \in y \;\wedge\; x_2 \in y \;\wedge\; (\forall z \in y)\,(z = x_1 \vee z = x_2)\\
y = (x_1, x_2) &\equiv (\exists z_1, z_2 \in y)\,(z_1 = \{x_1\} \wedge z_2 = \{x_1, x_2\} \wedge y = \{z_1, z_2\})\\
y = (x_1, \_) &\equiv (\exists z_1, z_2 \in y)\,(\exists x_2 \in z_2)\,(z_1 = \{x_1\} \wedge z_2 = \{x_1, x_2\} \wedge y = \{z_1, z_2\})\\
y = (\_, x_2) &\equiv (\exists z_1, z_2 \in y)\,(\exists x_1 \in z_2)\,(z_1 = \{x_1\} \wedge z_2 = \{x_1, x_2\} \wedge y = \{z_1, z_2\})
\end{aligned}
$$

Operations on sets

$$
\begin{aligned}
C = A \cup B &\equiv A \subseteq C \;\wedge\; B \subseteq C \;\wedge\; (\forall z \in C)\,(z \in A \vee z \in B)\\
C = A \cap B &\equiv C \subseteq A \;\wedge\; C \subseteq B \;\wedge\; (\forall z \in A)\,(z \in B \Rightarrow z \in C)\\
C = A \setminus B &\equiv (\forall x \in A)\,(x \notin B \Rightarrow x \in C) \;\wedge\; (\forall x \in C)\,(x \in A \wedge x \notin B)\\
C \subseteq A \times B &\equiv (\forall z \in C)\,(\exists x \in A)\,(\exists y \in B)\,(z = (x, y))\\
C = A \times B &\equiv C \subseteq A \times B \;\wedge\; (\forall x \in A)\,(\forall y \in B)\,(\exists z \in C)\,(z = (x, y))
\end{aligned}
$$

Functions

$$
\begin{aligned}
y = f(x) &\equiv (\exists z \in f)\,(z = (x, y))\\
A = \mathrm{dom}(f) &\equiv (\forall x \in A)\,(\exists z \in f)\,(z = (x, \_)) \;\wedge\; (\forall z \in f)\,(\exists x \in A)\,(z = (x, \_))\\
B = \mathrm{ran}(f) &\equiv (\forall y \in B)\,(\exists z \in f)\,(z = (\_, y)) \;\wedge\; (\forall z \in f)\,(\exists y \in B)\,(z = (\_, y))\\
f : A \rightharpoonup B &\equiv f \subseteq A \times B \;\wedge\; (\forall x \in A)\,(\forall y, y' \in B)\,(y = f(x) \wedge y' = f(x) \Rightarrow y = y')\\
f : A \to B &\equiv (f : A \rightharpoonup B) \;\wedge\; (\forall x \in A)\,(\exists y \in B)\,(y = f(x))\\
f : A \hookrightarrow B &\equiv f : A \to B \;\wedge\; (\forall x, x' \in A)\,(\forall y \in B)\,(y = f(x) \wedge y = f(x') \Rightarrow x = x')\\
f : A \twoheadrightarrow B &\equiv f : A \to B \;\wedge\; (\forall y \in B)\,(\exists x \in A)\,(y = f(x))\\
f : A \xrightarrow{\sim} B &\equiv f : A \hookrightarrow B \;\wedge\; f : A \twoheadrightarrow B\\
g = f \restriction X &\equiv g \subseteq f \wedge (\forall x \in X)(\forall z \in f)\,(z = (x, \_) \Rightarrow z \in g)
\end{aligned}
$$

Ordinals

$$
\begin{aligned}
y = x + 1 &\equiv x \subseteq y \;\wedge\; x \in y \;\wedge\; (\forall z \in y)\,(z \in x \vee z = x)\\
x \text{ transitive} &\equiv (\forall y \in x)\,(\forall z \in y)\,(z \in x)\\
\alpha \text{ ordinal} &\equiv \alpha \text{ transitive} \;\wedge\; (\forall x, y \in \alpha)\,(x \in y \vee x = y \vee y \in x)\\
\alpha \text{ limit ordinal} &\equiv \alpha \text{ ordinal} \;\wedge\; \alpha \neq \varnothing \;\wedge\; (\forall x \in \alpha)\,(\exists y \in \alpha)\,(x \in y)\\
\alpha \text{ successor ordinal} &\equiv \alpha \text{ ordinal} \;\wedge\; (\exists \beta \in \alpha)\,(\forall x \in \alpha)\,(x \in \beta \vee x = \beta)\\
\alpha \text{ infinite ordinal} &\equiv \alpha \text{ limit ordinal} \;\vee\; (\exists \beta \in \alpha)\,(\beta \text{ limit ordinal})\\
\alpha \in \omega &\equiv \alpha \text{ ordinal} \;\wedge\; \neg(\alpha \text{ infinite ordinal})\\
\alpha = \omega &\equiv \alpha \text{ limit ordinal} \;\wedge\; (\forall \beta \in \alpha)\,\neg(\beta \text{ limit ordinal})
\end{aligned}
$$

Figure 1.2: Abbreviations for some useful Δ_0-formulas

implication: given $\vec{x} \in C$, let us assume that $\phi^C(\vec{x})$. This means that there is $x_0 \in C$ such that $x_0 \in t(\vec{x})$ and $\phi_0^C(x_0, \vec{x})$. By IH, we get $\phi_0(x_0, \vec{x})$, hence $(\exists x_0 \in t(\vec{x}))\, \phi_0(x_0, \vec{x})$, which is precisely the formula $\phi(\vec{x})$. Converse implication: given $\vec{x} \in C$, let us now assume that $\phi(\vec{x})$, which means that $\phi_0(x_0, \vec{x})$ for some $x_0 \in t(\vec{x})$. But since $x \in C$ and since the constant/function symbols occurring in $t(\vec{x})$ are compatible with C, we have $t(\vec{x}) \in C$ and thus $x_0 \in C$ by transitivity. Since $x_0, \vec{x} \in C$, we can now apply IH to get $\phi_0^C(x_0, \vec{x})$. Hence $(\exists x_0 \in C)\, (x_0 \in t(\vec{x}) \land \phi_0^C(x_0, \vec{x}))$, which is precisely the formula $\phi^C(\vec{x})$. □

More generally:

Definition 1.28 (Strict Σ_n/Π_n-formulas) — For each (intuitive) natural number n, we define the classes of strict Σ_n/Π_n-formulas as follows:

- A formula ϕ is *strictly* Σ_0 or *strictly* Π_0 if it is strictly Δ_0.
- For all $n \geq 1$, a formula ϕ is *strictly* Σ_n (resp. *strictly* Π_n) if it is of the form $\phi \equiv \exists\vec{x}\,\phi$ (resp. $\phi \equiv \forall\vec{x}\,\phi$) where ϕ is a strict Π_{n-1}-formula (resp. a strict Σ_{n-1}-formula).

Proposition 1.29 (Relativizing strict Σ_1/Π_1-formulas to a transitive class) — *Let $\mathscr{T}$ be a standard set theory and C a transitive class of $\mathscr{T}$. Then for every formula $\phi(\vec{x})$ of the language of $\mathscr{T}$ whose constant/function symbols are compatible with C, we have:*

(1) $\mathscr{T} \vdash (\forall\vec{x}\in C)\,(\phi^C(\vec{x}) \Rightarrow \phi(\vec{x}))$ *if $\phi(\vec{x})$ is strictly* Σ_1

(2) $\mathscr{T} \vdash (\forall\vec{x}\in C)\,(\phi(\vec{x}) \Rightarrow \phi^C(\vec{x}))$ *if $\phi(\vec{x})$ is strictly* Π_1

Proof. (1) Let $\phi(\vec{x}) \equiv \exists\vec{y}\,\psi(\vec{y}, \vec{x})$, where $\psi(\vec{y}, \vec{x})$ is strictly Δ_0. Given $\vec{x} \in C$, let us assume that $\phi^C(\vec{x})$. This means that there are $\vec{y} \in C$ such that $\psi^C(\vec{y}, \vec{x})$. Since the latter formula is Δ_0, we get $\psi(\vec{y}, \vec{x})$ by Prop. 1.27. Hence $\exists\vec{y}\,\psi(\vec{y}, \vec{x})$, which is precisely the formula $\phi(\vec{x})$.

(2) This case is dual to (1). □

It is important to notice that the notions of strict $\Delta_0/\Sigma_n/\Pi_n$-formulas are defined on a purely syntactic basis, so that they are in general not closed under logical equivalence. By closing them under logical equivalence in ZF, we obtain Lévy's hierarchy:

Definition 1.30 (The Lévy Hierarchy) — For each intuitive natural number n, we call:

- A Σ_n*-formula* (resp. a Π_n*-formula*) any formula $\phi(\vec{x})$ that is equivalent in ZF to a strict Σ_n-formula $\psi(\vec{x})$ (resp. to a strict Π_n-formula $\psi(\vec{x})$), i.e. $\mathrm{ZF} \vdash \forall\vec{x}\,(\phi(\vec{x}) \Leftrightarrow \psi(\vec{x}))$.
- A Δ_n*-formula* any formula $\phi(\vec{x})$ that is both Σ_n and Π_n.

The logical complexity classes $\Sigma_n/\Pi_n/\Delta_n$ that form Lévy's hierarchy naturally enjoy inclusions that are depicted by arrows in the following diagram:

$$\begin{array}{ccccccccccccc}
 & & & \Sigma_1 & & \Sigma_2 & & & & \Sigma_n & & \\
 & & \nearrow & & \searrow\ \nearrow & & \searrow & & \nearrow & & \searrow & \\
\Delta_0 = \Sigma_0 = \Pi_0 & \rightarrow & \Delta_1 & & \Delta_2 & & \Delta_3 & \cdots & \Delta_n & & \Delta_{n+1} & \cdots \\
 & & \searrow & & \nearrow\ \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & & \Pi_1 & & \Pi_2 & & & & \Pi_n & &
\end{array}$$

Proposition 1.31 — *The classes $\Delta_n/\Sigma_n/\Pi_n$ are closed under the following operations:*

- *(Quantifier-free)* *If ϕ is quantifier-free, then ϕ is Δ_0 (and thus $\Delta_n/\Sigma_n/\Pi_n$ for all n).*
- *(Negation)* *If ϕ is Δ_n (resp. Σ_n, Π_n), then $\neg\phi$ is Δ_n (resp. Π_n, Σ_n).*
- *(Conjunction, disjunction)* *If ϕ and ψ are Δ_n (resp. Σ_n, Π_n), then the two formulas $\phi \wedge \psi$ and $\phi \vee \psi$ are Δ_n (resp. Σ_n, Π_n).*
- *(Bounded quantifications)* *If ϕ is Δ_n (resp. Σ_n, Π_n), then the two formulas $(\exists x \in t)\, \phi$ and $(\forall x \in t)\, \phi$ are Δ_n (resp. Σ_n, Π_n).*
- *(Existential quantification)* *If ϕ is Δ_n (resp. Σ_n, Π_n), then $\exists x\, \phi$ is Σ_n (resp. Σ_n, Σ_{n+1}).*
- *(Universal quantification)* *If ϕ is Δ_n (resp. Σ_n, Π_n), then $\forall x\, \phi$ is Π_n (resp. Π_{n+1}, Π_n).*

Proof. See Exercise 1.14. □

1.2.4 Absoluteness

In this section, we only consider standard set theories with no proper function symbol.

Definition 1.32 (Transitive model) — Let $\mathscr{T}_0$ and $\mathscr{T}$ be standard set theories with no proper function symbol. We say that a class $\mathscr{M}$ of $\mathscr{T}$ is a *transitive model* of $\mathscr{T}_0$ when $\mathscr{M}$ is transitive, nonempty, and when it is an inner model of $\mathscr{T}_0$, that is: $\mathscr{T}_0 \subseteq \mathscr{T}^{\mathscr{M}}$ (Def. 1.22).

Definition 1.33 (Absolute formula) — Let $\mathscr{T}$ be a standard set theory with no proper function symbol. We say that a formula $\phi(\vec{x})$ of the language of $\mathscr{T}$ is:

- *absolute* when $\mathscr{T} \vdash (\forall \vec{x} \in \mathscr{M})\, (\phi^{\mathscr{M}}(\vec{x}) \Leftrightarrow \phi(\vec{x}))$
  for any transitive model $\mathscr{M}$ of ZF in $\mathscr{T}$.
- *upwards absolute* when $\mathscr{T} \vdash (\forall \vec{x} \in \mathscr{M})\, (\phi^{\mathscr{M}}(\vec{x}) \Rightarrow \phi(\vec{x}))$
  for any transitive model $\mathscr{M}$ of ZF in $\mathscr{T}$.
- *downwards absolute* when $\mathscr{T} \vdash (\forall \vec{x} \in \mathscr{M})\, (\phi(\vec{x}) \Rightarrow \phi^{\mathscr{M}}(\vec{x}))$
  for any transitive model $\mathscr{M}$ of ZF in $\mathscr{T}$.

Intuitively, absolute formulas (of a standard set theory $\mathscr{T}$) are the formulas whose meaning is the same in all transitive models of ZF (in the theory $\mathscr{T}$) as in the whole universe (described by $\mathscr{T}$). Upwards absolute formulas are the formulas whose truth is preserved when going from a transitive model to the whole universe, whereas downwards absolute formulas are the formulas whose truth is preserved when going into the converse direction.

Proposition 1.34 (Absoluteness) — *In any standard set theory $\mathscr{T}$:*

(1) All Δ_1-formulas are absolute.

(2) All Σ_1-formulas are upwards absolute.

(3) All Π_1-formulas are downwards absolute.

Proof. Items (2) and (3) follow from Prop. 1.29. Item (1) follows from (2) and (3). □

Absoluteness of set-constructors Each set-constructor of ZF (pairing, union, powerset, etc.) is defined from a formula $\phi(\vec{x}, y)$ of the language of set theory such that

$$\mathrm{ZF} \vdash \forall \vec{x}\, \exists! y\, \phi(\vec{x}, y)\,.$$

(For instance, the powerset constructor is defined by the formula $\phi(x, y) \equiv \forall z\, [z \in y \Leftrightarrow z \subseteq x]$.) By extension, each set-constructor of ZF can be used in any standard extension $\mathscr{T} \supseteq \mathrm{ZF}$, writing $f_\phi(\vec{x})$ the unique set y such that $\phi(\vec{x}, y)$.

But when working in a transitive model $\mathscr{M}$ of ZF (in the theory $\mathscr{T}$), each set-constructor of ZF is also reflected within the sub-universe $\mathscr{M}$, due to the fact that

$$\mathscr{T} \vdash (\forall \vec{x} \in \mathscr{M})(\exists! y \in \mathscr{M})\, \phi^{\mathscr{M}}(\vec{x}, y)\,.$$

(since $\mathrm{ZF} \subseteq \mathscr{T}^{\mathscr{M}}$). In this situation, we shall systematically reuse the same symbol f_ϕ with the superscript $\mathscr{M}$ to denote the corresponding operation in $\mathscr{M}$, thus writing $f_\phi^{\mathscr{M}}(\vec{x})$ the unique $y \in \mathscr{M}$ such that $\phi^{\mathscr{M}}(\vec{x}, y)$, where $\vec{x} \in \mathscr{M}$.

The following proposition shows that both operations $f_\phi^{\mathscr{M}}(\vec{x})$ and $f_\phi(\vec{x})$ actually coincide on $\mathscr{M}$ as soon as the defining formula $\phi(\vec{x}, y)$ is Σ_1:

Proposition 1.35 (Absoluteness of Σ_1-set-constructors) — *If $y = f_\phi(\vec{x})$ is a set-constructor of ZF that is defined from a Σ_1-formula $\phi(\vec{x}, y)$, then in any standard set theory $\mathscr{T}$ and in any transitive model $\mathscr{M}$ of ZF in the theory $\mathscr{T}$, one has:*

$$\mathscr{T} \ \vdash \ (\forall \vec{x} \in \mathscr{M})\, (f_\phi^{\mathscr{M}}(\vec{x}) = f_\phi(\vec{x}))\,.$$

In particular, the sub-universe $\mathscr{M}$ is closed under the set-constructor f_ϕ:

$$\mathscr{T} \ \vdash \ (\forall \vec{x} \in \mathscr{M})\, f_\phi(\vec{x}) \in \mathscr{M}\,.$$

Proof. Given $\vec{x} \in \mathscr{M}$, we write y the unique set such that $\phi(\vec{x}, y)$ and y' the unique set in $\mathscr{M}$ such that $\phi^{\mathscr{M}}(\vec{x}, y')$. But since $\phi^{\mathscr{M}}(\vec{x}, y')$ implies $\phi(\vec{x}, y')$ (by upwards absoluteness, because ϕ is Σ_1), we have $\phi(\vec{x}, y')$ and thus $y = y'$ (by uniqueness). □

Therefore:

Corollary 1.36 (Absoluteness of elementary constructions) — *Let $\mathscr{T}$ be a standard set theory and $\mathscr{M}$ a transitive model of ZF in the theory $\mathscr{T}$. Then:*

$$\begin{array}{ll}
\mathscr{T} \ \vdash \ (\forall x, y \in \mathscr{M})\, \{x, y\}^{\mathscr{M}} = \{x, y\} & \mathscr{T} \ \vdash \ (\forall x, y \in \mathscr{M})\, (x \cup y)^{\mathscr{M}} = x \cup y \\
\mathscr{T} \ \vdash \ (\forall x, y \in \mathscr{M})\, (x, y)^{\mathscr{M}} = (x, y) & \mathscr{T} \ \vdash \ (\forall x, y \in \mathscr{M})\, (x \cap y)^{\mathscr{M}} = x \cap y \\
\mathscr{T} \ \vdash \ (\forall x \in \mathscr{M})\, (\bigcup x)^{\mathscr{M}} = \bigcup x & \mathscr{T} \ \vdash \ (\forall x, y \in \mathscr{M})\, (x \setminus y)^{\mathscr{M}} = x \setminus y \\
\mathscr{T} \ \vdash \ \varnothing^{\mathscr{M}} = \varnothing \ \land \ \omega^{\mathscr{M}} = \omega & \mathscr{T} \ \vdash \ (\forall x, y \in \mathscr{M})\, (x \times y)^{\mathscr{M}} = x \times y
\end{array}$$

Proof. Follows from Prop. 1.35 using the strict Δ_0-definitions of Fig. 1.2. □

The case of a set built by Separation is treated as follows:

Proposition 1.37 (Separation in a transitive model) — *Let $\mathscr{T}$ be a standard set theory, and $\mathscr{M}$ a transitive model of ZF in the theory $\mathscr{T}$. Then, for each formula $\phi(x, \vec{z})$ of the language of $\mathscr{T}$ whose function symbols are compatible with the class $\mathscr{M}$, we have:*

$$\mathscr{T} \ \vdash \ (\forall \vec{z} \in \mathscr{M})(\forall a \in \mathscr{M})\, \{x \in a : \phi(x, \vec{z})\}^{\mathscr{M}} = \{x \in a : \phi^{\mathscr{M}}(x, \vec{z})\}\,.$$

(The proof is left as an exercise to the reader.)

Finally, the set-constructors $X \mapsto \mathfrak{P}(X)$ (powerset) and $X, Y \mapsto Y^X$ (function space) are not defined by Σ_1-formulas, and they are not absolute. However, both versions of each constructor (i.e. in the whole universe and in the sub-universe $\mathscr{M}$) are related as follows:

Proposition 1.38 (Powersets and function spaces in transitive models) *— Let $\mathscr{T}$ be a standard set theory and $\mathscr{M}$ a transitive model of ZF in the theory $\mathscr{T}$. Then:*

$$\begin{aligned} \mathscr{T} &\vdash (\forall X \in \mathscr{M})\, \mathfrak{P}^{\mathscr{M}}(X) = \mathfrak{P}(X) \cap \mathscr{M} \\ \mathscr{T} &\vdash (\forall X, Y \in \mathscr{M})\, (Y^X)^{\mathscr{M}} = (Y^X) \cap \mathscr{M}\,, \end{aligned}$$

writing $A \cap \mathscr{M} = \{x \in A : x \in \mathscr{M}\}$.

Proof. Given $X, Y \in \mathscr{M}$, we have

$$\begin{aligned} (X = \mathfrak{P}(X))^{\mathscr{M}} &\Leftrightarrow (\forall Z\,(Z \in Y \Leftrightarrow Z \subseteq X))^{\mathscr{M}} \\ &\Leftrightarrow (\forall Z \in \mathscr{M})(Z \in Y \Leftrightarrow Z \subseteq X) && \text{(since the formula } Z \subseteq X \text{ is } \Delta_0) \\ &\Leftrightarrow Y \cap \mathscr{M} = \mathfrak{P}(X) \cap \mathscr{M} \\ &\Leftrightarrow Y = \mathfrak{P}(X) \cap \mathscr{M} && \text{(since } Y \subseteq \mathscr{M}) \end{aligned}$$

The case of the function space is treated similarly, using the fact that the formula $f : X \to Y$ ('f is a function from X to Y') is Δ_0 (Fig. 1.2). □

Absoluteness of finiteness Let $\mathscr{T}$ be a standard set theory and $\mathscr{M}$ a transitive model of ZF in the theory $\mathscr{T}$. We have seen (Corollary 1.36) that $\omega^{\mathscr{M}} = \omega$, which means that the set ω of finite ordinals (or natural numbers) is the same in $\mathscr{M}$ and in the whole universe described by $\mathscr{T}$. Finite ordinals (i.e. the elements of ω) are thus the same in $\mathscr{M}$ and in the whole universe:

Proposition 1.39 (Absoluteness of finite ordinals) *— Let $\mathscr{T}$ be a standard set theory and $\mathscr{M}$ a transitive model of ZF in the theory $\mathscr{T}$. Then:*

$$\mathscr{T} \vdash \forall n\,[n \in \omega \Leftrightarrow n \in \mathscr{M} \wedge (n \in \omega)^{\mathscr{M}}]\,.$$

More generally, we can show that the notion of a finite set is the same in $\mathscr{M}$ and in the whole universe. Note that since the formula 'X is finite' is Σ_1

$$X \text{ finite} \;\equiv\; \exists n\, \exists f\,(n \text{ finite ordinal} \wedge f : n \xrightarrow{\sim} X)\,,$$

every set that is finite in $\mathscr{M}$ is also finite in the whole universe (by Prop. 1.34 (2)). However, the converse implication is true, and we can actually show the stronger result:

Proposition 1.40 (Absoluteness of finiteness) *— Let $\mathscr{T}$ be a standard set theory and $\mathscr{M}$ a transitive model of ZF in the theory $\mathscr{T}$. Then:*

$$\mathscr{T} \vdash (\forall X \subseteq \mathscr{M})\,(X \textit{ finite} \;\Leftrightarrow\; X \in \mathscr{M} \wedge (X \textit{ finite})^{\mathscr{M}})\,.$$

Proof. (In the theory $\mathscr{T}$:) The converse implication is clear, since $\mathscr{M}$ is transitive and the formula 'X is finite' is Σ_1. For the direct implication, we prove by induction on $n \in \omega$ that:

$$\mathrm{IH}(n) \quad : \quad \forall X\, \forall f\,(X \subseteq \mathscr{M} \wedge f : n \xrightarrow{\sim} X \;\Rightarrow\; X \in \mathscr{M} \wedge f \in \mathscr{M})$$

The base case is trivial, since $X = f = \varnothing \in \mathscr{M}$. Assume that $\mathrm{IH}(n)$ holds for $n \in \omega$, and consider a set $X \subseteq \mathscr{M}$ with a bijection $f : (n+1) \xrightarrow{\sim} X$. By restriction, we have $(f \upharpoonright n) : n \xrightarrow{\sim} X \setminus \{f(n)\}$ so that $X \setminus \{f(n)\} \in \mathscr{M}$ and $(f \upharpoonright n) \in \mathscr{M}$ (by $\mathrm{IH}(n)$). Therefore, we get $X = (X \setminus \{f(n)\}) \cup \{f(n)\} \in \mathscr{M}$ and $f = (f \upharpoonright n) \cup \{(n, f(n))\} \in \mathscr{M}$ (from Coro. 1.36). □

From the above proposition, the formula ‘X is finite’ is thus absolute (although not Δ_0). Moreover, Prop. 1.38 implies that for every set $X \in \mathscr{M}$, the set $\mathfrak{P}_{\text{fin}}(X)$ of all finite subsets of X is the same in the sub-universe $\mathscr{M}$ and in the expanded universe:

Corollary 1.41 (Absoluteness of $\mathfrak{P}_{\text{fin}}$) — *Let $\mathscr{T}$ be a standard set theory and $\mathscr{M}$ a transitive model of ZF in the theory $\mathscr{T}$. Then:*

$$\mathscr{T} \ \vdash \ (\forall X \in \mathscr{M})\, \mathfrak{P}_{\text{fin}}^{\mathscr{M}}(X) = \mathfrak{P}_{\text{fin}}(X)\,.$$

In particular, we have $\mathfrak{P}^{\mathscr{M}}(X) = \mathfrak{P}(X)$ for every finite set $X \in \mathscr{M}$.

Absoluteness of ordinals In ZF, the Axiom of Regularity allows us to define the property $On(\alpha)$ (‘α is an ordinal’) using a Δ_0-formula (cf Exercise 1.9 (2)). From Prop. 1.34 we get:

Proposition 1.42 (Absoluteness of ordinals) — *Let $\mathscr{T}$ be a standard set theory and $\mathscr{M}$ a transitive model of ZF in the theory $\mathscr{T}$. Then:*

$$\mathscr{T} \ \vdash \ (\forall \alpha \in \mathscr{M})\,(On(\alpha) \ \Leftrightarrow \ On^{\mathscr{M}}(\alpha))\,.$$

The above proposition says that when working with a transitive model $\mathscr{M}$ of ZF (within a standard set theory), the classes On and $On^{\mathscr{M}}$ coincide on $\mathscr{M}$: $On^{\mathscr{M}} = On \cap \mathscr{M}$. On the other hand, it does not say anything about the presence or absence of ordinals outside $\mathscr{M}$. In the particular situation of forcing—where $\mathscr{M}$ is a transitive model of the ground theory $\mathscr{T}$ in the generic extension $\mathscr{T}^*$—we shall see that all ordinals actually belong to $\mathscr{M}$.

Exercises

On first-order theories

Exercise 1.1 (Inclusion of theories) — Let $\mathscr{T}$ and $\mathscr{T}'$ be two first-order theories that are respectively based on first-order languages $\mathscr{L}$ and $\mathscr{L}'$ such that $\mathscr{L} \subseteq \mathscr{L}'$. Prove that the two inclusions $Th(\mathscr{T}) \subseteq Th(\mathscr{T}')$ and $Ax(\mathscr{T}) \subseteq Th(\mathscr{T}')$ are equivalent.

Exercise 1.2 (Elimination of symbols) — The aim of this exercise it to prove that if a sequent S has a derivation built in a given first-order language $\mathscr{L}$ (following the rules of LK, cf Fig. 1.1), then the same sequent has another derivation built in the sub-language $\mathscr{L}_0 \subseteq \mathscr{L}$ formed by all the symbols occuring in S. In other words, we can always assume (without loss of generality) that a derivation only uses the symbols that occur in its conclusion.

The proof consists to define a retraction $(\_)^*$ from the language $\mathscr{L}$ to its sub-language $\mathscr{L}_0 \subseteq \mathscr{L}$, that is given by two functions $t \mapsto t^*$ and $\phi \mapsto \phi^*$ that associate to every term t (resp. to every formula ϕ) of the language $\mathscr{L}$ another term t^* (resp. another formula ϕ^*) of the language $\mathscr{L}_0 \subseteq \mathscr{L}$, such that $t^* \equiv t$ (resp. $\phi^* \equiv \phi$) when t (resp. ϕ) already belongs to $\mathscr{L}_0$. The two retractions $t \mapsto t^*$ and $\phi \mapsto \phi^*$ are defined as follows:

- If t already belongs to $\mathscr{L}_0$, then $t^* \equiv t$.
- If $t \equiv f(t_1, \ldots, t_k)$, where $f \in \mathscr{L}_0$, then $t^* \equiv f(t_1^*, \ldots, t_k^*)$.

- If $t \equiv f(t_1, \ldots, t_k)$, where $f \notin \mathscr{L}_0$, then $t^* \equiv x_0$, where x_0 is a fixed variable that parameterizes the definition of the two functions $t \mapsto t^*$ and $\phi \mapsto \phi^*$.
- If ϕ already belongs to $\mathscr{L}_0$, then $\phi^* \equiv \phi$.
- If $\phi \equiv t_1 = t_2$, then $\phi^* \equiv t_1^* = t_2^*$.
- If $\phi \equiv P(t_1, \ldots, t_k)$, where $P \in \mathscr{L}_0$, then $\phi^* \equiv P(t_1^*, \ldots, t_k^*)$.
- If $\phi \equiv P(t_1, \ldots, t_k)$, where $P \notin \mathscr{L}_0$, then $\phi^* \equiv \bot$.
- If $\phi \equiv \phi_1 \Rightarrow \phi_2$, then $\phi^* \equiv \phi_1^* \Rightarrow \phi_2^*$ (and similarly for $\wedge$, $\vee$, $\neg$).
- If $\phi \equiv \forall x\, \phi_1$, then $\phi^* \equiv \forall x\, \phi_1^*$ (and similarly for $\exists$).

(1) Given two terms t, u of the language $\mathscr{L}$ that contain no occurrence of x_0, and a variable $x \not\equiv x_0$, check that $(t\{x := u\})^* \equiv t^*\{x := u^*\}$.

(2) Given a term u and a formula ϕ of the language $\mathscr{L}$ that contain no occurrence of x_0, and a variable $x \not\equiv x_0$, check that $(\phi\{x := u\})^* \equiv \phi^*\{x := u^*\}$ (up to α-conversion).

The retraction $\phi \mapsto \phi^*$ is then extended in a structural (i.e. obvious) way to finite lists of formulas, to sequents and to finite trees labelled with sequents of the language $\mathscr{L}$. (Recall that derivations are a particular case of finite trees labelled with sequents.)

(3) Let d be a finite tree labelled with sequents of the language $\mathscr{L}$. Prove that if d is a (well-formed) derivation of a sequent S (in LK) and if x_0 has no occurrence in d, then d^* is a (well-formed) derivation of the sequent S^* (in LK) built in the language $\mathscr{L}_0$.

(4) Deduce that if a sequent S of the language $\mathscr{L}_0$ has a (well-formed) derivation built in the language $\mathscr{L}$, then it has also a (well-formed) derivation built in the language $\mathscr{L}_0$.

Exercise 1.3 (Proof of Prop. 1.13 p. 12) — Let $\mathscr{T}$ be a first-order theory, $\phi(y)$ a formula of the language of $\mathscr{T}$, and $\mathscr{T}'$ the corresponding Henkin extension (Def. 1.12 p. 12). To prove that $\mathscr{T}'$ is a conservative extension of $\mathscr{T}$, we consider a formula ψ of the language of $\mathscr{T}$ such that $\mathscr{T}' \vdash \psi$. From this assumption, there is a finite list $\Gamma \subseteq Ax(\mathscr{T}')$ and a derivation d built in the language of $\mathscr{T}'$ whose conclusion is the sequent $\Gamma \vdash \psi$.

(1) Using the structural rules of LK, show that without loss of generality, we can assume that $\Gamma \equiv \Gamma_0, \forall y\, (\phi(y) \Rightarrow \phi(c_\phi))$, where Γ_0 is a finite list of axioms of $\mathscr{T}$.

We now fix a variable x_0 that does not occur in d and consider the retraction $(\_)^*$ from the language of $\mathscr{T}'$ (with c_ϕ) to the language of $\mathscr{T}$ (without c_ϕ) such as defined in Exercise 1.2.

(2) Check that d^* is a (well-formed) derivation built in the language of $\mathscr{T}$ and whose conclusion is the sequent $\Gamma_0, \forall y\, (\phi(y) \Rightarrow \phi(x_0)) \vdash \psi$.

(3) Derive the sequent $\vdash \exists x_0\, \forall y\, (\phi(y) \Rightarrow \phi(x_0))$ (the drinkers' paradox) in LK.

(4) From (2) and (3), construct a derivation of the formula ψ in the theory $\mathscr{T}$.

Exercise 1.4 (Proof of Prop. 1.15 p. 13) — Let $\mathscr{T}$ be a first-order theory, $\phi(\vec{x}, y)$ a formula of the language of $\mathscr{T}$ such that $\mathscr{T} \vdash \forall\vec{x}\exists!y\,\phi(\vec{x}, y)$, and let $\mathscr{T}'$ be the corresponding definitional extension of $\mathscr{T}$ (Def. 1.14 p. 12). To every variable z and to every first-order term t of the language of $\mathscr{T}'$, we associate a formula of the language of $\mathscr{T}$ written $z =^* t$ and whose free variables are the free variables of t, plus the variable z. Intuitively, the formula $z =^* t$ expresses that 'z is equal to t', without referring to the symbol f_ϕ that may occur in t.

The definition of the formula $z =^* t$ proceeds as follows:

- If t contains no occurrence of f_ϕ, then $z =^* t$ is the formula $z = t$.
- If $t \equiv f_\phi(t_1, \dots, t_k)$ for some terms $t_1, \dots, t_k$, then $z =^* t$ is the formula $\exists z_1 \cdots \exists z_k (z_1 =^* t_1 \wedge \cdots \wedge z_k =^* t_k \wedge \phi(z_1, \dots, z_k, z))$, where $z_1, \dots, z_k$ are fresh variables.
- If $t \equiv f(t_1, \dots, t_n)$ for some terms $t_1, \dots, t_n$ and for some n-ary function symbol $f \not\equiv f_\phi$, then $z =^* t$ is the formula $\exists z_1 \cdots \exists z_n (z_1 =^* t_1 \wedge \cdots \wedge z_n =^* t_n \wedge z = f(z_1, \dots, z_n))$, where $z_1, \dots, z_k$ are fresh variables.

(1) Check that $\mathscr{T} \vdash \exists! z\,(z =^* t)$ for every variable z and for every term t of $\mathscr{T}'$.

(2) Given variables $x \not\equiv z$ and first-order terms t, u of the language of $\mathscr{T}'$ such that x does not occur free in u, check that $\mathscr{T} \vdash z =^* t\{x := u\} \Leftrightarrow \exists x\,((x =^* u) \wedge (z =^* t))$.

(3) Using the ideas presented above, define a translation $\psi \mapsto \psi^*$ that associates to every formula ψ of the language of $\mathscr{T}'$ a formula ψ^* of the language of $\mathscr{T}$ with the same free variables, and that has intuitively the same meaning as ψ. Moreover, design your translation in such a way that $\psi^* \equiv \psi$ as soon as the symbol f_ϕ does not occur in ψ.

(4) Given a variable x, a first-order term u and a formula ψ of the language of $\mathscr{T}$ such that x does not occur free in u, check that $\mathscr{T} \vdash (\psi\{x := u\})^* \Leftrightarrow \exists x\,((x =^* u) \wedge \psi^*)$.

(5) Show that if a sequent $\Gamma \vdash \Delta$ has a derivation in the language of $\mathscr{T}'$, then there is a finite list $\Gamma' \subseteq Ax(\mathscr{T})$ such that the sequent $\Gamma^*, \Gamma' \vdash \Delta^*$ has a derivation in the language of $\mathscr{T}$.

(6) Deduce that the function $\psi \mapsto \psi^*$ is a *logical translation* from $\mathscr{T}'$ to $\mathscr{T}$, in the sense that for every closed formula ψ of the language of $\mathscr{T}'$: if $\mathscr{T}' \vdash \psi$, then $\mathscr{T} \vdash \psi^*$. Conclude from this that $\mathscr{T}'$ is a conservative extension of $\mathscr{T}$.

(7) Prove that for every formula $\psi \equiv \psi(x_1, \dots, x_n)$ of the language of $\mathscr{T}'$ with free variables $x_1, \dots, x_n$, one has: $\mathscr{T}' \vdash \forall x_1 \cdots \forall x_n\,(\psi(x_1, \dots, x_n) \Leftrightarrow \psi^*(x_1, \dots, x_n))$.

Exercise 1.5 (Proof of Prop. 1.17 (1) p. 14) — Let $\mathscr{T}$ be a first-order theory and C a class of $\mathscr{T}$ that is provably nonempty. Prove that if a sequent $\Gamma \vdash \Delta$ is derivable in LK, and if the function symbols occuring in Γ, Δ are compatible with the class C, then

$$\mathscr{T}, x_1 \in C, \dots, x_n \in C, \Gamma^C \vdash \Delta^C\,,$$

where $x_1, \dots, x_n$ are the free variables of Γ, Δ. (*Hint:* use the result of Exercise 1.2). Deduce that if LK $\vdash \phi$, and if the symbols of ϕ are compatible with C, then $\mathscr{T} \vdash \phi^C$.

On set theories

Exercise 1.6 (Cantor-Bernstein-Schröder Theorem) — Let X be a set equipped with an injection $f : X \hookrightarrow X$ and $Y \subseteq X$ a subset such that $f(X) \subseteq Y$ (writing $f(X)$ the image of X by f). We want to prove that Y is equipotent to X (i.e. in bijection with X). For that, we consider the subset $Z_0 \subseteq X$ defined by $Z_0 = \bigcap\{Z \in \mathfrak{P}(X) : (X \setminus Y) \subseteq Z \wedge f(Z) \subseteq Z\}$.

(1) Prove that $(X \setminus Y) \subseteq Z_0$ and $f(Z_0) = Z_0 \cap Y$.

(2) Using the injection $f : X \hookrightarrow X$ and the subset $Z_0 \subseteq X$, construct a bijection $h : X \xrightarrow{\sim} Y$. (*Hint:* Define $h(x)$ by cases depending on whether $x \in Z_0$ or not.)

(3) From what precedes, deduce that if A and B are two sets with injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, then A and B are equipotent (Cantor-Bernstein-Schröder Theorem).

(4) Construct two injections $f : \mathfrak{P}(\omega) \hookrightarrow \mathbb{R}$ and $g : \mathbb{R} \hookrightarrow \mathfrak{P}(\omega)$, and deduce that the set $\mathfrak{P}(\omega)$ is equipotent to the real line $\mathbb{R}$.

Exercise 1.7 (Ordering and strict orderings) — Let A be a set. We say that a binary relation R on A is an *ordering* when R is reflexive, transitive and antisymmetric:

$$\begin{array}{rcll} (A,R) \text{ ordering} & \equiv & (\forall x \in A)(x\,R\,x) \ \wedge & \text{(reflexivity)} \\ & & (\forall x,y,z \in A)(x\,R\,y \wedge y\,R\,z \Rightarrow x\,R\,z) \ \wedge & \text{(transitivity)} \\ & & (\forall x,y \in A)(x\,R\,y \wedge y\,R\,x \Rightarrow x = y) & \text{(antisymmetry)} \end{array}$$

We say that R is a *strict ordering* when R is irreflexive and transitive:

$$\begin{array}{rcll} (A,R) \text{ strict ordering} & \equiv & (\forall x \in A)\,\neg(x\,R\,x) \ \wedge & \text{(irreflexivity)} \\ & & (\forall x,y,z \in A)(x\,R\,y \wedge y\,R\,z \Rightarrow x\,R\,z) & \text{(transitivity)} \end{array}$$

(1) Prove that if R is an ordering on the set A, then the binary relation R' on A defined by $x\,R'y \equiv x\,R\,y \wedge x \neq y$ (for all $x, y \in A$) is a strict ordering on A.

(2) Prove that if R' is a strict ordering on the set A, then the binary relation R on A defined by $x\,R\,y \equiv x\,R'y \vee x = y$ (for all $x, y \in A$) is an ordering on A.

(3) Deduce from (1) and (2) the existence of a simple bijection between the set of orderings on A and the set of strict orderings on A.

Exercise 1.8 (Characterizing strict well orderings) — Let R be a binary relation on a set A. We say that R is a *strict well ordering* if it is a strict ordering and if every non empty subset of A has a least element:

$$\begin{array}{rcl} (A,R) \text{ strict well ordering} & \equiv & (A,R) \text{ strict ordering } \wedge \\ & & \forall B\,[B \subseteq A \wedge B \neq \varnothing \ \Rightarrow \ (\exists x \in B)(\forall y \in B)\,(x\,R\,y \vee x = y)] \end{array}$$

(1) Prove that any strict well-ordering is a well-founded (or noetherian) relation:
(A,R) well-founded $\equiv \forall B\,[(\forall x \in A)((\forall y \in A)(y\,R\,x \Rightarrow y \in B) \Rightarrow x \in B) \ \Rightarrow \ A \subseteq B]$.

(2) Prove that any strict well-ordering is connex:
(A,R) connex $\equiv (\forall x,y \in A)\,(x\,R\,y \vee x = y \vee y\,R\,x)$.

(3) Conversely, prove that any connex and well-founded relation is a strict well-ordering.

Exercise 1.9 (Regularity and ∈-induction) — In this exercise, we work in $\mathrm{ZF}^- = \mathrm{ZF} - \mathrm{AR}$, where AR is the Axiom of Regularity (Section 1.2.1). We consider the following axiom scheme, called the *scheme of* ∈*-induction* (a.k.a. complete induction):

$$(\in\text{-Ind}) \qquad \forall\vec{z}\,[\forall x\,((\forall y \in x)\phi(y,\vec{z}) \;\Rightarrow\; \phi(x,\vec{z})) \;\Rightarrow\; \forall x\,\phi(x,\vec{z})]\,,$$

for every formula $\phi(x,\vec{z})$ of the language. Intuitively, this scheme of axioms expresses that the membership relation ∈ is well-founded on the whole universe.

(1) Prove that the two theories $\mathrm{ZF}^- + \mathrm{AR}$ (= ZF) and $\mathrm{ZF}^- + \in$-Ind are equivalent.

(2) Using the equivalence proved in Exercise 1.8, deduce that the formula $On(a)$ (Section 1.2.1) is provably equivalent in ZF to the strict Δ_0-formula (Def. 1.26):

$$\begin{aligned} On'(a) \;\equiv\;\; & (\forall x \in a)(\forall y \in x)(y \in a) \;\wedge & (a \text{ transitive}) \\ & (\forall x, y \in a)\,(x \in y \vee x = y \vee y \in x) & (\in \text{ connex on } a) \end{aligned}$$

Exercise 1.10 (Ordinal theory) — The aim of this exercise is to establish the standard results about the class On of ordinals (Section 1.2.1). For that, we work in $\mathrm{ZF}^- = \mathrm{ZF} - \mathrm{AR}$, where AR is the Axiom of Regularity. (Of course, all the following results still hold in ZF.)

(1) Prove (in ZF^-) that

(1.1) $0 = \varnothing$ is an ordinal

(1.2) The class On is transitive: $(\forall\alpha \in On)(\forall\beta \in \alpha)\,\beta \in On$

(1.3) $(\forall\alpha,\beta \in On)\,(\alpha \subseteq \beta \wedge \alpha \neq \beta \;\Leftrightarrow\; \alpha \in \beta)$

The class of ordinals is equipped with the ordering of inclusion, letting $\alpha \le \beta \equiv \alpha \subseteq \beta$ for all $\alpha,\beta \in On$. From (1.3), it is clear that the corresponding strict ordering is membership: $\alpha < \beta$ iff $\alpha \in \beta$ for all $\alpha,\beta \in On$. Moreover, $0 = \varnothing$ is the smallest ordinal (from (1.1)).

(2) Prove that:

(2.1) The ordering ≤ on On is linear: $(\forall\alpha,\beta \in On)\,(\alpha \le \beta \vee \beta \le \alpha)$
Or, which amounts to the same: $(\forall\alpha,\beta \in On)\,(\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha)$

(2.2) For every $\alpha \in On$: $\alpha = \{\beta \in On : \beta < \alpha\}$.

(2.3) If C is a nonempty class of ordinals, then the set $\alpha_0 = \bigcap C$ is an ordinal (so that $\alpha_0 = \inf C$) and $\alpha_0 \in C$ (so that $\alpha_0 = \min C$).

(2.4) For every $\alpha \in On$, the set $\alpha \cup \{\alpha\}$ is an ordinal, that is the smallest element of the class $\{\beta \in On : \beta > \alpha\}$. It is called the *successor of* α and written $\alpha + 1$.

(2.5) If X is a set of ordinals, then $\alpha_0 = \bigcup X$ is an ordinal (so that $\alpha_0 = \sup X$).

(2.6) There is no set of all ordinals, i.e. On is a proper class.

(3) Prove that every well-ordered set $(A, \le_A)$ is isomorphic to a unique ordinal α, through a unique isomorphism $f : A \xrightarrow{\sim} \alpha$ (i.e. such that $x \le_A y$ iff $f(x) \le f(y)$ for all $x, y \in A$).

Exercise 1.11 (Defining the cumulative hierarchy) — Throughout this exercise, we work in $ZF^- = ZF - AR$. The *cumulative hierarchy* is the transfinite sequence $(V_\alpha)_{\alpha\in On}$ that is defined by transfinite recursion on the ordinal α by $V_\alpha = \bigcup_{\beta<\alpha} \mathfrak{P}(V_\beta)$. The first aim of this exercise is to formalize the above definition in the language of set theory without Skolem symbols.

(1) Write down a formula $\phi(\alpha, f)$ of the language of set theory expressing that 'f is a function of domain α such that $f(\beta) = \bigcup_{\gamma<\beta} \mathfrak{P}(f(\gamma))$ for all $\beta < \alpha$'.

(2) Check that $\phi(\alpha, X)$ is functional in $\alpha \in On$, that is: $ZF^- \vdash (\forall\alpha \in On)\, \exists! f\, \phi(\alpha, f)$.
Hint: The proof requires to state and prove auxiliary lemmas.

(3) Deduce a formula $\psi(\alpha, X)$ meaning '$X = V_\alpha$', and check that it is functional in $\alpha \in On$.

Let V be the class defined by $V = \bigcup_{\alpha\in On} V_\alpha$. Formally, the (proper) class V is defined by the formula $V(x) \equiv (\exists\alpha \in On)\, \exists X\, (X = V_\alpha \wedge x \in X)$, writing $X = V_\alpha$ for $\psi(\alpha, X)$.

(4) Check that $ZF^- \vdash AR \Leftrightarrow \forall x\, (x \in V)$.

Exercise 1.12 (Equiconsistency of ZF and ZF^-) — The aim of this exercise is to prove that ZF is equiconsistent with $ZF^- = ZF - AR$, where AR is the Axiom of Regularity. For that, we consider the class $V = \bigcup_{\alpha\in On} V_\alpha$ such as defined in Exercise 1.11.

(1) Check that $ZF^- \vdash \phi^V$ for each axiom ϕ of ZF^-.

(2) Check that $ZF^- \vdash AR^V$. Deduce that V is an inner model of ZF in ZF^- (Def. 1.22).

(3) Conclude that both theories ZF and ZF^- are equiconsistent.

Exercise 1.13 (Collection Scheme) — The *Collection Scheme* is the axiom scheme obtained by dropping the uniqueness requirement on y in the hypothesis of Replacement:

(Collection) $$\forall\vec{z}\, \forall a\, [(\forall x \in a)\, \exists y\, \phi(x, y, \vec{z}) \;\Rightarrow\; \exists b\, (\forall x \in a)(\exists y \in b)\, \phi(x, y, \vec{z})]\,.$$

Since its hypothesis is weaker, the Collection Scheme is stronger than Replacement. The aim of this exercise is to show that the Collection Scheme holds in ZF, so that both axiom schemes are actually equivalent in the presence of the other axioms (especially Regularity).

From now on, we assume given a set a and a binary relation $\phi(x, y)$ (leaving implicit the parameters $\vec{z}$). We first define another relation $\phi'(x, y)$ expressing that 'y is a set of minimal rank such that $\phi(x, y)$' (cf Exercise 1.11 for the definition of V_α), letting:

$$\phi'(x, y) \;\equiv\; (\exists\alpha \in On)\, [y \in V_\alpha \wedge \phi(x, y) \wedge (\forall\beta < \alpha)(\forall y \in V_\beta)\, \neg\phi(x, y)]\,.$$

(1) Prove that $\forall x\, [\exists y\, \phi(x, y) \Leftrightarrow \exists y\, \phi'(x, y)]$

(2) Prove that $\forall x\, \exists! Y\, \forall y\, (y \in Y \Leftrightarrow \phi'(x, y))$

(3) Applying Replacement to the formula $\psi(x, Y) \equiv \forall y\, (y \in Y \Leftrightarrow \phi'(x, y))$, deduce the Collection Axiom associated to the formula $\phi(x, y)$.

Exercise 1.14 (Proof of Prop. 1.31 p. 21) — Prove Prop. 1.31 p. 21. *Hint:* The case of bounded quantifications requires to use the Collection Scheme (Exercise 1.13).

Exercise 1.15 (Skolem extensions and the Axiom of Choice) — The aim of this exercise is to show that Prop. 1.25 does not extend to arbitrary Skolem extensions (Def. 1.10). For that, let us consider the Skolem extension $\mathscr{T} \supseteq \mathrm{ZF}$ w.r.t. the formula $\phi(x, y) \equiv y \in x$. By Def. 1.10, $\mathscr{T}$ enriches ZF with a function symbol f and the Skolem axiom: $\forall x\, \forall y\, (y \in x \Rightarrow f(x) \in x)$.

(1) Check that the above Skolem axiom is equivalent to $\forall x\, (x \neq \varnothing \Rightarrow f(x) \in x)$.

We now consider the Separation formula ϕ_0 (containing the function symbol f) defined by

$$\phi_0 \;\equiv\; \forall a\, \exists b\, \forall z\, [z \in b \;\Leftrightarrow\; z \in a \wedge \exists x\, \exists y\, (z = (x, y) \wedge y = f(x))]\,.$$

(2) Prove (in the theory $\mathscr{T}$) that $\phi_0 \Rightarrow \mathrm{AC}$, where AC is the Axiom of Choice.

(3) Using the fact that $\mathrm{ZF} \nvdash \mathrm{AC}$ (provided ZF is consistent), deduce that $\mathscr{T} \nvdash \phi_0$, so that $\mathscr{T}$ is not a standard extension of ZF (provided ZF is consistent).