Realizing the axiom of dependent choices (DC)

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A small history of classical realizability

- Krivine 1994. *A general storage theorem for integers in call-by-name lambda-calculus*
- K. 2001. *Typed lambda-calculus in classical ZF set theory*
- K. 2003. *Dependent choice, 'quote' and the clock*
- K. 2004–2009. *Realizability in classical logic*
- K. 2011. *Realizability algebras: a program to well order* $\mathbb{R}$
- K. 2012. *Realizability algebras II: new models of ZF + DC*
- K. 2014. *On the structure of classical realizability models of ZF*
- K. 2016. *Bar recursion in classical realizability: dependent choice and continuum hypothesis*
- K. 2018. *Realizability algebras III: some examples*
- K. 2021. *A program for the full axiom of choice*

The axiom of choice in ZF

Recall: In ZF, the axiom of choice (AC) can be stated as follows:

(AC) Each set A has a choice function, that is: a function $h : \mathfrak{P}^*(A) \to A$ such that $h(X) \in X$ for all $X \in \mathfrak{P}^*(A)$

(Writing $\mathfrak{P}^*(A) := \mathfrak{P}(A) \setminus \{\varnothing\}$)

Proposition (Equivalent statements of AC)

The axiom of choice is equivalent to each of the following statements:

1. Each surjection $f : A \to B$ has a right-inverse, that is: a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$
2. Each equivalence relation $\sim$ on a set A has a system of representatives, that is: a subset $S \subseteq A$ that contains exactly one point of each equivalence class of $\sim$
3. The Cartesian product $\prod_{x \in A} B_x$ of a family $(B_x)_{x \in A}$ of nonempty sets (indexed by an arbitrary set A) is nonempty

Proof: Exercise

Weak forms of the axiom of choice

The axiom of denumerable choice (AC_ω)

$$\forall (A_n)_{n\in\mathbb{N}},\ (\forall n\in\mathbb{N},\ A_x \neq \varnothing)\ \Rightarrow\ \underbrace{\exists (u_n)_{n\in\mathbb{N}},\ \forall n\in\mathbb{N},\ u_n\in A_n}_{\prod_{n\in\mathbb{N}} A_n \neq \varnothing}$$

Note: This axiom is crucial to prove that non-finite sets are Dedekind-infinite (thus eliminating subinfinite sets, that are neither finite nor Dedekind-infinite)

The axiom of dependent choice (DC)

$$\begin{array}{l}\forall A,\ \forall R\subseteq A^2,\\ \quad (\forall x\in A,\ \exists y\in A,\ x\,R\,y)\ \Rightarrow\\ \qquad \forall x_0\in A,\ \exists (u_n)_{n\in\mathbb{N}}\in A^{\mathbb{N}},\ u_0 = x_0\ \wedge\ \forall n\in\mathbb{N},\ u_n\,R\,u_{n+1}\end{array}$$

Note: This axiom plays an important role in analysis, since it implies (and is actually equivalent to) Baire's category theorem

Proposition: $(AC)\ \Rightarrow\ (DC)\ \Rightarrow\ (AC_\omega)$ (in ZF)

But converse implications $(AC_\omega)\ \Rightarrow\ (DC)\ \Rightarrow\ (AC)$ are not derivable in ZF

Reformulating AC_ω and DC in the language of PA2

Although AC, AC_ω and DC are presented as axioms in ZF, they can be presented as axiom schemes in PA2:

The axiom scheme of denumerable choice (AC_ω)... in the language of PA2

For each formula $A[x, Y]$ of the language of PA2 depending on a 1st-order variable x and on a 2nd-order variable X of arity k:

$$(\forall x \in \mathbb{N})\, \exists Y\, A[x, Y] \;\Rightarrow\; \exists U\, (\forall x \in \mathbb{N})\, A[x, U(x)]$$

(where U is a 2nd-order variable of arity $k+1$)

The axiom scheme of dependent choice (DC)... in the language of PA2

For each formula $A[X, Y]$ of the language of PA2 depending on two 2nd-order variables X and Y of arity k:

$$\forall X\, \exists Y\, A[X, Y] \;\Rightarrow\; \forall X_0\, \exists U\, (U(0) = X_0 \;\wedge\; (\forall n \in \mathbb{N})\, A[U(n), U(n+1)])$$

(where X_0 and U are 2nd-order variables of arities k and $k+1$, respectively)

(The case of AC is more complex...)

Extensionality in 2nd-order logic (1/4)

- **Recall:** In 2nd-order logic, predicate equality is defined by:

$$P = Q \quad :\equiv \quad \forall \vec{x}\,(P(\vec{x}) \Leftrightarrow Q(\vec{x})) \qquad \text{(Extensional equality)}$$

- In *An introduction to Krivine realizability*, we saw that:

Proposition (Extensionality in 2nd-order logic)

For each 2nd-order formula $A[\vec{z}, \vec{Z}, X]$ depending on $\vec{z}, \vec{Z}, X$, we have:

$$\text{NJ2} \;\vdash\; \forall \vec{z}\, \forall \vec{Z}\, \forall X\, \forall Y \,\Big(X = Y \;\Rightarrow\; (A[\vec{z}, \vec{Z}, X] \Leftrightarrow A[\vec{z}, \vec{Z}, Y])\Big)$$

Proof. By structural induction on A

- By adequacy, this means that for each formula $A[\vec{z}, \vec{Z}, X]$ depending on $\vec{z}, \vec{Z}, X$, we have two (intuitionistic) proof-like terms:

$$\mathtt{ext}_{A/X} \;\Vdash\; \forall \vec{z}\, \forall \vec{Z}\, \forall X\, \forall Y\, (X = Y \Rightarrow A[\vec{z}, \vec{Z}, X] \Rightarrow A[\vec{z}, \vec{Z}, Y])$$
$$\mathtt{ext}'_{A/X} \;\Vdash\; \forall \vec{z}\, \forall \vec{Z}\, \forall X\, \forall Y\, (X = Y \Rightarrow A[\vec{z}, \vec{Z}, Y] \Rightarrow A[\vec{z}, \vec{Z}, X])$$

- We now want to make explicit the terms $\mathtt{ext}_{A/X}$ and $\mathtt{ext}'_{A/X}$...

Extensionality in 2^{nd}-order logic (2/4)

- For that, we introduce a new syntactic category of skeletons:

Skeletons $\sigma, \tau \ ::= \ \mathtt{V} \ | \ * \ | \ \sigma \to \tau$

Definition (Skeleton of a formula abstracted w.r.t. a 2^{nd}-order variable)

To each formula A abstracted w.r.t. a 2^{nd}-order variable X, we associate its skeleton $\mathbf{sk}(A/X)$, that is defined by:

$$\begin{array}{rcll} \mathbf{sk}(A \ / \ X) & :\equiv & * & (\text{if } X \notin FV(A)) \\ \mathbf{sk}(X(\vec{e}) \ / \ X) & :\equiv & \mathtt{V} & \\ \mathbf{sk}(A \Rightarrow B \ / \ X) & :\equiv & \mathbf{sk}(A/X) \to \mathbf{sk}(B/X) & \\ \mathbf{sk}(\forall x\, A \ / \ X) & :\equiv & \mathbf{sk}(A/X) & \\ \mathbf{sk}(\forall Y\, A \ / \ X) & :\equiv & \mathbf{sk}(A/X) & (\text{if } Y \not\equiv X) \end{array}$$

- **Note:** $\mathbf{sk}(A/X)$ is not sensitive to a substitution of a variable $\not\equiv X$:

$$\mathbf{sk}(A[x := e]/X) \ \equiv \ \mathbf{sk}(A[Y := P]/X) \ \equiv \ \mathbf{sk}(A/X) \qquad (Y \not\equiv X,\ X \notin FV(P))$$

Extensionality in 2^{nd}-order logic (3/4)

Definition (Terms $\mathtt{ext}_\sigma$ and $\mathtt{ext}'_\sigma$)

1. To each skeleton σ we associate two intuitionistic proof-like terms $t_\sigma[z]$ and $t'_\sigma[z]$ that only depend on a variable z, letting:

$$t_*[z] \;:\equiv\; \mathbf{I} \qquad\qquad t_{\mathbb{V}}[z] \;:\equiv\; z\,(\lambda x, y\,.\,x)$$
$$t'_*[z] \;:\equiv\; \mathbf{I} \qquad\qquad t'_{\mathbb{V}}[z] \;:\equiv\; z\,(\lambda x, y\,.\,y)$$

$$t_{\tau\to\sigma}[z] \;:\equiv\; \lambda f, x\,.\,t_\sigma[z]\,(f\,(t'_\tau[z]\,x))$$
$$t'_{\tau\to\sigma}[z] \;:\equiv\; \lambda f, x\,.\,t'_\sigma[z]\,(f\,(t_\tau[z]\,x))$$

2. For each skeleton σ, we finally let:

$$\mathtt{ext}_\sigma \;:\equiv\; \lambda z\,.\,t_\sigma[z] \qquad \text{and} \qquad \mathtt{ext}'_\sigma \;:\equiv\; \lambda z\,.\,t'_\sigma[z]$$

- Note that both proof-like terms $\mathtt{ext}_\sigma$ and $\mathtt{ext}'_\sigma$ only depend on a skeleton σ and not on a abstracted formula A/X...

  ... Nevertheless... (go to next slide)

Extensionality in 2^{nd}-order logic (4/4)

Proposition

For each 2^{nd}-order formula $A[\vec{z},\vec{Z},X]$ with parameters only depending on the variables $\vec{z}$, $\vec{Z}$, and X, we have:

$$\texttt{ext}_\sigma \ \Vdash\ \forall\vec{z}\,\forall\vec{Z}\,\forall X\,\forall Y\ \big(X = Y \Rightarrow A[\vec{z},\vec{Z},X] \Rightarrow A[\vec{z},\vec{Z},Y]\big)$$

$$\texttt{ext}'_\sigma \ \Vdash\ \forall\vec{z}\,\forall\vec{Z}\,\forall X\,\forall Y\ \big(X = Y \Rightarrow A[\vec{z},\vec{Z},Y] \Rightarrow A[\vec{z},\vec{Z},X]\big)$$

writing $\sigma :\equiv \mathbf{sk}(A[\vec{z},\vec{Z},X]/X)$

Proof: Exercise!

- So that in what follows, we shall write

$$\texttt{ext}_{A/X} :\equiv \texttt{ext}_{\mathbf{sk}(A/X)} \qquad \text{and} \qquad \texttt{ext}'_{A/X} :\equiv \texttt{ext}'_{\mathbf{sk}(A/X)}$$

  keeping in mind that the above realizers actually depend only on the skeleton of the abstracted formula A/X

Plan

1. Introduction
2. Realizing AC_ω & DC using quote
3. Realizing AC_ω & DC using the clock

Plan

The instructions quote and quote′

- **Numbering terms: the instruction quote:**

  Given an enumeration $(t_n)_{n\in\mathbb{N}}$ of all closed terms, we add the rule

$$\mathrm{quote} \star t \cdot u \cdot \pi \quad \succ \quad u \star \overline{\lceil t \rceil} \cdot \pi$$

  writing $\lceil t \rceil$ the smallest $n \in \mathbb{N}$ such that $t \equiv t_n$

- **Numbering stacks: the instruction quote′:**

  Given an enumeration $(\pi_n)_{n\in\mathbb{N}}$ of all stacks, we add the rule

$$\mathrm{quote}' \star u \cdot \pi \quad \succ \quad u \star \overline{\lceil \pi \rceil^{\perp}} \cdot \pi$$

  writing $\lceil \pi \rceil^{\perp}$ the smallest $n \in \mathbb{N}$ such that $\pi \equiv \pi_n$

Proposition

If there is a partial recursive function $f : \mathbb{N} \rightharpoonup \mathbb{N}$ such that $\lceil \pi \rceil^{\perp} = f(\lceil \mathrm{k}_\pi \rceil)$ for all $\pi \in \Pi$, then quote′ can be implemented from quote, letting:

$$\mathrm{quote}' \; :\equiv \; \lambda x\,.\,\text{œ}\,(\lambda k\,.\,\mathrm{quote}\; k\; (\lambda n\,.\,\check{f}\; n\, x))$$

Proof: Check it out!

A first choice principle (1/3)

Proposition 1 ("Type" of quote′)

Given a formula(∗) $A[X]$ that only depends on a 2nd-order variable X of arity $k \geq 0$, there is a falsity function $\Phi_A : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$ (depending on A and on the pole $\perp\!\!\!\perp$) such that:

$$\text{quote}' \Vdash (\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)] \Rightarrow \forall X\, A[X]$$

(∗) Here and in what follows: formula = formula of PA2 with parameters.

- Intuitively, the $(k+1)$-ary predicate $\dot{\Phi}_A$ represents a sequence $(\dot{\Phi}_A(n))_n$ of potential counter-examples to the predicate $A[X]$
- Since the converse implication trivially holds (proof: $\lambda z, \_ . z$), the resulting equivalence allows to replace any 2nd-order quantification $\forall X\, A[X]$ by a numeric quantification $(\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)]$
- And since quote′ can be implemented from quote, we also have:

$$\lambda x . \text{cc}\, (\lambda k . \text{quote}\ k\ (\lambda n . \check{f}\ n\, x)) \Vdash (\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)] \Rightarrow \forall X\, A[X]$$

A first choice principle (2/3)

Proof. Using meta-theoretic AC_ω, we associate to each $n \in \mathbb{N}$ a k-ary falsity function $\Phi_n : \mathbb{N}^k \to \mathfrak{P}(\Pi)$ defined by:

$$\Phi_n := \begin{cases} \text{Some function } F : \mathbb{N}^k \to \mathfrak{P}(\Pi) \text{ such that } \pi_n \in \|A[\dot{F}]\| \\ \qquad\qquad \text{if such a function } F \text{ exists} \\ \text{Any function } F : \mathbb{N}^k \to \mathfrak{P}(\Pi) \text{ otherwise} \end{cases}$$

(writing π_n the n^{th} element of the fixed enumeration of all stacks).
Then we define $\Phi_A : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$, letting $\Phi_A(n) := \Phi_n$ for all $n \in \mathbb{N}$.

We want to prove that $\mathrm{quote}' \Vdash (\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)] \Rightarrow \forall X\, A[X]$.

For that, pick a stack in $\big\|(\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)] \Rightarrow \forall X\, A[X]\big\|$, that is necessarily of the form $u \cdot \pi$ where $u \in \big|(\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)]\big|$ and $\pi \in \|A[\dot{F}]\|$ for some $F : \mathbb{N}^k \to \mathfrak{P}(\Pi)$, and let us prove that $\mathrm{quote}' \star u \cdot \pi \in \perp\!\!\!\perp$.

For that, write $n := \ulcorner \pi \urcorner^{\perp}$, so that $\pi \equiv \pi_n \in \|A[\dot{F}]\|$. From the def. of $\Phi_A(n) = \Phi_n$, we have $\pi \equiv \pi_n \in \|A[\dot{\Phi}_n]\| = \|A[\dot{\Phi}_A(n)]\|$.

Now we observe that $\mathrm{quote}' \star u \cdot \pi \succ u \star \overline{n} \cdot \pi$, so that it remains to prove that $u \star \overline{n} \cdot \pi \in \perp\!\!\!\perp$ (by anti-evaluation). This follows from the fact that:

$$u \in |(\forall x \in \mathbb{N}) A[\dot{\Phi}_A(x)]|, \qquad \overline{n} \in |n \in \mathbb{N}| \qquad \text{and} \qquad \pi \in \|A[\dot{\Phi}_A(n)]\|.$$

□

A first choice principle (3/3)

Changing the def. of Φ_A, we can build a simpler realizer based on quote:

Proposition 1.b (Variant of Prop. 1)

Given a formula $A[X]$ that only depends on a 2[nd]-order variable X of arity $k \geq 0$, there is a falsity function $\Phi'_A : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$ (depending on A and on the pole $\perp\!\!\!\perp$) such that:

$$\lambda x\,.\,\mathrm{quote}\,x\,x \ \Vdash\ (\forall n \in \mathbb{N})A[\dot{\Phi}'_A(n)] \ \Rightarrow\ \forall X\, A[X]$$

Def. of Φ'_A: For all $n \in \mathbb{N}$, write $S_n := \{\pi \in \Pi : t_n \star \overline{n} \cdot \pi \notin \perp\!\!\!\perp\}$ (where t_n is the nth element of the fixed enumeration of all closed terms). Using meta-theoretic AC_ω, we now associate to each $n \in \mathbb{N}$ a falsity function $\Phi'_n : \mathbb{N}^k \to \mathfrak{P}(\Pi)$ defined by:

$$\Phi'_n \ :=\ \begin{cases} \text{Some function } F : \mathbb{N}^k \to \mathfrak{P}(\Pi) \text{ such that } \|A[\dot{F}]\| \cap S_n \neq \varnothing \\ \qquad\qquad \text{if such a function } F \text{ exists} \\ \text{Any function } F : \mathbb{N}^k \to \mathfrak{P}(\Pi) \text{ otherwise} \end{cases}$$

Then we define $\Phi'_A : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$, letting $\Phi'_A(n) := \Phi'_n$ for all $n \in \mathbb{N}$.

Exercise: Prove that $\lambda x\,.\,\mathrm{quote}\,x\,x \ \Vdash\ (\forall n \in \mathbb{N})A[\dot{\Phi}'_A(n)] \ \Rightarrow\ \forall X\, A[X]$ □

(**Hint:** Reason by contradiction.)

Taking the contrapositive (1/2)

Considering the contrapositive of the first choice principle

$$(\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)] \ \Rightarrow \ \forall X\, A[X]$$

we get the following result:

Proposition 2

Given a formula $A[X]$ that only depends on a 2$^{\text{nd}}$-order variable X of arity $k \geq 0$, there is a falsity function $\Psi_A : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$ (depending on A and on the pole $\perp\!\!\!\perp$) such that:

$$\theta \ \Vdash \ \exists X\, A[X] \ \Rightarrow \ (\exists n \in \mathbb{N}) A[\dot{\Psi}_A(n)]\Big)$$

writing $\theta \ :\equiv \ \lambda z, f\,.\, \text{cc}\,(\lambda k\,.\, z\,(\text{quote}'\,(\lambda n, x\,.\, k\,(f\; n\, x))))$

- Intuitively, the $(k+1)$-ary predicate $\dot{\Psi}_A$ represents a sequence $(\dot{\Psi}_A(n))_{n \in \mathbb{N}}$ of potential witnesses of the predicate $A[X]$
- Since the converse implication trivially holds, the resulting equivalence allows to replace any 2$^{\text{nd}}$-order quantification $\exists X\, A[X]$ by a numeric quantification $(\exists n \in \mathbb{N}) A[\dot{\Psi}_A(n)]$

Taking the contrapositive (2/2)

Proof. Considering $\neg A$ instead of A, we know from Prop. 1 that there is a falsity function $\Phi_{\neg A} : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$ such that:

$$\text{quote}' \Vdash (\forall n \in \mathbb{N})\neg A[\dot{\Phi}_{\neg A}(n)] \Rightarrow \forall X\, \neg A[X]$$

so that writing $\Psi_A :\equiv \Phi_{\neg A}$, we get:

$$\text{quote}' \Vdash (\forall n \in \mathbb{N})\neg A[\dot{\Psi}_A(n)] \Rightarrow \forall X\, \neg A[X]$$

Now writing $t[q] :\equiv \lambda z, f \,.\, œ\,(\lambda k \,.\, z\,(q\,(\lambda n, x \,.\, k\,(f\; n\, x))))$, we observe that the following typing judgment is derivable in system λNK2:

$$q : (\forall n \in \mathbb{N})\neg A[\dot{\Psi}_A(n)] \Rightarrow \forall X\, \neg A[X]$$
$$\vdash \quad t[q] : \exists X\, A[X] \Rightarrow (\exists n \in \mathbb{N}) A[\dot{\Psi}_A(n)]$$

Therefore, by adequacy we get

$$\theta :\equiv t[\text{quote}'] \Vdash \exists X\, A[X] \Rightarrow (\exists n \in \mathbb{N}) A[\dot{\Psi}_A(n)].$$

□

The non-extensional axiom of choice (NEAC) (1/4)

Proposition 3 (The non-extensional axiom of choice – NEAC)

Given a formula $A[X]$ that only depends on a 2^{nd}-order variable X of arity $k \geq 0$, there is a falsity function $\mathcal{E}_A : \mathbb{N}^k \to \mathfrak{P}(\Pi)$ (depending on A and on the pole $\perp\!\!\!\perp$) such that:

$$\theta_\sigma \Vdash \exists X\, A[X] \Rightarrow A[\dot{\mathcal{E}}_A]$$

(where θ_σ is a closed proof-like depending on $\sigma :\equiv \mathbf{sk}(A[X]/X)$)

- The construction $\mathcal{E}_A$ is the 2^{nd}-order version of Hilbert's epsilon, that "chooses" for each 2^{nd}-order predicate $A[X]$ an object $\mathcal{E}_A$ that fulfills A, if such an object exists. (Otherwise, $\mathcal{E}_A$ is arbitrary)
- However, the symbol $\mathcal{E}$ is non-extensional, since in the realizability model we have in general:

$$\forall X\,(A[X] \Leftrightarrow B[X]) \not\Rightarrow \underbrace{\mathcal{E}_A = \mathcal{E}_B}_{\forall \vec{x}(\mathcal{E}_A(\vec{x}) \Leftrightarrow \mathcal{E}_B(\vec{x}))}$$

The non-extensional axiom of choice (NEAC) (2/4)

Proof. From Prop. 2, we know that there is a falsity function $\Psi_A : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$ such that: $\theta \Vdash \exists X\, A[X] \Rightarrow (\exists n \in \mathbb{N}) A[\dot{\Psi}_A(n)]$ (for some proof-like term θ).

Let us now consider the k-ary predicate P_A defined by:

$$P_A(x_1, \ldots, x_k) \;:\equiv\; (\exists n_0 \in \mathbb{N})\{\; A[\dot{\Psi}_A(n_0)] \;\wedge\; (\forall n < n_0) \neg A[\dot{\Psi}_A(n)] \;\wedge\; \dot{\Psi}_A(n_0, x_1, \ldots, x_k) \;\}$$

Using the fact that PA2 $\vdash$ "any nonempty subset of $\mathbb{N}$ has a smallest element", we easily construct a proof-term:

$$t_1 \;:\; (\exists n \in \mathbb{N}) A[\dot{\Psi}_A(n)] \;\Rightarrow\; (\exists n_0 \in \mathbb{N})\{\; A[\dot{\Psi}_A(n_0)] \wedge (\forall n < n_0) \neg A[\dot{\Psi}_A(n)] \;\}.$$

And from the def. of the k-ary predicate P_A combined with the uniqueness of the smallest $n \in \mathbb{N}$ such that $A[\dot{\Psi}_A(n)]$, we can build a proof-term:

$$t_2 \;:\; (\forall n_0 \in \mathbb{N})\{\; A[\dot{\Psi}_A(n_0)] \wedge \forall (n < n_0) \neg A[\dot{\Psi}_A(n)] \;\Rightarrow\; \dot{\Psi}_A(n_0) = P_A \;\}.$$

On the other hand, we know that $\mathtt{ext}_\sigma \Vdash \forall X \forall X' (X = X' \Rightarrow A[X] \Rightarrow A[X'])$, writing $\sigma :\equiv \mathbf{sk}(A[X]/X)$ the skeleton of $A[X]$ w.r.t. X (cf § Introduction).

Combining the proof-like terms θ, t_1, t_2 and $\mathtt{ext}_\sigma$, we easily deduce a proof-like term $\theta_\sigma \Vdash \exists X\, A[X] \Rightarrow A[P_A]$ (that only depends on $\sigma :\equiv \mathbf{sk}(A[X]/X)$).

To conclude it suffices to define $\mathcal{E}_A : \mathbb{N}^k \to \mathfrak{P}(\Pi)$ by $\mathcal{E}_A := \|P_A\|$. □

The non-extensional axiom of choice (NEAC) (3/4)

Due to the form of its realizer θ_σ, NEAC can be generalized to formulas $A[\vec{z}, \vec{Z}, X]$ that may depend on other (1^{st}- and 2^{nd}-order) variables $\vec{z}$, $\vec{Z}$:

Theorem 4 (General form of NEAC)

Given a formula $A[\vec{z}, \vec{Z}, X]$ that only depends on:

- p 1^{st}-order variables $\vec{z} :\equiv z_1, \ldots, z_p$,
- q 2^{nd}-order variables $\vec{Z} :\equiv Z_1, \ldots, Z_q$ of arities $k_1, \ldots, k_q \geq 0$ and
- a 2^{nd}-order variable X of arity $k \geq 0$,

there is a (3^{rd}-order) falsity function

$$\mathcal{E}_A \ : \ \underbrace{\mathbb{N}^p}_{z_1,\ldots,z_p} \times \underbrace{\mathfrak{P}(\Pi)^{\mathbb{N}^{k_1}} \times \cdots \times \mathfrak{P}(\Pi)^{\mathbb{N}^{k_q}}}_{Z_1,\ldots,Z_q} \to \underbrace{\mathfrak{P}(\Pi)^{\mathbb{N}^k}}_{X}$$

(depending on A and $\perp\!\!\!\perp$) such that:

$$\theta_\sigma \ \Vdash \ \forall \vec{z}\, \forall \vec{Z} \ \Big(\exists X\, A[\vec{z}, \vec{Z}, X] \ \Rightarrow \ A[\vec{z}, \vec{Z}, \dot{\mathcal{E}}_A(\vec{z}, \vec{Z})]\Big)$$

(where $\sigma :\equiv \mathbf{sk}(A[\vec{z}, \vec{Z}, X])$, and using the same realizer θ_σ as before)

The non-extensional axiom of choice (NEAC) (4/4)

Proof. For all parameters $\vec{m} = (m_1, \ldots, m_p) \in \mathbb{N}^p$ and $\vec{F} = F_1, \ldots, F_q$ (where $F_i : \mathbb{N}^{k_i} \to \mathfrak{P}(\Pi)$ for all $i \in [1..q]$), we choose (using meta-theoretic AC) a falsity function $\mathcal{E}_{A[\vec{m},\vec{F}]} : \mathbb{N}^k \to \mathfrak{P}(\Pi)$ such that

$$\theta_\sigma \Vdash \exists X\, A[\vec{m}, \vec{F}, X] \Rightarrow A[\vec{m}, \vec{F}, \dot{\mathcal{E}}_{A[\vec{m},\vec{F}]}]$$

(from Prop. 3), writing $\sigma :\equiv \mathbf{sk}(A[\vec{m}, \vec{F}, X]/X) \equiv \mathbf{sk}(A[\vec{z}, \vec{Z}, X]/X)$.
(Note that the skeleton σ does not depend on the parameters $\vec{m}$, $\vec{F}$.)

We now define the function $\mathcal{E}_A : \mathbb{N}^p \times \mathfrak{P}(\Pi)^{k_1} \times \cdots \times \mathfrak{P}(\Pi)^{k_q} \to \mathfrak{P}(\Pi)^{\mathbb{N}^k}$ by $\mathcal{E}_A(\vec{m}, \vec{F}) := \mathcal{E}_{A[\vec{m},\vec{F}]}$ for all $\vec{m} \in \mathbb{N}^p$ and $\vec{F} \in \mathfrak{P}(\Pi)^{\mathbb{N}^{k_1}} \times \cdots \times \mathfrak{P}(\Pi)^{\mathbb{N}^{k_q}}$.

For all parameters $\vec{m}$, $\vec{F}$, we thus have

$$\theta_\sigma \Vdash \exists X\, A[\vec{m}, \vec{F}, X] \Rightarrow A[\vec{m}, \vec{F}, \dot{\mathcal{E}}_A(\vec{m}, \vec{F})]$$

and since the realizer θ_σ is the same for all parameters $\vec{m}$, $\vec{F}$, we deduce that

$$\theta_\sigma \Vdash \forall \vec{z}\, \forall \vec{Z} \left(\exists X\, A[\vec{z}, \vec{Z}, X] \Rightarrow A[\vec{z}, \vec{Z}, \dot{\mathcal{E}}_A(\vec{z}, \vec{Z})]\right)$$

by an immediate generalization. □

On the importance of extensionality

- The non-extensional axiom of choice (NEAC) does not imply AC, since $\mathcal{E}$ is not extensional: $\forall X\,(A[X] \Leftrightarrow B[X]) \not\Rightarrow \mathcal{E}_A = \mathcal{E}_B$

Counter-example (Constructing a right-inverse of a surjective function?)

In 2nd-order logic, a 3rd-order function (i.e. from 2nd-order objects to themselves) is naturally represented as a formula $F[X,Y]$ such that:

(1) $\forall X\,\forall X'\,\forall Y\,(F[X,Y] \land X = X' \Rightarrow F[X',Y]) \land$
$\forall X\,\forall Y\,\forall Y'\,(F[X,Y] \land Y = Y' \Rightarrow F[X,Y'])$ (F is compatible)

(2) $\forall X\,\forall Y\,\forall Y'\,(F[X,Y] \land F[X,Y'] \Rightarrow Y = Y')$ (F is functional)

(3) $\forall X\,\exists Y\,F[X,Y]$ (F is total)

If moreover, we assume that:

(4) $\forall Y\,\exists X\,F[X,Y]$ (F is surjective)

it is natural to define a right-inverse G of F, letting: $G[Y,X] \;:\equiv\; (X = \mathcal{E}_{F[\cdot,Y]})$.

Then it is easy to realize that the function G is functional (2) and total (3), and moreover that $F \circ G = \mathrm{id}$, that is: $\forall X\,\forall Y\,\forall X'\,(F[X,Y] \land G[Y,X'] \Rightarrow X' = X)$.

Alas, we cannot realize that G is compatible (1), since $\mathcal{E}$ is not extensional

- Nevertheless, we shall see that NEAC implies both AC_ω and DC

Why NEAC implies AC_ω

Corollary 5 (Realizing AC_ω)

For each formula $A[x,Y]$ depending on a 1st-order variable x and on a 2nd-order variable Y of arity k, we have:

$$\xi_\sigma \ \Vdash\ \big((\forall x \in \mathbb{N})\,\exists Y\, A[x,Y]\big) \ \Rightarrow\ \exists U\,(\forall x \in \mathbb{N})\, A[x, U(x)]$$

(where ξ_σ is a closed proof-like term depending on $\sigma :\equiv \mathbf{sk}(A[x,Y]/Y)$, and where U is a 2nd-order variable of arity $k+1$)

Proof (idea). Let $\xi_\sigma \ :\equiv\ \lambda hf\,.\,f\,(\lambda n\,.\,\theta_\sigma\,(h\,n))$ and instanciate U by $\dot{F}$, where $F\ :\ \mathbb{N} \to \mathfrak{P}(\Pi)^{\mathbb{N}^k}\ :=\ \mathcal{E}_A$. (**Exercise:** write down the details.) □

Remark: Relativizations to $\mathbb{N}$ are actually useless. Indeed, if we replace ξ_σ by $\xi'_\sigma\ :\equiv\ \lambda hf\,.\,f\,(\theta_\sigma\,h)$, we realize the ι-indexed axiom of choice (AC_ι):

$$\xi'_A \ \Vdash\ \big(\forall x\, \exists Y\, A[x,Y]\big) \ \Rightarrow\ \exists U\, \forall x\ A[x, U(x)]$$

Exercise: write down the details

Why NEAC implies DC

Corollary 6 (Realizing DC)

For each formula $A[X, Y]$ depending on two 2nd-order variables X, Y of arity k, we have:

$$\eta_\sigma \Vdash \forall X\, \exists Y\, A[X, Y] \ \Rightarrow \\ \forall X_0\, \exists U\, (U(0) = X_0 \wedge (\forall n \in \mathbb{N})\, A[U(n), U(n+1)])$$

(where ξ_σ is a closed proof-like term depending on $\sigma :\equiv \mathbf{sk}(A[x, Y]/Y)$, and where X_0 and U are 2nd-order variables of arities k and $k+1$, respectively)

Proof. Assuming that X_0 is instanciated by a falsity function $F_0 : \mathbb{N}^k \to \mathfrak{P}(\Pi)$, we let $F_{n+1} := \mathcal{E}_A(F_n)$ for all $n \in \mathbb{N}$, and define the falsity function $G : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$ by $G(n) := F_n$ for all $n \in \mathbb{N}$. Letting $\zeta_\sigma :\equiv \lambda h\,.\,\langle\langle \mathbf{I}, \mathbf{I}\rangle, \lambda n\,.\,\theta_\sigma\, h\rangle$, we check that

$$\zeta_\sigma \Vdash \forall X\, \exists Y\, A[X, Y] \ \Rightarrow \ \dot{G}(0) = \dot{F}_0 \wedge (\forall n \in \mathbb{N})\, A[\dot{G}(n), \dot{G}(n+1)]$$

and letting $\eta_\sigma :\equiv \lambda xy\,.\,y\,(\zeta_\sigma\, x)$, we deduce that

$$\eta_\sigma \Vdash \forall X\, \exists Y\, A[X, Y] \ \Rightarrow \ \exists U\, (U(0) = \dot{F}_0 \wedge (\forall n \in \mathbb{N})\, A[U(n), U(n+1)])$$

We conclude by universally generalizing over the falsity function F_0. □

Plan

Introduction

Krivine (2003):

> *We observe that the application $n \mapsto t_n$ may be any surjective map from $\mathbb{N}$ onto Λ. The reduction rule for χ is then:*
>
> $$\chi \star u \cdot \pi \quad \succ \quad u \star \overline{n} \cdot \pi$$ [1]
>
> *where n is any integer such that $t_n \equiv u$. This suggests the following interpretation: χ is an input instruction and, when it comes in head position, the process $\chi \star u \cdot \pi$ waits for some integer n which is provided by some human operator or some external process. [...] The only constraint is that "u must be retrievable from n", i.e. the integers provided to the processes $\chi \star u \cdot \pi$ and $\chi \star u' \cdot \pi'$ with $u' \neq u$, must be different. A very simple and natural way to obtain this behaviour is to provide the integer n by means of a clock, since two different λ_c-terms cannot appear at the same time. [...]*

How to formalize (mathematically) this clock?

---

[1] Krivine's χ behaves as "$\lambda x \,.\, \mathrm{quote}\, x\, x$" (with the notations of the previous section)

Outline of the method (1/2)

How to retrieve the execution time from a process?

Naive method: Store the "current time" at the bottom of the current stack, and increment it at each evaluation step:

Push	$tu \star \vec{v} \cdot \alpha_n$	$\succ^1$	$t \star u \cdot \vec{v} \cdot \alpha_{n+1}$
Grab	$\lambda x . t \star u \cdot \vec{v} \cdot \alpha_n$	$\succ^1$	$t[x := u] \star \vec{v} \cdot \alpha_{n+1}$
Save	$\text{cc} \star u \cdot \vec{v} \cdot \alpha_n$	$\succ^1$	$u \star \mathsf{k}_{\vec{v}} \cdot \vec{v} \cdot \alpha_{n+1}$
Restore	$\mathsf{k}_{\vec{v}} \star u \cdot \vec{v}\,' \cdot \alpha_n$	$\succ^1$	$u \star \vec{v} \cdot \alpha_{n+1}$
Clock	$\text{clock} \star u \cdot \vec{v} \cdot \alpha_n$	$\succ^1$	$u \star \overline{n} \cdot \vec{v} \cdot \alpha_{n+1}$

Problem: Such a design of evaluation is completely incompatible with the adequacy lemma! (**Exercise:** Check it out!)

Morality: In classical realizability, we cannot tamper with stacks

Outline of the method (2/2)

How to retrieve the execution time from a process?

Simple solution: Store the "boot program" in the bottom of the stack, so that we can retrieve the current time by "subtraction". For that:

- Associate a stack constant α_θ to each $\theta \in \mathrm{PL}$ ("boot programs") and only consider deterministic evaluation sequences of the form:

$$\underbrace{\overbrace{\theta \star \alpha_\theta}^{\text{"boot process"}} \succ \cdots \succ t \star \vec{v} \cdot \alpha_\theta \succ t' \star \vec{v}\,' \cdot \alpha_\theta \succ \cdots}_{\text{thread of } \theta}$$

  (Such a thread may be linear-infinite, linear-finite or cyclic)

- Retrieve "current time" using an instruction "clock" with the rule

$$\mathsf{clock} \star t \cdot \vec{u} \cdot \alpha_\theta \succ t \star \overline{n} \cdot \vec{u} \cdot \alpha_\theta,$$

  where n is the smallest integer such that: $\theta \star \alpha_\theta \succ^n \mathsf{clock} \star t \cdot \vec{u} \cdot \alpha_\theta$

  (Note that when evaluation is cyclic, the clock is cyclic too)

A particular instance of the λ_c-calculus (1/3)

Recall: An instance of the λ_c-calculus is defined by:

- A set $\mathcal{K} = \{\text{c}\!\!\text{c}, \ldots\}$ of instructions (containing at least call/cc)
- A nonempty set Π_0 of stack constants (or stack bottoms)
- A preorder of evaluation $\succ$, that contains at least the four basic rules **Grab**, **Push**, **Save** and **Restore**

Definition of the λ_c-calculus with clock:

- Let: $\mathcal{K} \ := \ \{\text{c}\!\!\text{c}, \ \text{clock}\}$ (only two instructions: call-cc and clock)

  The set $\mathcal{K}$ determines the set of proof-like terms:

Proof-like terms $\theta, \phi \quad ::= \quad x \quad | \quad \lambda x\,.\,\theta \quad | \quad \theta\,\phi \quad | \quad \text{c}\!\!\text{c} \quad | \quad \text{clock}$

- Introducing a stack constant α_θ for each closed proof-like term $\theta \in \mathsf{PL}$, we let: $\Pi_0 \ := \ \{\alpha_\theta : \theta \in \mathsf{PL}\}$
- To each $\theta \in \mathsf{PL}$, we associate the boot process $\theta \star \alpha_\theta$

A particular instance of the λ_c-calculus (2/3)

- For each $\theta \in \mathrm{PL}$, we define a relation $\theta \triangleright^n p$ ("boot process $\theta \star \alpha_\theta$ evaluates to process p in n steps") from the inference rules:

$$\frac{}{\theta \triangleright^0 \theta \star \alpha_\theta}\ (\textbf{Init})$$

$$\frac{\theta \triangleright^n \lambda x\,.\,t \star u \cdot \pi}{\theta \triangleright^{n+1} t[x := u] \star \pi}\ (\textbf{Grab}) \qquad \frac{\theta \triangleright^n tu \cdot \pi}{\theta \triangleright^{n+1} t \star u \cdot \pi}\ (\textbf{Push})$$

$$\frac{\theta \triangleright^n \text{œ} \star u \cdot \pi}{\theta \triangleright^{n+1} u \star \mathrm{k}_\pi \cdot \pi}\ (\textbf{Save}) \qquad \frac{\theta \triangleright^n \mathrm{k}_\pi \star u \cdot \pi'}{\theta \triangleright^{n+1} u \star \pi}\ (\textbf{Restore})$$

$$\frac{\theta \triangleright^n \mathrm{clock} \star u \cdot \pi}{\theta \triangleright^{n+1} u \star \overline{n_0} \cdot \pi}\ (\textbf{Clock})$$

writing n_0 the smallest integer ($\leq n$) such that $\theta \triangleright^{n_0} \mathrm{clock} \star u \cdot \pi$

Lemma (Determinism of $\theta \triangleright$)

For all θ, n, p, p': $\theta \triangleright^n p$ and $\theta \triangleright^n p'$ imply $p \equiv p'$

A particular instance of the λ_c-calculus (3/3)

- We now define the relation $\succ^1$ of one step evaluation as follows:

Push	$tu \star \pi$	$\succ^1$	$t \star u \cdot \pi$
Grab	$\lambda x\,.\,t \star u \cdot \pi$	$\succ^1$	$t[x := u] \star \pi$
Save	$\text{cc} \star u \cdot \pi$	$\succ^1$	$u \star \mathrm{k}_\pi \cdot \pi$
Restore	$\mathrm{k}_\pi \star u \cdot \pi'$	$\succ^1$	$u \star \pi$
Clock	$\mathrm{clock} \star u \cdot \pi$	$\succ^1$	$u \star \overline{n} \cdot \pi$

writing $\pi \equiv \vec{v} \cdot \alpha_\theta$, and n the smallest integer such that $\theta \triangleright^n \mathrm{clock} \star u \cdot \pi$

- Let $(\succ^n) := (\succ^1)^n$ and $(\succ) := (\succ^1)^* = \bigcup_{n \in \mathbb{N}} (\succ^n)$

Lemma (Determinism of $\succ$ & characterization of $\theta \triangleright$)

1. For all p, p', p'': $p \succ^1 p'$ and $p \succ^1 p''$ imply $p' \equiv p''$
2. For all n, θ, p: $\theta \triangleright^n p$ iff $\theta \star \alpha_\theta \succ^n p$

Threads

- For each $\theta \in PL$, we define the thread of θ by:

$$\begin{aligned}\mathbf{thd}(\theta) &:= \{p : \theta \triangleright^n p \text{ for some } n \in \mathbb{N}\} \\ &= \{p : \theta \star \alpha_\theta \succ p\}\end{aligned}$$

- The thread of θ is either:
  - linear-infinite: $p_0 \succ p_1 \succ p_2 \succ \cdots \succ p_n \succ p_{n+1} \succ \cdots$
  - linear-finite: $p_0 \succ p_1 \succ p_2 \succ \cdots \succ p_n \not\succ^1$
  - cyclic: $p_0 \succ \cdots \succ p_k \succ \cdots \succ p_n \equiv p_k \quad (k < n)$

  The clock behaves accordingly (infinitely, finitely, cyclicly)

- Since $\mathbf{thd}(\theta)$ is (obviously) closed under evaluation, its complement

$$⫫_\theta := \mathbf{thd}(\theta)^{\complement} \qquad (\subseteq \Lambda \times \Pi)$$

  is closed under anti-evaluation, and can be used as a pole:

  $\Rightarrow$ local pole associated to the proof-like term θ

A first choice principle... again (1/2)

Proposition 5 ("Type" of the "clock")

Given a formula $A[X]$ that only depends on a 2^{nd}-order variable X of arity $k \geq 0$ and a local pole $\perp\!\!\!\perp_\theta := \mathbf{thd}(\theta)^{\complement}$ (for some $\theta \in \mathrm{PL}$), there is a falsity function $\Phi_A : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$ (depending on A and on θ) such that:

$$\text{clock} \Vdash (\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)] \Rightarrow \forall X\, A[X]$$

Proof: cf next slide

Note that the result only holds in local poles!

Remark: When working with the "clock", we thus need to replace

universal realizability (= realizability w.r.t. all poles)

by local realizability (= realizability w.r.t. all local poles $\perp\!\!\!\perp_\theta$)

The reader is invited to check the main results associated with universal realizability (e.g. witness extraction techniques) still holds with local realizability

A first choice principle... again (2/2)

Proof. Consider a pole of the form $\perp\!\!\!\perp_\theta := \mathbf{thd}(\theta)^{\complement}$ (for some $\theta \in \mathrm{PL}$). For each $n \in \mathbb{N}$, we let $S_n := \{\pi \in \Pi \ : \ \exists u \in \Lambda,\ \theta \triangleright^n \mathsf{clock} \star u \cdot \pi\}$. Since $\succ$ is deterministic, the set S_n contains at most one stack. Using meta-theoretic AC$_\omega$, we now associate to each $n \in \mathbb{N}$ a k-ary falsity function $\Phi_n : \mathbb{N}^k \to \mathfrak{P}(\Pi)$ defined by:

$$\Phi_n \ := \ \begin{cases} \text{Some function } \ F : \mathbb{N}^k \to \mathfrak{P}(\Pi) \ \text{ such that } \|A[\dot{F}]\| \cap S_n \neq \varnothing \\ \qquad\qquad \text{if such a function } F \text{ exists} \\ \text{Any function } \ F : \mathbb{N}^k \to \mathfrak{P}(\Pi) \ \text{ otherwise} \end{cases}$$

Then we define $\Phi_A : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$, letting $\Phi_A(n) := \Phi_n$ for all $n \in \mathbb{N}$.

We want to prove that $\mathsf{clock} \Vdash (\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)] \ \Rightarrow \ \forall X\, A[X]$, that is:
we want to prove that $\mathsf{clock} \star u \cdot \pi \in \perp\!\!\!\perp$ for all $u \in |(\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)]|$,
for all $F : \mathbb{N}^k \to \mathfrak{P}(\Pi)$ and for all $\pi \in \|A[\dot{F}]\|$.

Reasoning by contradiction, let us assume that $\mathsf{clock} \star u \cdot \pi \ \notin \ \perp\!\!\!\perp$. Hence we have $\mathsf{clock} \star u \cdot \pi \ \in \ \mathbf{thd}(\theta)$, so that $\mathsf{clock} \star u \cdot \pi \ \succ \ u \star \overline{n} \cdot \pi \ \notin \ \perp\!\!\!\perp$ (by evaluation), where n is the smallest integer such that $\theta \triangleright^n \mathsf{clock} \star u \cdot \pi$.

We now observe that $\pi \in \|A[\dot{F}]\| \cap S_n$, hence $\|A[\dot{\Phi}_A(n)]\| \cap S_n \neq \varnothing$ (from the def. of $\Phi_A(n) = \Phi_n$), and thus $\pi \in \|A[\dot{\Phi}_A(n)]\|$ (since $S_n = \{\pi\}$). Observing that

$$u \ \in \ |(\forall n \in \mathbb{N}) A[\dot{\Phi}_A(n)]|, \quad \overline{n} \in |n \in \mathbb{N}| \quad \text{and} \quad \pi \in \|A[\dot{\Phi}_A(n)]\|$$

we deduce that $u \star \overline{n} \cdot \pi \in \perp\!\!\!\perp$: contradiction! □

Realizing NEAC, AC_ω and DC

The same way as we did with the instruction "quote", we successively deduce from Prop. 5 the existence of:

- A function $\Psi_A : \mathbb{N}^{k+1} \to \mathfrak{P}(\Pi)$ and a term $\theta_0 \in PL$ such that:

$$\theta_0 \;\Vdash\; \exists X\, A[X] \;\Rightarrow\; (\exists n \in \mathbb{N}) A[\dot{\Psi}_A(n)]$$

- A function $\mathcal{E}_A : \mathbb{N}^k \to \mathfrak{P}(\Pi)$ and a term $\theta_\sigma \in PL$ such that:

$$\theta_\sigma \;\Vdash\; \exists X\, A[X] \;\Rightarrow\; A[\dot{\mathcal{E}}_A]$$

- And more generally for each formula $A[\vec{z}, \vec{Z}, X]$, a function $\mathcal{E}_A : \cdots \to \mathfrak{P}(\Pi)^{\mathbb{N}^k}$ and a term $\theta_\sigma \in PL$ such that:

NEAC: $$\theta_\sigma \;\Vdash\; \forall \vec{z}\, \forall \vec{Z} \left(\exists X\, A[\vec{z}, \vec{Z}, X] \;\Rightarrow\; A[\vec{z}, \vec{Z}, \dot{\mathcal{E}}_{A_0}(\vec{z}, \vec{Z})] \right)$$

The same way as before, AC_ω and DC are easily deduced from NEAC