Chapter 2

Constructible sets

In this chapter, we recall Gödel's construction of the constructive universe L and prove that both the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH) hold in this universe. Throughout this chapter, we informally work in ZF unless otherwise stated.

2.1 The set of formulas

In Sections 1.1.1 and 1.2.1, we defined the language of set theory at the meta-theoretic level. Up to now, formulas of ZF are syntactic entities that are external to set theory.

However, Gödel's construction relies on the central concept of definability (Section 2.2), that requires to manipulate formulas within the set-theoretic universe, the same way as we manipulate sets. For that, we now need to internalize the language of formulas of ZF in ZF, by constructing a particular denumerable set *Form* whose elements—called *internal formulas* or *codes of formulas*—are intended to represent external formulas as sets.

2.1.1 Construction of the set *Form*

The construction of the set *Form* closely follows the inductive definition of the syntactic category of formulas in the meta-language (Sections 1.1.1 and 1.2.1).

Variables are represented by natural numbers, letting $\mathit{Var} = \omega$. When they are used as (internal) variables, natural numbers are written using the typewriter letters $\mathtt{x}$, $\mathtt{y}$, $\mathtt{z}$, etc. (to prevent confusions with external variables, that are still written using italic letters x, y, z, etc.)

For simplicity, we only consider formulas that are constructed from equality, membership, negation, disjunction and existential quantification, since all the other constructions of first-order logic can be defined from the latter using De Morgan laws. The five syntactic constructs of the (simplified) language of formulas are represented following the correspondence:

$$\begin{array}{rcl@{\qquad}rcl}
\mathtt{x} \doteq \mathtt{x}' & = & (0, (\mathtt{x}, \mathtt{x}')) & \dot{\neg} f & = & (2, f) \\
\mathtt{x} \mathrel{\dot{\in}} \mathtt{x}' & = & (1, (\mathtt{x}, \mathtt{x}')) & f_1 \mathrel{\dot{\vee}} f_2 & = & (3, (f_1, f_2)) \\
 & & & (\dot{\exists}\,\mathtt{x})\, f & = & (4, (\mathtt{x}, f))
\end{array}$$

(Again, we use dotted symbols to prevent confusions with the corresponding meta-theoretical symbols.) Each formula is thus represented in set theory as an ordered pair whose first component indicates the nature of the toplevel syntactic construct, while its second component provides the corresponding arguments (quantified variable, immediate sub-formulas, etc.)

To define the set of all internal formulas, we write V_ω the set of all hereditarily finite sets (cf Exercise 1.11 p. 29) and consider the function $\Phi : \mathfrak{P}(V_\omega) \to \mathfrak{P}(V_\omega)$ defined by

$$\begin{array}{rcll}\Phi(X) & = & \{0\} \times (Var \times Var) & \cup \\ & & \{1\} \times (Var \times Var) & \cup \\ & & \{2\} \times Form & \cup \\ & & \{3\} \times (Form \times Form) & \cup \\ & & \{4\} \times (Var \times Form) & \end{array}$$

for all subsets $X \subseteq V_\omega$. Clearly, the function Φ is monotonic (for inclusion) so that from the Knaster-Tarski theorem (Exercise 2.1), it has a least fixpoint. By definition:

Definition 2.1 (The set of internal formulas) — We write *Form* the least fixpoint of the function $\Phi : \mathfrak{P}(V_\omega) \to \mathfrak{P}(V_\omega)$. Elements of this set are called *internal formulas*.

The reader is invited to check that, thanks to the Axiom of Regularity, the set *Form* is also the greatest fixpoint of the monotonic function Φ (Exercise 2.1 (3)). Moreover, this unique fixpoint can be simply defined as the denumerable union

$$Form \;=\; \bigcup_{n\in\omega} \Phi^n(\varnothing)\,,$$

where $\Phi^n(\varnothing)$ is the set of all (internal) formulas whose height is at most n.

Manipulating internal formulas In what follows, we shall manipulate internal formulas in the set-theoretic universe the same way as we manipulate external formulas in the meta-theory. For instance, we associate to every (internal) formula $f \in Form$ the set of its free variables, which we write $FV(f)$ (this is a finite subset of $Var = \omega$). For all $n \in \omega$, we also write

$$Form_n \;=\; \{f \in Form \;:\; FV(n) \subseteq \{\mathtt{x}_0, \ldots, \mathtt{x}_{n-1}\}\}$$

the set of all formulas whose free (internal) variables occur among $\mathtt{x}_0, \ldots, \mathtt{x}_{n-1}$ $(= 0, \ldots, n-1)$. In particular, the set $Form_0$ is the set of all closed internal formulas.

2.1.2 Encoding external formulas

To every external formula ϕ of the language of set theory, we can now associate a code in the set *Form*, which we write $\ulcorner\phi\urcorner$, following the correspondence depicted in Section 2.1.1. (The definition of $\ulcorner\phi\urcorner$ proceeds by an obvious external induction on the structure of ϕ.) For instance, the formula $\forall z\,(z \in x \Rightarrow z = y)$ is represented in the set *Form* by the code

$$\begin{array}{rcl}\ulcorner\forall z\,(z \in x \Rightarrow z = y)\urcorner & = & \dot\forall\,\mathtt{z}\,(\mathtt{z} \mathrel{\dot\in} \mathtt{x} \mathrel{\dot\Rightarrow} \mathtt{z} \mathrel{\dot=} \mathtt{y}) \\ & = & \dot\neg\,\dot\exists\,\mathtt{z}\;\dot\neg(\dot\neg\,\mathtt{z} \mathrel{\dot\in} \mathtt{x} \mathrel{\dot\vee} \mathtt{z} \mathrel{\dot=} \mathtt{y}) \\ & = & (2, (4, (\mathtt{z}, (2, (3, (2, (1, (\mathtt{z}, \mathtt{x}))), (0, (\mathtt{z}, \mathtt{y})))))))\end{array}$$

where $\mathtt{x}, \mathtt{y}, \mathtt{z} \in Var$ are the codes informally associated to the external variables x, y, z. The precise mapping between external variables and elements of *Var* is irrelevant; we only need that it remains consistent throughout the encoding process.

From the point of view of the meta-theory, the object $\ulcorner\phi\urcorner$ associated to the external formula ϕ is nothing but a definitional constant, or a closed term made of definitional functions and constants in some definitional extension of ZF. Notice that:

- For each external formula ϕ, we have $\quad \mathrm{ZF} \vdash \ulcorner\phi\urcorner \in \mathit{Form}$.
- The correspondence $\phi \mapsto \ulcorner\phi\urcorner$ is intuitively injective, since for any two external formulas ϕ and ψ that are syntactically different, one has $\quad \mathrm{ZF} \vdash \ulcorner\phi\urcorner \neq \ulcorner\psi\urcorner$.
- On the other hand, it is not possible to express within the language of ZF that the correspondence $\phi \mapsto \ulcorner\phi\urcorner$ is surjective, and there are conservative extensions of ZF in which this correspondence is actually *not* surjective (cf Exercise 2.2 p. 49).

2.1.3 Evaluating a formula

Definition 2.2 (Valuations) — Given a set X, we call a *valuation in* X any function $\rho \in X^{\mathit{Var}}$ mapping (codes of) variables to elements of X. Given a valuation $\rho \in X^{\mathit{Var}}$, a (code of a) variable $\mathtt{x}$ and an element $x \in X$, we write $\rho, \mathtt{x} \leftarrow x$ the valuation defined by

$$(\rho, \mathtt{x} \leftarrow x)(\mathtt{y}) \;=\; \begin{cases} x & \text{if } \mathtt{y} \equiv \mathtt{x} \\ \rho(\mathtt{y}) & \text{otherwise} \end{cases}$$

We can now define the evaluation function $\mathit{Val}_X : \mathit{Form} \to \mathfrak{P}(X^{\mathit{Var}})$ that associates to every formula $f \in \mathit{Form}$ its truth value relatively to X, that is: the set $\mathit{Val}_X(f) \subseteq X^{\mathit{Var}}$ of all valuations which make the formula true relatively to X. Formally:

Definition 2.3 (Truth value of a formula) — Let X be a set and f a formula. The *truth value of f relatively to X*, written $\mathit{Val}_X(f)$, is defined by induction on f using the equations:

$$\begin{array}{rcl} \mathit{Val}(\mathtt{x} \mathrel{\dot{=}} \mathtt{y}) & = & \{\rho \in X^{\mathit{Var}} : \rho(\mathtt{x}) = \rho(\mathtt{y})\} \\ \mathit{Val}(\mathtt{x} \mathrel{\dot{\in}} \mathtt{y}) & = & \{\rho \in X^{\mathit{Var}} : \rho(\mathtt{x}) \in \rho(\mathtt{y})\} \\ \mathit{Val}(\dot{\neg}\, f) & = & X^{\mathit{Var}} \setminus \mathit{Val}_X(f) \\ \mathit{Val}(f_1 \mathrel{\dot{\vee}} f_2) & = & \mathit{Val}_X(f_1) \cup \mathit{Val}_X(f_2) \\ \mathit{Val}((\dot{\exists}\, \mathtt{x}) f) & = & \{\rho \in X^{\mathit{Var}} : (\exists x \in X)\, ((\rho, \mathtt{x} \leftarrow x) \in \mathit{Val}_X(f)\} \end{array}$$

Notice that the above definition only inspects in valuations the bindings which correspond to the free variables of the formula f, the other bindings being simply ignored. Formally:

Fact 2.4 — *Given an internal formula $f \in \mathit{Form}$ and two valuations $\rho, \rho' \in X^{\mathit{Var}}$ such that $\rho \upharpoonright FV(f) = \rho' \upharpoonright FV(f)$, we have: $\quad \rho \in \mathit{Val}_X(f) \quad$ iff $\quad \rho' \in \mathit{Val}_X(f)$.*

Therefore, for every closed formula f, we have either:

- $\mathit{Val}_X(f) = X^{\mathit{Var}}$ (i.e. f is true relatively to X); or
- $\mathit{Val}_X(f) = \varnothing$ (i.e. f is false relatively to X).

The satisfaction predicate Given a formula $f = f(\mathtt{x}_1, \ldots, \mathtt{x}_n)$ whose free variables occur among $\mathtt{x}_1, \ldots, \mathtt{x}_n$, and given parameters $x_1, \ldots, x_n \in X$, we write:

$$X \models f(x_1, \ldots, x_n) \quad \equiv \quad (\exists \rho \in X^{\mathit{Var}})\, (\rho(\mathtt{x}_1) = x_1 \wedge \cdots \wedge \rho(\mathtt{x}_n) = x_n \wedge \rho \in \mathit{Val}_X(f))\,.$$

Note that when there is a valuation ρ mapping the variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$ to $x_1, \ldots, x_n \in X$ and such that $\rho \in Val_X(f)$, then for all valuations ρ mapping $\mathtt{x}_1, \ldots, \mathtt{x}_n$ to $x_1, \ldots, x_n \in X$, we actually have $\rho \in Val_X(f)$ (by Fact 2.4). Therefore, we can also let

$$X \models f(x_1, \ldots, x_n) \quad \equiv \quad (\forall \rho \in X^{Var})\,(\rho(\mathtt{x}_1) = x_1 \wedge \cdots \wedge \rho(\mathtt{x}_n) = x_n \Rightarrow \rho \in Val_X(f))$$

without changing the meaning of the definition.

The satisfaction predicate (relatively to a set X) defines an internal notion of truth that is related to the ordinary notion of truth (in ZF) by the following meta-theoretical result:

Proposition 2.5 (Internal truth vs. external truth) — *For every formula $\phi \equiv \phi(x_1, \ldots, x_n)$ of the language of ZFwith free variables $x_1, \ldots, x_n$, we have:*

$$ZF \ \vdash \ (\forall X)\,(\forall x_1, \ldots, x_n \in X)(X \models \phi(x_1, \ldots, x_n) \ \Leftrightarrow \ \phi^X(x_1, \ldots, x_n))\,,$$

where ϕ^X denotes the formula ϕ relativized to the set X.

Proof. The proof is constructed for each formula ϕ by external induction on ϕ. □

Theorem 2.6 (Löwenheim-Skolem, with AC) — *Given a set X and a subset $P \subseteq X$ (the set of parameters), there exists a set Q such that:*

(1) $P \subseteq Q \subseteq X$ and $|Q| \leq \max(|P|, \aleph_0)$;

(2) For all $f \in Form$ with free variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$:

$$(\forall x_1, \ldots, x_n \in P)\,\big(X \models f(x_1, \ldots, x_n) \ \Leftrightarrow \ Q \models f(x_1, \ldots, x_n)\big)\,.$$

Proof. Using AC, we consider a choice function $h : \mathfrak{P}(X) \setminus \{\varnothing\} \to X$ (cf Exercise 2.4). We now define a sequence of subsets $Q_k \subseteq X$ ($k \in \omega$), starting from $Q_0 = P$. For each $k \in \omega$, we define Q_{k+1} as the set of all elements of X of the form

$$h(\{x_0 \in X \ : \ X \models f(x_0, x_1, \ldots, x_n)\})$$

for some $n \in \omega$, $f \in Form_{n+1}$ and $x_1, \ldots, x_n \in X$ such that the set $\{x_0 \in X : X \models f(x_0, x_1, \ldots, x_n)\}$ is nonempty. The set Q_{k+1} actually contains Q_k as a subset, since for every $x \in Q_k$ we have

$$x \ = \ h(\{x\}) \ = \ h(\{x_0 \in X : X \models f(x_0, x)\}) \ \in \ Q_{k+1}\,, \quad \text{taking } f = (\mathtt{x}_0 \doteq \mathtt{x})$$

Moreover, it is clear that

$$|Q_{k+1}| \ \leq \ \sum_{n \in \omega} |Form_{n+1} \times Q_k^n| \ = \ \max(|Q_k|, \aleph_0)\,,$$

so that we get $|Q_k| \leq \max(|P|, \aleph_0)$ by a straightforward induction on $k \in \omega$. We now let $Q = \bigcup_{k \in \omega} Q_k$. Again, we have $|Q| \leq \max(|Q|, \aleph_0)$. Let us now prove that for each formula f with free variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$ and for all parameters $x_1, \ldots, x_n \in Q$ we have the equivalence

$$X \models f(x_1, \ldots, x_n) \ \Leftrightarrow \ Q \models f(x_1, \ldots, x_n)\,.$$

We reason by induction of f. The cases of atomic formulas are trivial, and the cases of internal negation and disjunction immediately follow from the induction hypothesis (IH). Let us now treat the case where $f = (\dot{\exists}\, \mathtt{x}_0)\, f_0$, fixing parameters $x_1, \ldots, x_n \in Q$. (Direct implication:) Let us assume that $X \models f(x_1, \ldots, x_n)$, which means that $X \models f_0(x, x_1, \ldots, x_n)$ for some $x \in X$. Since $x_1, \ldots, x_n \in Q$, we actually have $x_1, \ldots, x_n \in Q_k$ for some $k \in \omega$. If we now take $x' = h(\{x_0 \in X : X \models f(x_0, x_1, \ldots, x_n)\})$ instead of x, we still have $X \models f_0(x', x_1, \ldots, x_n)$, but now with $x' \in Q_{k+1}$ (from the definition of Q_{k+1}). Since all parameters $x', x_1, \ldots, x_n$ belong to Q, we get $Q \models f_0(x', x_1, \ldots, x_n)$ by IH. And since $x' \in Q$, we thus have $Q \models f(x_1, \ldots, x_n)$. (Converse implication:) Let us assume that $Q \models f(x_1, \ldots, x_n)$, which means that $Q \models f_0(x, x_1, \ldots, x_n)$ for some $x \in Q$. By IH we get $X \models f_0(x, x_1, \ldots, x_n)$, and thus $X \models f(x_1, \ldots, x_n)$. □

2.2 Definable subsets

2.2.1 Defining definable subsets

Definition 2.7 (Definable subsets) — Let X be a set. We say that a set Y is a *definable subset of X* when $Y \subseteq X$ and when there exists a formula $f \in Form_{n+1}$ (for some $n \in \omega$) with n elements $x_1, \ldots, x_n \in X$ such that for all $x \in X$ we have:

$$x \in Y \quad \text{iff} \quad X \models f(x, x_1, \ldots, x_n)$$

We then say that *Y is X-definable from the formula f with parameters* $x_1, \ldots, x_n$.

It is important to realize that the notion of definable subset is itself defined by a formula of ZF, namely, by the (external) formula:

$$\begin{aligned} Y \text{ definable subset of } X \quad \equiv \quad & Y \subseteq X \ \land\ (\exists n \in \omega)(\exists f \in Form_{n+1})(\exists (x_1, \ldots, x_n) \in X^n) \\ & (\forall x \in X)\, (x \in Y \ \Leftrightarrow\ X \models f(x, x_1, \ldots, x_n)) \,. \end{aligned}$$

Therefore, we can use the separation axiom with the formula 'Y is a definable subset of X' to define the set $\mathrm{Def}(X)$ of all definable subsets of X, letting:

$$\mathrm{Def}(X) \;=\; \{Y \in \mathfrak{P}(X) \;:\; Y \text{ definable subset of } X\}\,.$$

2.2.2 Properties of the set $\mathrm{Def}(X)$

Let X be a fixed set. By construction, the set $\mathrm{Def}(X)$ of definable subsets of X is a subset of the set $\mathfrak{P}(X)$, whose main property is that:

Proposition 2.8 — *For each external formula $\phi(x, x_1, \ldots, x_n)$ and for all $x_1, \ldots, x_n \in X$:*

$$\{x \in X \;:\; \phi^X(x, x_1, \ldots, x_n)\} \;\in\; \mathrm{Def}(X)\,.$$

Proof. Immediately follows from Prop. 2.5. □

Notice that the set $\mathrm{Def}(X)$ also contains all subsets of X that are definable from non standard formula codes (cf Exercise 2.2 p. 49), if such codes exist.

Proposition 2.9 (Sub-Boolean algebra) — *The set* $\mathrm{Def}(X) \subseteq \mathfrak{P}(X)$ *is a sub-Boolean algebra of* $\mathfrak{P}(X)$*: it contains the empty subset (*$\varnothing \in \mathrm{Def}(X)$*), the full subset (*$X \in \mathrm{Def}(X)$*), and it is closed under binary intersection, binary union and complementation:*

$$Y, Y' \in \mathrm{Def}(X) \quad \Rightarrow \quad (Y \cap Y'), (Y \cup Y'), (X \setminus Y) \in \mathrm{Def}(X)\,.$$

Proof. The set X (resp. $\varnothing$) is definable from the formula $\mathtt{x}_0 \doteq \mathtt{x}_0$ (resp. $\dot\neg(\mathtt{x}_0 \doteq \mathtt{x}_0)$), without parameter. Moreover, if a subset $Y \subseteq X$ is definable from a formula f with n parameters $x_1, \ldots, x_n$, then its complement set $X \setminus Y$ is definable from the formula $\dot\neg f$ with the same parameters. Finally, if another subset $Y' \subseteq X$ is definable from a formula f' with m parameters $x'_1, \ldots, x'_m$, then the union $Y \cup Y'$ is definable from the formula $f \dot\vee f''$ with the $n+m$ parameters $x_1, \ldots, x_n, x'_1, \ldots, x'_m$, where f'' is the formula obtained by replacing in f' every free occurrence of the variable $\mathtt{x}'_i$ ($i \in [1..m]$) by the variable $\mathtt{x}_{n+i}$. □

Proposition 2.10 — *The set* $\mathrm{Def}(X)$ *contains all finite and cofinite subsets of* X*:*

$$\mathfrak{P}_{\mathrm{fin}}(X) \subseteq \mathrm{Def}(X) \quad \textit{and} \quad \mathfrak{P}_{\mathrm{cofin}}(X) \subseteq \mathrm{Def}(X)\,.$$

Proof. From Prop. 2.9, it suffices to show that $\{x\} \in \mathrm{Def}(X)$ for every element $x \in X$. We then take the formula $f = (\mathtt{x}_0 \doteq \mathtt{x}_1)$ with the unique parameter $x_1 = x \in X$. □

Corollary 2.11 — *If* X *is finite, then* $\mathrm{Def}(X) = \mathfrak{P}(X)$.

If the Axiom of Choice (AC) holds, we can also prove the converse implication, which immediately follows from the following result:

Proposition 2.12 (With the Axiom of Choice) — *If* X *is infinite, then the set* $\mathrm{Def}(X)$ *has the same cardinal as* X*:* $|\mathrm{Def}(X)| = |X|$. *Therefore the inclusion* $\mathrm{Def}(X) \subset \mathfrak{P}(X)$ *is strict.*

Proof. The set $\mathrm{Def}(X)$ is actually constructed as the image of the function

$$\begin{array}{rcl} \mathrm{def}_X \;:\; \sum_{n\in\omega}(\mathit{Form}_{n+1} \times X^n) & \to & \mathfrak{P}(X) \\ (n, f, x_1, \ldots, x_n) & \mapsto & \{x \in X : X \models f(x, x_1, \ldots, x_n)\} \end{array}$$

Therefore we have $|\mathrm{Def}(X)| = |\mathrm{Im}(\mathrm{def}_X)| \leq |\sum_{n\in\omega}(\mathit{Form}_{n+1} \times X^n)| = |X|$ (since X is infinite), the converse inequality $|X| \leq |\mathrm{Def}(X)|$ being obvious since $\mathrm{Def}(X)$ contains all singletons. Moreover we have $|\mathrm{Def}(X)| = |X| < 2^{|X|}$ (from Cantor's paradox), and thus $\mathrm{Def}(X) \subset \mathfrak{P}(X)$. □

Even without the Axiom of Choice, we can nevertheless turn any well-ordering over the set X into a well-ordering over the set $\mathrm{Def}(X)$, and this in a canonical way:

Proposition 2.13 (Without the Axiom of Choice) — *If* X *is well-orderable, then* $\mathrm{Def}(X)$ *is well-orderable too. Moreover, there is (in ZF) a functional relation that associates to every set* X *equipped with a well-ordering* $\leq_X$ *a well-ordering* $\leq_{\mathrm{Def}(X)}$ *over the set* $\mathrm{Def}(X)$.

Proof. Let $\leq_{Form}$ be a fixed well-ordering over the denumerable set *Form*. Given a set X equipped with a well-ordering $(\leq_X) \subseteq X^2$, we first endow the set $S = \sum_{n\in\omega}(Form_{n+1} \times X^n)$ with the well-ordering $\leq_S$ defined by

$$\begin{array}{rcl}(n, f, x_1, \ldots, x_n) \leq_S (m, g, y_1, \ldots, y_m) & \Leftrightarrow & n < m \ \vee \ (n = m \wedge f <_{Form} g) \ \vee \\ & & (n = m \wedge f = g \wedge (x_1, \ldots, x_n) \leq_{X^n} (y_1, \ldots, y_m))\end{array}$$

(writing $\leq_{X^n}$ the lexicographic ordering over X^n induced by $\leq_X$). Finally we transport the well ordering $\leq_S$ to $\mathrm{Def}(X)$ via the surjection $\mathrm{def}_X : S \twoheadrightarrow \mathrm{Def}(X)$ (by converse image), letting

$$Y \leq_{\mathrm{Def}(X)} Z \quad \Leftrightarrow \quad \min_S(\mathrm{def}_X^{-1}\{Y\}) \leq_S \min_S(\mathrm{def}_X^{-1}\{Z\})$$

for all $Y, Z \in \mathrm{Def}(X)$. Clearly, this construction of $\leq_{\mathrm{Def}(X)}$ is functional in $(X, \leq)$. □

2.2.3 Non-monotonicity of the correspondence $X \mapsto \mathrm{Def}(X)$

The correspondence $X \mapsto \mathrm{Def}(X)$ is in general non-monotonic, since we may have $X \subseteq Y$ whereas $\mathrm{Def}(X) \nsubseteq \mathrm{Def}(Y)$. Indeed, it suffices to take $X \subseteq Y$ such that X is a non definable subset of Y, so that $X \in \mathrm{Def}(X)$ whereas $X \notin \mathrm{Def}(Y)$.

However, the desired property of monotonicity holds in the following simple situation:

Proposition 2.14 — *If $X \subseteq Y$ and $X \in Y$, then* $\mathrm{Def}(X) \subseteq \mathrm{Def}(Y)$.

The proof of this result relies on the following construction that internalizes the (external) notion of relativization in the set *Form*. Formally, to each formula $f \in Form$ and to each variable $\mathtt{z} \in Var$ we associate the formula $f^{\mathtt{z}}$ that is defined by induction on f as follows:

$$\begin{array}{rcll}(\mathtt{x} \dot{=} \mathtt{y})^{\mathtt{z}} & = & \mathtt{x} \dot{=} \mathtt{y} & \\ (\mathtt{x} \dot{\in} \mathtt{y})^{\mathtt{z}} & = & \mathtt{x} \dot{\in} \mathtt{y} & \\ (\dot{\neg} f)^{\mathtt{z}} & = & \dot{\neg}(f^{\mathtt{z}}) & \\ (f_1 \dot{\vee} f_2)^{\mathtt{z}} & = & f_1^{\mathtt{z}} \dot{\vee} f_2^{\mathtt{z}} & \\ ((\dot{\exists}\mathtt{x}) f)^{\mathtt{z}} & = & (\dot{\exists}\mathtt{x})(\mathtt{x} \dot{\in} \mathtt{z} \mathbin{\dot{\wedge}} f^{\mathtt{z}}) & (\text{if } \mathtt{x} \neq \mathtt{z})\end{array}$$

(In the last equation, we can always ensure that $\mathtt{x} \neq \mathtt{z}$ by renaming the bound variable $\mathtt{x}$.)

Lemma 2.15 (Truth of the formula $f^{\mathtt{z}}$) — *Let X, Y be two sets such that $X \subseteq Y$ and $X \in Y$. Then for every formula $f \in Form$ with free variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$, for every variable $\mathtt{z}$ distinct from $\mathtt{x}_1, \ldots, \mathtt{x}_n$ and for all parameters $x_1, \ldots, x_n \in X$, we have the equivalence*

$$X \models f(x_1, \ldots, x_n) \ \Leftrightarrow \ Y \models (f^{\mathtt{z}})(x_1, \ldots, x_n, X)$$

(where in the right-hand side, the extra variable $\mathtt{z}$ is bound to the extra parameter $X \in Y$).

Proof. By induction on the structure of the internal formula f. □

We can now present the proof of Prop. 2.14:

Proof of Prop. 2.14. Take X and Y such that $X \subseteq Y$ and $X \in Y$, and consider a set $Z \in \mathrm{Def}(X)$ that is definable from a formula $f \in Form_{n+1}$ with n parameters $x_1, \ldots, x_n \in X$. For all $x \in Y$, we have the equivalences:

$$\begin{array}{rcll}x \in Z & \Leftrightarrow & x \in X \wedge X \models f(x, x_1, \ldots, x_n) & \\ & \Leftrightarrow & x \in X \wedge Y \models (f^{\mathtt{x}_{n+1}})(x, x_1, \ldots, x_n, X) & (\text{by Lemma 2.15}) \\ & \Leftrightarrow & Y \models f'(x, x_1, \ldots, x_n, X) & \end{array}$$

by taking $f' = \mathtt{x}_0 \dot{\in} \mathtt{x}_{n+1} \mathbin{\dot{\wedge}} f^{\mathtt{x}_{n+1}}$. Therefore $Z \in \mathrm{Def}(Y)$. □

2.3 Constructible sets

2.3.1 Constructing constructible sets

Similarly to the definition of the cumulative hierarchy $(V_\alpha)_{\alpha\in On}$ (cf Exercise 1.11 p. 29), we now define a transfinite sequence $(L_\alpha)_{\alpha\in On}$, letting

$$L_\alpha \;=\; \bigcup_{\beta<\alpha} \mathrm{Def}(L_\beta)$$

for all $\alpha \in On$. It is clear from the definition that the sequence $\alpha \mapsto L_\alpha$ is monotonic. From this, we deduce that for all $\alpha \in On$:

$$L_0 = \varnothing, \qquad L_{\alpha+1} = \mathrm{Def}(L_\alpha) \qquad \text{and} \qquad L_\alpha = \bigcup_{\beta<\alpha} L_\beta \quad \text{if } \alpha \text{ limit ordinal.}$$

Definition 2.16 (Constructible set) – We call the *constructible universe* and write L the class that is defined as the transfinite union of all L_α when $\alpha \in On$, letting:

$$x \in L \;\equiv\; (\exists \alpha \in On)\,(x \in L_\alpha)\,.$$

The elements of the class L are called the *constructible sets*.

It is clear that $L_\alpha \subseteq V_\alpha$ for all $\alpha \in On$. For each set $x \in L$, we call the *order* of x and write $\mathrm{ord}(x)$ the smallest ordinal α such that $x \in L_\alpha$. Note that this ordinal can be neither zero nor a limit ordinal; it is thus necessarily a successor ordinal.

It is also clear that L_n is finite for all $n \in \omega$, so we have $L_n = V_n$ for all $n \in \omega$. Taking the union, we even get $L_\omega = V_\omega$, which means that L_ω is the set of all hereditarily finite sets. However, we have $L_{\omega+1} \neq V_{\omega+1}$, since $L_{\omega+1} = \mathrm{Def}(L_\omega)$ is denumerable from Prop. 2.12.

Proposition 2.17 — *The set L_α is transitive for all $\alpha \in On$, as well as the class L.*

Proof. The transitivity of the set L_α is proved by transfinite induction on $\alpha \in On$, and the transitivity of the class L immediately follows. □

Proposition 2.18 — *Every ordinal is constructible, and for all $\alpha \in On$ we have* $\mathrm{ord}(\alpha) = \alpha + 1$.

Proof. We first check that $\alpha \notin L_\alpha$ for all $\alpha \in On$. If this is not the case, there is a smallest $\alpha \in On$ such that $\alpha \in L_\alpha = \bigcup_{\beta<\alpha} \mathrm{Def}(L_\beta)$, so that $\alpha \in \mathrm{Def}(L_\beta)$ for some $\beta < \alpha$. From this we get $\alpha \subseteq L_\beta$ and thus $\beta \in L_\beta$, which contradicts the minimality of α. Therefore $\alpha \notin L_\alpha$, from which we deduce that $\beta \notin L_\alpha$ for all $\beta \geq \alpha$ (by transitivity of L_α). Let us now prove by transfinite induction that $\alpha \in L_{\alpha+1}$ for all $\alpha \in On$. For that, take $\alpha \in On$ and assume that $\beta \in L_{\beta+1}$ for all $\beta < \alpha$. From this assumption we get $\beta \in L_\alpha$ for all $\beta < \alpha$, hence $\alpha \subseteq L_\alpha$. Since we already proved that $\beta \notin L_\alpha$ for all $\beta \geq \alpha$, we even have $\alpha = L_\alpha \cap On = L_\alpha \cap On'$ (using the equivalence of Exercise 1.9 p. 28). Therefore:

$$\alpha \;=\; \{x \in L_\alpha : On'(\alpha)\} \;=\; \{x \in L_\alpha : {On'}^{L_\alpha}(\alpha)\} \;\in\; \mathrm{Def}(L_\alpha) \;=\; L_{\alpha+1}$$

by Prop. 1.27, since the formula On' is strictly Δ_0. □

Since the class L contains all ordinals, it is clear that:

Corollary 2.19 — *The class L is a proper class.*

2.3.2 The axiom of constructibility

The *axiom of constructibility* states that 'every set is constructible', that is:

$$(\forall x)(\exists\alpha \in On)\,(x \in L_\alpha)\,.$$

In the literature, this axiom is often abbriged as $V = L$.

The rest of Section 2.3 is devoted to the proof that the axiom $V = L$ is consistent relatively to ZF (Def. 1.4 p. 8). For that, the strategy is to prove that the class L is a transitive model of ZF that fulfills the axiom $V = L$, in the sense that we have the inclusion of theories

$$\mathrm{ZF} + (V = L)\ \subseteq\ \mathrm{ZF}^L\,.$$

(We shall thus conclude by Corollary 1.20 p. 15.) But before proving this result, we first need to introduce an important tool of set theory: the reflection scheme.

2.3.3 The reflection scheme

Let $(W_\alpha)_{\alpha\in On}$ be an increasing transfinite sequence of sets (defined in ZF from a functional relation written $y = W_\alpha$) such that $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$ for every limit ordinal α. We already know two examples of such transfinite sequences: the cumulative hierarchy $(V_\alpha)_{\alpha\in On}$ (cf Exercise 1.11 p. 29) and the constructible hierarchy $(L_\alpha)_{\alpha\in On}$.

The reflection scheme is the following scheme of theorems:

Theorem 2.20 (Reflection scheme) — *For every (external) formula $\phi(\vec{x})$ with free variables $\vec{x}$, the following formula is provable in ZF:*

$$(\forall\alpha \in On)(\exists\beta \text{ limit} > \alpha)(\forall\vec{x}\in W_\beta)(\phi^W(\vec{x})\ \Leftrightarrow\ \phi^{W_\beta}(\vec{x}))\,.$$

Before proving the theorem, let us first notice that the theorem holds vacuously when the class W is empty (since all W_β are empty as well). Without loss of generality, we can thus assume that W is nonempty, and even that none of the sets W_β is empty. (If not: reindex the transifinite sequence $(W_\alpha)_{\alpha\in On}$ from the first ordinal α_0 such that $W_{\alpha_0} \neq \varnothing$.) But under this assumption, we can also restrict to the case where ϕ is in prenex form, namely, of the form

$$\phi(\vec{x})\ \equiv\ (Q_1y_1)\cdots(Q_my_m)\,\psi(y_1,\ldots,y_m,\vec{x})$$

where Q_i is $\forall$ or $\exists$ for all $i \in [1..m]$, and where the formula ψ is quantifier-free. Indeed, every formula is tautologically equivalent to a formula in prenex form, and from Prop. 1.17 p. 14, this equivalence still holds when relativized to the (nonempty) class W as well as to all (nonempty) sets W_α. From now on, we thus only consider the case where $\phi \equiv \phi(\vec{x})$ is in prenex form.

Let us now introduce some terminology. Given a formula $\phi \equiv \phi(\vec{x})$ in prenex form, we say that an ordinal β is *suitable* for ϕ when

$$(\forall\vec{x}\in W_\beta)\,(\phi^W(\vec{x})\ \Leftrightarrow\ \phi^{W_\beta}(\vec{x}))\,.$$

(Notice that this statement is a formula of ZF whose only free variable is β.)

Moreover, we say that the ordinal β is *recursively suitable* for ϕ when the ordinal β is suitable for every subformula of ϕ, including the formula ϕ itself. (Notice that this statement is also a formula of ZF, since ϕ has finitely many subformulas, in the external sense.)

We now prove the following lemma:

Lemma 2.21 — *Let $\phi \equiv \phi(\vec{x})$ be a formula of ZF in prenex form and $(\beta_k)_{k\in\omega}$ an increasing sequence of ordinals. If for all $k \in \omega$ the ordinal β_k is recursively suitable for the formula ϕ, then the (limit) ordinal $\beta = \sup_{k\in\omega}\beta_k$ is recursively suitable for the formula ϕ too.*

Proof. By external induction on the formula ϕ, distinguishing the following cases:

- ϕ is quantifier-free. This case is trivial.
- $\phi \equiv \phi(\vec{x}) \equiv \exists x_0\, \phi_0(x_0, \vec{x})$. For all $k \in \omega$, the ordinal β_k is recursively suitable for the formula ϕ, hence it is recursively suitable for the subformula ϕ_0. By IH, we deduce that $\beta = \sup_{k\in\omega}\beta_k$ is recursively suitable for ϕ_0 too. To prove that β is recursively suitable for ϕ, it remains to prove that

$$(\forall \vec{x} \in W_\beta)\,(\phi^W(\vec{x}) \Leftrightarrow \phi^{W_\beta}(\vec{x}))\,.$$

  Direct implication: given $\vec{x} \in W_\beta$, assume that $\phi^W(\vec{x})$, and take $k \in \omega$ such that $\vec{x} \in W_{\beta_k}$ (since $W_\beta = \bigcup_{k\in\omega} W_{\beta_k}$, by continuity). Since β_k is suitable for ϕ, we have $\phi^{W_{\beta_k}}(\vec{x})$, which means that $\phi_0^{W_{\beta_k}}(x_0, \vec{x})$ for some $x_0 \in W_{\beta_k}$. Hence $\phi_0^W(x_0, \vec{x})$ (since β_k is suitable for ϕ_0) and thus $\phi_0^{W_\beta}(x_0, \vec{x})$ (since β is suitable for ϕ_0, by IH, and since $x_0, \vec{x} \in W_\beta$). From this we deduce the formula $(\exists x_0 \in W_\beta)\, \phi_0^{W_\beta}(x_0, \vec{x})$, that is: $\phi^{W_\beta}(\vec{x})$. Converse implication: given $\vec{x} \in W_\beta$, assume that $\phi^{W_\beta}(\vec{x})$, which means that $\phi_0^{W_\beta}(x_0, \vec{x})$ for some $x_0 \in W_\beta$. Hence we get $\phi_0^W(x_0, \vec{x})$ (since β is suitable for ϕ_0, by IH), from which we deduce that $(\exists x_0 \in W)\, \phi_0^W(x_0, \vec{x})$, that is: $\phi^W(\vec{x})$.
- $\phi \equiv \phi(\vec{x}) \equiv \forall x_0\, \phi_0(x_0, \vec{x})$. This case is treated similarly. □

We can now present the

Proof of Theorem 2.20. We actually prove the stronger result

$$(\forall \alpha \in On)(\exists \beta \text{ limit} > \alpha)(\beta \text{ is recursively suitable for } \phi)\,,$$

reasoning by external induction on the formula ϕ. We distinguish the following cases:

- ϕ is quantifier-free. Take $\beta = \alpha + \omega$.
- $\phi \equiv \phi(\vec{x}) \equiv \exists x_0\, \phi_0(x_0, \vec{x})$. Given $\vec{x} \in W$, we write $\gamma_{\vec{x}}$ the smallest ordinal γ such that, if there is $x \in W$ such that $\phi_0^W(x, \vec{x})$, then there is $x_0 \in W_\gamma$ such that $\phi_0^W(x_0, \vec{x})$. We define by induction on $k \in \omega$ an increasing sequence of limit ordinals $(\beta_k)_{k\in\omega}$ as follows.
  - The ordinal β_0 is defined as the first limit ordinal greater than α that is recursively suited to ϕ_0. (This ordinal exists by IH.)
  - Assuming that β_k has been defined for some $k \in \omega$, we first let

$$\beta'_k = \sup_{x\in W_{\beta_k}} \gamma_{\vec{x}} \qquad \text{(by Replacement)}$$

    and then define β_{k+1} as the smallest limit ordinal greater than $\max(\beta_k, \beta'_k)$ that is recursively suited for the formula ϕ_0. (Again, this ordinal exists by IH.)

By construction, all ordinals $\beta_k > \alpha$ ($k \in \omega$) are recursively suitable for the formula ϕ_0, hence the (limit) ordinal $\beta = \sup_{k\in\omega} \beta_k > \alpha$ is recursively suitable for the formula ϕ_0, from Lemma 2.21. It now remains to prove that β is suitable for the formula $\phi \equiv \exists x_0\, \phi_0$ too. Given $\vec{x} \in W_\beta$, we take $k \in \omega$ such that $\vec{x} \in W_{\beta_k}$ and notice that

$$\begin{aligned} \phi^W(\vec{x}) \quad &\Leftrightarrow \quad (\exists x \in W)\, \phi_0^W(x, \vec{x}) \\ &\Leftrightarrow \quad (\exists x_0 \in W_{\gamma_{\vec{x}}})\, \phi_0^W(x_0, \vec{x}) \\ &\Leftrightarrow \quad (\exists x_0 \in W_\beta)\, \phi_0^W(x_0, \vec{x})\,, \end{aligned}$$

from the definition of $\gamma_{\vec{x}}$, and since $W_{\gamma_{\vec{x}}} \subseteq W_{\beta_{k+1}} \subseteq W_\beta \subset W$. And since β is suited for the formula ϕ_0, we get:

$$\begin{aligned} \phi^W(\vec{x}) \quad &\Leftrightarrow \quad (\exists x_0 \in W_\beta)\, \phi_0^W(x_0, \vec{x}) \\ &\Leftrightarrow \quad (\exists x_0 \in W_\beta)\, \phi_0^{W_\beta}(x_0, \vec{x}) \;\equiv\; \phi^{W_\beta}(\vec{x})\,. \end{aligned}$$

- $\phi \equiv \phi(\vec{x}) \equiv \forall x_0\, \phi_0(x_0, \vec{x})$. This case is treated similarly. □

2.3.4 The class L is a transitive model of ZF (in ZF)

Let us now prove that the (proper) class L is a transitive model of ZF (inside ZF). Since we already know that the class L is transitive and nonempty, we only need to check that:

Proposition 2.22 — *Each axiom ϕ of ZF is satisfied in L: ZF $\vdash \phi^L$.*

Proof. We treat the case of each axiom (or axiom scheme) separately.

Extensionality We want to prove the axiom of Extensionality relativized to L, that is:

$$(\forall a, b \in L)[(\forall x \in L)(x \in a \Leftrightarrow x \in b) \;\Rightarrow a = b]\,.$$

This formula immediately follows from the axiom of Extensionality (in ZF) using the fact that the class L is transitive.

Pairing We want to prove the formula

$$(\forall a, b \in L)(\exists c \in L)(\forall x \in L)\,(x \in c \;\Leftrightarrow\; x = a \vee x = b)\,.$$

Clearly, this amounts to prove that for all constructible sets a and b, the unordered pair $c = \{a, b\}$ is constructible too. For that, we take $a, b \in L$ and consider an ordinal α such that $a, b \in L_\alpha$. Letting $f = (\mathbf{x}_0 \doteq \mathbf{x}_1 \mathbin{\dot\vee} \mathbf{x}_0 \doteq \mathbf{x}_2)$, we see that

$$\{a, b\} \;=\; \{x \in L_\alpha : L_\alpha \models f(x, a, b)\} \;\in\; \mathrm{Def}(L_\alpha) \;=\; L_{\alpha+1}\,.$$

Separation Given a formula $\phi(x, \vec{z})$ of the language of ZF, we want to prove that

$$(\forall \vec{z} \in L)(\forall a \in L)(\exists b \in L)(\forall x \in L)\,[x \in b \;\Leftrightarrow\; x \in a \wedge \phi^L(x, \vec{z})]\,.$$

Again, this amounts to prove that for all $\vec{z} \in L$ and $a \in L$, the set $b = \{x \in a : \phi^L(x, \vec{z})\}$ (defined by Separation in ZF) is constructible too. For that, we take $\vec{z} \in L$ and $a \in L$, and consider an ordinal α such that $\vec{z} \in L_\alpha$ and $a \in L_\alpha$. From the Reflection Scheme (Thm. 2.20), we know that there is a limit ordinal $\beta > \alpha$ such that for all $x \in L_\beta$ we have $\phi^L(x, \vec{z}) \Leftrightarrow \phi^{L_\beta}(x, \vec{z})$ ($\Leftrightarrow L_\beta \models \phi(x, \vec{z})$). Therefore we have

$$\{x \in a : \phi^L(x, \vec{z})\} = a \cap \{x \in L_\beta : L_\beta \models \phi(x, \vec{z})\} \;\in\; \mathrm{Def}(L_\beta) \;=\; L_{\beta+1}\,.$$

Union This case is treated similarly to the case of Pairing.

Powerset We want to prove the formula $(\forall a\, \exists b\, \forall x\, (x \in b \Leftrightarrow x \subseteq a))^L$, which simplifies to

$$(\forall a \in L)(\exists b \in L)(\forall x \in L)\,(x \in b \ \Leftrightarrow\ x \subseteq a)$$

(using the transitivity of L to simplify $(x \subseteq a)^L$ to $x \subseteq a$). This amounts to prove that for every constructible set a, the set $b = \mathfrak{P}(a) \cap L$ of all constructible subsets of a is constructible too. For that, we take $a \in L_\alpha$ for some $\alpha \in On$, and we notice that the class of ordinals $\{\mathrm{ord}(x) : x \in \mathfrak{P}(a) \cap L\}$ is a set (by Replacement), so that we can find $\beta \geq \alpha$ such that $\mathrm{ord}(x) \leq \beta$ for all $x \in \mathfrak{P}(a) \cap L$, which means that $\mathfrak{P}(a) \cap L \subseteq L_\beta$. Letting $f = (\dot\forall\, \mathsf{y})(\mathsf{y} \mathrel{\dot\in} \mathsf{x}_0 \mathrel{\dot\Rightarrow} \mathsf{y} \mathrel{\dot\in} \mathsf{x}_1)$, we see that

$$\mathfrak{P}(a) \cap L \ = \ a \cap \{x \in L_\beta : x \subseteq a\} \ = \ a \cap \{x \in L_\beta : L_\beta \models f(x, a)\} \ \in \ \mathrm{Def}(L_\beta) \ = \ L_{\beta+1}\,.$$

Infinity We already know that $\omega \in L$. We conclude by noticing that the formula 'α is the smallest limit ordinal' is Δ_0.

Replacement This case is similar to the Separation Scheme.

Regularity Obvious from Prop. 1.29, since this axiom is stricly Π_1. □

2.3.5 The axiom $V = L$ is satisfied in L

It now remains to prove that the axiom $V = L$ is satisfied in the universe L. This result is essentially a consequence of the fact that all the formulas that are involved in the definition of the functional relation '$y = L_\alpha$' (including the formula '$y = L_\alpha$' itself) are Σ_1.

For that, we first need to establish some facts about Σ_1-formulas. In what follows, we shall often use the following fact implicitly:

Fact 2.23 (Composing a Σ_1-formula with a Σ_1-functional relation) — *Let $\psi(\vec{x}, y, \vec{z})$ be a Σ_1-formula of ZF and $y = f_\phi(\vec{x})$ a (provably) functional relation (in ZF) that is itself defined from a Σ_1-formula $\phi(\vec{x}, y)$ of ZF. Then the following formula is Σ_1:*

$$\psi(\vec{x}, f_\phi(\vec{x}), \vec{z}) \ \equiv\ \exists y\, (\psi(\vec{x}, y, \vec{z}) \land \phi(\vec{x}, y))\,.$$

Proof. Obvious from Prop. 1.31. □

In particular, the composition of a k-ary functional relation $z = f_\psi(y_1, \ldots, y_k)$ defined from a Σ_1-formula $\psi(y_1, \ldots, y_k, z)$ with k functional relations $y_1 = f_{\phi_1}(\vec{x}), \ldots, y_k = f_{\phi_k}(\vec{x})$ defined from Σ_1-formulas $\phi_1(\vec{x}, y_1), \ldots, \phi_k(\vec{x}, y_k)$ yields a functional relation

$$z = f_\psi(f_{\phi_1}(\vec{x}), \ldots, f_{\phi_k}(\vec{x})) \ \equiv\ \exists y_1 \cdots \exists y_k\, (\phi_1(\vec{x}, y_1) \land \cdots \land \phi_k(\vec{x}, y_k) \land \psi(y_1, \ldots, y_k, z))$$

that is itself defined from a Σ_1-formula. For instance, the functional relations $y = 0$, $y = 1$, $y = 2$, $y = 3$, $y = 4$, $y = \omega$, $y = \{x\}$, $y = x_1 \times x_2$ and $y = x_1 \cup x_2$ are defined from Δ_0- and thus Σ_1-formulas, so that the composed functional relation

$$\begin{aligned} Y = \Phi(X) \quad \equiv \quad & \{0\} \times (\omega \times \omega) \ \cup\ \{1\} \times (\omega \times \omega) \ \cup \\ & \{2\} \times X \ \cup\ \{3\} \times (X \times X) \ \cup\ \{4\} \times (\omega \times X) \end{aligned}$$

is defined from a Σ_1-formula. Therefore:

Fact 2.24 — *The formula $Y = Form$ ('Y is the set of internal formulas') is Σ_1.*

Proof. Since the set *Form* is the unique fixpoint of the functional relation $Y = \Phi(X)$ (cf Exercise 2.1), it suffices to let: $Y = Form \;\equiv\; Y = \Phi(Y)$. □

Fact 2.25 — *The functional relation $y = FV(f)$ ('y is the set of free variables of the internal formula f') is Σ_1.*

Proof. The function $h : Form \to \mathfrak{P}_{\text{fin}}(\omega)$ that associates to every internal formula $f \in Form$ the set of its free variables $h(f) = FV(f)$ is defined from the Σ_1-formula $\phi(h)$ given by

$$\begin{array}{rcll} \phi(h) & \equiv & h \text{ function} \ \land\ \mathrm{dom}(h) = Form & \land \\ & & (\forall v_1, v_2 \in \omega)\, h(v_1 \doteq v_2) = \{v_1, v_2\} & \land \\ & & (\forall v_1, v_2 \in \omega)\, h(v_1 \mathrel{\dot\in} v_2) = \{v_1, v_2\} & \land \\ & & (\forall f \in Form)\, h(\dot\neg f) = h(f) & \land \\ & & (\forall f_1, f_2 \in Form)\, h(f_1 \mathrel{\dot\lor} f_2) = h(f_1) \cup h(f_2) & \land \\ & & (\forall v \in \omega)\, (\forall f \in Form)\, h((\dot\exists\, v)\, f) = h(f) \setminus \{v\} & \end{array}$$

(using the fact that the definitional symbols involved in the above formula are all defined from Δ_0- or Σ_1-formulas). Therefore we can let: $y = FV(f) \;\equiv\; \exists h\, [\phi(h) \land y = h(f)]$. □

We now need to establish that the satisfaction predicate $X \models (f, \rho)$ is Σ_1. The only difficulty is that the set of valuations $X^{Var} = X^\omega$ cannot be defined from a Σ_1-formula, since the construction $X \mapsto X^\omega$ is not absolute. For this reason, we shall not consider full valuations $\rho \in X^{Var}$ (in the sense of Def. 2.2), but only *finite valuations* $\rho \in X^{FV(f)}$, that is: finite functions ρ ranging over the (finite) set of free variables of the internal formula f under consideration.

Fact 2.26 — *The functional relation $Y = X^{FV(f)}$ ('Y is the set of all functions from the set of free variables of f to the set X') is Σ_1.*

Proof. Given a finite set V of variables and an extra variable $v \notin V$, we notice that

$$X^{V \cup \{v\}} \;=\; \mathrm{distr}(X^V, v, X) \;=\; \{\rho \cup \{(v, x)\} \;:\; \rho \in X^V \land x \in X\},$$

where the functional relation $Z = \mathrm{distr}(Y, v, X)$ is defined from the Σ_1-formula

$$\begin{array}{rcl} Z = \mathrm{distr}(Y, v, X) & \equiv & (\forall \rho \in Y)(\forall x \in X)\, \rho \cup \{(v, x)\} \in Z \ \land \\ & & (\forall \rho' \in Z)(\exists \rho \in Y)(\exists x \in X)\, \rho' = \rho \cup \{(v, x)\}. \end{array}$$

We then let:

$$\begin{array}{rcl} Y = X^{FV(f)} & \equiv & \exists N\, \exists g\, \exists h\, [N \in \omega \ \land\ g : N \tilde{\to} FV(f) \ \land \\ & & \quad h \text{ function} \ \land\ \mathrm{dom}(h) = N + 1 \ \land\ h(N) = Y \ \land \\ & & \quad h(0) = \{\varnothing\} \ \land\ (\forall n \in N)\, h(n+1) = \mathrm{distr}(h(n), g(n), X)]. \end{array}$$ □

Similarly, the functional relation $Y = Val_X(f)$ ('Y is the truth value of the internal formula f w.r.t. the set X', cf Def. 2.3) cannot be defined from a Σ_1-formula. Instead, we shall consider the *reduced truth value* $Val'_X(f) \subseteq X^{FV(f)}$ that is defined by:

$$Val'_X(f) \;=\; \{\rho \in X^{FV(f)} : X \models (f, \rho)\} \;=\; \{\rho \upharpoonright FV(f) \;:\; \rho \in Val_X(f)\}.$$

Fact 2.27 — *The functional relation $Y = Val'_X(f)$ is Σ_1.*

Proof. Given a set X, the function h that associates to every formula $f \in Form$ the reduced truth value $h(f) = Val'_X(f)$ is defined from the Σ_1-formula $\phi(X, h)$ given by:

$$\begin{array}{rcll} \phi(X,h) & \equiv & h \text{ function } \wedge \ \mathrm{dom}(h) = Form & \wedge \\ & & (\forall v_1, v_2 \in \omega)\,(\forall \rho \in X^{FV(v_1 \dot{=} v_2)})\,[\rho \in h(v_1 \dot{=} v_2) \Leftrightarrow \rho(v_1) = \rho(v_2)] & \wedge \\ & & (\forall v_1, v_2 \in \omega)\,(\forall \rho \in X^{FV(v_1 \dot{\in} v_2)})\,[\rho \in h(v_1 \dot{\in} v_2) \Leftrightarrow \rho(v_1) \in \rho(v_2)] & \wedge \\ & & (\forall f \in Form)\,(\forall \rho \in X^{FV(\dot{\neg} f)})\,[\rho \in h(\dot{\neg} f) \Leftrightarrow \rho \in h(f)] & \wedge \\ & & (\forall f_1, f_2 \in Form)\,(\forall \rho \in X^{FV(f_1 \dot{\vee} f_2)}) & \\ & & \quad [\rho \in h(f_1 \dot{\vee} f_2) \ \Leftrightarrow \ (\rho \upharpoonright FV(f_1)) \in h(f_1) \vee (\rho \upharpoonright FV(f_2)) \in h(f_2)] & \wedge \\ & & (\forall v \in \omega)\,(\forall f \in Form)\,(\forall \rho \in X^{FV((\dot{\exists} v) f)}) & \\ & & \quad [\rho \in h((\dot{\exists} v) f) \ \Leftrightarrow \ (\exists x \in X)\,(\rho \cup \{(v, x)\}) \in h(f)] & \end{array}$$

We then let: $\quad Y = Val'_X(f) \ \equiv \ \exists h\,[\phi(X,h) \wedge h(f) = Y]$. □

Fact 2.28 — *The functional relation $Y = \mathrm{Def}(X)$ is Σ_1.*

Proof. We successively let:

$$Z = \mathrm{def}_X(f, \rho) \ \equiv \ Z \subseteq X \ \wedge \ (\forall x \in X)\,[x \in Z \Leftrightarrow (\rho \cup \{(\mathbf{x}_0, x)\}) \in Val'_X(f)]$$

$$\begin{array}{rcl} Y = \mathrm{Def}(X) & \equiv & (\forall f \in Form)\,(\forall \rho \in X^{FV((\dot{\exists} \mathbf{x}_0) f)})\,\mathrm{def}_X(f, \rho) \in Y \ \wedge \\ & & (\forall Z \in Y)\,(\exists f \in Form)\,(\exists \rho \in X^{FV((\dot{\exists} \mathbf{x}_0) f)})\,Z = \mathrm{def}_X(f, \rho)\,. \end{array}$$ □

We can now conclude that:

Fact 2.29 — *The functional relation $y = L_\alpha$ is Σ_1.*

Proof. Given an ordinal α, the function h of domain α that associates to each ordinal $\beta < \alpha$ the set $h(\beta) = L_\beta$ is defined from the Σ_1-formula $\phi(\alpha, h)$ given by:

$$\begin{array}{rcl} \phi(\alpha, h) & \equiv & h \text{ function } \wedge \ \mathrm{dom}(h) = \alpha \ \wedge \\ & & (\forall \beta < \alpha)\,(\forall \gamma < \beta)\,\mathrm{Def}(h(\gamma)) \subseteq h(\beta) \ \wedge \\ & & (\forall \beta < \alpha)\,(\forall x \in h(\beta))\,(\exists \gamma < \beta)\,x \in \mathrm{Def}(h(\gamma))\,. \end{array}$$

We then let: $\quad y = L_\alpha \ \equiv \ On(\alpha) \wedge \exists h\,[\phi(\alpha + 1, h) \wedge y = h(\alpha)]$. □

Proposition 2.30 (Absoluteness of $\alpha \mapsto L_\alpha$) — *The operation $\alpha \mapsto L_\alpha$ is absolute, in the sense that in any standard set theory $\mathscr{T}$ and in any transitive model $\mathscr{M}$ of ZF in $\mathscr{T}$, we have:*

$$\mathscr{T} \ \vdash \ (\forall \alpha \in (On \cap \mathscr{M}))\,\forall y\,(y = L_\alpha \ \Leftrightarrow \ y \in \mathscr{M} \wedge (y = L_\alpha)^{\mathscr{M}})\,.$$

Proof. The relation $y = L_\alpha$ is provably functional on the class On (in ZF), that is:

$$\mathrm{ZF} \ \vdash \ (\forall \alpha \in On)\,\exists! y\,(y = L_\alpha)\,.$$

So that in any standard set theory $\mathscr{T}$ and in any transitive model $\mathscr{M}$ of ZF in $\mathscr{T}$, we have

$$\mathscr{T} \ \vdash \ (\forall \alpha \in On^{\mathscr{M}})\,(\exists! y \in \mathscr{M})\,(y = L_\alpha)^{\mathscr{M}}\,.$$

Therefore, for each ordinal $\alpha \in (On \cap \mathscr{M}) = On^{\mathscr{M}}$ (Prop. 1.42) there is a unique set y ($\in V$) such that $y = L_\alpha$ as well as a unique set $y' \in \mathscr{M}$ such that $(y' = L_\alpha)^{\mathscr{M}}$. But since the formula '$y' = L_\alpha$' is Σ_1 and thus upwards absolute, the latter implies that $y' = L_\alpha$ (in V) so that $y = y'$ by uniqueness. The desired equivalence immediately follows from this remark. □

Corollary 2.31 (Satisfaction of $V = L$ in L) — *The axiom $V = L$ is satisfied in L.*

Proof. The formula $(V = L)^L$ holds in ZF, since

$$\begin{array}{rcll} (V = L)^L & \equiv & (\forall x\, (\exists \alpha \in On)\, \exists y\, (y = L_\alpha \wedge x \in y))^L & \\ & \Leftrightarrow & (\forall x \in L)(\exists \alpha \in On^L)(\exists y \in L)\, ((y = L_\alpha)^L \wedge x \in y) & \\ & \Leftrightarrow & (\forall x \in L)(\exists \alpha \in On)(\exists y \in L)\, ((y = L_\alpha)^L \wedge x \in y) & (\text{since } On^L = On) \\ & \Leftrightarrow & (\forall x \in L)(\exists \alpha \in On)\exists y\, (y = L_\alpha \wedge x \in y) & (\text{Prop. 2.30}) \\ & \Leftrightarrow & (\forall x \in L)\, x \in L & \square \end{array}$$

From Prop. 2.22 and Coro. 2.31 we get:

Theorem 2.32 — *The class L is a transitive model of ZF + $(V = L)$ in ZF.*

Therefore:

Corollary 2.33 — *Both theories ZF and ZF + $(V = L)$ are equiconsistent.*

Another fundamental property of the constructible universe is that the class L is the smallest transitive model of ZF that contains On, and this in any standard set theory:

Proposition 2.34 (Minimality of L) — *Let $\mathscr{T}$ be a standard set theory and $\mathscr{M}$ a transitive model of ZF in $\mathscr{T}$. If $\mathscr{T} \vdash On \subseteq \mathscr{M}$, then $\mathscr{T} \vdash L \subseteq \mathscr{M}$.*

Proof. (In $\mathscr{T}$:) Since $On^{\mathscr{M}} = On \cap \mathscr{M} = On$ (by hypothesis), we have $L_\alpha \in \mathscr{M}$ for all $\alpha \in On$ (from Prop. 2.30) and thus $L_\alpha \subseteq \mathscr{M}$ by transitivity. Therefore $L \subseteq \mathscr{M}$. $\square$

2.4 Consequences of the axiom of constructibility

In this section, we prove that the axiom of constructibility $V = L$ implies both the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH).

2.4.1 The Axiom of Choice

Let us recall that the Axiom of Choice states that:

(AC) Every set X has a choice function $f : \mathfrak{P}^*(X) \to X$

(cf exercises p. 49 for the definition of the notion of a choice function). In ZF, the Axiom of Choice is equivalent to Zermelo's Axiom (Exercise 2.7), which states that

(Zermelo's Axiom) Every set X is well-orderable.

The implication $(V = L) \Rightarrow$ AC is thus a consequence of the following result:

Proposition 2.35 (Well-ordering on L_α) — *In ZF, there is a functional relation $\alpha \mapsto \leq_{L_\alpha}$ that associates to every ordinal $\alpha \in On$ a well ordering $\leq_{L_\alpha}$ on the set L_α.*

Proof. The transfinite family of well orderings $(\leq_{L_\alpha})_{\alpha\in On}$ is constructed by induction on α. Assuming that the well orderings $\leq_{L_\beta}$ on the sets L_β have been defined for all $\beta < \alpha$, we define the well ordering $\leq_{L_\alpha}$ on the set L_α as follows. For every $x \in L_\alpha = \bigcup_{\beta<\alpha} \mathrm{Def}(L_\beta)$, we write β_x the smallest ordinal $\beta < \alpha$ such that $x \in \mathrm{Def}(L_\beta)$. Given $x, y \in L_\alpha$, we then let

$$x \leq_{L_\alpha} y \quad \equiv \quad \beta_x < \beta_y \ \vee \ (\beta_x = \beta_y \wedge x \leq_{\mathrm{Def}(L_{\beta_x})} y)\,,$$

where $\leq_{\mathrm{Def}(L_{\beta_x})}$ is the well ordering on $\mathrm{Def}(L_{\beta_x})$ that is canonically associated to the well ordering $\leq_{L_{\beta_x}}$ on L_{β_x} as shown in Prop. 2.13. Clearly, the relation $\leq_{L_\alpha}$ is a well ordering. □

Theorem 2.36 ($V = L$ implies AC) — *ZF* $\vdash (V = L) \Rightarrow AC$.

Proof. Given an arbitrary set X, the axiom $V = L$ implies that $X \in L_\alpha$ for some $\alpha \in On$, and thus $X \subseteq L_\alpha$ (by transitivity). But since the set L_α is well-orderable (Prop. 2.35), the subset $X \subseteq L_\alpha$ is well-orderable too. □

Theorem 2.36 shows that we have the inclusions of theories

$$\mathrm{ZF} \quad \subseteq \quad \mathrm{ZFC}\ (= \mathrm{ZF} + \mathrm{AC}) \quad \subseteq \quad \mathrm{ZF} + (V = L)\,.$$

From Coro. 2.33 we deduce that:

Corollary 2.37 — *The theories ZF and ZFC are equiconsistent.*

2.4.2 The Generalized Continuum Hypothesis

Let $(\aleph_\alpha)_{\alpha\in On}$ be the transfinite sequence of infinite cardinals (cf Exercise 2.15). The Generalized Continuum Hypothesis states that for each infinite cardinal $\aleph_\alpha$ ($\alpha \in On$), the cardinal of its powerset $\mathfrak{P}(\aleph_\alpha)$ is the next infinite cardinal:

(GCH) $$(\forall \alpha \in On)\, 2^{\aleph_\alpha} = \aleph_{\alpha+1}\,.$$

Notice that the above formulation implicitly relies on AC, whose presence is required to define the cardinal $2^{\aleph_\alpha} = |\mathfrak{P}(\aleph_\alpha)|$ (using Zermelo's Lemma, cf exercises p. 53). In this section, we shall thus assume the Axiom of Choice (that is already implied by the axiom $V = L$, by Theorem 2.36) to benefit from the standard results of cardinal arithmetic.

Before proving that $V = L$ implies GCH as well, let us introduce some terminology.

Definition 2.38 (Extensional sets) — We say that a set X is *extensional* when the Extensionality Axiom relativized to X holds, that is, when:

$$(\forall x, y \in X)\, (x \cap X = y \cap X \Rightarrow x = y)\,.$$

It is clear that any transitive set is extensional. The converse is false in general, but we can nevertheless turn every extensional set into a transitive set as shown by the following result:

Lemma 2.39 — *For every extensional set X, there is a unique transitive set Y and a unique isomorphism $u : X \xrightarrow{\sim} Y$, that is: a unique bijection $u : X \xrightarrow{\sim} Y$ such that:*

$$(\forall x, y \in X)\, \big(y \in x \ \Leftrightarrow \ u(y) \in u(x)\big)\,.$$

Proof. It suffices to notice that u is an isomorphism between X and a transitive set if and only if u is a function of domain X such that for all $x \in X$:

$$u(x) = \{u(y) : y \in (x \cap X)\}.$$

The existence and uniqueness of u and Y immediately follows, by $\in$-induction. □

Proposition 2.40 (Cardinal of Σ_1-operations) — *Let $y = f_\phi(x)$ be a functional relation that is defined in ZFC from a Σ_1-formula $\phi(x, y)$. Then for every set x in the domain of f_ϕ, one has $|f_\phi(x)| \leq \max(|\mathrm{Cl}(x)|, \aleph_0)$, where $\mathrm{Cl}(x)$ denotes the transitive closure of x.*

(Recall that the transitive closure of a set x, written $\mathrm{Cl}(x)$, is the smallest transitive set y such that $x \subseteq y$. This set can be defined by Replacement, letting $\mathrm{Cl}(x) = \bigcup_{n\in\omega} \bigcup^n x$.)

Proof. Up to logical equivalence (in ZF), we can assume that the formula $\phi(x, y)$ that defines the functional relation $y = f_\phi(x)$ is strictly Σ_1. Let x be a set in the domain of f_ϕ, and take $\alpha \in On$ such that $x \in V_\alpha$. Since $\exists! y\, \phi(x, y)$, there is an ordinal $\beta > \alpha$ such that

$$(\exists! y \in V_\beta)\, \phi^{V_\beta}(x, y)$$

from the Reflection Scheme (Theorem 2.20). Let us now write $P = \mathrm{Cl}(\{x\}) = \mathrm{Cl}(x) \cup \{x\}$ (so that $|P| = |\mathrm{Cl}(x)| + 1$). From the Löwenheim-Skolem theorem (Theorem 2.6), there is a set Q such that $P \subseteq Q \subseteq V_\beta$ and $|Q| \leq \max(|P|, \aleph_0)$, and such that

$$(\forall x_1, \ldots, x_n \in P)\,(V_\beta \models f(x_1, \ldots, x_n) \Leftrightarrow Q \models f(x_1, \ldots, x_n))$$

for every internal formula f with free variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$. Since the set V_β is transitive, the internal formula $f = \ulcorner \forall x\, \forall y\, (\forall z\, (z \in x \Leftrightarrow z \in y) \Rightarrow x = y) \urcorner$ (Extensionality) is true in V_β. Therefore it is true in Q too, which means that the set Q is extensional. From Lemma 2.39, there exists a transitive set Q' together with an isomorphism $u : Q \xrightarrow{\sim} Q'$. By (recursively) replacing each parameter $x \in Q$ by its image $u(x) \in Q'$, we deduce that

$$(\forall x_1, \ldots, x_n \in P)\,(V_\beta \models f(x_1, \ldots, x_n) \Leftrightarrow Q' \models f(u(x_1), \ldots, u(x_n)))$$

for every internal formula f. (This result is proved by structural induction on f.) Moreover, we easily deduce from the transitivity of P that $u(x) = x$ for all $x \in P$ (by $\in$-induction on x). So that the above equivalence actually simplifies to

$$(\forall x_1, \ldots, x_n \in P)\,(V_\beta \models f(x_1, \ldots, x_n) \Leftrightarrow Q' \models f(x_1, \ldots, x_n)).$$

If we now apply the latter to the internal formula $f = \ulcorner (\exists! y)\, \phi(x, y) \urcorner$ (with the sole parameter $x \in P$), we deduce that there is a unique $y \in Q'$ such that $\phi^{Q'}(x, y)$. But since the formula ϕ is strictly Σ_1 and since the set Q' is transitive, we get $\phi(x, y)$ from Prop. 1.29, so that $y = f_\phi(x)$. By transitivity (of Q'), we thus have $f_\phi(x) = y \subseteq Q'$, hence:

$$|f_\phi(x)| \leq |Q'| = |Q| \leq \max(|P|, \aleph_0) = \max(|\mathrm{Cl}(x)|, \aleph_0).$$ □

From this we deduce the following useful lemma:

Lemma 2.41 — *In ZFC:*

(1) For all $\alpha \geq \omega$, we have $|L_\alpha| = |\alpha|$.

(2) For all $x \in L$, we have $|\mathrm{ord}(x)| \leq \max(|\mathrm{Cl}(x)|, \aleph_0)$.

Proof. (1) From Prop. 2.40 we have $|L_\alpha| \leq \max(|\mathrm{Cl}(\alpha)|, \aleph_0) = |\alpha|$ for every ordinal $\alpha \geq \omega$, since the functional relation $y = L_\alpha$ is Σ_1. But we also have $|L_\alpha| \geq |\alpha|$ since $\alpha \subseteq L_\alpha$.

(2) We first notice that the functional relation

$$\alpha = \mathrm{ord}(x) \quad \equiv \quad On(\alpha) \ \wedge \ \exists y\,(y = L_\alpha \wedge x \in y) \ \wedge \ (\forall \beta < \alpha)\,\exists y'\,(y' = L_\beta \ \wedge \ x \notin y')$$

is Σ_1 (cf Prop. 1.31 p. 21). The desired inequality immediately follows from Prop. 2.40. □

We now have all the ingredients to prove that:

Theorem 2.42 ($V = L$ implies GCH) — *ZFC* $\vdash (V = L) \Rightarrow$ *GCH.*

Proof. Given $\alpha \in On$, we want to prove that $|\mathfrak{P}(\aleph_\alpha)| = \aleph_{\alpha+1}$. For that, let us consider a subset $a \subseteq \aleph_\alpha$. Since $a \in L$ (thanks to axiom $V = L$), we have $|\mathrm{ord}(a)| \leq \max(|\mathrm{Cl}(a)|, \aleph_0) \leq \aleph_\alpha$ from Lemma 2.41 (2). Therefore $\mathrm{ord}(a) < \aleph_{\alpha+1}$ and thus $a \in L_{\aleph_{\alpha+1}}$. We have shown that $a \in L_{\aleph_{\alpha+1}}$ for all $a \in \mathfrak{P}(\aleph_\alpha)$, hence $\mathfrak{P}(\aleph_\alpha) \subseteq L_{\aleph_{\alpha+1}}$. Therefore $|\mathfrak{P}(\aleph_\alpha)| \leq |L_{\aleph_{\alpha+1}}| = \aleph_{\alpha+1}$ by Lemma 2.41 (1). The converse inequality $|\mathfrak{P}(\aleph_\alpha)| \geq \aleph_{\alpha+1}$ follows from Cantor's diagonal argument. □

This shows that we have the inclusion of theories

$$\mathrm{ZF} \quad \subseteq \quad \mathrm{ZFC} \quad \subseteq \quad \mathrm{ZFC} + \mathrm{GCH} \quad \subseteq \quad \mathrm{ZF} + (V = L)\,.$$

From Coro. 2.33 we deduce that:

Corollary 2.43 — *The theories ZF, ZFC and ZFC + GCH are equiconsistent.*

Exercises

Miscellaneous

Exercise 2.1 (Knaster-Tarski theorem) — Recall that a poset $(A, \leq)$ is a complete lattice when every subset of A has a greatest lower bound and a least upper bound in A. (In particular, the poset A has extremal elements $\bot = \bigwedge A = \bigvee \varnothing$ and $\top = \bigvee A = \bigwedge \varnothing$.)

Given a complete lattice $(A, \leq)$ and a monotonic function $f : A \to A$, we write $\mathrm{Fix}(f) = \{x \in A : f(x) = x\}$ the set of all fixpoints of f in A.

(1) Show that the set $\mathrm{Fix}(f)$ has a smallest element and a largest element. *Hint:* One may consider the two sets $L = \{x \in A : x \leq f(x)\}$ and $U = \{x \in A : f(x) \leq x\}$.

We now consider the particular case where $(A, \leq) = (\mathfrak{P}(V_\omega), \subseteq)$ and where $f : A \to A$ is the function $\Phi : \mathfrak{P}(V_\omega) \to \mathfrak{P}(V_\omega)$ such as defined in section 2.1.1.

(2) Prove that $\Phi(\bigcup_{i\in\omega} X_i) = \bigcup_{i\in\omega} \Phi(X_i)$ for every increasing sequence $(X_i)_{i\in\omega} \in A^\omega$, and deduce that the least fixpoint $Form = \min(\mathrm{Fix}(\Phi))$ is given by $Form = \bigcup_{i\in\omega} \Phi^i(\varnothing)$.

(3) Using the Axiom of Regularity, prove that $X \subseteq Form$ for every fixpoint X of Φ. Deduce that $Form$ is the only fixpoint of Φ.

Exercise 2.2 (Non standard formula codes) — Consider the extension $\mathscr{T} \supseteq$ ZF whose language is the language of ZF enriched with a fresh constant symbol c, and whose axioms are the axioms of ZF enriched with the closed formulas

$$c \in \mathit{Form} \qquad \text{and} \qquad c \neq \ulcorner\phi\urcorner \qquad\qquad \text{(for every formula } \phi \text{ of ZF)}$$

Intuitively, the theory $\mathscr{T}$ extends ZF by adding a non standard formula code, that is, an internal formula $c \in \mathit{Form}$ that corresponds to no external formula. Prove that $\mathscr{T}$ is a conservative extension of ZF (and thus equiconsistent with ZF).

Exercise 2.3 (Canonical ordering on On^2) — Let On^2 be the class of pairs of ordinals. The class On^2 is equipped with the ordering $(\alpha_1,\beta_1) \leq (\alpha_2,\beta_2)$ defined by:

$$\begin{aligned}(\alpha_1,\beta_1) \leq (\alpha_2,\beta_2) \quad \equiv \quad & \max(\alpha_1,\beta_1) < \max(\alpha_2,\beta_2) \ \lor \\ & (\max(\alpha_1,\beta_1) = \max(\alpha_2,\beta_2) \land \alpha_1 < \alpha_2) \ \lor \\ & (\max(\alpha_1,\beta_1) = \max(\alpha_2,\beta_2) \land \alpha_1 = \alpha_2 \land \beta_1 \leq \beta_2)\end{aligned}$$

(1) Prove that the ordering $(\alpha_1,\beta_1) \leq (\alpha_2,\beta_2)$ on On^2 is a well ordering, that is:

   (*i*) For all $(\alpha,\beta) \in On^2$, the class $\mathrm{seg}_{(\alpha,\beta)} = \{(\alpha',\beta') \in On^2 : (\alpha',\beta') < (\alpha,\beta)\}$ is a set.

   (*ii*) Every nonempty set $X \subset On^2$ has a smallest element.

   (This well-ordering is called the *canonical well-ordering on On^2*.)

For every $(\alpha,\beta) \in On^2$, we write $\gamma_{(\alpha,\beta)}$ the unique ordinal that is isomorphic to the well-ordered set $\mathrm{seg}_{(\alpha,\beta)} = \{(\alpha',\beta') \in On^2 : (\alpha',\beta') < (\alpha,\beta)\}$.

(2) Prove that: $\quad (\forall \alpha_1,\beta_1,\alpha_2,\beta_2 \in On)\,[(\alpha_1,\beta_1) \leq (\alpha_2,\beta_2) \ \Leftrightarrow\ \gamma_{(\alpha_1,\beta_1)} \leq \gamma_{(\alpha_2,\beta_2)}]$

(3) Prove that the operation $(\alpha,\beta) \mapsto \gamma_{(\alpha,\beta)}$ defines an isomorphism between On^2 and On.

Let $(\omega_\alpha)_{\alpha\in On}$ be the transfinite sequence of limit ordinals defined by $\omega_0 = \omega$, $\omega_{\alpha+1} = \omega_\alpha + \omega = \sup_{n\in\omega} \omega_\alpha + n$ (for all $\alpha \in On$) and $\omega_\alpha = \sup_{\beta<\alpha} \omega_\beta$ (if α is a limit ordinal).

(4) Prove that: $\quad (\forall \lambda \in On)\,(\lambda \text{ limit ordinal} \ \Leftrightarrow\ (\exists \alpha \in On)\,\lambda = \omega_\alpha)$.

(5) Prove that: $\quad (\forall \alpha \in On)\,\gamma_{(\omega_\alpha,\omega_\alpha)} = \omega_\alpha$.

(6) Deduce that for every infinite ordinal α, the set $\alpha^2 = \alpha \times \alpha$ is equipotent to α.

The Axiom of Choice

Given a set X and writing $\mathfrak{P}^*(X) = \mathfrak{P}(X) \setminus \{\varnothing\}$, we call a *choice function for X* any function $h : \mathfrak{P}^*(X) \to X$ such that $h(Y) \in Y$ for all $Y \in \mathfrak{P}^*(X)$. The Axiom of Choice (AC) states that:

Axiom of Choice Every set X has a choice function $h : \mathfrak{P}^*(X) \to X$

In what follows, we write ZFC = ZF + AC.

The Axiom of Choice has many equivalent formulations in ZF, and we shall now present some of them. For that, let us introduce some terminology.

- Given an equivalence relation $\sim$ on a set A, we call a *system of representatives of* $\sim$ any subset $S \subseteq A$ whose intersection with every equivalence class of $(A, \sim)$ is a singleton:

$$S \text{ system of representatives of } (A, \sim) \quad \equiv \quad S \subseteq A \ \land \ (\forall x \in A)(\exists! y \in S)\, x \sim y$$

- Given a family of sets $(A_i)_{i \in I}$, we call the *Cartesian product of the family* $(A_i)_{i \in I}$ and write $\prod_{i \in I} A_i$ the set of all functions f of domain I and such that $f(i) \in A_i$ for all $i \in I$:

$$\prod_{i \in I} A_i \ = \ \{f \ : \ \mathrm{dom}(f) = I \ \land \ (\forall i \in I)\, f(i) \in A_i\}\,.$$

  (This set exists by Separation, as a subset of the set A^I, where $A = \bigcup_{i \in I} A_i$.)

Exercise 2.4 (Equivalent formulations of the Axiom of Choice) — Prove in ZF that the following three formulas are equivalent:

(1) Every set has a choice function (AC).

(2) Every equivalence relation has a system of representatives.

(3) The Cartesian product of a family of nonempty sets is nonempty.

Exercise 2.5 (Zorn's lemma) — Let $(A, \leq)$ be a poset. We call a *chain* of A any subset $C \subseteq A$ that is linearly ordered by $\leq$, namely: $(\forall x, y \in C)\,(x \leq y \lor y \leq x)$. We say that the poset $(A, \leq)$ is *inductive* when every chain of A has an upper bound (not necessarily a least upper bound).

The aim of this exercise is to prove (using AC) that every inductive poset has a maximal element (Zorn's Lemma). For that, we consider an inductive poset $(A, \leq)$ and fix a choice function $h : \mathfrak{P}^*(A) \to A$. For every chain $C \subseteq A$, we write $\mathrm{sub}(C) = \{x \in A : (\forall y \in C)\, y < x\}$ the set of all *strict upper bounds* of C in A.

(1) Check that if $\mathrm{sub}(C) = \varnothing$, then the chain C has a maximum $\max(C)$, and that this maximum is a maximal element of A.

For each chain $C \subseteq A$, we let $\mathrm{next}(C) = h(\mathrm{sub}(C))$ if $\mathrm{sub}(C) \neq \varnothing$, and $\mathrm{next}(C) = \max(C)$ otherwise. By transfinite induction on $\alpha \in On$, we let $x_\alpha = \mathrm{next}(\{x_\beta : \beta < \alpha\})$ for all $\alpha \in On$.

(2) Check that the transfinite sequence $(x_\alpha)_{\alpha \in On}$ is well-defined and monotonic.

(3) Prove that the transfinite sequence $(x_\alpha)_{\alpha \in On}$ is stationary from some ordinal, and deduce from it that A has a maximal element.

Exercise 2.6 (Zermelo's lemma) — The aim of this exercise is to prove that every set has a well-ordering (Zermelo's Lemma) using Zorn's Lemma (Exercise 2.5). For that, we consider a set A and write $\mathcal{W}_A$ the set of all pairs $(X, \leq_X)$ where $X \subseteq A$ and where $\leq_X$ is a well-ordering on X. The set $\mathcal{W}_A$ is ordered by the binary relation $(X, \leq_X) \preceq (Y, \leq_Y)$ defined by

$$\begin{aligned}(X, \leq_X) \preceq (Y, \leq_Y) \ \equiv \ & X \subseteq Y \ \land \ (\leq_X) = (\leq_Y) \cap X^2 \ \land \\ & (\forall x \in X)\,(\forall y \in Y)\,(y \leq_Y x \Rightarrow y \in X)\end{aligned}$$

(This ordering means that $(X, \leq_X)$ is an *initial segment* of $(Y, \leq_Y)$.)

(1) Prove that the set $(\mathcal{W}_A, \preceq)$ is inductive, and thus has a maximal element $(X, \leq_X)$.

(2) Conclude by showing that, if $(X, \leq_X)$ is a maximal element of $(\mathcal{W}_A, \preceq)$, then $X = A$.

(3) Give an alternative and more direct proof of Zermelo's lemma (using AC) by adapting the technique of Exercise 2.5.

Exercise 2.7 (Equivalent formulations of AC) — Deduce from what precedes that Zorns's Lemma (Exercise 2.5) and Zermelo's Lemma (Exercise 2.6) are equivalent to AC in ZF.

Weaker forms of the Axiom of Choice

There are many weak forms of the Axiom of Choice, that are implied by AC without being equivalent. Most of them are useful in Analysis, especially the Axiom of Dependent Choices (DC) and the Countable Axiom of Choice (CAC):

(DC) $$(\forall X \neq \varnothing)(\forall R \subseteq X^2)[(\forall x \in X)(\exists y \in X)\,(x\,R\,y) \Rightarrow (\exists (x_n)_{n\in\omega} \in X^\omega)(\forall n \in \omega)\,(x_n\,R\,x_{n+1})]$$

(CAC) $$\forall I\,(\forall (A_i)_{i\in I})\,[I \text{ denumerable} \wedge (\forall i \in I)\,A_i \neq \varnothing \Rightarrow \prod_{i\in I} A_i \neq \varnothing]$$

(Here, 'countable' means 'denumerable', that is: in one-to-one correspondence with ω.)

Exercise 2.8 (AC, DC and CAC)

(1) Prove in ZF that AC $\Rightarrow$ DC and DC $\Rightarrow$ CAC. Explain why, intuitively, the converse implications do not hold.

(2) Prove in ZF *without CAC* that: $\forall I\,(\forall (A_i)_{i\in I})\,[I \text{ finite} \wedge (\forall i \in I)\,A_i \neq \varnothing \Rightarrow \prod_{i\in I} A_i \neq \varnothing]$.
(*Hint:* Reason by induction on the cardinal of I.)

Exercise 2.9 (An alternative formulation of DC) — It is also possible to formulate the Axiom of Dependent Choices (DC) by explicitly fixing the first term x_0 the resulting sequence $(x_n)_{n\in\omega}$ should start from:

(DC′) $$\forall X\,(\forall R \subseteq X^2)[(\forall x \in X)(\exists y \in X)\,(x\,R\,y) \Rightarrow (\forall x \in X)(\exists (x_n)_{n\in\omega} \in X^\omega)\,(x_0 = x \wedge (\forall n \in \omega)\,(x_n\,R\,x_{n+1}))]$$

(1) Prove in ZF that DC $\Leftrightarrow$ DC′.

(*Hint:* To prove DC′ from DC, it is necessary to change X and R.)

Exercise 2.10 (CAC and infinite sets) — Recall that a set X is:

- *Dedekind-infinite* when there exists an injection $f : X \hookrightarrow X$ that is not surjective;
- *denumerable* when there exists a bijection $f : \omega \xrightarrow{\sim} X$;
- *finite* when there exists $n \in \omega$ with a bijection $f : n \xrightarrow{\sim} X$;
- *infinite* when X is not finite.

(1) Prove in ZF that: X Dedekind-infinite $\Leftrightarrow (\exists Y \subseteq X)\, X$ denumerable.

(2) Prove in ZF that: X Dedekind-infinite $\Rightarrow X$ infinite.

(3) Prove in ZF that: X infinite $\Rightarrow$ $(\forall n \in \omega)\,(\exists f : n \to X)\, f$ injective.

(4) Using (3), prove in ZF + CAC that: X infinite $\Rightarrow X$ Dedekind-infinite.
(*Hint:* The proof requires to glue astutely—without any form of choice—an arbitrary sequence of injections $(f_n)_{n\in\omega} \in \prod_{n\in\omega} \mathrm{Inj}(n, X)$ into an injection $f : \omega \hookrightarrow X$.)

Remark: The implication *X infinite* $\Rightarrow X$ Dedekind-infinite (4) cannot be proved in ZF (without CAC), so that in ZF there may exist sets that are neither Dedekind-infinite nor finite.

Exercise 2.11 (The Baire Category Theorem) — The Axiom of Dependent Choices (DC) is typically needed to prove the *Baire Category Theorem* (BCT) in Analysis. Recall that a *Baire space* is a topological space in which the intersection of any denumerable family of open dense subsets is dense. Prove in ZF + DC that:

- (BCT1) Every complete metric space is a Baire space.
- (BCT2) Every locally compact Hausdorff space is a Baire space.

Basic properties of cardinals (without AC)

Recall that in ZF, a *cardinal* is an ordinal $\kappa \in On$ such that there is no bijection from κ onto an ordinal $\alpha < \kappa$. Formally, the formula $Cn(\kappa)$ ('κ is a cardinal') is thus given by:

$$Cn(\kappa) \;\equiv\; On(\kappa) \wedge (\forall \alpha < \kappa) \forall f\, \neg(f : \kappa \xrightarrow{\sim} \alpha) \qquad (\Pi_0\text{-formula})$$

Exercise 2.12 (Basic properties) — Prove in ZF (without AC) that:

(1) All natural numbers $n \in \omega$ are cardinals.

(2) The ordinal ω is a cardinal.

(3) If $(\kappa_i)_{i\in I}$ is a family of cardinals, then the ordinal $\sup_{i\in I} \kappa_i = \bigcup_{i\in I} \kappa_i$ is a cardinal.

From the definition, it is clear that two cardinals are equal if and only if they are equipotent (i.e. in bijection). This equivalence can be refined as follows:

Exercise 2.13 (Ordering on cardinals) — Prove in ZF that for any two cardinals κ and μ, the following three formulas are equivalent:

(*i*) $\kappa \leq \mu$

(*ii*) There is an injection $f : \kappa \hookrightarrow \mu$

(*iii*) $\kappa = 0$ or there is a surjection $f : \mu \twoheadrightarrow \kappa$.

(*Hint:* Use the Cantor-Bernstein-Schröder Theorem, Exercise 1.6.)

Exercise 2.14 (The proper class *Cn*) — The aim of this exercise is to prove in ZF (without AC) that the class *Cn* of cardinals is a proper class.

(1) Prove that every ordinal $\alpha \in On$ is equipotent to a unique cardinal (written $|\alpha|$).

(2) Prove that for every cardinal κ, the class of ordinals equipotent to κ is a set.

(3) Deduce that the class *Cn* of cardinals is a proper class.

The above exercise implies that each cardinal $\kappa \in Cn$ has a *successor*, that is defined as the smallest cardinal greater than κ and written κ^+. Moreover, since all cardinals $\kappa \geq \omega$ are infinite, it is clear that the class Cn_∞ of infinite cardinals is a proper class too.

Exercise 2.15 (The sequence $(\aleph_\alpha)_{\alpha\in On}$) — By transfinite induction on $\alpha \in On$, we define the cardinal $\aleph_\alpha$ as the smallest infinite cardinal greater than $\aleph_\beta$ for all $\beta < \alpha$.

(1) Check that $\aleph_0 = \omega$ and $\aleph_{\alpha+1} = (\aleph_\alpha)^+$ for all $\alpha \in On$.

(2) Prove that $\aleph_\alpha = \sup_{\beta<\alpha} \aleph_\beta$ for every limit ordinal α.

(3) Prove that each infinite cardinal appears in the transfinite sequence $(\aleph_\alpha)_{\alpha\in On}$.

Cardinal arithmetic (with AC)

In ZFC, every set X is well-orderable (Exercise 2.6) so that it is equipotent to some cardinal κ (from Exercise 2.14 (1)). From the definition of the notion of a cardinal, the cardinal κ that is equipotent with X is unique; it is called the *cardinal of X* and written $|X|$.

Given two cardinals κ and μ, we respectively write $\kappa + \mu$, $\kappa \cdot \mu$ and κ^μ the cardinals of the disjoint union $\kappa + \mu$, of the Cartesian product $\kappa \times \mu$ and of the set of all functions from μ to κ. (From this definition, it is clear that addition and multiplication are associative and commutative, and that multiplication distributes over addition.) Note that 2^μ is also the cardinal of the powerset $\mathfrak{P}(\mu)$, that is equipotent with the set of all functions from μ to 2.

Exercise 2.16 (Properties of cardinal exponentation) — Check that the following equalities hold for all cardinals κ, μ and ν:

$$\begin{array}{rclrcl} \kappa^0 &=& 1 & 0^\mu &=& 0 \quad (\text{if } \mu \geq 1) \\ \kappa^1 &=& \kappa & 1^\mu &=& 1 \\ \kappa^{\mu+\nu} &=& \kappa^\mu \cdot \kappa^\nu & (\kappa\cdot\mu)^\nu &=& \kappa^\nu \cdot \mu^\nu \\ \kappa^{\mu\cdot\nu} &=& (\kappa^\mu)^\nu \;=\; (\kappa^\nu)^\mu & & & \end{array}$$

Exercise 2.17 (Infinite cardinal arithmetic) — From Exercise 2.3 (6), it is clear that $\kappa^2 = \kappa$ for every infinite cardinal κ. From this, deduce that for all infinite cardinals κ and μ:

(1) $\kappa + \mu = \kappa \cdot \mu = \max(\kappa, \mu)$

(2) $\max(\kappa, 2^\mu) \leq \kappa^\mu \leq \max(2^\kappa, 2^\mu)$

Given a family of cardinals $(\kappa_i)_{i\in I}$, we more generally write $\sum_{i\in I} \kappa_i$ (resp. $\prod_{i\in I} \kappa_i$) the cardinal of the disjoint union $\sum_{i\in I} \kappa_i$ (resp. of the Cartesian product $\prod_{i\in I} \kappa_i$).

Exercise 2.18 (König's Theorem) — Let $(\kappa_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ be two families of cardinals indexed by a same set I. The aim of this exercise is to prove that:

$$(\forall i \in I)\, \kappa_i < \mu_i \;\Rightarrow\; \sum_{i\in I} \kappa_i < \prod_{i\in I} \mu_i \qquad \text{(König's theorem)}$$

For that, we assume that $\kappa_i < \mu_i$ for all $i \in I$, and write $S = \sum_{i\in I} \kappa_i$ the disjoint union of the family $(\kappa_i)_{i\in I}$ and $P = \prod_{i\in I} \mu_i$ the Cartesian product of the family $(\mu_i)_{i\in I}$. Given a function $f : S \to P$, we define for each $i \in I$ the function $f_i : \kappa_i \to \mu_i$ by $f_i(\alpha) = f(i, \alpha)(i)$ for all $\alpha \in \kappa_i$.

(1) Prove the existence of an element $p \in P$ such that $p(i) \notin \mathrm{img}(f_i)$ for all $i \in I$.

(2) Using (1), prove that f is not surjective, and deduce that $|S| < |P|$.

Cofinality and regular cardinals

Exercise 2.19 (Cofinality) — Let α and β be two ordinals. We say that α *is cofinal to* β and write $\alpha \triangleleft \beta$ when there is an increasing function $f : \alpha \to \beta$ such that every element of β is bounded by an element of the image of f. In symbols:

$$\alpha \triangleleft \beta \quad \equiv \quad (\exists f : \alpha \to \beta)\, [(\forall x, y \in \alpha)\, (x < y \Rightarrow f(x) < f(y)) \;\wedge\; (\forall y \in \beta)(\exists x \in \alpha)\, y \le f(x)]$$

(This is a Σ_1-formula.)

(1) Prove that $\alpha \triangleleft \beta$ implies $\alpha \le \beta$.

(2) Check that the binary relation $\alpha \triangleleft \beta$ is an ordering on the class On, that is not linear. (*Hint:* Prove that 0, 1 and ω are pariwise incomparable w.r.t. the ordering $\alpha \triangleleft \beta$.)

Given an ordinal α, we call the *cofinality of* α and write $\mathrm{cof}(\alpha)$ the smallest ordinal β that is cofinal to α. Since $\alpha \triangleleft \alpha$, we thus have $\mathrm{cof}(\alpha) \le \alpha$.

(3) Prove that $\mathrm{cof}(0) = 0$, $\mathrm{cof}(\alpha + 1) = 1$ (for all $\alpha \in On$) and $\mathrm{cof}(\omega) = \omega$.

(4) Prove that $\mathrm{cof}(\mathrm{cof}(\alpha)) = \mathrm{cof}(\alpha)$ for all $\alpha \in On$.

Regular and singular ordinals We say that an ordinal α is *regular* when $\mathrm{cof}(\alpha) = \alpha$ (cf Exercise 2.19), and that it is *singular* otherwise. The aim of the following exercise is to prove that every regular ordinal is a cardinal:

Exercise 2.20 (Regularity 1) — Let α and β be ordinals, and $f : \beta \twoheadrightarrow \alpha$ a surjection from β onto α. We consider the subset $B \subseteq \beta$ that is defined by $B = \{x \in \beta : (\forall y < x)\, f(y) < f(x)\}$.

(1) Prove that $f \upharpoonright B$ is increasing (strictly).

(2) Prove that for all $y \in \alpha$, there exists $x \in \beta$ such that $y \le f(x)$.

(3) Deduce that $\mathrm{cof}(\alpha) \le \beta$.

(4) From what precedes, deduce that every regular ordinal is a cardinal.

Regular and singular cardinals We say that a cardinal κ is *regular* (resp. *singular*) when it is *regular* (resp. *singular*) as an ordinal, that is: when $\mathrm{cof}(\kappa) = \kappa$ (resp. $\mathrm{cof}(\kappa) < \kappa$). From the results established in Exercise 2.19, it is clear that:

(*i*) The only finite regular cardinals are 0 and 1, and the cardinal $\aleph_0 = \omega$ is regular.

(*ii*) For every ordinal α, the ordinal $\mathrm{cof}(\alpha)$ is a regular cardinal.

Exercise 2.21 (Regularity 2) — The aim of this exercise is to prove that the cardinal $\aleph_{\alpha+1}$ is regular for all $\alpha \in On$. For that, we take $\alpha \in On$ and reason by contradiction, assuming that $\mathrm{cof}(\aleph_{\alpha+1}) < \aleph_{\alpha+1}$. We let $\rho = \mathrm{cof}(\aleph_{\alpha+1})$ and consider an increasing function $f : \rho \to \aleph_{\alpha+1}$ such that $(\forall y \in \aleph_{\alpha+1})(\exists x \in \rho)\, y \leq f(x)$.

(1) Check that $\aleph_{\alpha+1} = \bigcup_{x\in\rho} f(x)$, and deduce that $\aleph_{\alpha+1} \leq \sum_{x\in\rho} |f(x)| \leq \aleph_\alpha^2$.

(2) Deduce from (2) that the assumption $\rho = \mathrm{cof}(\aleph_{\alpha+1}) < \aleph_{\alpha+1}$ is absurd.

(3) What is the smallest infinite singular cardinal? What is its cofinality?

Exercise 2.22 (Regularity 3) — Let κ be an infinite cardinal.

(1) Prove that if ρ is an infinite regular cardinal, we have $\mathrm{cof}(\kappa) = \rho$ if and only if there exists an increasing sequence of cardinals $\kappa_x < \kappa$ indexed by $x \in \rho$ such that $\kappa = \sum_{x\in\rho} \kappa_x$.

(2) Using König's Lemma (Exercise 2.18), deduce that $\mathrm{cof}(2^\kappa) > \kappa$ and $\kappa^{\mathrm{cof}(\kappa)} > \kappa$.

Exercise 2.23 (Cardinal exponentation under GCH) — The aim of this exercise is to determine the rules of cardinal exponentation under the Generalized Continuum Hypothesis. For that, we work in ZFC + GCH and consider two infinite cardinals $\kappa, \mu \in Cn_\infty$.

(1) Prove that if $\mu^+ \geq \kappa$, then $\kappa^\mu = \mu^+$.

(2) Prove that if $\mu^+ < \kappa$ and $\mu \geq \mathrm{cof}(\kappa)$, then $\kappa^\mu = \kappa^+$.

From now on, we consider the case where $\mu^+ < \kappa$ and $\mu < \mathrm{cof}(\kappa)$. From these assumptions, it is clear that $\kappa > \aleph_0$, so that either $\kappa = \lambda^+$ for some $\lambda \in Cn_\infty$ (i.e. κ is the successor of an infinite cardinal), or $\kappa = \sup\{\lambda \in Cn_\infty : \lambda < \kappa\}$ (i.e. κ is a limit cardinal).

(3) Prove that if κ is a successor cardinal, then $\kappa^\mu = \kappa$.

From now on, we also assume that κ is a limit cardinal.

(4) Given a function $f : \mu \to \kappa$, prove that $\sup_{x\in\mu} |f(x)|^+ < \kappa$, and deduce that there is an infinite cardinal $\lambda < \kappa$ such that $\mathrm{ran}(f) \subseteq \lambda$.

(5) Deduce that $\kappa^\mu = \sup_{\lambda<\kappa} \lambda^\mu$, so that $\kappa^\mu = \kappa$.

(6) From what precedes, deduce that:

$$\mathrm{ZFC} + \mathrm{GCH} \ \vdash \ (\forall \kappa, \mu \in Cn_\infty)\ \kappa^\mu = \begin{cases} \kappa & \text{if } \mu^+ < \kappa \text{ and } \mu < \mathrm{cof}(\kappa) \\ \kappa^+ & \text{if } \mu^+ < \kappa \text{ and } \mu \geq \mathrm{cof}(\kappa) \\ \mu^+ & \text{if } \mu^+ \geq \kappa \end{cases}$$