Chapter 3

Generic extensions

In this chapter, we show how to transform a particular standard set theory $\mathscr{T}$, called a *ground theory* (Section 3.1), into another standard set theory $\mathscr{T}^*$, called the *generic extension of* $\mathscr{T}$ (Section 3.2). The study of the properties of the theory $\mathscr{T}^*$—and in particular, the fact that it is equiconsistent with $\mathscr{T}$ is postponed to Chapter 7.

3.1 The ground theory $\mathscr{T}$

Definition 3.1 (Ground theory for forcing) — A *ground theory (for forcing)* is a standard set theory $\mathscr{T}$ whose language contains no (proper) function symbol, and that is equipped with two constant symbols P and $(\leq)$ such that: $\mathscr{T} \vdash P \neq \varnothing \ \land \ (\leq)$ is an ordering on P.

Here, the purpose of the two constant symbols P and $(\leq)$ is to explicitly *name* the particular nonempty poset (i.e. its carrier and ordering) that is intended to be used as the forcing poset in the construction of the generic extension $\mathscr{T}^*$. In practice, a ground theory is always constructed from a given standard extension $\mathscr{T}_0 \supseteq \mathrm{ZF}$ (with no function symbol) by Skolemizing (in the sense of Def. 1.12) the theorem expressing the existence of the desired nonempty poset:

$$\mathscr{T}_0 \ \vdash \ \exists P\, \exists(\leq)\, [P \neq \varnothing \ \land \ (\leq) \text{ is an ordering on } P \ \land \ \ldots \text{ specification of } P \text{ and } \leq \ldots]\,.$$

The Henkin extension $\mathscr{T} \supseteq \mathscr{T}_0$ we get (with two extra constant symbols P and $(\leq)$) is then a suitable ground theory for forcing. Note that by Prop. 1.13 the extension $\mathscr{T} \supset \mathscr{T}_0$ is conservative, so that the ground theory $\mathscr{T}$ is equiconsistent with the initial theory $\mathscr{T}_0$.

Example 3.2 (Ground theory for forcing ¬CH) — Let $\mathscr{T}$ be the ground theory based on the minimal language $\{\in, P, (\leq)\}$ whose axioms are the axioms of ZFC, plus the axiom

$$P = \mathrm{Fin}(\aleph_2 \times \omega, 2) \ \land \ (\leq) = \{(p, q) \in P^2 : p \supseteq q\}\,,$$

writing $\mathrm{Fin}(E, F)$ the set of finite (partial) functions from E to F. By construction, the ground theory $\mathscr{T}$ is a conservative extension of ZFC, and we shall see in Chapter 4 that in the corresponding generic extension $\mathscr{T}^*$ we have $\mathscr{T}^* \vdash \neg\mathrm{CH}$.

The same technique can be refined to force $2^{\aleph_0} = \aleph_n$, and this for every (external) integer $n \geq 1$. In this case, we must design the ground theory $\mathscr{T}$ is such a way that $\mathscr{T} \vdash \mathrm{GCH}$.

Example 3.3 (Ground theory for forcing $2^{\aleph_0} = \aleph_n$) — Let $\mathscr{T}$ be the ground theory based on the language $\{\in, P, (\leq)\}$ whose axioms are the axioms of ZFC + GCH, plus the axiom

$$P = \mathrm{Fin}(\aleph_n \times \omega, 2) \ \wedge \ (\leq) = \{(p,q) \in P^2 : p \supseteq q\}.$$

By construction, $\mathscr{T}$ is a conservative extension of ZFC + GCH, so that it is equiconsistent with ZF from the results of Chapter 2. In Chapter 4, we shall see that in the corresponding generic extension $\mathscr{T}^*$, we have $\mathscr{T}^* \vdash 2^{\aleph_0} = \aleph_n$.

3.2 The generic extension $\mathscr{T}^*$

From now on, we work with a fixed ground theory $\mathscr{T}$ (Def. 3.1).

3.2.1 Presentation

From the ground theory $\mathscr{T}$ (Section 3.1) we now define another first-order theory $\mathscr{T}^*$—called the *generic extension* of $\mathscr{T}$—as follows:

- The language of $\mathscr{T}^*$ is the language of $\mathscr{T}$ enriched with:
  - A class symbol (i.e. a unary predicate) $\mathscr{M}(x)$ representing a transitive model of the ground theory $\mathscr{T}$ (Def. 1.32) within the theory $\mathscr{T}^*$. As for any class, we use the shorthand $x \in \mathscr{M} \equiv \mathscr{M}(x)$ (keeping in mind that $\mathscr{M}$ is a proper class in $\mathscr{T}^*$).
  - An extra constant symbol G that represents an $\mathscr{M}$-generic filter $G \subseteq P$ (cf Section 3.2.6). Intuitively, the $\mathscr{M}$-generic filter G is a new subset of P that we add to the theory to generate the larger universe described by $\mathscr{T}^*$.
- The axioms of $\mathscr{T}^*$ are divided into five groups that will be presented and discussed in the rest of Section 3.2:
  - the Axioms of ZF (Section 3.2.2),
  - the Axioms of Transitivity (Section 3.2.3),
  - the Axioms of $\mathscr{T}$ relativized to $\mathscr{M}$ (Section 3.2.4),
  - the Axiom of Genericity (Section 3.2.6), and
  - the Axiom of Naming (Section 3.2.7).

3.2.2 Axioms of ZF

The very first axioms of the generic extension $\mathscr{T}^*$ are the axioms of ZF such as given in Section 1.2.1, without restriction on the formulas used in Separation and Replacement:

Axiom 1 (Axioms of ZF) — $\mathscr{T}^* \vdash \phi$ for every axiom ϕ of ZF. All formulas of the language of $\mathscr{T}^*$ are allowed in the axioms of Separation and Replacement.

So that by construction:

Fact 3.4 — *$\mathscr{T}^*$ is a standard extension of ZF (with no proper function symbol).*

Since the new class symbol $\mathscr{M}$ can be used in any instance of Separation, we can associate to every set A the set $A \cap \mathscr{M} = \{x \in A : x \in \mathscr{M}\}$, that represents the intersection of the set A with the (proper) class $\mathscr{M}$—the result of this operation being a set.

On the other hand, the other axioms of the ground theory $\mathscr{T}$ are not imported in the target theory $\mathscr{T}^*$, so that strictly speaking, $\mathscr{T}^*$ is not an extension of $\mathscr{T}$ (in the sense of Def. 1.5). Due to this, it may be the case that a formula is provable in $\mathscr{T}$ while its negation is provable in $\mathscr{T}^*$. For instance, we shall consider in Section 4.4 an example of a ground theory $\mathscr{T}$ such that $\mathscr{T} \vdash 2^{\aleph_0} = \aleph_1$ (CH) whereas $\mathscr{T}^* \vdash 2^{\aleph_0} = \aleph_2$ ($\Rightarrow \neg$CH).

3.2.3 Axioms of Transitivity

The first Axiom of Transitivity expresses that the class $\mathscr{M}$ is transitive:

Axiom 2 (Trans-$\mathscr{M}$) — $\mathscr{T}^* \vdash (\forall x \in \mathscr{M})(\forall y \in x)(y \in \mathscr{M})$.

The following Axioms of Transitivity express that the class $\mathscr{M}$ contains every set represented by a constant symbol in the language of $\mathscr{T}$, including the symbols P and $(\leq)$:

Axiom 3 (Trans-c) — $\mathscr{T}^* \vdash c \in \mathscr{M}$, for each constant symbol c of $\mathscr{T}$.

(Note that there is no such axiom for G.) An immediate consequence of the latter axioms is that the transitive class $\mathscr{M}$ is provably nonempty in the theory $\mathscr{T}^*$:

Fact 3.5 — $\mathscr{T}^* \vdash \exists x\,(x \in \mathscr{M})$.

Proof. Follows from (Trans-P): $\mathscr{T}^* \vdash P \in \mathscr{M}$.

Since the language of $\mathscr{T}$ may provide extra relation symbols R (other than $\in$ and $=$), we add specific axioms to constrain such relations within the class $\mathscr{M}$ in the theory $\mathscr{T}^*$:

Axiom 4 (Trans-R) — $\mathscr{T}^* \vdash \forall x_1 \cdots \forall x_k\,(R(x_1, \ldots, x_k) \Rightarrow x_1 \in \mathscr{M} \land \cdots \land x_k \in \mathscr{M})$, for each k-ary predicate symbol R of the language of $\mathscr{T}$ that is not $\in$ nor $=$.

Remark 3.6 — The axioms (Trans-R) are not strictly necessary, and we only introduce them for completeness. Without these axioms, the extra relations R introduced by the language of $\mathscr{T}$ would be specified only in the class $\mathscr{M}$ (thanks to the axioms of $\mathscr{T}$ relativized to $\mathscr{M}$, cf Section 3.2.4), but their behaviour would be unknown outside $\mathscr{M}$. However, these axioms are just a convenience, since we can always replace the relation $R(x_1, \ldots, x_k)$ by the relation $R'(x_1, \ldots, x_k) \equiv x_1 \in \mathscr{M} \land \cdots \land x_k \in \mathscr{M} \land R(x_1, \ldots, x_k)$ to achieve the same effect.

3.2.4 Axioms of $\mathscr{T}$ relativized to $\mathscr{M}$

The next axioms of the theory $\mathscr{T}^*$ are the axioms of $\mathscr{T}$ relativized to $\mathscr{M}$:

Axiom 5 (Rel-ϕ) — $\mathscr{T}^* \vdash \phi^{\mathscr{M}}$, for every axiom ϕ of $\mathscr{T}$.

Since the class $\mathscr{M}$ is (provably) nonempty, we deduce from Prop. 1.17 that:

Fact 3.7 (Transitive model) — *The class $\mathscr{M}$ is a transitive model of $\mathscr{T}$ within $\mathscr{T}^*$: if $\mathscr{T} \vdash \phi$, then $\mathscr{T}^* \vdash \phi^{\mathscr{M}}$ (for every closed formula ϕ of the language of $\mathscr{T}$).*

From Prop. 1.34, we immediately deduce the following result, that allows us to transfer $\Delta_1/\Sigma_1/\Pi_1$-properties from/to the sub-universe $\mathscr{M}$:

Proposition 3.8 (Transferring properties from/to $\mathscr{M}$) — *Let $\phi \equiv \phi(x_1, \ldots, x_n)$ be a formula of the language of $\mathscr{T}$ with free variables $x_1, \ldots, x_n$. Then:*

$$\begin{aligned}
&\mathscr{T}^* \vdash (\forall x_1, \ldots, x_n \in \mathscr{M})\,(\phi^{\mathscr{M}}(x_1, \ldots, x_n) \Leftrightarrow \phi(x_1, \ldots, x_n)) && (\textit{if } \phi \textit{ is } \Delta_1)\\
&\mathscr{T}^* \vdash (\forall x_1, \ldots, x_n \in \mathscr{M})\,(\phi^{\mathscr{M}}(x_1, \ldots, x_n) \Rightarrow \phi(x_1, \ldots, x_n)) && (\textit{if } \phi \textit{ is } \Sigma_1)\\
&\mathscr{T}^* \vdash (\forall x_1, \ldots, x_n \in \mathscr{M})\,(\phi(x_1, \ldots, x_n) \Rightarrow \phi^{\mathscr{M}}(x_1, \ldots, x_n)) && (\textit{if } \phi \textit{ is } \Pi_1)
\end{aligned}$$

In Section 1.2.4, we have already studied the relationship between the usual set-theoretic constructors (pairing, union, Separation, Cartesian product, powerset, function space, etc.) and their relativized form w.r.t. an arbitrary transitive model $\mathscr{M}$ of ZF. Transposing Corollary 1.36, Prop. 1.37 and Prop. 1.38 to the present situation, we get:

Proposition 3.9 (Transferring constructions) — *In the theory $\mathscr{T}^*$, we have:*

$$\begin{aligned}
&\mathscr{T}^* \vdash (\forall x, y \in \mathscr{M})\,\{x, y\}^{\mathscr{M}} = \{x, y\} && \mathscr{T}^* \vdash (\forall x, y \in \mathscr{M})\,(x \cup y)^{\mathscr{M}} = x \cup y\\
&\mathscr{T}^* \vdash (\forall x, y \in \mathscr{M})\,(x, y)^{\mathscr{M}} = (x, y) && \mathscr{T}^* \vdash (\forall x, y \in \mathscr{M})\,(x \cap y)^{\mathscr{M}} = x \cap y\\
&\mathscr{T}^* \vdash (\forall x \in \mathscr{M})\,\left(\textstyle\bigcup x\right)^{\mathscr{M}} = \textstyle\bigcup x && \mathscr{T}^* \vdash (\forall x, y \in \mathscr{M})\,(x \setminus y)^{\mathscr{M}} = x \setminus y\\
&\mathscr{T}^* \vdash \varnothing^{\mathscr{M}} = \varnothing \land \omega^{\mathscr{M}} = \omega && \mathscr{T}^* \vdash (\forall x, y \in \mathscr{M})\,(x \times y)^{\mathscr{M}} = x \times y
\end{aligned}$$

$$\mathscr{T}^* \vdash (\forall \vec{z} \in \mathscr{M})(\forall a \in \mathscr{M})\,\{x \in a : \phi(x, \vec{z})\}^{\mathscr{M}} = \{x \in a : \phi^{\mathscr{M}}(x, \vec{z})\}$$
for every formula $\phi(x, \vec{z})$ of the language of $\mathscr{T}$

$$\mathscr{T}^* \vdash (\forall X \in \mathscr{M})\,\mathfrak{P}^{\mathscr{M}}(X) = \mathfrak{P}(X) \cap \mathscr{M} \qquad \mathscr{T}^* \vdash (\forall X, Y \in \mathscr{M})\,(Y^X)^{\mathscr{M}} = (Y^X) \cap \mathscr{M}$$

And transposing Prop. 1.40 and Prop. 1.42, we also get:

Proposition 3.10 (Finite sets and ordinals) — *In the theory $\mathscr{T}^*$, we have:*

$$\begin{aligned}
&\mathscr{T}^* \vdash (\forall X \subseteq \mathscr{M})\,(X \textit{ finite} \Leftrightarrow X \in \mathscr{M} \land (X \textit{ finite})^{\mathscr{M}})\\
&\mathscr{T}^* \vdash (\forall \alpha \in \mathscr{M})\,(On(\alpha) \Leftrightarrow On^{\mathscr{M}}(\alpha))
\end{aligned}$$

Note that for the moment, we can only prove that the two classes On (the class of ordinals in the whole universe) and $On^{\mathscr{M}}$ (the class of ordinals in sub-universe $\mathscr{M}$) coincide on $\mathscr{M}$: $On^{\mathscr{M}} = On \cap \mathscr{M}$. In Section 3.2.7, we shall refine this result by proving (using the Axiom of Naming) that all ordinals actually belong to $\mathscr{M}$, so that $On = On^{\mathscr{M}}$.

3.2.5 Some terminology about the poset $(P, \leq)$

Before presenting the next axiom of $\mathscr{T}^*$—the Axiom of Genericity—we first need to recall some terminology pertaining to the forcing poset $(P, \leq)$, such as the notions of downwards/upwards closure, of compatible/incompatible elements, the notion of an atom, the notions of dense and predense subsets as well as the notion of a filter.

All these notions have one thing in common, which is that they are defined by the means of Δ_0-formulas, as shown in Fig. 3.1. In practice, this means that:

(1) All these notions can be used in the theory $\mathscr{T}$ or in the theory $\mathscr{T}^*$ indifferently.

$$
\begin{array}{rcl}
p \leq q & \equiv & (\exists z \in (\leq))\,(z = (p, q)) \\
(\leq) \text{ order} & \equiv & (\leq) \subseteq P \times P \ \wedge\ (\forall p \in P)\,(p \leq p)\ \wedge \\
& & (\forall p, q \in P)\,(p \leq q \wedge q \leq p \Rightarrow p = q) \wedge \\
& & (\forall p, q, r \in P)\,(p \leq q \wedge q \leq r \Rightarrow p \leq r) \\
p \top q & \equiv & (\exists r \in P)\,(r \leq p \wedge r \leq q) \\
p \perp q & \equiv & \neg(p \top q) \\
p \text{ atom} & \equiv & p \in P\ \wedge\ (\forall q_1, q_2 \in P)\,(q_1 \leq p \wedge q_2 \leq p \Rightarrow q_1 \top q_2) \\
P \text{ atomless} & \equiv & (\forall p \in P)\,\neg(p \text{ atom}) \\
F \text{ filter of } P & \equiv & F \subseteq P\ \wedge\ F \neq \varnothing \wedge \\
& & (\forall p \in F)\,(\forall q \in P)\,(p \leq q \Rightarrow q \in F) \wedge \\
& & (\forall p, q \in F)\,(\exists r \in F)\,(p \leq r \wedge q \leq r) \\
D \text{ dense subset of } P & \equiv & D \subseteq P\ \wedge\ (\forall p \in P)\,(\exists q \in D)\,(q \leq p) \\
D \text{ predense subset of } P & \equiv & D \subseteq P\ \wedge\ (\forall p \in P)\,(\exists q \in D)\,(q \top p) \\
A \text{ antichain of } P & \equiv & A \subseteq P\ \wedge\ (\forall p, q \in A)\,(p \neq q \Rightarrow p \perp q)
\end{array}
$$

Figure 3.1: Δ_0-notions about $(P, \leq)$

(2) These notions are absolute (Prop. 3.8), so that their meaning is the same in the sub-universe $\mathscr{M}$ (which is described by the axioms of $\mathscr{T}$ relativized to $\mathscr{M}$) and in the whole universe (which is described by the axioms of $\mathscr{T}^*$).

Definition 3.11 (Downwards/upwards closure) — Given a subset $X \subseteq P$, we say that:

- *X is upwards closed* when $(\forall p, q \in P)\,(p \in X \wedge q \geq p \Rightarrow q \in X)$;
- *X is downwards closed* when $(\forall p, q \in P)\,(p \in X \wedge q \leq p \Rightarrow q \in X)$.

Given an arbitrary subset $X \subseteq P$, we respectively write $\uparrow X$ and $\downarrow X$ the *upwards closure* and the *downwards closure* of X in P, that are defined by

$$\uparrow X \ = \ \{p \in P : (\exists q \in X)\,(q \leq p)\} \quad \text{and} \quad \downarrow X \ = \ \{p \in P : (\exists q \in X)\,(q \geq p)\}\,.$$

By definition, the set $\uparrow X$ (resp. $\downarrow X$) is the smallest upwards closed subset of P (resp. the smallest downwards closed subset of P) that contains X as a subset. Both operations $X \mapsto \uparrow X$ and $X \mapsto \downarrow X$ are absolute in the sense of Prop. 1.35:

$$
\begin{array}{l}
\mathscr{T}^* \ \vdash\ (\forall X \in \mathfrak{P}(P))\,[X \in \mathscr{M} \ \Rightarrow\ \uparrow X \in \mathscr{M}\ \wedge\ \uparrow X = (\uparrow X)^{\mathscr{M}})\ \wedge \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow X \in \mathscr{M}\ \wedge\ \downarrow X = (\downarrow X)^{\mathscr{M}})\quad]
\end{array}
$$

Definition 3.12 (Compatible/incompatible conditions) Given two conditions $p, q \in P$, we say that p and q are *compatible* when there is a condition $r \in P$ such that $r \leq p$ and $r \leq q$, which we write $p \top q$. When it is not the case, we say that the two conditions p and q are *incompatible*, which we write $p \perp q$:

$$
\begin{array}{rcl}
p \top q & \equiv & (\exists r \in P)\,(r \leq p \wedge r \leq q) \\
p \perp q & \equiv & \neg(\exists r \in P)\,(r \leq p \wedge r \leq q)
\end{array}
$$

Definition 3.13 (Atoms) — A condition $p \in P$ is an *atom* when the conditions $\leq p$ are pairwise compatible: p atom $\equiv$ $(\forall q_1, q_2 \in P)\,(q_1 \leq p \wedge q_2 \leq p \Rightarrow q_1 \top q_2)$.
Finally, we say that the forcing poset $(P, \leq)$ is *atomless* when P contains no atom.

Again, the formulas '$p \top q$', '$p \perp q$', '$p \in P$ is an atom' and 'P is atomless' are Δ_0 (Fig. 3.1) and thus absolute.

Definition 3.14 (Filters of P) — We call a *filter of P* any subset $F \subseteq P$ such that:

- (*i*) F is nonempty: $F \neq \varnothing$
- (*ii*) F is upwards closed: $(\forall p, q \in P)(p \in F \wedge p \leq q \Rightarrow q \in F)$
- (*iii*) F is downwards directed: $(\forall p, q \in F)(\exists r \in F)(r \leq p \wedge r \leq q)$

Moreover, we say that a filter $F \subseteq P$ is *principal* when F has a smallest element p_0. Principal filters of P are exactly the subsets of P of the form $F = \uparrow\{p_0\}$, where $p_0 \in P$.

The formula 'F is a filter of P' is Δ_0 (Fig. 3.1) and thus absolute. However, the expanded universe (described by $\mathscr{T}^*$) may contain filters $F \subseteq P$ that do not belong to $\mathscr{M}$—the typical example of such an 'extra' filter being given by G (Section 3.2.6).

Definition 3.15 (Dense and predense subsets) — We say that a subset $D \subseteq P$ is

- *dense* when $(\forall p \in D)\,(\exists q \in P)\,(q \leq p)$;
- *predense* when $(\forall p \in D)\,(\exists q \in P)\,(q \top p)$.

From the above definition, it is clear that a subset $D \subseteq P$ is dense if and only if its upwards closure is P, that is, iff $\uparrow D = P$. Similarly, a subset $D \subseteq P$ is predense if and only if its downwards closure $\downarrow D$ is dense, that is, iff $\uparrow\downarrow D = P$.

Remark 3.16 — The above notion of 'density' implicitly refers to the topology on P whose open subsets are the downwards closed subsets of P and whose closed subsets are the upwards closed subsets of P. This topology is $\mathbf{T}_1$ but in general not $\mathbf{T}_2$ (unless ordering is discrete on P). A property of this topology is that open subsets (resp. closed subsets) are closed under arbitrary intersections (resp. under arbitrary unions). In this framework, we can notice that two conditions $p, q \in P$ are incompatible if and only if we can separate them with two open subsets U and V such that $p \in U$, $q \in V$ and $U \cap V = \varnothing$.

Again, the formulas 'D is dense' and 'D is predense' are Δ_0 (Fig. 3.1) and thus absolute. (Of course, not all dense/predense subsets of P belong to $\mathscr{M}$.)

Definition 3.17 (Separativeness) — We say that the forcing poset $(P, \leq)$ is *separative* when for all $p, q \in P$ such that $p \nleq q$, there is $p' \leq p$ such that $p' \perp q$.

(Again this property is Δ_0 and thus absolute.)

In Chapter 5, we shall see that this property—that has no specific impact on the structure of the expanded universe—implies that the canonical mapping $p \mapsto p^{\perp\perp}$ from the poset $(P, \leq)$ to the Boolean algebra $(\mathcal{B}, \subseteq)$ generated by P is an embedding.

3.2.6 Axiom of Genericity

Definition 3.18 ($\mathscr{M}$-generic filters) — We say that a filter $F \subseteq P$ is *$\mathscr{M}$-generic* when F meets all dense subsets $D \subseteq P$ that belong to the sub-universe $\mathscr{M}$.

The Axiom of Genericity expresses that the set G is an *$\mathscr{M}$-generic filter* of P:

Axiom 6 (Genericity) — $\mathscr{T}^* \vdash G$ filter of $P \wedge (\forall D \in \mathfrak{P}^{\mathscr{M}}(P))\,(D$ dense $\Rightarrow D \cap G \neq \varnothing)$.

An important property of the $\mathscr{M}$-generic filter G is that it more generally meets all *predense* subsets $D \subseteq P$ that belong to $\mathscr{M}$:

Lemma 3.19 — $\mathscr{T}^* \vdash (\forall D \in \mathfrak{P}^{\mathscr{M}}(P))\,(D \textit{ predense} \Rightarrow D \cap G \neq \varnothing)$.

Proof. Let $D \in \mathscr{M}$ such that D is predense. Then its downwards closure $\downarrow D$ is dense, and still belongs to $\mathscr{M}$: $\downarrow D = (\downarrow D)^{\mathscr{M}} \in \mathscr{M}$. Since G is $\mathscr{M}$-generic, there is $p \in G$ such that $p \in \downarrow D$. And since $p \in \downarrow D$, there is $q \in D$ such that $p \leq q$. But since G is a filter, G is upwards closed so that $q \in G$ too. Therefore $q \in D \cap G$. □

The case of trivial generic filters To understand the significance of the notion of $\mathscr{M}$-generic filter, it is important to study the particular case of trivial generic filters.

Given a condition $p_0 \in P$, the set $\uparrow\!\downarrow p_0 = \uparrow\!\downarrow\{p_0\}$ is by definition the set of all conditions that are compatible with p_0: $\uparrow\!\downarrow p_0 = \{p \in P : p \top p_0\}$. And by construction, this set always belongs to $\mathscr{M}$, since it is defined using Δ_0-Separation (cf Prop. 3.9).

In the particular case where the condition p_0 is an atom, we can prove that the set $\uparrow\!\downarrow p_0$ is a filter of P, and that it meets all dense subsets $D \subseteq P$, even when $D \notin \mathscr{M}$. In other words, the subset $\uparrow\!\downarrow p_0 \subseteq P$ (that belongs to $\mathscr{M}$) is a *generic filter*—and not only an *$\mathscr{M}$-generic filter*:

Proposition 3.20 — *For every atom $a_0 \in P$, the set $\uparrow\!\downarrow p_0$ is a generic filter:*

$$\mathscr{T}^* \vdash (\forall p_0 \in P)\,[p_0 \textit{ atom} \Rightarrow \uparrow\!\downarrow p_0 \textit{ filter of } P \wedge (\forall D \subseteq P)\,(D \textit{ dense} \Rightarrow D \cap \uparrow\!\downarrow p_0 \neq \varnothing)]$$

Proof. We first check that $\uparrow\!\downarrow p_0$ is a filter:

(*i*) $\uparrow\!\downarrow p_0$ is nonempty, since $p_0 \in \uparrow\!\downarrow p_0$.

(*ii*) $\uparrow\!\downarrow p_0$ is upwards closed (by construction).

(*iii*) $\uparrow\!\downarrow p_0$ is downwards directed. Let $p_1, p_2 \in G$. From the definition of $\uparrow\!\downarrow p_0$, there are $q_1, q_2 \leq p_0$ such that $q_1 \leq p_1$ and $q_2 \leq p_2$. But since p_0 is an atom, q_1 and q_2 are compatible, so there is $r \in P$ such that $r \leq q_1$ and $r \leq q_2$. It is then clear that $r \in \uparrow\!\downarrow p_0$ (since $r \leq p_0$), and that $r \leq p_1$ and $r \leq p_2$.

To show that $\uparrow\!\downarrow p_0$ is generic, take a dense subset $D \subseteq P$. By density, there is $q \in D$ such that $q \leq p_0$. So that $q \in \uparrow\!\downarrow p_0$. □

In what follows, the generic filters of the form $\updownarrow\!\!\downarrow p_0$ (where $p_0 \in P$ is an atom) will be called the *trivial generic filters*. (Note that all trivial generic filters belong to $\mathscr{M}$.) The following proposition shows that G belongs to $\mathscr{M}$ if and only if G is trivial:

Proposition 3.21 — $\mathscr{T}^* \vdash G \in \mathscr{M} \iff (\exists p_0 \in P)\,(p_0 \textit{ atom} \wedge G = \updownarrow\!\!\downarrow p_0)$.

Proof. We only prove the direct implication (the converse being trivial). For that, let us assume that $G \in \mathscr{M}$. From this we get $G^c \in \mathscr{M}$, writing $G^c = P \setminus G$. But since G is $\mathscr{M}$-generic and since $G^c \cap G = \varnothing$, it follows from Lemma 3.19 that G^c is not predense. Therefore, there is a condition $p_0 \in P$ that is incompatible with every element of G^c, so that $\updownarrow\!\!\downarrow p_0 \cap G^c = \varnothing$, and thus: $\updownarrow\!\!\downarrow p_0 \subseteq G$. In particular, we have $\downarrow\{p_0\} \subseteq G$, which proves that p_0 is an atom (since the elements of G are pairwise compatible). We have proved that $\updownarrow\!\!\downarrow p_0 \subseteq G$. The converse inclusion $G \subseteq \updownarrow\!\!\downarrow p_0$ is obvious, since all elements of G are compatible with $p_0 \in G$. □

From this, we immediately get a sufficient condition ensuring that the $\mathscr{M}$-generic filter G does not belong to $\mathscr{M}$:

Corollary 3.22 — $\mathscr{T}^* \vdash P \textit{ atomless} \Rightarrow G \notin \mathscr{M}$

In other words: if in the ground theory $\mathscr{T}$ the poset $(P, \leq)$ is provably atomless, then in the corresponding generic extension one has $\mathscr{T}^* \vdash G \notin \mathscr{M}$ (since the formula 'P is atomless' is Δ_0). In practice, all the interesting forcing posets are (provably) atomless (in the ground theory $\mathscr{T}$), so that we shall only consider generic extensions $\mathscr{T}^*$ where $G \notin \mathscr{M}$.

3.2.7 Axiom of Naming

In forcing, we frequently need to work at the boundary of the sub-universe $\mathscr{M}$, by considering sets A whose elements are all in $\mathscr{M}$ (i.e. such that $A \subseteq \mathscr{M}$) but that do not necessarily belong themselves to $\mathscr{M}$. In what follows, we write $\mathfrak{P}(\mathscr{M})$ the class of all sets A such that $A \subseteq \mathscr{M}$. By construction, the class $\mathfrak{P}(\mathscr{M})$ is transitive and contains $\mathscr{M}$ as a sub-class. Typical examples of sets in $\mathfrak{P}(\mathscr{M})$ (but not necessarily in $\mathscr{M}$) are:

- The $\mathscr{M}$-generic filter $G \subseteq P$,
- All subsets $Y \subseteq X$ of a given set $X \in \mathscr{M}$,
- All functions $f : X \to Y$ for given sets $X, Y \in \mathscr{M}$, etc.

In particular, the difference $\mathfrak{P}(\mathscr{M}) \setminus \mathscr{M}$ contains all 'new' subsets $Y \subseteq X$ of 'old' sets $X \in \mathscr{M}$, as it contains all 'new' functions $f : X \to Y$ between 'old' sets $X, Y \in \mathscr{M}$.

Among all sets in the class $\mathfrak{P}(\mathscr{M})$, some of them can be represented by particular elements of $\mathscr{M}$—called *P-names*—using the properties of the $\mathscr{M}$-generic filter $G \subseteq P$ as follows:

Definition 3.23 (*P*-name) — Let $A \in \mathfrak{P}(\mathscr{M})$. We say that a set $N \in \mathscr{M}$ is a *P-name for A* when the elements of A are exactly the sets x such that $(x, p) \in N$ for some $p \in G$:

$$N \text{ is a } P\text{-name for } A \quad \equiv \quad N \in \mathscr{M} \ \wedge\ \forall x\,(x \in A \ \Leftrightarrow\ (\exists p \in G)\,(x, p) \in N)\,.$$

Remarks 3.24 — (1) The formula 'N is a P-name for A' implies the inclusion $A \subseteq \bigcup\bigcup N$, from the definition of ordered pairs: $(x, p) = \{\{x\}, \{x, p\}\}$. But since the set $\bigcup\bigcup N$ belongs to $\mathscr{M}$ (recall that $N \in \mathscr{M}$ and that the operation $X \mapsto \bigcup X$ is absolute), this implies that each set in $\mathfrak{P}(\mathscr{M})$ that has a P-name in $\mathscr{M}$ is actually bounded by a set in $\mathscr{M}$.

(2) A set $A \subseteq \mathscr{M}$ may have several P-names $N \in \mathscr{M}$, but every set $N \in \mathscr{M}$ is always the P-name of a unique set $A \in \mathfrak{P}(\mathscr{M})$, that is given by

$$\begin{aligned} A = I_G(N) &\stackrel{\Delta}{=} \{x \in \textstyle\bigcup\bigcup N : (\exists p \in G)\,(x, p) \in N\} \\ &= \pi_1(N \cap (\mathscr{M} \times G)) \end{aligned}$$

(writing π_1 the first projection of a set of pairs.) The operation $N \mapsto I_G(N)$ is called the *interpretation*—or the *decoding*—of a P-name $N \in \mathscr{M}$ into the set $I_G(N) \in \mathfrak{P}(\mathscr{M})$.

(3) The only relevant elements of a P-name are its elements of the form (x, p), where $p \in P$. So that if N is a P-name for A, then the subset

$$N' = N \cap (\mathscr{M} \times P) = \{(x, p) \in N : p \in P\} \subseteq N$$

is still a P-name for the same set $A \in \mathfrak{P}(\mathscr{M})$. In practice, this means that P-names can be taken in the sub-class $\mathfrak{P}^{\mathscr{M}}(\mathscr{M} \times P) = \mathfrak{P}(\mathscr{M} \times P) \cap \mathscr{M} \subseteq \mathscr{M}$ rather than in $\mathscr{M}$.

Moreover, it is important to notice that many sets in $\mathfrak{P}(\mathscr{M})$ have P-names:

- First, every set $A \in \mathscr{M}$ has a P-name in $\mathscr{M}$, for instance the set $N_A = A \times P$. Notice that we cannot take $N_A = A \times G$, since this set is in general not in $\mathscr{M}$.
- The $\mathscr{M}$-generic filter $G \subseteq P$ (which is in general outside $\mathscr{M}$) has also a P-name $N_G \in \mathscr{M}$, which is given by $N_G = \{(p, p) : p \in P\}$ ($\in \mathscr{M}$). In what follows, this set $N_G \in \mathscr{M}$ is called the *canonical P-name for G*.
- The reader is invited to check that if two sets $A, B \in \mathfrak{P}(\mathscr{M})$ have P-names in $\mathscr{M}$, then the sets $A \cup B$, $A \cap B$, $A \setminus B$ have P-names in $\mathscr{M}$ too (Exercise ???).

The Axiom of Naming generalizes the existence of P-names to all sets in $\mathfrak{P}(\mathscr{M})$, by stating that every set $A \in \mathfrak{P}(\mathscr{M})$ has a P-name $B \in \mathscr{M}$:

Axiom 7 (Naming) — $\mathscr{T}^* \vdash (\forall A \in \mathfrak{P}(\mathscr{M}))\,(\exists N \in \mathscr{M})\,\forall x\,(x \in A \Leftrightarrow (\exists p \in G)\,(x, p) \in N)$.

Or, with a more compact formulation:

(Axiom of Naming) $\qquad (\forall A \in \mathfrak{P}(\mathscr{M}))\,(\exists N \in \mathscr{M})\,A = I_G(N)$

Intuitively, the Axiom of Naming expresses that the expanded universe (described by $\mathscr{T}^*$) is generated from the 'new' subset $G \subseteq P$ around the initial universe (described by $\mathscr{T}$ and reflected in the class $\mathscr{M}$ in the theory $\mathscr{T}^*$). This intuition will become more clear with Prop. 3.28 below, and it is fully developped in Exercise 3.6.

Technically, the Axiom of Naming is also an extremely powerful tool to speak about all sets in $\mathfrak{P}(\mathscr{M})$—and thus about all subsets of a given set $X \in \mathscr{M}$—within the sub-universe $\mathscr{M}$. In Section 3.3, we shall introduce a (recursive) generalization of the notion of a P-name that can be more generally used to represent the whole universe inside the sub-universe $\mathscr{M}$.

The Bounding Lemma The most obvious consequence of the Axiom of Naming is that every set $A \in \mathfrak{P}(\mathscr{M})$ is bounded by a set $A' \in \mathscr{M}$:

Lemma 3.25 (Bounding Lemma) — $\mathscr{T}^* \vdash (\forall A \in \mathfrak{P}(\mathscr{M}))(\exists A' \in \mathscr{M})\, A \subseteq A'$.

Proof. Taking a P-name $N \in \mathscr{M}$ for A, we let $A' = \bigcup\bigcup N$. □

The main consequence of the Bounding Lemma is that ordinals are the same in $\mathscr{M}$ and in the expanded universe:

Proposition 3.26 (Ordinals) — $\mathscr{T}^* \vdash \forall\alpha\,(On(\alpha) \Leftrightarrow \alpha \in \mathscr{M} \wedge On^{\mathscr{M}}(\alpha))$.

Proof. From Prop. 3.8 we know that $(\forall\alpha \in \mathscr{M})\,(On(\alpha) \Leftrightarrow On^{\mathscr{M}}(\alpha))$, so that we only need to show that $\forall\alpha\,(On(\alpha) \Rightarrow \alpha \in \mathscr{M})$. We reason by contradiction, assuming that α is the smallest ordinal such that $\alpha \notin \mathscr{M}$. From this we get $\alpha \subseteq \mathscr{M}$ (since $\beta \in \mathscr{M}$ for all $\beta < \alpha$), hence there is $A \in \mathscr{M}$ such that $\alpha \subseteq A$ (from the Bounding Lemma). Let $A' = \{\beta \in A : On(\beta)\}$. By construction, we have $A' \in \mathscr{M}$ and $\alpha \subseteq A'$. Since $\alpha \notin \mathscr{M}$, the inclusion $\alpha \subset A'$ is strict, so there is $\beta \in A'$ such that $\beta \notin \alpha$. But either $\alpha \in \beta \in \mathscr{M}$, which is absurd (by transitivity), or $\alpha = \beta \in \mathscr{M}$, which is absurd too. □

Therefore $On \subseteq \mathscr{M}$, so that $\mathscr{M}$ is a proper class.

Cardinals in the expanded universe A consequence of Prop. 3.26 is that all cardinals of the expanded universe—that are also ordinals—actually belong to the sub-universe $\mathscr{M}$. Moreover, since the formula $Cn(\kappa)$ ('κ is a cardinal') is Π_1

$$Cn(\kappa) \quad\equiv\quad On(\kappa) \ \wedge\ (\forall\alpha<\kappa)\,\forall f\,\neg(f : \alpha \xrightarrow{\sim} \kappa)\,,$$

we deduce from Prop. 3.8 that every cardinal κ in the expanded universe is actually a cardinal in the sub-universe $\mathscr{M}$:

Proposition 3.27 (Cardinals) — $\mathscr{T}^* \vdash \forall\kappa\,(Cn(\kappa) \Rightarrow \kappa \in \mathscr{M} \wedge Cn^{\mathscr{M}}(\kappa))$.

The converse implication is in general no true, which means that the expanded universe may have fewer cardinals than the sub-universe $\mathscr{M}$ (see the discussion in the Introduction of this course), and we shall see striking examples of this phenomenon of *cardinal decimation* in Chapter 6. On the other hand, we shall see in Section 4.1.1 that when the forcing poset $(P, \leq)$ fulfils the so-called *countable chain condition*, then the cardinals of the expanded universe are exactly the same as in the sub-universe $\mathscr{M}$.

The case of a trivial generic filter G We have seen (Prop. 3.21) that the $\mathscr{M}$-generic filter G belongs to the sub-universe $\mathscr{M}$ if and only if it is trivial, that is, iff $G = \uparrow\!\downarrow p_0$ for some atom $p_0 \in P$. The Axiom of Naming implies that in this case, the whole universe collapses to $\mathscr{M}$:

Proposition 3.28 (Collapsing) — $\mathscr{T}^* \vdash G \in \mathscr{M} \Rightarrow \forall x\,(x \in \mathscr{M})$.

Proof. Assuming that $G \in \mathscr{M}$, we first show that $\forall A\,(A \subseteq \mathscr{M} \Rightarrow A \in \mathscr{M})$. For that, consider $A \subseteq \mathscr{M}$ and pick a P-name N for A, that is: a set $N \in \mathscr{M}$ such that

$$A \ =\ I_G(N) \ =\ \{x \in \bigcup\bigcup N : (\exists p \in G)\,(x, p) \in N\}\,.$$

But since $G \in \mathscr{M}$, the right-hand side of the above equality is in $\mathscr{M}$, so that $A \in \mathscr{M}$. From the implication $\forall A\,(A \subseteq \mathscr{M} \Rightarrow A \in \mathscr{M})$, we immediately deduce that all sets are in the sub-universe $\mathscr{M}$ by a straightforward $\in$-induction. □

3.3 Recursive P-names

From now on, the *P-names* introduced in Section 3.2.7 are called *simple P-names*, as opposed to the more sophisticated notion of a recursive P-name we introduce in this section.

3.3.1 Intuitions

In Section 3.2.7, we introduced the notion of *P-name* (together with the corresponding axiom) as a way to represent any set $A \subset \mathscr{M}$ as a set of pairs $N \in \mathscr{M}$ whose first and second components range over $\mathscr{M}$ and P (provided we simplify N as shown in Remark 3.24 (3)). Simple naming is thus nothing but a way to reflect the class $\mathfrak{P}(\mathscr{M})$ within the sub-class

$$\mathfrak{P}^{\mathscr{M}}(\mathscr{M} \times P) \quad \subseteq \quad \mathscr{M}\,,$$

the process of decoding being given by the operation $\quad I_G : \mathfrak{P}^{\mathscr{M}}(\mathscr{M} \times P) \to \mathfrak{P}(\mathscr{M})$.

However, this process can be iterated. For instance, if we want to represent elements of $\mathfrak{P}(\mathfrak{P}(\mathscr{M}))$ in the sub-universe $\mathscr{M}$, it suffices to use P-names rather than elements of $\mathscr{M}$ as first components, and to apply the decoding function I_G twice:

$$\mathfrak{P}^{\mathscr{M}}(\mathfrak{P}^{\mathscr{M}}(\mathscr{M} \times P) \times P) \;\stackrel{I_G}{\longrightarrow}\; \mathfrak{P}(\mathfrak{P}^{\mathscr{M}}(\mathscr{M} \times P)) \;\stackrel{\mathfrak{P}I_G}{\longrightarrow}\; \mathfrak{P}(\mathfrak{P}(\mathscr{M}))\,.$$

In this way, we reflect the class $\mathfrak{P}(\mathfrak{P}(\mathscr{M}))$ within the sub-class $\mathfrak{P}^{\mathscr{M}}(\mathfrak{P}^{\mathscr{M}}(\mathscr{M} \times P) \times P) \subseteq \mathscr{M}$.

By iterating this technique transfinitely many times, we shall see that it is more generally possible to reflect the *whole universe* (described by $\mathscr{T}^*$) within a sub-class $\mathscr{M}^{(P)} \subseteq \mathscr{M}$ that satisfies the (recursive) equation

$$\mathscr{M}^{(P)} \;=\; \mathfrak{P}^{\mathscr{M}}(\mathscr{M}^{(P)} \times P) \qquad\qquad (\subseteq \mathscr{M})$$

and whose elements are called *recursive P-names*.

3.3.2 Constructing the class $\mathscr{M}^{(P)}$

Technically, the (proper) class $\mathscr{M}^{(P)}$ is built as the union of a transfinite sequence $(\mathscr{M}^{(P)}_{\alpha})_{\alpha \in On}$ defined similarly to the cumulative hierarchy $(V_{\alpha})_{\alpha \in On}$ (cf Exercise 1.11) by

$$\mathscr{M}^{(P)}_{\alpha} \;=\; \bigcup_{\beta<\alpha} \mathfrak{P}^{\mathscr{M}}(\mathscr{M}^{(P)}_{\beta} \times P)\,.$$

However, some care must be taken to ensure that all sets $\mathscr{M}_{\alpha}$ belong to $\mathscr{M}$, as well as all truncated sequences $(\mathscr{M}^{(P)}_{\beta})_{\beta<\alpha}$, where $\alpha \in On$. For this reason, we shall first construct the class of P-names in the ground theory $\mathscr{T}$—where we shall write it $V^{(P)}$ rather than $\mathscr{M}^{(P)}$.

Construction in the ground theory $\mathscr{T}$ Working in the ground theory $\mathscr{T}$, we consider the transfinite sequence $(V^{(P)}_{\alpha})_{\alpha \in On}$ that is defined similarly to the sequence $(V_{\alpha})_{\alpha \in On}$, letting

$$V^{(P)}_{\alpha} \;=\; \bigcup_{\beta<\alpha} \mathfrak{P}(V^{(P)}_{\beta} \times P) \qquad\qquad (\text{for all } \alpha \in On)$$

Fact 3.29 — *(In $\mathscr{T}$:) The transfinite sequence $(V^{(P)})_{\alpha\in On}$ is monotonic:*

(*i*) *If $\alpha < \beta$, then $V^{(P)}_\alpha \subseteq V^{(P)}_\beta$;*

Moreover:

(*ii*) $V^{(P)}_0 = \varnothing$;

(*iii*) $V^{(P)}_{\alpha+1} = \mathfrak{P}(V^{(P)}_\alpha \times P)$ *for every ordinal α;*

(*iv*) $V^{(P)}_\alpha = \bigcup_{\beta<\alpha} V^{(P)}_\beta$ *for every limit ordinal α.*

The class of *recursive P-names* is written $V^{(P)}$ (in $\mathscr{T}$) and defined as the transfinite union of the sequence $(V^{(P)}_\alpha)_{\alpha\in On}$, letting $x \in V^{(P)} \equiv (\exists\alpha \in On)\, x \in V^{(P)}_\alpha$.

Construction in the generic extension $\mathscr{T}^*$ In the ground theory $\mathscr{T}$, we have defined a functional relation $y = V^{(P)}_\alpha$ whose domain is the class *On* of ordinals, so that we have:

$$\mathscr{T} \;\vdash\; (\forall\alpha \in On)(\exists! y)\, y = V^{(P)}_\alpha\,.$$

From Fact 3.7, we thus have in the generic extension $\mathscr{T}^*$:

$$\mathscr{T}^* \;\vdash\; (\forall\alpha \in On)(\exists! y \in \mathscr{M})\, \big(y = V^{(P)}_\alpha\big)^{\mathscr{M}}$$

(using the fact that $On^{\mathscr{M}} = On$).

By definition, we write $\mathscr{M}^{(P)}_\alpha$ the unique set $y \in \mathscr{M}$ such that $\big(y = V^{(P)}_\alpha\big)^{\mathscr{M}}$ (for all $\alpha \in On$). In other words, the set $\mathscr{M}^{(P)}_\alpha$ is nothing but the set $V^{(P)}_\alpha$ (initially constructed in $\mathscr{T}$) seen in the theory $\mathscr{T}^*$ as an element of the sub-universe $\mathscr{M}$, that is: $\mathscr{M}^{(P)}_\alpha = (V^{(P)}_\alpha)^{\mathscr{M}}$.

Relativizing the defining equation $V^{(P)}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{P}(V^{(P)}_\beta \times P)$ of the sequence $(V^{(P)}_\alpha)_{\alpha\in On}$ (in $\mathscr{T}$) to the sub-universe $\mathscr{M}$ (in $\mathscr{T}^*$), we get:

$$\mathscr{T}^* \;\vdash\; (\forall\alpha \in On)\; \mathscr{M}^{(P)}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{P}^{\mathscr{M}}(\mathscr{M}^{(P)}_\beta \times P)\,.$$

And relativizing Fact 3.29 to $\mathscr{M}$ as well:

Fact 3.30 — *(In $\mathscr{T}^*$:) The transfinite sequence $(\mathscr{M}^{(P)})_{\alpha\in On}$ is monotonic:*

(*i*) *If $\alpha < \beta$, then $\mathscr{M}^{(P)}_\alpha \subseteq \mathscr{M}^{(P)}_\beta$*

Moreover:

(*ii*) $\mathscr{M}^{(P)}_0 = \varnothing$;

(*iii*) $\mathscr{M}^{(P)}_{\alpha+1} = \mathfrak{P}^{\mathscr{M}}(\mathscr{M}^{(P)}_\alpha \times P)$ *for every ordinal α;*

(*iv*) $\mathscr{M}^{(P)}_\alpha = \bigcup_{\beta<\alpha} \mathscr{M}^{(P)}_\beta$ *for every limit ordinal α.*

Definition 3.31 (The class $\mathscr{M}^{(P)}$ of recursive P-names) — The class $\mathscr{M}^{(P)}$ is defined (in the generic extension $\mathscr{T}^*$) as the union of all sets $\mathscr{M}_\alpha^{(P)}$ (where $\alpha \in On$)

$$x \in \mathscr{M}^{(P)} \equiv (\exists \alpha \in On)\,(x \in \mathscr{M}_\alpha^{(P)}),$$

and the elements of $\mathscr{M}^{(P)}$ are called *recursive P-names*. Given a recursive P-name $u \in \mathscr{M}^{(P)}$, we call the *rank of u* and write $\mathrm{rk}^{(P)}(u)$ the smallest $\alpha \in On$ such that $u \in \mathscr{M}_\alpha^{(P)}$.

It is clear that the class $\mathscr{M}^{(P)}$ of recursive P-names is nothing but the class $V^{(P)}$ (such as defined in the ground theory $\mathscr{T}$) relativized to the class $\mathscr{M}$ (in the generic extension $\mathscr{T}^*$):

$$\mathscr{T}^* \vdash \forall u\,(u \in \mathscr{M}^{(P)} \Leftrightarrow u \in \mathscr{M} \wedge (u \in V^{(P)})^{\mathscr{M}}).$$

Moreover, the class $\mathscr{M}^{(P)}$ is characterized as follows:

Lemma 3.32 — $\mathscr{T}^* \vdash \forall u\,[u \in \mathscr{M}^{(P)} \Leftrightarrow u \in \mathscr{M} \wedge u \subseteq \mathscr{M}^{(P)} \times P]$.

Proof. The direct implication is obvious from the definition of the class $\mathscr{M}^{(P)}$ of recursive P-names. Conversely, let us assume that $u \in \mathscr{M}$ is a set of pairs (v, p) such that $v \in \mathscr{M}^{(P)}$ and $p \in P$. Since the transfinite sequence $(\mathscr{M}_\alpha^{(P)})_{\alpha \in On}$ is monotonic, we can find a sufficiently large ordinal α such that $v \in \mathscr{M}_\alpha^{(P)}$ for all $(v, p) \in u$. Therefore $u \subseteq \mathscr{M}_\alpha^{(P)} \times P$, and since $u \in \mathscr{M}$, we get $u \in \mathfrak{P}^{\mathscr{M}}(\mathscr{M}_\alpha^{(P)} \times P) = \mathscr{M}_{\alpha+1}^{(P)}$. □

In other words, we have: $\mathscr{M}^{(P)} = \mathfrak{P}^{\mathscr{M}}(\mathscr{M}^{(P)} \times P) = \mathfrak{P}(\mathscr{M}^{(P)} \times P) \cap \mathscr{M}$.

3.3.3 Interpreting recursive P-names

To every recursive P-name $u \in \mathscr{M}^{(P)}$ we now associate a set $I_G^\infty(u)$ (in the expanded universe) that is defined by induction on $\mathrm{rk}^{(P)}(u)$, letting:

$$I_G^\infty(u) = \{I_G^\infty(v) : (\exists p \in G)\,(v, p) \in u\}.$$

(This definition is well-founded, since $\mathrm{rk}^{(P)}(v) < \mathrm{rk}^{(P)}(u)$ as soon as $(v, p) \in u$ for some $p \in P$.)

Definition 3.33 (Recursive P-name of a set) — Given an arbitrary set x and a recursive P-name $u \in \mathscr{M}^{(P)}$, we say that *u is a recursive P-name for x* when $x = I_G^\infty(u)$.

By a transfinite application of the Naming axiom, we can now prove that every set *of the expanded universe* has a recursive P-name in the class $\mathscr{M}^{(P)} \subseteq \mathscr{M}$:

Proposition 3.34 — $\mathscr{T}^* \vdash \forall x\,(\exists u \in \mathscr{M}^{(P)})\,x = I_G^\infty(u)$

Proof. Reasoning by $\in$-induction on x, let us assume that every element of x has a recursive P-name in $\mathscr{M}^{(P)}$. We construct a recursive P-name for x in $\mathscr{M}^{(P)}$ as follows:

- For all $y \in x$, let us write α_y the smallest ordinal such that y has at least one recursive P-name in $\mathscr{M}_{\alpha_y}^{(P)}$, and let $A_y = \{v \in \mathscr{M}_{\alpha_y}^{(P)} : y = I_G^\infty(v)\}$. Writing $A = \bigcup_{y \in x} A_y$, we have $A \subseteq \mathscr{M}^{(P)} \subseteq \mathscr{M}$. Moreover, we have:

$$\forall y\,[y \in x \Leftrightarrow (\exists v \in A)\,(y = I_G^\infty(v))].$$

- From the Naming axiom, there is $N \in \mathscr{M}$ such that $A = I_G(N)$. Let us now consider the set $u \in \mathscr{M}$ that is defined by

$$u \;=\; N \cap (\mathscr{M}^{(P)} \times P) \;=\; \{z \in N \;:\; (\exists v \in \mathscr{M}^{(P)})\,(\exists p \in P)\, z = (v, p)\} \qquad (\in \mathscr{M})$$

so that by Lemma 3.32 we have $u \in \mathscr{M}^{(P)}$.

- Since $u \subseteq N$, we have $I_G(u) \subseteq I_G(N) = A$. To show the converse inclusion, take $v \in A$. Since N is a P-name for A, there is $p \in G$ such that $(v, p) \in N$. And since $v \in \mathscr{M}^{(P)}$, we have $(v, p) \in u$, so that $v \in I_G(u)$. Therefore $A = I_G(u)$.

- For all y, we finally check that

$$\begin{array}{rcll} y \in x & \Leftrightarrow & (\exists v \in A)\,(y = I_G^\infty(v)) & \text{(By construction of } A\text{)} \\ & \Leftrightarrow & \exists v\,(\exists p \in G)\,((v, p) \in u \wedge y = I_G^\infty(v)) & \text{(Since } A = I_G(u)\text{)} \\ & \Leftrightarrow & y \in I_G^\infty(x) & \text{(Def. of } I_G^\infty(x)\text{)} \end{array}$$

from which we deduce that $x = I_G^\infty(u)$. □

3.3.4 Preservation of the Axiom of Choice

An important consequence of the existence of recursive P-names for all sets in the expanded universe is that for every set X, we can find a set Y *in the sub-universe* $\mathscr{M}$ that is at least as large as X, in the sense that there is a surjection from Y onto X:

Lemma 3.35 (Surjection Lemma) — $\mathscr{T}^* \vdash \forall X\,(\exists Y \in \mathscr{M})\,\exists f\,(f : Y \twoheadrightarrow X)$.

Proof. This is obvious when $X = \varnothing$: take $Y = f = \varnothing$. We now consider the case where $X \neq \varnothing$, and pick an element $x_0 \in X$. From Prop. 3.34, there is a recursive P-name $Y \in \mathscr{M}^{(P)} \subseteq \mathscr{M}$ such that $I_G^\infty(Y) = X$. We define the function $f : Y \to X$ by

$$f(v, p) \;=\; \begin{cases} I_G^\infty(v) & \text{if } p \in G \\ x_0 & \text{otherwise} \end{cases}$$

for all $(v, p) \in Y$. By construction, f is surjective. □

The Surjection Lemma (Lemma 3.35) implies that if the Axiom of Choice (AC) is true in the sub-universe $\mathscr{M}$, then it is true in the expanded universe:

Proposition 3.36 (Axiom of Choice) — $\mathscr{T}^* \vdash AC^{\mathscr{M}} \Rightarrow AC$.

Proof. Assume that AC holds in the sub-universe $\mathscr{M}$, and take a set X. From the Surjection Lemma, there is $Y \in \mathscr{M}$ with a surjection $f : Y \twoheadrightarrow X$. By $\text{AC}^{\mathscr{M}}$, there are $\alpha, g \in \mathscr{M}$ such that $(On(\alpha) \wedge g : \alpha \xrightarrow{\sim} Y)^{\mathscr{M}}$. By absoluteness, this means that α is an ordinal and that $g : \alpha \xrightarrow{\sim} Y$ is a bijection from α onto Y. We now define $h : X \to \alpha$ by $h(x) = \min\{\beta \in \alpha : f(g(\beta)) = x\}$ for all $x \in X$. By construction the function $h : X \hookrightarrow \alpha$ is injective, so that X can be well-ordered. □

From this, it is clear that if $\mathscr{T} \vdash \text{AC}$, then $\mathscr{T}^* \vdash \text{AC}$.

Exercises

Exercise 3.1 (Naming) — In this exercise, we work in the generic extension $\mathscr{T}^*$ without the Axiom of Naming (Section 3.2.7). We consider two sets $A, B \in \mathfrak{P}(\mathscr{M})$ with respective names $N_A, N_B \in \mathscr{M}$, so that $A = I_G(N_A)$ and $B = I_G(N_B)$.

(1) Prove that $N_A \cup N_B$ is a name for $A \cup B$, that is: $I_G(N_A \cup N_B) = A \cup B$.

(2) Explain why, in general, the intersection $N_A \cap N_B$ is not a name for $A \cap B$.

We now consider the two sets $N'_A, N'_B \in \mathscr{M}$ defined by:

$$\begin{array}{rcl} N'_A &=& \{(x,p) \in (\bigcup\bigcup N_A) \times P \ :\ (\exists q \geq p)\,(x,q) \in N_A\} \\ N'_B &=& \{(x,p) \in (\bigcup\bigcup N_B) \times P \ :\ (\exists q \geq p)\,(x,q) \in N_B\} \end{array}$$

(3) Prove that $I_G(N'_A) = I_G(N_A) = A$ and $I_G(N'_B) = I_G(N_B) = B$.

(4) Deduce that the set $N'_A \cap N'_B \in \mathscr{M}$ is a name for $A \cap B$: $I_G(N'_A \cap N'_B) = A \cap B$.

Exercise 3.2 (Orthogonal of a set of conditions) — In this exercise, we work in the generic extension $\mathscr{T}^*$. Given a set of conditions $X \subseteq P$, we call the *orthogonal of* X and write $X^\perp$ the set of conditions defined by $X^\perp = \{q \in P : (\forall p \in X)\, p \perp q\}$. The bi- and tri-orthogonal of X are respectively defined by $X^{\perp\perp} = (X^\perp)^\perp$ and $X^{\perp\perp\perp} = (X^{\perp\perp})^\perp$.

(1) Check that for all $X, Y \subseteq P$:

(1.1) $X \subseteq Y$ implies $Y^\perp \subseteq X^\perp$;

(1.2) $X \subseteq X^{\perp\perp}$;

(1.3) $X^{\perp\perp\perp} = X^\perp$;

(1.4) $X^\perp$ is downwards closed;

(1.5) $X \cap X^\perp = \varnothing$

(1.6) $X \cup X^\perp$ is predense.

(2) Deduce from (v) and (vi) that if $X \in \mathfrak{P}^{\mathscr{M}}(P)$, then X meets G iff $X^{\perp\perp}$ meets G:

$$\mathscr{T}^* \ \vdash\ (\forall X \in \mathfrak{P}^{\mathscr{M}}(P))\,(X \cap G \neq \varnothing \Leftrightarrow X^{\perp\perp} \cap G \neq \varnothing)\,.$$

Exercise 3.3 (The Boolean algebra $\mathscr{B}$) — In the generic extension $\mathscr{T}^*$, we let

$$\mathscr{B} \ =\ \{X \in \mathfrak{P}^{\mathscr{M}}(P) : X = X^{\perp\perp}\}\,.$$

(1) Check that $\mathscr{B} \in \mathscr{M}$, and $\mathscr{B} = \{X \in \mathfrak{P}(P) : X = X^{\perp\perp}\}^{\mathscr{M}}$.

(2) Prove that the poset $(\mathscr{B}, \subseteq)$ is a lattice, where $\bot_{\mathscr{B}} = \varnothing$ and $\top_{\mathscr{B}} = P$, and where binary meets and joins are given by $X \wedge_{\mathscr{B}} Y = X \cap Y$ and $X \vee_{\mathscr{B}} Y = (X \cup Y)^{\perp\perp}$ for all $X, Y \in \mathscr{B}$.

(3) Prove that for all downwards closed subsets $X, Y \subseteq P$:

(3.1) $X^\perp = \{p \in P : (\forall q \leq p)\, q \notin X\}$

(3.2) $X^{\perp\perp} = \{p \in P : (\forall q \leq p)(\exists r \leq q)\, r \in X\}$

(3.3) $(X \cap Y)^{\perp\perp} = X^{\perp\perp} \cap Y^{\perp\perp}$

(4) Deduce that the lattice $(\mathcal{B}, \subseteq)$ is distributive, in the sense that

$$\begin{array}{rcl} (X \vee_{\mathcal{B}} Y) \wedge_{\mathcal{B}} Z &=& (X \wedge_{\mathcal{B}} Z) \vee_{\mathcal{B}} (Y \wedge_{\mathcal{B}} Z) \\ (X \wedge_{\mathcal{B}} Y) \vee_{\mathcal{B}} Z &=& (X \vee_{\mathcal{B}} Z) \wedge_{\mathcal{B}} (Y \vee_{\mathcal{B}} Z) \end{array} \qquad (\text{for all } X, Y, Z \in \mathcal{B})$$

(5) Conclude that the poset $(\mathcal{B}, \subseteq)$ is a Boolean algebra.

Exercise 3.4 ($\mathcal{B}$-names) — In this exercise, we work in the generic extension $\mathscr{T}^*$, and write $\mathcal{B}$ the Boolean algebra defined in Exercise 3.3. We call a *$\mathcal{B}$-name* any set $u \in \mathscr{M}$ such that u is a function whose range is included in $\mathcal{B}$:

$$u \text{ is a } \mathcal{B}\text{-name} \;\equiv\; u \in \mathscr{M} \;\wedge\; u \text{ function} \;\wedge\; \mathrm{ran}(u) \subseteq \mathcal{B}$$

Given a $\mathcal{B}$-name u, the notation $u(x)$ (function application) is extended to all $x \in \mathscr{M}$, letting $u(x) = \varnothing = 0_{\mathcal{B}}$ when $x \notin \mathrm{dom}(u)$. Similarly to the decoding function I_G from $\mathscr{M}$ to $\mathfrak{P}(\mathscr{M})$, we define a decoding function J_G from the class of $\mathcal{B}$-names to $\mathfrak{P}(\mathscr{M})$, letting

$$J_G(u) \;=\; \{x \in \mathrm{dom}(u) : u(x) \cap G \neq \varnothing\}$$

for each $\mathcal{B}$-name u. When $J_G(u) = A$ $(\in \mathfrak{P}(\mathscr{M}))$, we say that *$u$ is a $\mathcal{B}$-name for A*.

(1) Without using the Axiom of Naming, prove that a set $A \in \mathfrak{P}(\mathscr{M})$ has a $\mathcal{B}$-name if and only if it has a P-name. (*Hint:* use the result of Exercise 3.2 (2).)

(2) Deduce that the Axiom of Naming is equivalent to the formula

$$(\forall A \in \mathfrak{P}(\mathscr{M}))(\exists u \in \mathscr{M})\,(u \text{ is a } \mathcal{B}\text{-name} \;\wedge\; J_G(u) = A)\,.$$

(3) Without using the Axiom of Naming, prove that if two sets $A, B \in \mathfrak{P}(\mathscr{M})$ have $\mathcal{B}$-names u_A, u_B, respectively, then the sets $A \cap B$, $A \cup B$ and $A \setminus B$ have $\mathcal{B}$-names too.

(4) Using the Axiom of Naming, prove that for each set $X \in \mathscr{M}$, the correspondence $u \mapsto J_G(u)$ defines a surjection $J_{G,X} : (\mathcal{B}^X)^{\mathscr{M}} \twoheadrightarrow \mathfrak{P}(X)$.

(5) Assuming that the Axiom of Choice holds in both theories $\mathscr{T}$ and $\mathscr{T}^*$, deduce that for each set $X \in \mathscr{M}$, the cardinal of the powerset $\mathfrak{P}(X)$ (in the expanded universe) is bounded by the cardinal of the function set $\mathcal{B}^X$ in the sub-universe $\mathscr{M}$:

$$\mathscr{T}^* \;\vdash\; (\forall X \in \mathscr{M})\, |\mathfrak{P}(X)| \leq (|\mathcal{B}^X|)^{\mathscr{M}}\,.$$

Exercise 3.5 (Maximal antichains) — In this exercise, we work indifferently in the ground theory $\mathscr{T}$ or the generic extension $\mathscr{T}^*$. We call an *antichain of P* any subset $A \subseteq P$ whose elements are pairwise incompatible. In symbols:

$$A \text{ antichain of } P \;\equiv\; A \subseteq P \;\wedge\; (\forall p, q \in A)\,(p \neq q \Rightarrow p \perp q)\,.$$

Given a subset $X \subseteq P$, we say that an antichain A of P is *maximal in X* when $A \subseteq X$ and when there is no antichain A' of P such that $A \subset A' \subseteq X$.

(1) Assuming that AC holds in the considered theory ($\mathscr{T}$ or $\mathscr{T}^*$), prove that for every subset $X \subseteq P$, there exists an antichain of P that is maximal in X.

(2) Prove that for every subset $X \subseteq P$ and for every antichain A of P that is maximal in X, we have: $X^{\perp\perp} = A^{\perp\perp}$ (cf Exercise 3.2). *Hint:* It suffices to prove that $X^{\perp} = A^{\perp}$.

From now on, we assume that the Axiom of Choice holds in $\mathscr{T}$ and work in the corresponding generic extension $\mathscr{T}^*$ (that fulfils AC too). We write $\mathfrak{A}_P$ the set of all antichains of P in $\mathscr{M}$:

$$\mathfrak{A}_P \;=\; \{A \subseteq P : A \text{ antichain of } P\}^{\mathscr{M}} \;=\; \{A \in \mathfrak{P}^{\mathscr{M}}(P) : A \text{ antichain of } P\} \qquad (\in \mathscr{M})$$

(3) Prove that there is *in the sub-universe* $\mathscr{M}$ a surjection $f : (\mathfrak{A}_P \twoheadrightarrow \mathcal{B})^{\mathscr{M}}$.

(4) Deduce that $\mathscr{T}^* \vdash (|\mathfrak{A}_P| \leq |\mathcal{B}|)^{\mathscr{M}}$ and $\mathscr{T}^* \vdash |\mathfrak{A}_P| \leq |\mathcal{B}|$.

(5) Using the results of Exercise 3.4, prove that

$$\mathscr{T}^* \;\vdash\; (\forall X \in \mathscr{M})\, |\mathfrak{P}(X)| \leq (|\mathfrak{A}_P^X|)^{\mathscr{M}}\,.$$

Exercise 3.6 ($\mathscr{T}^*$ is generated from $\mathscr{M}$ and G) — The aim of this exercise is to show that the expanded universe described by $\mathscr{T}^*$ is generated from the sub-universe $\mathscr{M}$ (that reflects the ground theory $\mathscr{T}$ in $\mathscr{T}^*$) and the $\mathscr{M}$-generic filter $G \subseteq P$. For that, we first notice that the notation $I_G^\infty(u)$ (Section 3.3.3) can be actually defined in ZF for all sets G and u, letting

$$I_G^\infty(u) \;=\; \{I_G^\infty(v) \;:\; (v, p) \in u \text{ for some } p \in G\}\,.$$

(The definition proceeds by induction on u, using G as an arbitrary parameter.)

(1) Check that the functional relation '$x = I_G^\infty(u)$' (with free variables u, G and x) can be defined in the language of ZF using a Σ_1-formula.

Working in the generic extension $\mathscr{T}^*$, we now consider a transitive model C of ZF (in $\mathscr{T}^*$) such that we have $\mathscr{T}^* \vdash \mathscr{M} \subseteq C$ ('C contains $\mathscr{M}$') and $\mathscr{T}^* \vdash G \in C$ ('C contains G').

(2) Prove that $\mathscr{T}^* \;\vdash\; (\forall u \in C)\, I_G^\infty(u) \in C$.
(*Hint:* use the fact that the functional relation $x = I_G^\infty(u)$ is Σ_1).

(3) Using Prop. 3.34, conclude that: $\mathscr{T}^* \vdash \forall x\, (x \in C)$ (i.e. $\mathscr{T}^* \vdash V = C$)