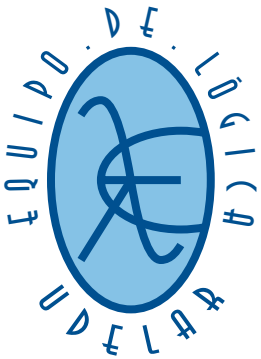
Kleene realizability and negative translations

Alexandre Miquel



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Plan

Plan

The language of realizers (recall)

Terms of PCF (= λ-calculus + primitive pairs & integers)

Terms $t, u ::= x \mid \lambda x\,.\,t \mid tu \mid \mathtt{pair} \mid \mathtt{fst} \mid \mathtt{snd} \mid \mathtt{0} \mid \mathtt{S} \mid \mathtt{rec}$

Syntactic worship: Free & bound variables. Renaming. Work up to α-conversion. Set of free variables: $FV(t)$. Capture-avoiding substitution: $t\{x := u\}$

- **Notations:** $\langle t_1, t_2\rangle := \mathtt{pair}\, t_1\, t_2, \qquad \bar{n} := \mathtt{S}^n\,\mathtt{0} \quad (n \in \mathbb{N})$

Reduction rules

$$(\lambda x\,.\,t)\,u \succ t\{x := u\}$$

$$\mathtt{fst}\ \langle t_1, t_2\rangle \succ t_1 \qquad \mathtt{snd}\ \langle t_1, t_2\rangle \succ t_2$$

$$\mathtt{rec}\ t_0\ t_1\ \mathtt{0} \succ t_0 \qquad \mathtt{rec}\ t_0\ t_1\ (\mathtt{S}\,u) \succ t_1\ u\ (\mathtt{rec}\ t_0\ t_1\ u)$$

- Grand reduction written $t \succ^* u$ (reflexive, transitive, context-closed)

Definition of the relation $t \Vdash A$ (recall)

- **Recall:** For each closed FO-term e, we write $e^{\mathbb{N}}$ its denotation in $\mathbb{N}$

Definition of the realizability relation $t \Vdash A$ (t, A closed)

$$
\begin{array}{lcl}
t \Vdash e_1 = e_2 & \equiv & e_1^{\mathbb{N}} = e_2^{\mathbb{N}} \ \wedge \ t \succ^* 0 \\
t \Vdash \bot & \equiv & \bot \\
t \Vdash \top & \equiv & t \succ^* 0 \\
t \Vdash A \Rightarrow B & \equiv & \forall u \, (u \Vdash A \ \Rightarrow \ tu \Vdash B) \\
t \Vdash A \wedge B & \equiv & \exists t_1 \, \exists t_2 \, (t \succ^* \langle t_1, t_2 \rangle \ \wedge \ t_1 \Vdash A \ \wedge \ t_2 \Vdash B) \\
t \Vdash A \vee B & \equiv & \exists u \, ((t \succ^* \langle \bar{0}, u \rangle \ \wedge \ u \Vdash A) \ \vee \ (t \succ^* \langle \bar{1}, u \rangle \ \wedge \ u \Vdash B)) \\
t \Vdash \forall x \, A(x) & \equiv & \forall n \, (t \, \bar{n} \Vdash A(n)) \\
t \Vdash \exists x \, A(x) & \equiv & \exists n \, \exists u \, (t \succ^* \langle \bar{n}, u \rangle \ \wedge \ u \Vdash A(n))
\end{array}
$$

Lemma (closure under anti-evaluation)

If $t \succ^* t'$ and $t' \Vdash A$, then $t \Vdash A$

The main Theorem (recall)

Lemma (Adequacy)

Let $d : (A_1, \ldots, A_n \vdash B)$ be a derivation in NJ. Then:

- for all valuations ρ,
- for all realizers $t_1 \Vdash A_1[\rho], \ldots, t_n \Vdash A_n[\rho]$,

we have: $$d^*[\rho]\{z_1 := t_1, \ldots, z_n := t_n\} \Vdash B[\rho]$$

Lemma

All axioms of HA are realized

Theorem (Soundness)

If $\mathrm{HA} \vdash A$, then $t \Vdash A$ for some closed PCF-term t

Harrop formulas (1/2)

The class of Harrop formulas

Harrop formulas $H ::= e_1 = e_2 \mid \top \mid \bot \mid H_1 \wedge H_2 \mid A \Rightarrow H \mid \forall x\, H$

- **Intuition:** Harrop formulas do not contain the two "problematic" constructions $\vee$ and $\exists$, except on the left-hand side of implications
- Therefore, Harrop formulas are classical:

Proposition

For each Harrop formula $H(\vec{x})$:

$$\mathrm{HA} \vdash \forall \vec{x}\,(H(\vec{x}) \Leftrightarrow \neg\neg H(\vec{x}))$$

Proof. By structural induction on $H(\vec{x})$.

Harrop formulas (2/2)

- To each (possibly open) Harrop formula H, we associate a closed PCF-term t_H that is computationally trivial:

$$\begin{array}{rcl@{\qquad}rcl} \tau_H & := & 0 \quad (H \text{ atomic}) & \tau_{A \Rightarrow H} & = & \lambda_{-}.\,\tau_H \\ \tau_{H_1 \wedge H_2} & = & \langle \tau_{H_1}, \tau_{H_2} \rangle & \tau_{\forall x\, H} & = & \lambda_{-}.\,\tau_H \end{array}$$

Theorem

For all closed Harrop formulas H:

$$\text{If} \quad H \text{ is realized}, \quad \text{then} \quad \tau_H \Vdash H$$

Moreover, all realizers of H are "computationally equivalent" to τ_H

- **Intuition:** Harrop formulas have computationally irrelevant realizers, that can be replaced by the trivial realizers τ_H
  - Useful for optimizing extracted programs (e.g. Fermat's last theorem)
  - But shows that Harrop formulas are computationally irrelevant

Plan

How to cope with classical logic?

- Kleene realizability is definitely incompatible with classical logic:

$$\begin{array}{rcl} & \not\Vdash & \forall x\,(\mathsf{Halt}(x) \vee \neg\mathsf{Halt}(x)) \\ \mathsf{any\_term} & \Vdash & \neg\forall x\,(\mathsf{Halt}(x) \vee \neg\mathsf{Halt}(x)) \end{array}$$

(The same holds for all variants of Kleene realizability)

- Two possible solutions:
  1. Compose Kleene realizability with a negative translation from classical logic (LK) to intuitionistic logic (LJ) (next slide)
  2. Reformulate the principles of realizability to make them compatible with classical logic: Krivine classical realizability (next talk)

The Gödel-Gentzen negative translation

- **Idea:** Turn positive constructions (atomic formulas, $\vee$, $\exists$) into negative constructions ($\bot$, $\neg$, $\Rightarrow$, $\wedge$, $\forall$) using De Morgan laws
- Every formula A is translated into a formula A^G defined by:

$$\begin{array}{rcl@{\qquad}rcl} \top^G & := & \top & \bot^G & := & \bot \\ (A \Rightarrow B)^G & := & A^G \Rightarrow B^G & (e_1 = e_2)^G & := & \neg\neg(e_1 = e_2) \\ (A \wedge B)^G & := & A^G \wedge B^G & (A \vee B)^G & := & \neg(\neg A^G \wedge \neg B^G) \\ (\forall x\, A)^G & := & \forall x\, A^G & (\exists x\, A)^G & := & \neg\forall x\, \neg A^G \end{array}$$

writing: $\neg A := A \Rightarrow \bot$

Theorem (Soundness)

1. $\mathrm{LK} \vdash A^G \Leftrightarrow A$
2. If $\mathrm{PA} \vdash A$, then $\mathrm{HA} \vdash A^G$

Realizing translated formulas

- **Strategy:**
  1. Build a derivation d of A (in PA)
  2. Turn it into a derivation d^G of A^G (in HA)
  3. Turn d^G into a Kleene realizer (program extraction)
- Does not work! Failure comes from:

Proposition (Realizability collapse)

For every closed formula A:

1. A^G is a Harrop formula (computationally irrelevant)
2. Kleene's semantics for A^G mimics Tarski's semantics for A:

$$A^G \text{ is realized} \quad \text{iff} \quad \tau_{A^G} \Vdash A^G \quad \text{iff} \quad \mathbb{N} \models A$$

Proof. By structural induction on A.

- **Conclusion:** Kleene $\circ$ Gödel-Gentzen $=$ Tarski

Friedman's R-translation

(called A-translation by Friedman)

- **Principle:** In Gödel-Gentzen translation, replace each occurrence of $\bot$ (absurdity) by a fixed formula R, called the return formula
- Every formula A is translated into a formula A^F defined by:

$$\begin{array}{rclcrcl} \top^F & := & \top & & \bot^F & := & R \\ (A \Rightarrow B)^F & := & A^F \Rightarrow B^F & & (e_1 = e_2)^F & := & \neg_R\neg_R(e_1 = e_2) \\ (A \wedge B)^F & := & A^F \wedge B^F & & (A \vee B)^F & := & \neg_R(\neg_R A^F \wedge \neg_R B^F) \\ (\forall x\, A)^F & := & \forall x\, A^F & & (\exists x\, A)^F & := & \neg_R\forall x\, \neg_R A^F \end{array}$$

writing: $\neg_R A := A \Rightarrow R$

Theorem (Soundness)

If $\mathrm{PA} \vdash A$, then $\mathrm{HA} \vdash A^F$ (independently from the formula R)

Beware! The formulas A and A^F are no more classically equivalent (in general)

Friedman's trick

Theorem (Kreisel-Friedman)

PA conservatively extends HA over Π^0_2-formulas:

If $\quad \mathrm{PA} \vdash \forall x\, \exists y\, f(x,y) = 0, \quad$ then $\quad \mathrm{HA} \vdash \forall x\, \exists y\, f(x,y) = 0$

Proof. Assume that $\mathrm{PA} \vdash \forall x\, \exists y\, f(x,y) = 0$. We have:

$$\begin{array}{ll}
\mathrm{HA} \vdash \forall x \neg_R \forall y \neg_R \neg_R \neg_R f(x,y) = 0 & \text{(by } R\text{-translation)} \\
\mathrm{HA} \vdash \forall x \neg_R \forall y \neg_R f(x,y) = 0 & \text{(since } \neg_R \neg_R \neg_R \Leftrightarrow \neg_R\text{)} \\
\mathrm{HA} \vdash \neg_R \forall y \neg_R f(x_0,y) = 0 & (\forall\text{-elim, } x_0 \text{ fresh}) \\
\mathrm{HA} \vdash \forall y\, (f(x_0,y) = 0 \Rightarrow R) \;\Rightarrow\; R & \text{(def. of } \neg_R\text{)}
\end{array}$$

We now let: $R := \exists y_0\, f(x_0,y_0) = 0$ (Friedman's trick!) From the def. of R:

$$\mathrm{HA} \vdash \forall y\, (f(x_0,y) = 0 \Rightarrow \exists y_0\, f(x_0,y_0) = 0) \;\Rightarrow\; \exists y_0\, f(x_0,y_0) = 0$$

But the premise of the above implication is provable

$$\mathrm{HA} \vdash \forall y\, (f(x_0,y) = 0 \Rightarrow \exists y_0\, f(x_0,y_0) = 0) \qquad (\exists\text{-intro})$$

hence we get

$$\begin{array}{ll}
\mathrm{HA} \vdash \exists y_0\, f(x_0,y_0) = 0 & \text{(modus ponens)} \\
\mathrm{HA} \vdash \forall x_0\, \exists y_0\, f(x_0,y_0) = 0 & (\forall\text{-intro})
\end{array}$$

□

Realizing translated formulas, again

- **Strategy:**
  1. Build a derivation d of a Π^0_2-formula A (in PA)
  2. Turn it into a derivation F-trick(d^F) of A (in HA)
  3. Turn F-trick(d^F) into a Kleene realizer of A (program extraction)

- This technique perfectly works in practice. However:
  - The formula A^F is not a Harrop formula (in general), even when A is.
    Possible fix: Introduce specific optimization techniques, e.g.:

    Refined Program Extraction [Berger et al. 2001]

  - The translation $A \mapsto A^F$ completely changes the structure of the underlying proof. **Possible fix:** cf next part

Plan

The Lafont-Reus-Streicher negative translation

- **Idea:** Translate each formula A into the (relative) negation of a formula $A^\perp$ already representing the negation of A:

$$A^{LRS} \;:=\; \neg_R A^\perp \;\equiv\; A^\perp \Rightarrow R \qquad (A^\perp \text{ defined by induction on } A)$$

(Again, this translation is parameterized by a return formula R)

- To every predicate symbol p (source language) we associate a predicate symbol $\bar{p}$ representing its negation (target language)
- Definition of the translations $A \mapsto A^\perp$ and $A \mapsto A^{LRS}$:

$$\begin{array}{rcl@{\qquad}rcl} (p(e_1,\ldots,e_k))^\perp & := & \bar{p}(e_1,\ldots,e_k) & \perp^\perp & := & \top \\ (A \Rightarrow B)^\perp & := & A^{LRS} \wedge B^\perp & (\forall x\, A)^\perp & := & \exists x\, A^\perp \end{array}$$

$$A^{LRS} \;:=\; \neg_R A^\perp \;\equiv\; A^\perp \Rightarrow R$$

Theorem (Soundness)

1. When $R \equiv \perp$: $\mathsf{LK} \vdash A^\perp \Leftrightarrow \neg A$ and $\mathsf{LK} \vdash A^{LRS} \Leftrightarrow A$
2. If $\mathsf{LK} \vdash A$, then $\mathsf{LJ} \vdash A^{LRS}$ (independently from the formula R)

Computational interpretation

- **Intuition:** The translated formula $A^{\perp}$ represents the type of stacks opposing (classical) terms of type A:

$$(A_1 \Rightarrow \cdots \Rightarrow A_n \Rightarrow B)^{\perp} \equiv A_1^{LRS} \wedge \cdots \wedge A_n^{LRS} \wedge B^{\perp}$$
$$(A_1 \to \cdots \to A_n \to B)^{\perp} \equiv A_1^{LRS} \times \cdots \times A_n^{LRS} \times B^{\perp}$$

- To analyze the computational contents of the LRS-translation, we need to work across to λ-calculi:

  - A source calculus to represent classical proofs:

    $\lambda_{\text{source}} = \lambda_{\to} + \text{cc} : ((A \to B) \to A) \to A$ (Peirce's law)

    (Polymorphic constant cc introduces classical reasoning)

  - An intuitionistic target calculus to represent translated proofs:

    $\lambda_{\text{target}} = \lambda_{\to,\times}$

    (In this calculus, pairs are used to represent stacks)

The source λ-calculus ($\{\bot, \Rightarrow, \forall\}$-fragment of LK)

Syntax

Types $A, B \quad ::= \quad \bot \quad | \quad p(e_1, \ldots, e_k) \quad | \quad A \Rightarrow B \quad | \quad \forall x\, A$

Proof-terms $t, u \quad ::= \quad z \quad | \quad \lambda z\,.\,t \quad | \quad tu \quad | \quad \text{cc}$

- Classical logic obtained by introducing an inert constant cc (call/cc) for Peirce's law (taken as an axiom)
- Constructions $\top$, $\wedge$, $\vee$, $\exists$ encoded using De Morgan laws (= full LK)

Typing rules

$$\frac{}{\Gamma \vdash z : A}\ {\scriptstyle (z:A)\in\Gamma} \qquad \frac{}{\Gamma \vdash \text{cc} : ((A \Rightarrow B) \Rightarrow A) \Rightarrow A}$$

$$\frac{\Gamma, z : A \vdash t : B}{\Gamma \vdash \lambda z\,.\,t : A \Rightarrow B} \qquad \frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t : \forall x\, A}\ {\scriptstyle x \notin FV(\Gamma)} \qquad \frac{\Gamma \vdash t : \forall x\, A}{\Gamma \vdash t : A\{x := e\}} \qquad \frac{\Gamma \vdash t : \bot}{\Gamma \vdash t : A}$$

Note: $\forall$ is treated uniformly: $\forall x\, A(x) \approx \bigcap_x A(x)$ (no function argument!)

The target λ-calculus ($\{\top, \Rightarrow, \wedge, \exists\}$-fragment of LJ)

Syntax

Types $A, B \quad ::= \quad \top \mid \bar{p}(e_1, \ldots, e_k) \mid A \Rightarrow B \mid A \wedge B \mid \exists x\, A$

Proof-terms $t, u \quad ::= \quad z \mid \lambda z\,.\,t \mid tu \mid \langle t, u \rangle \mid \mathtt{fst}(t) \mid \mathtt{snd}(t)$

$+$ usual reduction rules for proof-terms

Typing rules

$$\frac{}{\Gamma \vdash z : A}\ {\scriptstyle (z:A)\in\Gamma} \qquad \frac{}{\Gamma \vdash t : \top}\ {\scriptstyle FV(t)\subseteq \mathrm{dom}(\Gamma)}$$

$$\frac{\Gamma, z : A \vdash t : B}{\Gamma \vdash \lambda z\,.\,t : A \Rightarrow B} \qquad \frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \langle t, u \rangle : A \wedge B} \qquad \frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \mathtt{fst}(t) : A} \qquad \frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \mathtt{snd}(t) : B}$$

$$\frac{\Gamma \vdash t : A\{x := e\}}{\Gamma \vdash t : \exists x\, A} \qquad \frac{\Gamma \vdash t : (\exists x\, A) \Rightarrow B}{\Gamma \vdash t : \forall x\, (A \Rightarrow B)}\ {\scriptstyle x\notin FV(B)}$$

Note: $\exists$ treated uniformly: $\exists x\, A(x) \approx \bigcup_x A(x)$ (no witness!)

Remark: Uniform vs non-uniform quantifiers (1/2)

- In the Curry-Howard correspondence (and in realizability), there are two different ways to interpret quantifiers:

Quantifier	**Uniform** (ML/Haskell style)	**Non-uniform** (Type Theory style)
$\forall x\, A(x)$	$\bigcap_{x\in D} A(x)$ (intersection type)	$\prod_{x\in D} A(x)$ (type of dependent functions)
$\exists x\, A(x)$	$\bigcup_{x\in D} A(x)$ (union type)	$\sum_{x\in D} A(x)$ (type of dependent pairs)

- **Remark:** Tarski/Kripke/Heyting/Cohen models do not distinguish the two interpretations: difference only appears in realizability

Remark: Uniform vs non-uniform quantifiers (2/2)

- 1st-, 2nd- and higher-order logic support both interpretations
  (But uniform interpretation is more concise & natural)
- The same holds for impredicative set theories: ZF, IZF_C, IZF_R
- Arithmetic (PA/HA) only supports the non-uniform interpretation
  (due to the induction principle)
- But in all cases, the non-uniform interpretation can be encoded from the uniform interpretation, using a relativization:

$$\text{(non-uniform) } \forall x\, A(x) \quad := \quad \text{(uniform) } \forall x\, (\underbrace{x \in D \Rightarrow A(x)}_{\text{type of functions}})$$

$$\text{(non-uniform) } \exists x\, A(x) \quad := \quad \text{(uniform) } \exists x\, (\underbrace{x \in D \wedge A(x)}_{\text{type of pairs}})$$

- This is why we shall prefer the uniform interpretation (in what follows)

The Lafont-Reus-Streicher logical translation

- The logical translation $A \mapsto A^{LRS}$

$$\begin{array}{rcl@{\qquad}rcl} (p(e_1,\ldots,e_k))^{\perp} & := & \bar{p}(e_1,\ldots,e_k) & \perp^{\perp} & := & \top \\ (A \Rightarrow B)^{\perp} & := & A^{LRS} \wedge B^{\perp} & (\forall x\, A)^{\perp} & := & \exists x\, A^{\perp} \end{array}$$

$$A^{LRS} \quad := \quad \neg_R A^{\perp}$$

corresponds to a program transformation on untyped proof terms, called a continuation-passing style (CPS) translation:

$$\begin{array}{rcl@{\qquad}rcl} (z)^{LRS} & := & \lambda s\,.\,z\,s & (\mathrm{cc})^{LRS} & := & \lambda\langle z, s\rangle\,.\,z\,\langle \mathrm{k}_s, s\rangle \\ (\lambda z\,.\,t)^{LRS} & := & \lambda\langle z, s\rangle\,.\,t^{LRS} & \text{where } \mathrm{k}_s & := & \lambda\langle z, \_\rangle\,.\,z\,s \\ (tu)^{LRS} & := & \lambda s\,.\,t^{LRS}\,\langle u^{LRS}, s\rangle & & & \end{array}$$

Note: $\lambda\langle z, s\rangle\,.\,t$ defined as $\lambda z_0\,.\,(\lambda z s\,.\,t)\,(\mathtt{fst}(z_0))\,(\mathtt{snd}(z_0))$

Theorem (Soundness)

If $\Gamma \vdash t : A$ (in the source λ-calculus)

then $\Gamma^{LRS} \vdash t^{LRS} : A^{LRS}$ (in the target λ-calculus)

Towards the Krivine abstract machine

- From the computational behavior of translated proof terms t^{LRS}...

$$\begin{array}{rcl}
(\lambda z\,.\,t)^{LRS} \mathbin{@} \langle u, s\rangle & \succ & t^{LRS}\{z := u\} \mathbin{@} s \\
(tu)^{LRS} \mathbin{@} s & \succ & t^{LRS} \mathbin{@} \langle u^{LRS}, s\rangle \\
(\mathfrak{cc})^{LRS} \mathbin{@} \langle u, s\rangle & \succ & u \mathbin{@} \langle \mathsf{k}_s, s\rangle \\
\mathsf{k}_s \mathbin{@} \langle u, s'\rangle & \succ & u \mathbin{@} s
\end{array}$$

... we deduce evaluation rules for classical proof terms:

Krivine Abstract Machine (KAM)

$$\begin{array}{lrcl}
\textbf{Grab} & \lambda z\,.\,t \star u \cdot \pi & \succ & t\{z := u\} \star \pi \\
\textbf{Push} & tu \star \pi & \succ & t \star u \cdot \pi \\
\textbf{Save} & \mathfrak{cc} \star u \cdot \pi & \succ & u \star \mathsf{k}_\pi \cdot \pi \\
\textbf{Restore} & \mathsf{k}_\pi \star u \cdot \pi' & \succ & u \star \pi
\end{array}$$

- Reformulating Kleene realizability through the LRS translation (and its CPS), we get Krivine classical realizability (cf next talk)