Classical realizability and forcing
Part 3: A cardinals' heresy in classical realizability

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Logic Colloquium (LC'14)
Vienna Summer of Logic – July 19th, 2014 – Vienna

Plan

The language of classical realizers

Terms, stacks and processes

Terms	t, u	$::=$	$x \mid \lambda x\,.\,t \mid tu \mid \kappa \mid \mathrm{k}_\pi$	$(\kappa \in \mathcal{K})$
Stacks	π, π'	$::=$	$\alpha \mid t \cdot \pi$	$(\alpha \in \Pi_0,\ t \text{ closed})$
Processes	p, q	$::=$	$t \star \pi$	$(t \text{ closed})$

Krivine Abstract Machine (KAM)

Push	$tu \star \pi$	$\succ$	$t \star u \cdot \pi$
Grab	$\lambda x\,.\,t \star u \cdot \pi$	$\succ$	$t\{x := u\} \star \pi$
Save	$\text{cc} \star u \cdot \pi$	$\succ$	$u \star \mathrm{k}_\pi \cdot \pi$
Restore	$\mathrm{k}_\pi \star u \cdot \pi'$	$\succ$	$u \star \pi$
	$\ldots$		$\ldots$

(+ reflexivity & transitivity)

Interpreting closed formulas with parameters

Let A be a closed formula (with parameters)

- Falsity value $\|A\|$ defined by induction on A:

$$
\begin{aligned}
\|\dot{F}(e_1,\ldots,e_n)\| &= F([\![e_1]\!],\ldots,[\![e_n]\!]) \\
\|A \Rightarrow B\| &= |A|\cdot\|B\| \;=\; \{t\cdot\pi \;:\; t\in|A|,\ \ \pi\in\|B\|\} \\
\|\forall x\ A\| &= \bigcup_{n\in\mathbb{N}} \|A\{x:=n\}\| \\
\|\forall X\ A\| &= \bigcup_{F:\mathbb{N}^n\to\mathfrak{P}(\Pi)} \|A\{X:=\dot{F}\}\|
\end{aligned}
$$

- Truth value $|A|$ defined by orthogonality:

$$
|A| \;=\; \|A\|^{\perp\!\!\!\perp} \;=\; \{t\in\Lambda \;:\; \forall\pi\in\|A\| \quad t\star\pi\in\perp\!\!\!\perp\}
$$

The realizability relation

Falsity value $\|A\|$ and truth value $|A|$ depend on the pole $\perp\!\!\!\perp$

$\rightsquigarrow$ write them (sometimes) $\|A\|_{\perp\!\!\!\perp}$ and $|A|_{\perp\!\!\!\perp}$ to recall the dependency

Realizability relations

$$t \Vdash A \quad \equiv \quad t \in |A|_{\perp\!\!\!\perp} \qquad \text{(Realizability w.r.t. } \perp\!\!\!\perp\text{)}$$

$$t \Vvdash A \quad \equiv \quad \forall \perp\!\!\!\perp \quad t \in |A|_{\perp\!\!\!\perp} \qquad \text{(Universal realizability)}$$

Theorem (Adequacy)

If A is a theorem of classical 2nd-order logic, then:

$$\theta \Vvdash A \qquad \text{for some } \theta \in \mathrm{PL}$$

More connectives (1/2)

- Add binary intersection types

Formulas $A, B \quad ::= \quad \cdots \quad | \quad A \cap B \quad | \quad \top$

letting	$\|A \cap B\| = \|A\| \cup \|B\|$	and	$\|\top\| = \varnothing$
so that	$|A \cap B| = |A| \cap |B|$	and	$|\top| = \Lambda$

- Intersection type is a strong form of conjunction:

$$\lambda xz\,.\,z\,x\,x \ \Vdash\ A \cap B \Rightarrow A \wedge B$$

But converse implication not realized in general

More connectives (2/2)

- Add equational implication:

Formulas: $A, B \quad ::= \quad \cdots \quad | \quad e_1 = e_2 \mapsto A$

Letting $$\|e_1 = e_2 \mapsto A\| \;=\; \begin{cases} \|A\| & \text{if } [\![e_1]\!] = [\![e_2]\!] \\ \varnothing & \text{if } [\![e_1]\!] \neq [\![e_2]\!] \end{cases}$$

Proposition (equivalence of $e_1 = e_2 \mapsto A$ and $e_1 = e_2 \Rightarrow A$)

$$\begin{array}{rcl} \lambda xy\,.\,yx & \Vdash & (e_1 = e_2 \mapsto A) \Rightarrow (e_1 = e_2 \Rightarrow A) \\ \lambda x\,.\,x\,\mathbf{I} & \Vdash & (e_1 = e_2 \Rightarrow A) \Rightarrow (e_1 = e_2 \mapsto A) \end{array}$$

- **Example:** $e_1 \neq e_2 \;\equiv\; (e_1 = e_2 \mapsto \bot)$ (disequality)
  - Denotation of $e_1 \neq e_2$ much simpler than $\neg(e_1 = e_2)$
  - But $e_1 \neq e_2$ equivalent to $\neg(e_1 = e_2)$ (in the sense of realizability)

Plan

Plan

The theory induced by the realizability model $\mathscr{M}_{\perp\!\!\!\perp}$

- Recall that:
  - When $\perp\!\!\!\perp = \varnothing$: $\mathscr{M}_{\perp\!\!\!\perp}$ collapses to $\mathscr{M}$ (Tarski model)
  - When $\perp\!\!\!\perp \neq \varnothing$: every truth value $|A|$ is inhabited

$\rightsquigarrow$ Restrict to proof-like terms (treat k_π as paraproof)

Definition (Theory induced by $\mathscr{M}_{\perp\!\!\!\perp}$)

1. A is realized in $\mathscr{M}_{\perp\!\!\!\perp}$ $\equiv$ $|A| \cap \mathrm{PL} \neq \varnothing$ (notation: $\mathscr{M}_{\perp\!\!\!\perp} \Vdash A$)
2. Formulas A that are realized in $\mathscr{M}_{\perp\!\!\!\perp}$ form the theory induced by $\mathscr{M}_{\perp\!\!\!\perp}$

Properties of the induced theory

1. The theory induced by $\mathscr{M}_{\perp\!\!\!\perp}$ is closed under logical consequence in the sense of classical 2nd-order logic
2. Peano axioms 3 and 4 are realized in $\mathscr{M}_{\perp\!\!\!\perp}$ (not induction)
3. More generally: Horn formulas that are true in $\mathscr{M}$ are realized in $\mathscr{M}_{\perp\!\!\!\perp}$
4. If $\mathscr{M} \models \mathrm{AC}$ and quote $\in \mathcal{K}$, then $\mathscr{M}_{\perp\!\!\!\perp} \Vdash \mathrm{DC}$

The problem of consistency

- Is the theory (induced by) $\mathscr{M}_{⫫}$ consistent?

$$\begin{aligned} \mathscr{M}_{⫫} \not\Vdash \bot \quad & \Leftrightarrow \quad |\bot| \cap \mathsf{PL} = \varnothing \\ & \Leftrightarrow \quad \forall \theta \in \mathsf{PL} \;\; \theta \not\Vdash \bot \\ & \Leftrightarrow \quad \forall \theta \in \mathsf{PL} \;\; \exists \pi \in \Pi \;\; \theta \star \pi \notin ⫫ \end{aligned}$$

Definition (coherent pole)

$⫫$ coherent $\equiv$ $\forall \theta \in \mathsf{PL} \;\; \exists \pi \in \Pi \;\; \theta \star \pi \notin ⫫$

- **By definition:** $\mathscr{M}_{⫫}$ consistent (as a theory) iff $⫫$ coherent
- **Examples of coherent poles:**
  - The empty pole $⫫ = \varnothing$ (but in this case: $\mathscr{M}_{\varnothing}$ collapses to $\mathscr{M}$)
  - The pole of threads: cf later

The problem of induction

- In 2nd-order logic, the set of natural numbers is defined by

$$x \in \mathbb{N} \quad \equiv \quad \forall Z\,[Z(0) \Rightarrow \forall y\,(Z(y) \Rightarrow Z(y+1)) \Rightarrow Z(x)]$$

  Induction axiom is the formula: $\forall x\,(x \in \mathbb{N})$

- **Problem:** this axiom is in general not realized (by a proof-like term)

  Moreover, there are coherent poles $\perp\!\!\!\perp$ such that:

$$\mathscr{M}_{\perp\!\!\!\perp} \quad \Vdash \quad \neg\forall x\,(x \in \mathbb{N})$$

  so that: $$\mathscr{M}_{\perp\!\!\!\perp} \quad \Vdash \quad \exists x\,(x \notin \mathbb{N})$$

- Need to establish a strong distinction between
  - individuals (all 1st-order objects), and
  - natural numbers (individuals x such that $x \in \mathbb{N}$)

- Problem is traditionally put under the carpet, by relativizing all 1st-order quantifications to $\mathbb{N}$. But what happens if we don't?

Existence of unnamed elements

- In Tarski/Boolean-valued/forcing models, all elements are named:

  If $\mathscr{M} \models \exists x\, A(x)$, then $\mathscr{M} \models A(v)$ for some $v \in \mathscr{M}$

- Not the case anymore in classical realizability models $\mathscr{M}_{\perp\!\!\perp}$!
  In some models, one can find formulas $A(x)$ such that

  $$\mathscr{M}_{\perp\!\!\perp} \Vdash \exists x\, A(x)$$

  whereas $\mathscr{M}_{\perp\!\!\perp} \Vdash \neg A(n)$ for all $n \in \mathbb{N}$

  - Due to uniform interpretation of $\forall$
  - Typical example: $A(x) \equiv x \notin \mathbb{N}$

- Existence of unnamed elements
  - The theory induced by $\mathscr{M}_{\perp\!\!\perp}$ lacks the witness property
  - Recover some fundamental incompleteness of classical theories

Realizing true Horn formulas (again)

Definition (Horn formulas)

1. A (positive/negative) literal is a formula L of the form
$$L \;\equiv\; e_1 = e_2 \qquad \text{or} \qquad L \;\equiv\; e_1 \neq e_2$$
2. A Horn formula is a closed formula H of the form
$$H \;\equiv\; \forall\vec{x}\,[L_1 \Rightarrow \cdots \Rightarrow L_p \Rightarrow L_{p+1}] \qquad (p \geq 0)$$
where $L_1, \ldots, L_p$ are positive; L_{p+1} positive or negative

Theorem (Realizing true Horn formulas)

If $\mathscr{M} \models H$, then $\mathscr{M}_{\perp\!\!\!\perp} \Vdash H$

- **Beware!** The meaning of H is not the same in $\mathscr{M}$ and $\mathscr{M}_{\perp\!\!\!\perp}$
  - In $\mathscr{M}$, quantifications range over natural numbers
  - In $\mathscr{M}_{\perp\!\!\!\perp}$, quantifications range over all individuals
- Theorem does not extend to arbitrary clauses

Plan

The model of threads $\mathscr{M}_{\mathbf{thd}}$

- From now on, we assume that:
  - There are only two instructions œ and quote $(\mathcal{K} = \{\text{œ}, \text{quote}\})$
  - The set Π_0 of stack constants is denumerable
- Evaluation rules are:

Push	$tu \star \pi$	$\succ$	$t \star u \cdot \pi$
Grab	$\lambda x \,.\, t \star u \cdot \pi$	$\succ$	$t\{x := u\} \star \pi$
Save	$\text{œ} \star u \cdot \pi$	$\succ$	$u \star \mathrm{k}_\pi \cdot \pi$
Restore	$\mathrm{k}_\pi \star u \cdot \pi'$	$\succ$	$u \star \pi$
Quote	$\text{quote} \star t \cdot u \cdot \pi$	$\succ$	$u \star \overline{\lceil t \rceil} \cdot \pi$

Properties of evaluation

1. Evaluation is deterministic:
   If $\quad p \succ_1 p'_1 \quad$ and $\quad p \succ_1 p'_2, \quad$ then $\quad p'_1 \equiv p'_2$
2. Stack constants cannot be generated during evaluation:
   Let $\alpha \in \Pi_0$. If $p \succ p'$ and α occurs in p', then α occurs in p

The model of threads $\mathscr{M}_{\mathbf{thd}}$

- The thread of a proof-like term $\theta \in \mathrm{PL}$
  - Consider a bijection $\theta \mapsto \alpha_\theta$ from PL to Π_0
  - Let: $\mathbf{thd}(\theta) = \{p \in \Lambda \star \Pi \ :\ \theta \star \alpha_\theta \succ p\}$ (thread of θ)
  - Remark: if $\theta \not\equiv \theta'$, then $\mathbf{thd}(\theta) \cap \mathbf{thd}(\theta') = \varnothing$

- **The pole of threads:**
  - **Idea:** to build a coherent pole, exclude all $\theta \star \alpha_\theta$ (for $\theta \in \mathrm{PL}$)
  - Let $\perp\!\!\!\perp_{\mathbf{thd}} = \Big(\bigcup_{\theta \in \mathrm{PL}} \mathbf{thd}(\theta)\Big)^c$ (pole of threads)

Proposition: The pole $\perp\!\!\!\perp_{\mathbf{thd}}$ is coherent and nonempty

- **The model of threads:** $\mathscr{M}_{\mathbf{thd}} = \mathscr{M}_{\perp\!\!\!\perp_{\mathbf{thd}}}$

Proposition (Characterizing the realizers of $\bot$)

(For all $t \in \Lambda$) $t \Vdash \bot$ iff t never appears in head position in a thread

Negating the type of the parallel 'or'

- Write:

$$B_1 \equiv \bot \Rightarrow \top \Rightarrow \bot \qquad \text{(realized by } \lambda xy\,.\,x)$$
$$B_2 \equiv \top \Rightarrow \bot \Rightarrow \bot \qquad \text{(realized by } \lambda xy\,.\,y)$$

- **Intuition:** Formula $B_1 \cap B_2$ is the type of the parallel 'or'

Proposition

For all $\pi \in \Pi$ and $u, u' \in \Lambda$ distinct: $\omega\, u\, k_\pi \Vdash \bot$ or $\omega\, u'\, k_\pi \Vdash \bot$
(writing $\omega \equiv (\lambda x\,.\,xx)(\lambda x\,.\,xx)$)

Proof by contradiction, using the fact that in a sequential calculus, a process can enter an infinite loop at most once.

Corollary

$$\theta_1 \equiv \lambda x\,.\,\text{cc}\,(\lambda k\,.\,x\,(\omega\,\bar{0}\,k)\,(\omega\,\bar{1}\,k)) \Vdash \neg(B_1 \cap B_2)$$

(Internalizes the fact that in a sequential world, there is no parallel 'or')

- Shows that in $\mathscr{M}_{\mathbf{thd}}$: $A \wedge B \not\Leftrightarrow A \cap B$

Negating the type of the parallel 'or' (variant)

- Write:

$$\begin{array}{llll} \mathsf{B}_1 & \equiv & \bot \Rightarrow \top \Rightarrow \bot & \text{(realized by } \lambda xy\,.\,x\text{)} \\ \mathsf{B}_2 & \equiv & \top \Rightarrow \bot \Rightarrow \bot & \text{(realized by } \lambda xy\,.\,y\text{)} \\ \mathsf{B}_3 & \equiv & \bot \Rightarrow \bot \Rightarrow \bot & \text{(realized by both)} \end{array}$$

Proposition

For all $\pi \in \Pi$ $u \in \Lambda$ and $v, v', v'' \in \Lambda$ pairwise distinct:

$$\mathsf{k}_\pi\, u\, v \Vdash \bot \quad \text{or} \quad \mathsf{k}_\pi\, u\, v' \Vdash \bot \quad \text{or} \quad \mathsf{k}_\pi\, u\, v'' \Vdash \bot$$

Proof by contradiction, using a similar argument as before.

Corollary

$$\begin{array}{lll} \theta_2 & \equiv & \lambda xy\,.\,\text{œ}\,(\lambda k\,.\,y\,(k\,x\,\bar{0})\,(y\,(k\,x\,\bar{1})\,(k\,x\,\bar{2}))) \\ & \Vdash & \neg(\bot \Rightarrow \mathsf{B}_3 \Rightarrow \bot) \cap (\top \Rightarrow (\mathsf{B}_1 \cap \mathsf{B}_2 \Rightarrow \bot)) \end{array}$$

Plan

Ordering over individuals

- Let $x \leq y \;\equiv\; x - y = 0$ (where $x - y$ is truncated subtraction in $\mathbb{N}$)

Proposition (Ordering)

In $\mathscr{M}_{\mathbf{thd}}$: $x \leq y$ is an ordering over the set of all individuals, with smallest element 0, and no maximal element:

$$\begin{array}{lll} \mathscr{M}_{\mathbf{thd}} & \Vdash & \forall x\,(0 \leq x) \\ \mathscr{M}_{\mathbf{thd}} & \Vdash & \forall x\,(x \leq x) \\ \mathscr{M}_{\mathbf{thd}} & \Vdash & \forall x\,\forall y\,(x \leq y \Rightarrow y \leq x \Rightarrow x = y) \\ \mathscr{M}_{\mathbf{thd}} & \Vdash & \forall x\,\forall y\,\forall z\,(x \leq y \Rightarrow y \leq z \Rightarrow x \leq z) \end{array}$$

$$\begin{array}{lll} \mathscr{M}_{\mathbf{thd}} & \Vdash & \forall x\,(x \leq s(x)) \\ \mathscr{M}_{\mathbf{thd}} & \Vdash & \forall x\,(s(x) \neq x) \end{array}$$

Proof: Horn formulas, that are all true in the ground model $\mathscr{M}$

- Extends the usual ordering on $\mathbb{N}$ (in the ground model $\mathscr{M}$) to the set of all individuals (in the theory induced by $\mathscr{M}_{\mathbf{thd}}$)
- Are all properties of $\leq$ (in $\mathbb{N}$) still valid for individuals in $\mathscr{M}_{\mathbf{thd}}$?

Entering THE TWILIGHT ZONE

- Formula expressing the totality of ordering is not a Horn formula:

$$\forall x \, \forall y \, (x \leq y \vee y \leq x)$$
$$[\Leftrightarrow \quad \forall x \, \forall y \, (x \not\leq y \Rightarrow y \not\leq x \Rightarrow \bot)]$$

(writing $\quad x \not\leq y \equiv (x - y = 0 \mapsto \bot)$, $\quad$ equivalent to $\neg(x \leq y)$)

Proposition (Non-totality of ordering)

In $\mathscr{M}_{\mathbf{thd}}$: ordering $x \leq y$ is non total (over the set of individuals)

$$\theta_1 \ \Vdash \ \neg\forall x \, \forall y \, (x \not\leq y \Rightarrow y \not\leq x \Rightarrow \bot)$$

where $\quad \theta_1 \ \equiv \ \lambda x \, . \, \text{œ} \, (\lambda k \, . \, x \, (\omega \, \bar{0} \, k) \, (\omega \, \bar{1} \, k)))$

Proof: formula has the same semantics as $\neg(\mathrm{B}_1 \cap \mathrm{B}_2)$

- On the other hand, ordering is total on $\mathbb{N}$:

$$\mathscr{M}_{\mathbf{thd}} \ \Vdash \ (\forall x, y \in \mathbb{N}) \, (x \leq y \vee y \leq x)$$

Corollary: $\quad \mathscr{M}_{\mathbf{thd}} \ \Vdash \ \exists x \, (x \notin \mathbb{N})$ $\qquad$ ('there is an individual outside $\mathbb{N}$')

Lattice structure

- Consider the binary function symbols $\curlywedge$ and $\curlyvee$ interpreted in $\mathscr{M}$ by

$$n \curlywedge^{\mathscr{M}} m \;=\; \min(n,m) \qquad \text{and} \qquad n \curlyvee^{\mathscr{M}} m \;=\; \max(n,m)$$

Proposition (Lattice structure)

In $\mathscr{M}_{\mathbf{thd}}$: The set of individuals is an unbounded distributive lattice:

- Any two individuals x and y have a meet $x \curlywedge y$:
  $\forall x\,\forall y\,(x \curlywedge y \le x)$, $\forall x\,\forall y\,(x \curlywedge y \le y)$, $\forall x\,\forall y\,\forall z\,(z \le x \Rightarrow z \le y \Rightarrow z \le x \curlywedge y)$
- Any two individuals x and y have a join $x \curlyvee y$:
  $\forall x\,\forall y\,(x \le x \curlyvee y)$, $\forall x\,\forall y\,(y \le x \curlyvee y)$, $\forall x\,\forall y\,\forall z\,(x \le z \Rightarrow y \le z \Rightarrow x \curlyvee y \le z)$
- The two operations $x \curlywedge y$ and $x \curlyvee y$ distribute w.r.t. each other

Proof: Horn formulas, that are all true in the ground model $\mathscr{M}$

- **Beware:** In general, $x \curlywedge y$ does not represent the min:

$$\mathscr{M}_{\perp\!\!\!\perp} \;\;\not\Vdash\;\; \forall x\,\forall y\,[(x \curlywedge y) = x \vee (x \curlywedge y) = y]$$

(Reason: not a Horn formula)

More on the non totality of ordering

- Relation "z_1 and z_2 are between x and y" expressed by

$$\mathbf{b}(x, y, z_1, z_2) \ \equiv \ (x - z_1) + (z_1 - y) + (x - z_2) + (z_2 - y) = 0$$

Proposition (Ordering is densely non total)

In $\mathscr{M}_{\mathbf{thd}}$: Between distinct individuals $x \neq y$ such that $x \leq y$, one can find two individuals z_1, z_2 that cannot be compared:

$$\theta_2 \ \Vdash \ \forall x \, \forall y \, [x \neq y \Rightarrow \forall z_1 \, \forall z_2 \, (z_1 \not\leq z_2 \Rightarrow z_2 \not\leq z_1 \Rightarrow \bar{\mathbf{b}}(x, y, z_1, z_2)) \Rightarrow x \not\leq y] \, ,$$

where $\theta_2 \ \equiv \ \lambda xy \, . \, \text{œ} \, (\lambda k \, . \, y \, (k \, x \, \bar{0}) \, (y \, (k \, x \, \bar{1}) \, (k \, x \, \bar{2})))$

Proof: Formula has the same semantics as $(\bot \Rightarrow B_3 \Rightarrow \bot) \cap (\top \Rightarrow (B_1 \cap B_2) \Rightarrow \bot)$

Proposition

In $\mathscr{M}_{\mathbf{thd}}$: For every individual $x \neq 0$, there is an individual y that cannot be compared with x:

$$\theta_2 \ \Vdash \ \forall x \, (x \neq 0 \Rightarrow \neg \forall y \, (x \not\leq y \Rightarrow y \not\leq x \Rightarrow \bot))$$

Proof: Formula is a super-type of $(\bot \Rightarrow B_3 \Rightarrow \bot) \cap (\top \Rightarrow (B_1 \cap B_2) \Rightarrow \bot)$

Non-Horn clauses

Proposition (Non-Horn clauses) [Geoffroy & M. 2014]

Consider a clause

$$C(\vec{x}) \;\equiv\; \bigvee_{i=1}^{p} P_i(\vec{x}) \;\vee\; \bigvee_{i=1}^{n} N_i(\vec{x})$$

such that:

1. $P_1, \ldots, P_p$ positive $(p \geq 2)$, $N_1, \ldots, N_n$ negative literals
2. $C(\vec{x})$ is universally true in $\mathscr{M}$: $\quad \mathscr{M} \models \forall \vec{x}\, C(\vec{x})$
3. For all $i \in \{1..p\}$: $\quad \mathscr{M} \not\models \forall \vec{x} \Big(C(\vec{x}) \Leftrightarrow P_i(\vec{x}) \vee \bigvee_{i=1}^{n} N_i(\vec{x}) \Big)$

Then: $\quad \mathscr{M}_{\mathbf{thd}} \Vdash \exists \vec{x}\, \neg C(\vec{x})$

Initial elements

- Initial element $\equiv$ individual x such that $x \not\geq 1$

Proposition

In $\mathscr{M}_{\mathbf{thd}}$:

$$x \not\geq 1 \;\Leftrightarrow\; x \neq (x-1)+1 \quad (x \text{ not the succ. of its pred.})$$
$$\Leftrightarrow\; \forall y\,(s(y) \neq x) \quad (x \text{ not a successor})$$

Proof: The three formulas have the same denotation

Proposition

In $\mathscr{M}_{\mathbf{thd}}$: Every individual is decomposed in a unique way as the sum of an initial element and a natural number:

$$\forall x\,(\exists! y \not\geq 1)(\exists! n \in \mathbb{N})\,(x = y + n)$$

Proof: Existence: By well-founded induction on the relation $x = s(y)$ (well-founded induction principle realized by $\mathbf{Y}$). Uniqueness: follows from totality of ordering on $\mathbb{N}$

- Decomposition is not algebraic! Initial elements are not closed under $+$.

Plan

The sets ∇a

- Write $x \ll y \equiv x + 1 \leq y$ (x is way below y)
  $\nabla a \equiv \{x : x \ll a\}$ (written $\daleth a$ by Krivine)
- **Intuition:** In $\mathscr{M}$, we have $\nabla n = \{0..n-1\}$ (for all $n \in \mathbb{N}$)
  but in the theory $\mathscr{M}_{\mathbf{thd}}$, these sets are much larger!

Proposition

In $\mathscr{M}_{\mathbf{thd}}$: For every individual $a > 1$, the set $\nabla a = \{x : x \ll a\}$ is Dedekind-infinite

Proof: Follows from density of $\leq$ using DC

Proposition ($\nabla(ab) \approx \nabla a \times \nabla b$)

In $\mathscr{M}_{\mathbf{thd}}$: for all individuals a, b: $\nabla(ab)$ is equipotent with $\nabla a \times \nabla b$

Proof: Consider the (prim. rec.) bijection from $\{0..ab-1\}$ to $\{0..a-1\} \times \{0..b-1\}$ in the ground model $\mathscr{M}$. This extends to a bijection from $\nabla(ab)$ to $\nabla a \times \nabla b$ in $\mathscr{M}_{\mathbf{thd}}$, since the property of being a bijection is expressed using Horn formulas

Cardinality of the sets ∇a

- The sets ∇a are infinite (for $a > 1$)...
  ... but they keep some properties of finite sets

  (Recall that in the ground model $\mathscr{M}$: $\nabla n = \{0, ..., n-1\}$)

Theorem

In $\mathscr{M}_{\mathbf{thd}}$: For all individuals a, b such that $a \ll b$, there is no surjection from ∇a onto ∇b:

$$\theta \Vdash \forall a \forall b \forall Z \, [a \ll b \mapsto \forall x \, \forall y \, \forall y' \, (Z(x,y) \Rightarrow Z(x,y') \Rightarrow y \neq y' \Rightarrow \bot) \Rightarrow \forall y \, (y \ll b \mapsto \neg \forall x \, (x \ll a \mapsto \neg Z(x,y))) \Rightarrow \bot]$$

where $\theta \equiv \lambda x_1 x_2 \, . \, \text{cc} \, (\lambda k \, . \, x_2 \, (\lambda z \, . \, x_1 \, z \, z \, (\omega \, z \, k)))$

Proof: By contradiction, the problem reduces to the pigeonhole principle from $\{0, \ldots, b-1\}$ to $\{0, \ldots, a-1\}$ for some $a, b \in \mathbb{N}$ such that $a < b$.

Entering deeper THE TWILIGHT ZONE

- **In particular:** Since $2 \ll 4$ (in $\mathscr{M}_{\mathbf{thd}}$), there is no surjection from the (infinite) set $\nabla 2$ onto $\nabla 4 \ldots \approx \nabla 2 \times \nabla 2$

Proposition

In $\mathscr{M}_{\mathbf{thd}}$: There is an infinite set of individuals (i.e. $\nabla 2$) which is not in bijection with its Cartesian square

Corollary

In $\mathscr{M}_{\mathbf{thd}}$:

1. The set $\nabla 2$ is not well-orderable (as well as the set of all individuals)
2. The set $\nabla 2$ is not countable (ditto)

- Actually: $\nabla 2$ can be embedded into the real line (cf later)

The set $\nabla 2 = \{x : x \leq 1\}$ as a Boolean algebra

Proposition (Boolean algebra $\nabla 2$)

In $\mathscr{M}_{\mathbf{thd}}$: The operation $x \mapsto 1 - x$ is a negation in the lattice $\nabla 2$:

$$\begin{array}{lcl}
\mathscr{M}_{\mathbf{thd}} & \Vdash & (\forall x \in \nabla 2)((1-x) \in \nabla 2) \\
\mathscr{M}_{\mathbf{thd}} & \Vdash & (\forall x \in \nabla 2)((1-(1-x)) = x) \\
\mathscr{M}_{\mathbf{thd}} & \Vdash & (\forall x, y \in \nabla 2)(x \leq y \Rightarrow 1 - y \leq 1 - x) \\
\mathscr{M}_{\mathbf{thd}} & \Vdash & (\forall x, y \in \nabla 2)(1 - (x \curlywedge y) = (1-x) \curlyvee (1-y)) \\
\mathscr{M}_{\mathbf{thd}} & \Vdash & (\forall x, y \in \nabla 2)(1 - (x \curlyvee y) = (1-x) \curlywedge (1-y)) \\
\mathscr{M}_{\mathbf{thd}} & \Vdash & (\forall x \in \nabla 2)(x \curlywedge (1-x) = 0) \\
\mathscr{M}_{\mathbf{thd}} & \Vdash & (\forall x \in \nabla 2)(x \curlyvee (1-x) = 1)
\end{array}$$

Hence $\nabla 2$ is a Boolean algebra

Note: In $\nabla 2$, the 3 operations $x \curlywedge y$ (meet), $x \times y$ (ordinary multiplication) and $x \times_2 y$ (multiplication modulo 2) coincide

- In particular
  - The Boolean algebra $\nabla 2$ is not countable (since $\not\approx \nabla 2 \times \nabla 2$)
  - The Boolean algebra $\nabla 2$ is atomless (due to density)

Embedding $\nabla 2$ into the real line

- Add a unary function symbol δ interpreted in $\mathscr{M}$ by

$$\delta(n) \;=\; \begin{cases} 0 & \text{if } \lfloor n \rfloor \Vdash \bot \\ 1 & \text{if } \lfloor n \rfloor \not\Vdash \bot \end{cases}$$

- The image of $\mathbb{N}$ by δ is a countable dense subset of $\nabla 2$:

Proposition (Density of $\delta(\mathbb{N})$ in $\nabla 2$)

1. $\mathscr{M}_{\mathbf{thd}} \;\Vdash\; (\forall n \in \mathbb{N})\,(\delta(n) \in \nabla 2)$
2. $\mathscr{M}_{\mathbf{thd}} \;\Vdash\; (\forall x \in \nabla 2)\,(x \neq 0 \;\Rightarrow\; (\exists n \in \mathbb{N})\,(\delta(n) \neq 0 \wedge \delta(n) \leq x))$

(1): Obvious (Horn). (2): Relies on quote

Corollary (Embedding $\nabla 2$ into $\mathbb{R}$)

Write: $\Phi(x) = \{n \in \mathbb{N} \;:\; \delta(n) \leq x\}$

1. $\mathscr{M}_{\mathbf{thd}} \;\Vdash\; (\forall x \in \nabla 2)\,(\Phi(x) \subseteq \mathbb{N})$ (i.e. $\Phi(x) \in \mathbb{R}$)
2. $\mathscr{M}_{\mathbf{thd}} \;\Vdash\; (\forall x, y \in \nabla 2)\,(\Phi(x) = \Phi(y) \Rightarrow x = y)$ (i.e. Φ is into)

$\nabla 2$ as a Boolean algebra of cardinals

- Pushing further these techniques, Krivine proved the following:

Theorem

In $\mathscr{M}_{\mathbf{thd}}$: for all $a, b \in \nabla 2$, the following are equivalent:

1. $a \leq b$
2. There is an injection $F : {\downarrow}\{a\} \hookrightarrow {\downarrow}\{b\}$
3. There is a surjection $F' : {\downarrow}\{b\} \twoheadrightarrow {\downarrow}\{a\}$

writing ${\downarrow}\{a\} = \{x : x \leq a\}$ (prime ideal of a)

- **Intuition:** $\nabla 2$ is a (nontrivial) Boolean algebra of cardinals!
- Moreover, all these phenomena can be exported to the real line $\mathbb{R}$ via the embedding $\Phi : \nabla 2 \hookrightarrow \mathbb{R}$

Plan

Conclusion

- Using the method of threads, we constructed a particular realizability model of 2nd-order logic in which:
  - There are (infinitely) many more individuals than natural numbers
  - There is a sequence $(\nabla n)_{n \in \mathbb{N}}$ of sets of individuals such that
    1. $\nabla(np) \approx \nabla n \times \nabla p$ (for all $n, p \in \mathbb{N}$)
    2. There is no surjection from ∇n onto $\nabla(n+1)$ (for all $n \in \mathbb{N}$)
    3. $\nabla 0 = \varnothing$, $\nabla 1 = \{0\}$ and ∇n is infinite (for all $n \geq 2$)
  - The set $\nabla 2$ is a non-countable atomless Boolean algebra of cardinals: $a \leq b \ (\in \nabla 2) \quad \Leftrightarrow \quad \downarrow\{a\} \hookrightarrow \downarrow\{b\}$
  - There are embeddings $\Phi_n : \nabla n \hookrightarrow \mathbb{R}$ (for all $n \in \mathbb{N}$)
- The same results can be formulated in ZF [Krivine 12]
  - All phenomena that deal with individuals are intensional (they are observed with intensional membership only)
  - But via the embeddings $\Phi_n : \nabla n \hookrightarrow \mathbb{R}$, they become extensional (they can be observed in the usual $=/\in$ language of ZF)

Classical realizability models of ZF

- **What we currently know:**
  - Classical realizability generalizes the method of Cohen forcing
  - Generalization is strict, since classical realizability model construction can be used to break AC (impossible with forcing alone)
  - Equivalent to forcing when: $\mathscr{M}_{\perp\!\!\perp} \Vdash \nabla 2 = \{0; 1\}$
  - The ground model $\mathscr{M}$ does not appear trivially as a submodel of $\mathscr{M}_{\perp\!\!\perp}$ (unlike forcing), but it induces a Boolean-valued model $\nabla\mathscr{M}$ over the Boolean algebra $\nabla 2$ (within the theory $\mathscr{M}_{\perp\!\!\perp}$), which is elementarily equivalent to the Tarski model $\mathscr{M}$
  - The Boolean algebra $\nabla 2$ has a canonical ultrafilter [Krivine 14]
  - Therefore (by quotient + Mostowski collapse), $\mathscr{M}$ and $\mathscr{M}_{\perp\!\!\perp}$ have the same constructible sets: Schoenfield's absoluteness theorem applies
- **What we don't know:** How to use it! (Generic filter?)