Forcing as a program transformation

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April 8th, 14th, 2022

Different notions of models

- **Tarski models:** $[\![A]\!] \in \{0,1\}$
  - Interprets classical provability (correctness/completeness)
- **Intuitionistic realizability:** $[\![A]\!] \in \mathfrak{P}(\Lambda)$ [Kleene '45]
  - Interprets intuitionistic proofs
  - Independence results, in intuitionistic theories
  - Definitely incompatible with classical logic
- **Cohen forcing:** $[\![A]\!] \in \mathfrak{P}(P)$ [Cohen '63]
  - Independence results, in classical theories
    (Negation of the continuum hypothesis, Solovay's axiom, etc.)
- **Classical realizability:** $[\![A]\!] \in \mathfrak{P}(\Pi)$ [Krivine '94, '01, '09, ...]
  - Interprets classical proofs
  - Generalizes Tarski models... and forcing

Plan

Plan

What is forcing?

- A technique invented by Cohen ('63) to prove the independence of the continuum hypothesis (CH) w.r.t. ZFC:

The continuum hypothesis (CH), Hilbert's 1st problem

For every infinite subset $S \subseteq \mathbb{R}$:

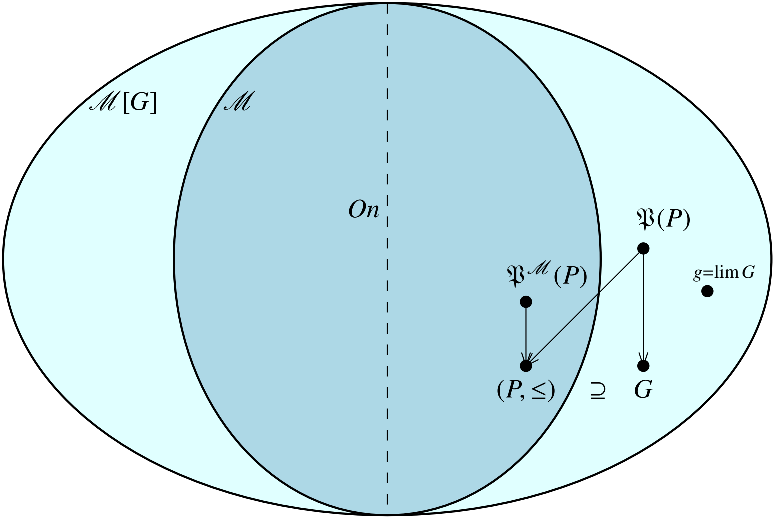
- Either S is denumerable (i.e. in bijection with $\mathbb{N}$)
- Either S has the power of continuum (i.e. is in bijection with $\mathbb{R}$)

In symbols: $$2^{\aleph_0} = \aleph_1$$

- Gödel ('38) proved $\mathrm{ZFC} \nvdash \neg\mathrm{CH}$ introducing constructible sets
- Cohen ('63) proved $\mathrm{ZFC} \nvdash \mathrm{CH}$ introducing forcing
- Related to Boolean-valued models [Scott, Solovay, Vopěnka]
- Used to prove the consistency/independence of many axioms [Solovay, Shelah, Woodin, etc.]

How does forcing work?

Exploit the underspecification of the powerset $\mathfrak{P}(X)$ (X infinite)



An analogy with algebra

Set theory	**Algebra**
Start from a ground model $\mathscr{M}$	Start from a ground field F
We want to add a new set approximated by the elements of a given forcing poset $(P, \leq) \in \mathscr{M}$	We want to add a new point that should be a root of a given polinomial $P \in F[X]$
This defines a ficticious generic filter $G \subseteq P$ (outsize $\mathscr{M}$)	This defines a ficticious root α of P (outsize F)
which generates around $\mathscr{M}$ a generic extension $\mathscr{M}[G]$	which generates around F a field extension $F[\alpha]$
Construction: $\mathscr{M}[G] := \mathscr{M}^{(P)}/{\sim_{\mathsf{Ext}}}$	Construction: $F[\alpha] \ := \ F[X]/PF[X]$

Example: forcing ¬CH

- **Aim:** Force the existence of an injection $h : \aleph_2 \to \mathfrak{P}(\omega)$

  We shall build it as a characteristic function $g : \aleph_2 \times \omega \to 2$

- The ideal object g is approximated in the ground model $\mathscr{M}$ by elements of $(P, \leq) \;:=\; (\mathsf{Fin}(\aleph_2 \times \omega, 2),\; \supseteq)$ (forcing poset)

- **Forcing invocation:** Let $\mathscr{M}[G]$ be the generic extension generated by an $\mathscr{M}$-generic filter $G \subseteq P$ (always exists!)

- In $\mathscr{M}[G]$, we let: $g \;:=\; \lim G \;=\; \bigcup G \quad (: \aleph_2 \times \omega \rightharpoonup 2)$

  Using the $\mathscr{M}$-genericity of the filter $G \subseteq P$, we prove that:

  - Partial function $g : \aleph_2 \times \omega \to 2$ is actually total
  - Corresponding function $h : \aleph_2 \to \mathfrak{P}(\omega)$ is actually injective

Technicalities (countable chain condition) under the carpet

Compared properties of $\mathscr{M}$ and $\mathscr{M}[G]$

Forcing theorem: Given a model $\mathscr{M}$ and a forcing poset $(P, \leq) \in \mathscr{M}$, the generic extension $\mathscr{M}[G]$ always exists

- $\mathscr{M}$ and $\mathscr{M}[G]$ have the very same ordinals
- If Axiom of Choice (AC) holds in $\mathscr{M}$, then it holds in $\mathscr{M}[G]$ too
- Finite cardinals and $\aleph_0$ $(= \omega)$ are the same in $\mathscr{M}$ and in $\mathscr{M}[G]$
- $\mathscr{M}[G]$ has in general fewer cardinals than $\mathscr{M}$
    - **Intuition:** new bijections may appear in $\mathscr{M}[G]$ between sets in $\mathscr{M}$, thus identifying their cardinals in $\mathscr{M}[G]$
    - Cardinals are preserved if P fulfils the countable chain condition (This was the case for $P = \mathrm{Fin}(E, 2)$ used for forcing ¬CH)
    - But in some circumstances, one may use forcing to kill cardinals: Levy collapse, Solovay's axiom, etc.

The proof-theoretic point of view

- Construction of $\mathscr{M}[G]$ parameterized by a forcing poset $(P, \leq)$, whose elements are called forcing conditions
  - $p \leq q$ reads: 'p is stronger than q'
- Internally relies on a logical translation

$$A \quad \mapsto \quad p \Vdash A \qquad \text{('}p\text{ forces }A\text{')}$$

where p is a fresh variable (representing a condition)

  - Complex definition by induction on A, using the poset $(P, \leq)$

Properties

1. $\vdash A$ entails $\vdash (\forall p \in P)(p \Vdash A)$
2. But $\vdash (\forall p \in P)(p \Vdash A)$ for more formulas A (depending on P)
3. $\vdash (\forall p \in P)(p \nVdash \bot)$ (consistency)

- **Remark:** Forcing commutes with $\bot$, $\top$, $\wedge$ and $\forall$, but not with $\Rightarrow$, $\neg$, $\vee$, $\exists$

Kripke forcing versus Cohen forcing

Kripke models for (classical) modal logic (S4)

$$p \Vdash A \Rightarrow B \quad\equiv\quad (p \Vdash A) \Rightarrow (p \Vdash B)$$

$$p \Vdash \Box A \quad\equiv\quad \forall q \leq p\ (q \Vdash A)$$

$$\frac{p \Vdash A \Rightarrow B \qquad p \Vdash A}{p \Vdash B}$$

$\Uparrow$

Gödel's translation from LJ to S4

$(A \Rightarrow B)^\dagger \equiv \Box(A^\dagger \Rightarrow B^\dagger)$

Kripke models for intuitionistic logic (LJ)

$$p \Vdash A \Rightarrow B \quad\equiv\quad \forall q \leq p\ ((q \Vdash A) \Rightarrow (q \Vdash B))$$

$$\frac{p \Vdash A \Rightarrow B \qquad q \Vdash A}{q \Vdash B}\ q \leq p$$

$\Uparrow$

¬¬-translation from LK to LJ

(tricky!)

Forcing in classical logic (LK)

$$p \Vdash A \Rightarrow B \quad\equiv\quad \forall q\ ((q \Vdash A) \Rightarrow \forall r \leq p, q\ (r \Vdash B))$$

$$\frac{p \Vdash A \Rightarrow B \qquad q \Vdash A}{r \Vdash B}\ r \leq p, q$$

Cohen forcing versus classical realizability

Cohen forcing	**Classical realizability**
$[\![A]\!] \in \mathfrak{P}(P)$	$\lvert A\rvert \in \mathfrak{P}(\Lambda_c)$
$p \Vdash A$	$t \Vdash A$
$\dfrac{p \Vdash A \Rightarrow B \qquad q \Vdash A}{\underbrace{pq}_{\text{g.l.b.}} \Vdash B}$	$\dfrac{t \Vdash A \Rightarrow B \qquad u \Vdash A}{\underbrace{tu}_{\text{application}} \Vdash B}$
$\dfrac{p \Vdash A \qquad q \Vdash B}{pq \Vdash A \wedge B}$	$\dfrac{t \Vdash A \qquad u \Vdash B}{\langle t, u\rangle \Vdash A \wedge B}$
$A \wedge B \ = \ A \cap B$	$A \wedge B \ \neq \ A \cap B$

- **Slogan:** Classical realizability $=$ Non commutative forcing

Combining Cohen forcing with classical realizability

- **Forcing in classical realizability** [Krivine '09]
  - Introduce realizability algebras, generalizing the λ_c-calculus
  - Discover the program transformation underlying forcing
  - Extend iterated forcing to classical realizability
  - Show how to force the existence of a well-ordering over $\mathbb{R}$ (while keeping evaluation deterministic)

- **Computational analysis of forcing** [M. '11]
  - Focus on the underlying program transformation (no generic filter)
  - Hard-wire the program transformation into the abstract machine

Underlying methodology

Translation of formulas & proofs $\rightsquigarrow$ Classical program transformation $\rightsquigarrow$ New abstract machine (no transformation)

Plan

1. Cohen forcing
2. Higher-order arithmetic (tuned)
3. The forcing transformation
4. The forcing machine
5. Realizability algebras
6. Conclusion

Higher-order arithmetic (PAω^+)

- A multi-sorted language that allows to express
  - Individuals (sort ι)
  - Propositions (sort o)
  - Functions over individuals ($\iota \to \iota$, $\iota \to \iota \to \iota$, ...)
  - Predicates over individuals ($\iota \to o$, $\iota \to \iota \to o$, ...)
  - Predicates over predicates... ($(\iota \to o) \to o$, ...)

Syntax of sorts (kinds) and higher-order terms

$$\begin{array}{llcl}
\textbf{Sorts} & \tau, \sigma & ::= & \iota \quad | \quad o \quad | \quad \tau \to \sigma \\
\textbf{Terms} & M, N, A, B & ::= & x^\tau \quad | \quad \lambda x^\tau \,.\, M \quad | \quad M N \quad | \quad 0 \quad | \quad \mathsf{s} \quad | \quad \mathsf{rec}_\tau \\
 & & | & A \Rightarrow B \quad | \quad \forall x^\tau \, A \quad | \quad M = M' \mapsto A \\
\textbf{Proof terms} & t, u & ::= & \text{(postponed)}
\end{array}$$

- Equational implication: $M = M' \mapsto A$
  - Means: $\begin{cases} A & \text{if } M = M' \\ \top & \text{otherwise} \end{cases}$ (equality of denotations) ($\top$ = type of all proofs)
  - Provably equivalent to: $M =_\tau M' \Rightarrow A$ (Leibniz equality)

Encodings

- The logic of PAω^+ is ultimately based on $\Rightarrow$ and $\forall$.
  Other constructions of logic are encoded as follows:

$$\begin{array}{lcll}
\bot & \equiv & \forall z^o z & \text{(Absurdity)} \\
\neg A & \equiv & A \Rightarrow \bot & \text{(Negation)} \\
A \wedge B & \equiv & \forall z^o((A \Rightarrow B \Rightarrow z) \Rightarrow z) & \text{(Conjunction)} \\
A \vee B & \equiv & \forall z^o((A \Rightarrow z) \Rightarrow (B \Rightarrow z) \Rightarrow z) & \text{(Disjunction} \\
\exists x^\tau A(x) & \equiv & \forall z^o(\forall x^\tau (A(x) \Rightarrow z) \ \Rightarrow \ z) & (\exists \text{ at sort } \tau) \\
M =_\tau M' & \equiv & \forall z^{\tau \to o}(z\, M \Rightarrow z\, M') & \text{(Leibniz equality)}
\end{array}$$

- $M = M' \mapsto A$ (equational implication) provably equivalent to $M =_\tau M' \Rightarrow A$ (combination of Leibniz equality and implication), but has much more compact proof terms

- Top proposition: $\top \ :\equiv \ (\mathsf{tt} = \mathsf{ff} \mapsto \bot)$ (type of all proof-terms)
  where $\mathsf{tt} \equiv \lambda x^o y^o \, . \, x$, $\mathsf{ff} \equiv \lambda x^o y^o \, . \, y$ and $\bot \equiv \forall z^o \, z$

Conversion (1/2)

- Conversion $M \cong_{\mathcal{E}} M'$ parameterized by a (finite) set of equations
$$\mathcal{E} \equiv M_1 = M'_1, \ldots, M_k = M'_k \qquad \text{(non oriented, well sorted)}$$

- Reflexivity, symmetry, transitivity + base case:

$$\frac{}{M \cong_{\mathcal{E}} M'} \; {\scriptstyle (M=M')\in\mathcal{E}}$$

- β-conversion, recursion:

$$\begin{array}{rcl} (\lambda x^\tau \,.\, M)N & \cong_{\mathcal{E}} & M[x := N] \\ \mathsf{rec}_\tau\; M\; M'\; 0 & \cong_{\mathcal{E}} & M \\ \mathsf{rec}_\tau\; M\; M'\,(\mathsf{s}\, N) & \cong_{\mathcal{E}} & M'\, N\,(\mathsf{rec}_\tau\; M\; M'\; N) \end{array}$$

- Usual context rules + extended rule for $M = M' \mapsto A$:

$$\frac{A \;\cong_{\mathcal{E},M=M'}\; A'}{M = M' \mapsto A \;\cong_{\mathcal{E}}\; M = M' \mapsto A'}$$

Conversion (2/2)

- Rules for identifying computationally equivalent propositions, according to Curry-style proof terms (def. postponed):

$$\begin{array}{rcll}
\forall x^\tau\, \forall y^\sigma\, A & \cong_{\mathcal{E}} & \forall y^\sigma\, \forall x^\tau\, A & \\
\forall x^\tau\, A & \cong_{\mathcal{E}} & A & (\text{if } x^\tau \notin FV(A)) \\
A \Rightarrow \forall x^\tau\, B & \cong_{\mathcal{E}} & \forall x^\tau\, (A \Rightarrow B) & (\text{if } x^\tau \notin FV(A)) \\
M = M' \mapsto N = N' \mapsto A & \cong_{\mathcal{E}} & N = N' \mapsto M = M' \mapsto A & \\
M = M \mapsto A & \cong_{\mathcal{E}} & A & \\
A \Rightarrow (M = M' \mapsto B) & \cong_{\mathcal{E}} & M = M' \mapsto (A \Rightarrow B) & \\
\forall x^\tau (M = M' \mapsto A) & \cong_{\mathcal{E}} & M = M' \mapsto \forall x^\tau A & (\text{if } x^\tau \notin FV(M,M'))
\end{array}$$

- Example: $\top \quad :\equiv \quad (\mathsf{tt} = \mathsf{ff} \mapsto \bot)$ (type of all proof-terms)

  where $\mathsf{tt} \equiv \lambda x^o y^o\,.\,x$, $\mathsf{ff} \equiv \lambda x^o y^o\,.\,y$ and $\bot \equiv \forall z^o\, z$

  we can derive that: $(A \Rightarrow \top) \;\cong\; \top$ (A any proposition)

Deduction system (typing)

- Proof terms: $t, u \;::=\; x \;\mid\; \lambda x\,.\,t \;\mid\; tu \;\mid\; \mathbf{c\!\!c}$ (Curry-style)
- Contexts: $\Gamma \;::=\; x_1 : A_1, \ldots, x_n : A_n$ (A_i of sort o)

Deduction/typing rules

$$\dfrac{}{\mathcal{E};\Gamma \vdash x : A}\;{\scriptstyle (x:A)\in\Gamma} \qquad\qquad \dfrac{\mathcal{E};\Gamma \vdash t : A}{\mathcal{E};\Gamma \vdash t : A'}\;{\scriptstyle A\cong_{\mathcal{E}} A'}$$

$$\dfrac{\mathcal{E};\Gamma, x : A \vdash t : B}{\mathcal{E};\Gamma \vdash \lambda x\,.\,t : A \Rightarrow B} \qquad\qquad \dfrac{\mathcal{E};\Gamma \vdash t : A \Rightarrow B \qquad \mathcal{E};\Gamma \vdash u : A}{\mathcal{E};\Gamma \vdash tu : B}$$

$$\dfrac{\mathcal{E}, M = M';\Gamma \vdash t : A}{\mathcal{E};\Gamma \vdash t : M = M' \mapsto A} \qquad\qquad \dfrac{\mathcal{E};\Gamma \vdash t : M = M \mapsto A}{\mathcal{E};\Gamma \vdash t : A}$$

$$\dfrac{\mathcal{E};\Gamma \vdash t : A}{\mathcal{E};\Gamma \vdash t : \forall x^\tau A}\;{\scriptstyle x^\tau \notin FV(\mathcal{E};\Gamma)} \qquad\qquad \dfrac{\mathcal{E};\Gamma \vdash t : \forall x^\tau A}{\mathcal{E};\Gamma \vdash t : A[x := N^\tau]}$$

$$\dfrac{}{\mathcal{E};\Gamma \vdash \mathbf{c\!\!c} : ((A \Rightarrow B) \Rightarrow A) \Rightarrow A}$$

Remark: All proof-terms have type $\top :\equiv (\mathrm{tt} = \mathrm{ff} \mapsto \bot)$ (normalization fails)

From operational semantics...

- Krivine's λ_c-calculus
  - λ-calculus with call/cc and continuation constants:

$$t, u \quad ::= \quad x \quad | \quad \lambda x\,.\,t \quad | \quad tu \quad | \quad \mathrm{c\!c} \quad | \quad \mathsf{k}_\pi$$

  - An abstract machine with explicit stacks:
    - Stack $=$ list of closed terms (notation: π, π')
    - Process $=$ closed term $\star$ stack

- Evaluation rules (weak head normalization, call by name)

$$\begin{array}{llcllcrcl}
(\textbf{Grab}) & \lambda x\,.\,t & \star & u \cdot \pi & \succ & t[x := u] & \star & \pi \\
(\textbf{Push}) & tu & \star & \pi & \succ & t & \star & u \cdot \pi \\
(\textbf{Call/cc}) & \mathrm{c\!c} & \star & t \cdot \pi & \succ & t & \star & \mathsf{k}_\pi \cdot \pi \\
(\textbf{Resume}) & \mathsf{k}_\pi & \star & t \cdot \pi' & \succ & t & \star & \pi
\end{array}$$

… to classical realizability semantics

- Interpreting higher-order terms:
  - Individuals interpreted as natural numbers $[\![\iota]\!] \;:=\; \mathbb{N}$
  - Propositions interpreted as falsity values $[\![o]\!] \;:=\; \mathfrak{P}(\Pi)$
  - Functions interpreted set-theoretically $[\![\tau \to \sigma]\!] \;:=\; [\![\sigma]\!]^{[\![\tau]\!]}$
- Parameterized by a pole $\perp\!\!\!\perp \subseteq \Lambda_c \star \Pi$ (closed under anti-evaluation)
- Interpreting logical constructions:

$$[\![\forall x^\tau A]\!] \;=\; \bigcup_{v \in [\![\tau]\!]} [\![A[x := \dot{v}]]\!] \qquad\qquad [\![A \Rightarrow B]\!] \;=\; [\![A]\!]^{\perp\!\!\!\perp} \cdot [\![B]\!]$$

$$[\![M = M' \mapsto A]\!] \;=\; \begin{cases} [\![A]\!] & \text{if } [\![M]\!] = [\![M']\!] \\ \varnothing & \text{otherwise} \end{cases}$$

Adequacy

If
- $\mathcal{E}; x_1 : A_1, \ldots, x_n : A_n \vdash t : B$ (in PAω^+)
- $\rho \models \mathcal{E}$, $\quad u_1 \Vdash A_1[\rho]$, …, $u_n \Vdash A_n[\rho]$

then: $t[x_1 := u_1, \ldots, x_n := u_n] \Vdash B[\rho]$

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Representing conditions

- **Intuition:** Represent the set of conditions as an upwards closed subset of a meet-semilattice

- Take:
  - A sort κ of conditions, equipped with
  - A binary product $(p, q) \mapsto pq$ (of sort $\kappa \to \kappa \to \kappa$)
  - A unit 1 (of sort κ)
  - A predicate $p \mapsto C[p]$ of well-formedness (of sort $\kappa \to o$)

- **Typical example:** finite functions from τ to σ are modelled by
  - $\kappa \;:\equiv\; \tau \to \sigma \to o$ (binary relations $\subseteq \tau \times \sigma$)
  - $pq \;:\equiv\; \lambda x^\tau y^\sigma \,.\, p\,x\,y \vee q\,x\,y$ (union of relations p and q)
  - $1 \;:\equiv\; \lambda x^\tau y^\sigma \,.\, \bot$ (empty relation)
  - $C[p] \;:\equiv\;$ "p is a finite function from τ to σ"

Combinators

- The forcing translation is parameterized by
  - The sort κ + closed terms $\cdot$, 1, C (logical level)
  - 9 closed proof terms $\alpha_*, \alpha_1, \ldots, \alpha_8$ (computational level)

$$\begin{array}{lcl}
\alpha_* & : & C[1] \\
\alpha_1 & : & \forall p^\kappa\ \forall q^\kappa\ (C[pq] \Rightarrow C[p]) \\
\alpha_2 & : & \forall p^\kappa\ \forall q^\kappa\ (C[pq] \Rightarrow C[q]) \\
\alpha_3 & : & \forall p^\kappa\ \forall q^\kappa\ (C[pq] \Rightarrow C[qp]) \\
\alpha_4 & : & \forall p^\kappa\ (C[p] \Rightarrow C[pp]) \\
\alpha_5 & : & \forall p^\kappa\ \forall q^\kappa\ \forall r^\kappa\ (C[(pq)r] \Rightarrow C[p(qr)]) \\
\alpha_6 & : & \forall p^\kappa\ \forall q^\kappa\ \forall r^\kappa\ (C[p(qr)] \Rightarrow C[(pq)r]) \\
\alpha_7 & : & \forall p^\kappa\ (C[p] \Rightarrow C[p1]) \\
\alpha_8 & : & \forall p^\kappa\ (C[p] \Rightarrow C[1p])
\end{array}$$

This set is not minimal. One can take $\alpha_*, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_7$ and define:
$\alpha_2 :\equiv \alpha_1 \circ \alpha_3, \quad \alpha_6 :\equiv \alpha_3 \circ \alpha_5 \circ \alpha_3 \circ \alpha_5 \circ \alpha_3, \quad \alpha_8 :\equiv \alpha_3 \circ \alpha_7$

Derived combinators

- The combinators $\alpha_1, \ldots, \alpha_8$ can be composed:

  Example: $\alpha_1 \circ \alpha_6 \circ \alpha_3 \quad : \quad \forall p^\kappa \; \forall q^\kappa \; \forall r^\kappa \; (C[(pq)r] \Rightarrow C[rp])$

- We will also use the following derived combinators:

$$\begin{array}{lcllcl}
\alpha_9 & :\equiv & \alpha_3 \circ \alpha_1 \circ \alpha_6 \circ \alpha_3 & : & \forall p^\kappa \; \forall q^\kappa \; \forall r^\kappa \; (C[(pq)r] \Rightarrow C[pr]) \\
\alpha_{10} & :\equiv & \alpha_2 \circ \alpha_5 & : & \forall p^\kappa \; \forall q^\kappa \; \forall r^\kappa \; (C[(pq)r] \Rightarrow C[qr]) \\
\alpha_{11} & :\equiv & \alpha_9 \circ \alpha_4 & : & \forall p^\kappa \; \forall q^\kappa \; (C[pq] \Rightarrow C[p(pq)]) \\
\alpha_{12} & :\equiv & \alpha_5 \circ \alpha_3 & : & \forall p^\kappa \; \forall q^\kappa \; \forall r^\kappa \; (C[p(qr)] \Rightarrow C[q(rp)]) \\
\alpha_{13} & :\equiv & \alpha_3 \circ \alpha_{12} & : & \forall p^\kappa \; \forall q^\kappa \; \forall r^\kappa \; (C[p(qr)] \Rightarrow C[(rp)q]) \\
\alpha_{14} & :\equiv & \alpha_5 \circ \alpha_3 \circ \alpha_{10} \circ \alpha_4 \circ \alpha_2 & : & \forall p^\kappa \; \forall q^\kappa \; \forall r^\kappa \; (C[p(qr)] \Rightarrow C[q(rr)]) \\
\alpha_{15} & :\equiv & \alpha_9 \circ \alpha_3 & : & \forall p^\kappa \; \forall q^\kappa \; \forall r^\kappa \; (C[p(qr)] \Rightarrow C[qp])
\end{array}$$

- **Important remark:**
  - $C[pq] \Rightarrow C[p] \wedge C[q]$, but $C[p] \wedge C[q] \not\Rightarrow C[pq]$ (in general)
  - Two conditions p and q are compatible when $C[pq]$

Ordering

- Let $p \leq q \quad :\equiv \quad \forall r^{\kappa}(C[pr] \Rightarrow C[qr])$
- $\leq$ is a preorder with greatest element 1:

$$\begin{array}{lcl} \lambda c\,.\,c & : & \forall p^{\kappa}\ (p \leq p) \\ \lambda xyc\,.\,y(xc) & : & \forall p^{\kappa}\ \forall q^{\kappa}\ \forall r^{\kappa}\ (p \leq q \Rightarrow q \leq r \Rightarrow p \leq r) \\ \alpha_8 \circ \alpha_2 & : & \forall p^{\kappa}\ (p \leq 1) \end{array}$$

- Product pq is the g.l.b. of p and q:

$$\begin{array}{lcl} \alpha_9 & : & \forall p^{\kappa}\ \forall q^{\kappa}\ (pq \leq p) \\ \alpha_{10} & : & \forall p^{\kappa}\ \forall q^{\kappa}\ (pq \leq q) \\ \lambda xy\,.\,\alpha_{13} \circ y \circ \alpha_{12} \circ x \circ \alpha_{11} & : & \forall p^{\kappa}\ \forall q^{\kappa}\ \forall r^{\kappa}\ (r \leq p \Rightarrow r \leq q \Rightarrow r \leq pq) \end{array}$$

- C (set of 'good' conditions) is upwards closed:

$$\lambda xc\,.\,\alpha_1\,(x\,(\alpha_7\,c)) \quad : \quad \forall p^{\kappa}\ \forall q^{\kappa}\ (p \leq q \Rightarrow C[p] \Rightarrow C[q])$$

- Bad conditions are smallest elements:

$$\lambda xc\,.\,x\,(\alpha_1\,c) \quad : \quad \forall p^{\kappa}\ (\neg C[p] \Rightarrow \forall q^{\kappa}\ p \leq q)$$

The auxiliary translation $(\_)^*$

- Translating sorts: $\tau \rightsquigarrow \tau^*$

$$\iota^* \;:\equiv\; \iota \qquad\qquad o^* \;:\equiv\; \kappa \to o \qquad\qquad (\tau \to \sigma)^* \;:\equiv\; \tau^* \to \sigma^*$$

Intuition: Propositions become sets of conditions

- Translating terms: $M \rightsquigarrow M^*$

$$\begin{aligned}
(x^\tau)^* &:\equiv x^{\tau^*} & \mathsf{0}^* &:\equiv \mathsf{0}\\
(\lambda x^\tau \,.\, M)^* &:\equiv \lambda x^{\tau^*} \,.\, M^* & \mathsf{s}^* &:\equiv \mathsf{s}\\
(MN)^* &:\equiv M^* N^* & \mathsf{rec}_\tau^* &:\equiv \mathsf{rec}_{\tau^*}
\end{aligned}$$

$$\begin{aligned}
(\forall x^\tau A)^* &:\equiv \lambda r^\kappa \,.\, \forall x^{\tau^*} A^* r\\
(M_1 = M_2 \mapsto A)^* &:\equiv \lambda r^\kappa \,.\, M_1^* = M_2^* \mapsto (A^* r)\\
(A \Rightarrow B)^* &:\equiv \lambda r^\kappa \,.\, \forall q^\kappa \forall r'^\kappa (r = qr' \mapsto \forall s^\kappa (C[qs] \Rightarrow A^* s) \Rightarrow B^* r')
\end{aligned}$$

Lemma

- $(M[x^\tau := N])^* \;\equiv\; M^*[x^{\tau^*} := N^*]$ (substitutivity)
- If $M_1 \cong_{\mathcal{E}} M_2$, then $M_1^* \cong_{\mathcal{E}^*} M_2^*$ (compatibility with conversion)

The forcing translation

- Given a proposition A and a condition p, let:

$$p \Vdash A \quad :\equiv \quad \forall r^{\kappa}(C[pr] \Rightarrow A^{*}r)$$

  - The forcing translation is trivial on $\forall$ and $\_ = \_ \mapsto \_$:

$$\begin{array}{rcl} p \Vdash \forall x^{\tau} A & \cong_{\varnothing} & \forall x^{\tau^{*}} (p \Vdash A) \\ p \Vdash M_1 = M_2 \mapsto A & \cong_{\varnothing} & M_1^{*} = M_2^{*} \mapsto (p \Vdash A) \end{array}$$

  - All the complexity lies in implication! (cf next slide)

General properties

$$\begin{array}{lcll} \beta_1 & :\equiv & \lambda xyc \,.\, y\,(x\,c) & : \quad \forall p^{\kappa}\ \forall q^{\kappa}\ (q \leq p \Rightarrow (p \Vdash A) \Rightarrow (q \Vdash A)) \\ \beta_2 & :\equiv & \lambda xc \,.\, x\,(\alpha_1\,c) & : \quad \forall p^{\kappa}\ (\neg C[p] \Rightarrow p \Vdash A) \\ \beta_3 & :\equiv & \lambda xc \,.\, x\,(\alpha_9\,c) & : \quad \forall p^{\kappa}\ \forall q^{\kappa}\ ((p \Vdash A) \Rightarrow (pq \Vdash A)) \\ \beta_4 & :\equiv & \lambda xc \,.\, x\,(\alpha_{10}\,c) & : \quad \forall p^{\kappa}\ \forall q^{\kappa}\ ((q \Vdash A) \Rightarrow (pq \Vdash A)) \end{array}$$

Forcing an implication

- Definition of $p \Vdash A \Rightarrow B$ looks strange:

$$\begin{array}{rcl} p \Vdash A \Rightarrow B & \equiv & \forall r^{\kappa}(C[pr] \Rightarrow (A \Rightarrow B)^{*} r) \\ & \cong_{\varnothing} & \forall r^{\kappa}(C[pr] \Rightarrow \forall q^{\kappa} \forall r'^{\kappa}(r = qr' \mapsto (q \Vdash A) \Rightarrow B^{*} r')) \end{array}$$

- But it is equivalent to

$$\forall q\,((q \Vdash A) \Rightarrow (pq \Vdash B)) \qquad \left(\textbf{Hint:}\quad \frac{p \Vdash A \Rightarrow B \qquad q \Vdash A}{pq \Vdash B}\right)$$

Coercions between $p \Vdash A \Rightarrow B$ and $\forall q\,((q \Vdash A) \Rightarrow (pq \Vdash B))$

$$\begin{array}{llll} \gamma_1 & :\equiv & \lambda xcy\,.\,x\,y\,(\alpha_6\,c) & : \quad (\forall q\,((q \Vdash A) \Rightarrow (pq \Vdash B)) \;\Rightarrow\; p \Vdash A \Rightarrow B) \\ \gamma_2 & :\equiv & \lambda xyc\,.\,x\,(\alpha_5\,c)\,y & : \quad (p \Vdash A \Rightarrow B) \;\Rightarrow\; \forall q\;((q \Vdash A) \Rightarrow (pq \Vdash B)) \\ \gamma_3 & :\equiv & \lambda xyc\,.\,x\,(\alpha_{11}\,c)\,y & : \quad (p \Vdash A \Rightarrow B) \;\Rightarrow\; (p \Vdash A) \;\Rightarrow\; (p \Vdash B) \\ \gamma_4 & :\equiv & \lambda xcy\,.\,x\,(y\,(\alpha_{15}\,c)) & : \quad \neg A^{*}\,p \;\Rightarrow\; p \Vdash A \Rightarrow B \end{array}$$

The meaning of the definition of "$p \Vdash A$" (1/2)

Where does the definition of "$p \Vdash A$" come from?

$$p \Vdash A \quad :\equiv \quad \forall r^{\kappa}(C[pr] \Rightarrow A^{*}r)$$

The Boolean algebra generated by the forcing structure

- Given p^{κ}, r^{κ} let: $p \perp r \ :\equiv \ \neg C[pr]$ ("p and r are incompatible")
- To each set of conditions $S^{\kappa \to o}$ we associate its orthogonal
$$S^{\perp} \ := \ \{p^{\kappa} \ : \ \forall r^{\kappa}\,(S\,r \Rightarrow p \perp r)\} \qquad (: \kappa \to o)$$
- Write: $\mathcal{B} \ := \ \{S^{\kappa \to o} \ : \ S = S^{\perp\perp}\} \ : \ (\kappa \to o) \to o$
  the set of all sets that are bi-orthogonally closed

Proposition

The poset $(\mathcal{B}, \subseteq)$ is a complete Boolean algebra

$(\mathcal{B}, \subseteq)$ is the Boolean algebra generated by the forcing structure $(\kappa, \cdot, 1, C)$

The meaning of the definition of “$p \Vdash A$” (2/2)

- Given $p, r : \kappa$, $S : \kappa \to o$, recall that:

$$\begin{array}{rcl} p \perp r & :\equiv & \neg C[pr] \\ S^{\perp} & :\equiv & \{p^{\kappa} \ : \ \forall r^{\kappa}\,(S\,r \Rightarrow p \perp r)\} \\ \mathcal{B} & :\equiv & \{S^{\kappa \to o} \ : \ S = S^{\perp\perp}\} \end{array}$$

Proposition

The poset $(\mathcal{B}, \subseteq)$ is a complete Boolean algebra

- **Fact:** For each set $S^{\kappa \to o}$, we have $S^{\perp} = S^{\perp\perp\perp}$, hence $S^{\perp} \in \mathcal{B}$
- Recall that the translation $M \mapsto M^*$ turns each proposition $A : o$ into a set $A^* : \kappa \to o$. Then we observe that:

$$\begin{array}{rcl} \{p^{\kappa} \ : \ p \Vdash A\} & = & \{p^{\kappa} \ : \ \forall r^{\kappa}\,(C[pr] \Rightarrow A^* r)\} \\ & = & \{p^{\kappa} \ : \ \forall r^{\kappa}\,(\neg A^*\,r \Rightarrow p \perp r)\} \\ & = & ((A^*)^c)^{\perp} \ \in \ \mathcal{B} \end{array}$$

Translating proof-terms

- Krivine's program transformation $t \mapsto t^*$:

$$\begin{array}{rcl}
x^* & \equiv & x \qquad\qquad \text{\oe}^* \equiv \lambda c x \,.\, \text{\oe}\,(\lambda k \,.\, x\,(\alpha_{14}\, c)\,(\gamma_4\, k)) \\
(t\, u)^* & \equiv & \gamma_3\, t^*\, u^* \\
(\lambda x \,.\, t)^* & \equiv & \gamma_1\, (\lambda x \,.\, t^*\, \underbrace{[x := \beta_4\, x]}_{\text{bounded var}}\, \underbrace{[x_i := \beta_3\, x_i]_{i=1}^{n}}_{\text{other free vars of } t})
\end{array}$$

$$\begin{array}{l}
\gamma_4 \equiv \lambda x c y \,.\, x\,(y\,(\alpha_{15}\, c)) \\
\gamma_3 \equiv \lambda x y c \,.\, x\,(\alpha_{11}\, c)\, y \\
\gamma_1 \equiv \lambda x c y \,.\, x\, y\,(\alpha_6\, c) \\
\beta_3 \equiv \lambda x c \,.\, x\,(\alpha_9\, c) \\
\beta_4 \equiv \lambda x c \,.\, x\,(\alpha_{10}\, c)
\end{array}$$

  - The translation inserts: γ_3 ("apply") in front of each app.
    γ_1 ("fold") in front of each λ
  - A bound occurrence of x in t is translated as $\beta_3^k\,(\beta_4\, x)$, where k is the de Bruijn index of this occurrence

Soundness (in PAω^+)

If $\mathcal{E};\ x_1 : A_1,\ \ldots,\ x_n : A_n \ \vdash\ t\ :\ B$
then $\mathcal{E}^*;\ x_1 : (p \Vdash A_1),\ \ldots,\ x_n : (p \Vdash A_n) \ \vdash\ t^*\ :\ (p \Vdash B)$

Translating proof-terms (optimized)

- The latter program transformation creates bureaucratic β-redexes due to the macros β_3, β_4, γ_3, γ_1 and γ_4
- If we reduce them, we get the following transformation:

$$x^* \equiv x \qquad \text{œ}^* \equiv \lambda cx\,.\,\text{œ}\,(\lambda k\,.\,x\,(\alpha_{14}\,c)\,(\lambda cx\,.\,k\,(x\,(\alpha_{15}\,c))))$$

$$(t\,u)^* \equiv \lambda c\,.\,t^*\,(\alpha_6\,c)\,u^*$$

$$(\lambda x\,.\,t)^* \equiv \lambda cx\,.\,t^*\,\underbrace{[x := \lambda c\,.\,x\,(\alpha_{10}\,c)]}_{\text{bounded var}}\,\underbrace{[x_i := \lambda c\,.\,x_i\,(\alpha_9\,c)]_{i=1}^{n}}_{\text{other free vars of } t}\,(\alpha_{11}\,c)$$

Soundness (in PAω^+)

If $\mathcal{E};\ x_1 : A_1,\ \ldots,\ x_n : A_n \ \vdash\ t\ :\ B$

then $\mathcal{E}^*;\ x_1 : (p \Vdash A_1),\ \ldots,\ x_n : (p \Vdash A_n)\ \vdash\ t^*\ :\ (p \Vdash B)$

Computational meaning of the transformation

- A proof of $p \Vdash A \;\equiv\; \forall r^\kappa (C[pr] \Rightarrow A^* r)$ is a function waiting an argument $c : C[pr]$ (for some r) $\rightsquigarrow$ computational condition

$$\begin{array}{rcllcrcl}
(\lambda x\,.\,t)^* & \star & c \cdot u \cdot \pi & \succ & t^*[x := \lambda c'\,.\,u\,(\alpha_{10}\,c')] & \star & \alpha_6\,c \cdot \pi \\
(tu)^* & \star & c \cdot \pi & \succ & t^* & \star & \alpha_{11}\,c \cdot u^* \cdot \pi \\
\mathrm{c\!c}^* & \star & c \cdot t \cdot \pi & \succ & t & \star & \alpha_{14}\,c \cdot \mathsf{k}_\pi^* \cdot \pi \\
\mathsf{k}_\pi^* & \star & c \cdot t \cdot \pi' & \succ & t & \star & \alpha_{15}\,c \cdot \pi
\end{array}$$

where: $\mathsf{k}_\pi^* \;\equiv\; \gamma_4\,\mathsf{k}_\pi \quad (\approx \lambda c x\,.\,\mathsf{k}_\pi\,(x\,(\alpha_{15}\,c)))$

Evaluation combinators

$$\begin{array}{rcccc}
\alpha_{10} & : & C[(pq)r] & \Rightarrow & C[qr] \\
\alpha_6 & : & C[p(qr)] & \Rightarrow & C[(pq)r] \\
\alpha_{11} & : & C[pr] & \Rightarrow & C[p(pr)] \\
\alpha_{14} & : & C[p(qr)] & \Rightarrow & C[q(rr)] \\
\alpha_{15} & : & C[p(qr)] & \Rightarrow & C[qp]
\end{array}$$

Plan

Krivine Forcing Abstract Machine (KFAM)

$$
\begin{array}{llcllllllll}
\textbf{Terms} & t, u & ::= & x & \mid & \lambda x\,.\,t & \mid & tu & \mid & \text{\oe} \\
\textbf{Environments} & e & ::= & \emptyset & \mid & e, x = c & & & & \\
\textbf{Closures} & c & ::= & t[e] & \mid & \mathsf{k}_\pi & \mid & \underbrace{t[e]^* \quad \mid \quad \mathsf{k}_\pi^*}_{\text{forcing closures}} & & \\
\textbf{Stacks} & \pi & ::= & \diamond & \mid & c \cdot \pi & & & &
\end{array}
$$

- Evaluation rules: real mode:

$$
\begin{array}{rcllcll}
x[e, y = c] & \star & \pi & \succ & x[e] & \star & \pi & (y \not\equiv x) \\
x[e, x = c] & \star & \pi & \succ & c & \star & \pi & \\
(\lambda x\,.\,t)[e] & \star & c \cdot \pi & \succ & t[e, x = c] & \star & \pi & \\
(tu)[e] & \star & \pi & \succ & t[e] & \star & u[e] \cdot \pi & \\
\text{\oe}[e] & \star & c \cdot \pi & \succ & c & \star & \mathsf{k}_\pi \cdot \pi & \\
\mathsf{k}_\pi & \star & c \cdot \pi' & \succ & c & \star & \pi &
\end{array}
$$

- Evaluation rules: forcing mode:

$$
\begin{array}{rcllcll}
x[e, y = c]^* & \star & c_0 \cdot \pi & \succ & x[e]^* & \star & \alpha_9\, c_0 \cdot \pi & (y \not\equiv x) \\
x[e, x = c]^* & \star & c_0 \cdot \pi & \succ & c & \star & \alpha_{10}\, c_0 \cdot \pi & \\
(\lambda x\,.\,t)[e]^* & \star & c_0 \cdot c \cdot \pi & \succ & t[e, x = c]^* & \star & \alpha_6\, c_0 \cdot \pi & \\
(tu)[e]^* & \star & c_0 \cdot \pi & \succ & t[e]^* & \star & \alpha_{11}\, c_0 \cdot u[e]^* \cdot \pi & \\
\text{\oe}[e]^* & \star & c_0 \cdot c \cdot \pi & \succ & c & \star & \alpha_{14}\, c_0 \cdot \mathsf{k}_\pi^* \cdot \pi & \\
\mathsf{k}_\pi^* & \star & c_0 \cdot c \cdot \pi' & \succ & c & \star & \alpha_{15}\, c_0 \cdot \pi &
\end{array}
$$

Adequacy in real and forcing modes

- New abstract machine means:
  - New classical realizability model (based on the KFAM)
  - New adequacy results

Adequacy (real mode)

If
- $\mathcal{E}; x_1 : A_1, \ldots, x_n : A_n \vdash t : B$ (in PAω^+)
- $\rho \models \mathcal{E}, \quad c_1 \Vdash A_1[\rho], \ \ldots, \ c_n \Vdash A_n[\rho]$

then: $t[x_1 = c_1, \ldots, x_n = c_n] \ \Vdash \ B[\rho]$ (real mode)

- Assuming that $\alpha_i \Vdash$ type of α_i (for $i = 6, 9, 10, 11, 14, 15$)

Adequacy (forcing mode)

If
- $\mathcal{E}; x_1 : A_1, \ldots, x_n : A_n \vdash t : B$ (in PAω^+)
- $\rho \models \mathcal{E}^*, \quad c_1 \Vdash (p_1 \ \mathsf{IF} \ A_1[\rho]), \ \ldots, \ c_n \Vdash (p_n \ \mathsf{IF} \ A_n[\rho])$

then: $t[x_1 = c_1, \ldots, x_n = c_n]^* \ \Vdash \ ((p_0 p_1) \cdots p_n \ \mathsf{IF} \ B[\rho])$ (forcing mode)

Program extraction in presence of forcing

- Assume that:
  1. We got a proof of B under some axiom A
     $$x : A \vdash u : B \qquad \text{(user program)}$$
  2. Axiom A is not provable, but it can be forced using a suitable set of forcing conditions $(C, \leq)$:
     $$\vdash s : (1 \Vdash A) \qquad \text{(system program)}$$
- Then:
  1. We have $$u[x = s[]]^* \ \Vdash \ (1 \Vdash B)$$
  2. If moreover B is an arithmetic formula
     $$(\xi_B\ z)[z = u[x = s[]]^*] \ \Vdash \ B$$
     using a suitable wrapper $\xi_B \Vdash (1 \Vdash B) \Rightarrow B$

Plan

Realizability algebras

[Krivine '10, M. '11]

Definition

A realizability algebra $\mathscr{A}$ is given by:

- 3 sets $\mathbf{\Lambda}$ ($\mathscr{A}$-terms), $\mathbf{\Pi}$ ($\mathscr{A}$-stacks), $\mathbf{\Lambda} \star \mathbf{\Pi}$ ($\mathscr{A}$-processes)
- 3 functions $(\cdot) : \mathbf{\Lambda} \times \mathbf{\Pi} \to \mathbf{\Pi}$, $(\star) : \mathbf{\Lambda} \times \mathbf{\Pi} \to \mathbf{\Lambda} \star \mathbf{\Pi}$, $(\mathsf{k}_{\_}) : \mathbf{\Pi} \to \mathbf{\Lambda}$
- A compilation function $(t, \sigma) \mapsto t[\sigma]$ that takes:
  - an open proof term t
  - a $\mathbf{\Lambda}$-substitution σ closing t

  and returns an $\mathscr{A}$-term $t[\sigma] \in \mathbf{\Lambda}$
- A set of $\mathscr{A}$-processes $\bot\!\!\!\bot \subseteq \mathbf{\Lambda} \star \mathbf{\Pi}$ such that:

$$\begin{array}{rlcrl}
\sigma[x] \star \pi & \in \bot\!\!\!\bot & \text{implies} & x[\sigma] \star \pi & \in \bot\!\!\!\bot \\
t[\sigma, x := a] \star \pi & \in \bot\!\!\!\bot & \text{implies} & (\lambda x \,.\, t)[\sigma] \star a \cdot \pi & \in \bot\!\!\!\bot \\
t[\sigma] \star u[\sigma] \cdot \pi & \in \bot\!\!\!\bot & \text{implies} & (tu)[\sigma] \star \pi & \in \bot\!\!\!\bot \\
a \star \mathsf{k}_{\pi} \cdot \pi & \in \bot\!\!\!\bot & \text{implies} & \mathsf{cc}[\sigma] \star a \cdot \pi & \in \bot\!\!\!\bot \\
a \star \pi & \in \bot\!\!\!\bot & \text{implies} & \mathsf{k}_{\pi} \star a \cdot \pi' & \in \bot\!\!\!\bot
\end{array}$$

Realizability model of PAω^+ (general case)

- Parameterized by a realizability algebra $\mathscr{A} = (\mathbf{\Lambda}, \mathbf{\Pi}, \mathbf{\Lambda} \star \mathbf{\Pi}, \ldots, \perp\!\!\!\perp)$
- Interpreting higher-order terms:
  - Individuals interpreted as natural numbers $[\![\iota]\!] \ := \ \mathbb{N}$
  - Propositions interpreted as $\mathscr{A}$-falsity values $[\![o]\!] \ := \ \mathfrak{P}(\mathbf{\Pi})$
  - Functions interpreted set-theoretically $[\![\tau \to \sigma]\!] \ := \ [\![\sigma]\!]^{[\![\tau]\!]}$
- Interpreting logical constructions

$$[\![\forall x^\tau \, A]\!] \ = \bigcup_{v \in [\![\tau]\!]} [\![A[x := \dot{v}]]\!] \qquad [\![A \Rightarrow B]\!] \ = \ [\![A]\!]^{\perp\!\!\!\perp} \cdot [\![B]\!]$$

$$[\![M = M' \mapsto A]\!] \ = \ \begin{cases} [\![A]\!] & \text{if } [\![M]\!] = [\![M']\!] \\ \varnothing & \text{otherwise} \end{cases}$$

Adequacy

If
- $\mathcal{E}; x_1 : A_1, \ldots, x_k : A_k \vdash t : B$ (in PAω^+)
- $\rho \models \mathcal{E}, \quad u_1 \Vdash A_1[\rho], \ldots, u_k \Vdash A_k[\rho]$

then: $t[x_1 := u_1, \ldots, x_k := u_k] \Vdash B[\rho]$

Examples (1/2)

- From an implementation of the λ_c-calculus:

Standard realizability algebra

- $\mathbf{\Lambda} := \Lambda, \quad \mathbf{\Pi} := \Pi, \quad \mathbf{\Lambda} \star \mathbf{\Pi} := \Lambda \star \Pi$
- $k_\pi, \quad t \cdot \pi, \quad t \star \pi$ defined as themselves
- Compilation function $(t, \sigma) \mapsto t[\sigma]$ defined as substitution
- $\perp\!\!\!\perp$:= any saturated set of processes

- We can do the same for all classical λ-calculi :
  - Parigot's $\lambda\mu$-calculus
  - Curien-Herbelin's $\bar{\lambda}\mu$-calculus (CBN or CBV)
  - Barbanera-Berardi's symmetric λ-calculus ($\pitchfork$ comes for free)

Examples (2/2)

- From a meet semi-lattice L:

- $\mathbf{\Lambda} = \mathbf{\Pi} = \mathbf{\Lambda} \star \mathbf{\Pi} := L$
- $k_\pi := \pi, \quad t \star \pi = t \star \pi := t\pi$ (product in L)
- Compilation function $(t, \sigma) \mapsto t[\sigma]$:

$$t[\sigma] := \prod_{x \in FV(t)} \sigma(x)$$

- $\perp\!\!\!\perp := $ any ideal of L

- Corresponding realizability model isomorphic to the Boolean valued model on the complete Boolean algebra $\mathcal{B}(L)/\perp\!\!\!\perp$

KFAM: The realizability algebra of real mode

- From a saturated set $\perp\!\!\!\perp$ in the KFAM:

The realizability algebra $\mathscr{A} \ := \ (\mathbf{\Lambda}, \mathbf{\Pi}, \mathbf{\Lambda} \star \mathbf{\Pi}, \ldots, \perp\!\!\!\perp)$

- $\mathbf{\Lambda}$, $\mathbf{\Pi}$, $\mathbf{\Lambda} \star \mathbf{\Pi}$ $:=$ sets of closures, stacks, processes of the KFAM
- k_π (real mode), $t \cdot \pi$, $t \star \pi$ defined as in the KFAM
- Compilation function: $(t, [\sigma]) \mapsto t[\sigma]$ $:=$ closure formation (real mode)
- $\perp\!\!\!\perp$ $:=$ itself

- Adequacy w.r.t. the algebra $\mathscr{A}$ $=$

  Adequacy in the KFAM in real mode (w.r.t. the pole $\perp\!\!\!\perp$)

KFAM: The realizability algebra of forcing mode

- Given $\mathscr{A} = (\mathbf{\Lambda}, \mathbf{\Pi}, \mathbf{\Lambda} \star \mathbf{\Pi}, \ldots, \perp\!\!\!\perp)$ (cf prev. slide)
  $+$ a forcing structure $(\kappa, C, \cdot, 1)$

The realizability algebra $\mathscr{A}^* := (\mathbf{\Lambda}^*, \mathbf{\Pi}^*, \mathbf{\Lambda}^* \star \mathbf{\Pi}^*, \ldots, \perp\!\!\!\perp^*)$

- $\mathbf{\Lambda}^* := \mathbf{\Lambda} \times [\![\kappa]\!]$, $\mathbf{\Pi}^* := \mathbf{\Pi} \times [\![\kappa]\!]$, $\mathbf{\Lambda}^* \star \mathbf{\Pi}^* := (\mathbf{\Lambda} \star \mathbf{\Pi}) \times [\![\kappa]\!]$
- $\mathsf{k}_{(\pi,p)} := (\mathsf{k}^*_\pi,\, p)$ (forcing mode)
- $(t,\, p) \cdot (\pi,\, q) := (t \cdot \pi,\, pq)$
- $(t,\, p) \star (\pi,\, q) := (t \star \pi,\, pq)$
- Compilation function:

$$t[x_1 := (c1,\, p_1), \ldots, x_k := (c_k,\, p_k)] \;:\equiv\; (t[x_1 := c_1, \ldots, x_k := c_k]^*,\, ((1p_1)\cdots)p_k)$$

(forcing mode)

- $\perp\!\!\!\perp^* := \{(t \star \pi, p) \;:\; \forall c \in \mathbf{\Lambda}\; ((c \Vdash_{\mathscr{A}} C[p]) \Rightarrow (t \star c \cdot \pi) \in \perp\!\!\!\perp)$

The connection lemma

- Write $[\![\_]\!]$ (resp. $[\![\_]\!]^*$) the interpretation w.r.t. $\mathscr{A}$ (resp. w.r.t. $\mathscr{A}^*$)
- Notice that: $[\![o]\!]^* \;=\; \mathfrak{P}(\Pi \star [\![\kappa]\!]) \;\simeq\; (\mathfrak{P}(\Pi))^{[\![\kappa]\!]} \;=\; [\![o^*]\!]$

Connection lemma

1. There exists an iso: $\psi_\tau \;:\; [\![\tau^*]\!] \overset{\sim}{\to} [\![\tau]\!]^*$
2. For all closed M of sort τ: $[\![M]\!]^* \;=\; \psi_\tau([\![M^*]\!])$
3. Given a closed formula A and a pair $(c, p) \in \mathbf{\Lambda}^*$ $(= \mathbf{\Lambda} \star [\![\kappa]\!])$

$$(c, p) \Vdash_{\mathscr{A}^*} A \qquad \text{iff} \qquad c \Vdash_{\mathscr{A}} (p \mathrel{\mathrm{I\!F}} A)$$

- Connection lemma + Adequacy w.r.t. the algebra $\mathscr{A}^*$ =

  Adequacy in the KFAM in forcing mode (w.r.t. the pole $\perp\!\!\!\perp$)

To sum up

- **From syntax...**
  - The program transform $t \mapsto t^*$ underlying Cohen forcing :

    $$\vdash t : A \qquad \rightsquigarrow \qquad \vdash t^* : (p \Vdash A)$$

  - A new machine (KFAM) with two execution modes such that

    $$t[]^* \quad \text{has the same behavior as} \quad t^*[]$$

- **... to semantics : iterated forcing**
  - Two realizability algebras $\mathscr{A}$ and $\mathscr{A}^*$ related by

    $$(c, p) \Vdash_{\mathscr{A}^*} A \qquad \text{iff} \qquad c \Vdash_{\mathscr{A}} (p \Vdash A)$$

  - Two adequacy lemmas (real/forcing) as instances of the general lemma of adequacy

Plan

Conclusion

Underlying methodology

Translation of formulas & proofs ⇝ Classical program transformation ⇝ New abstract machine (no transformation)

- This methodology applies to the forcing translation (Cohen)
    - Computational meaning of the underlying program transformation
    - A new abstract machine: the KFAM
    - Reminiscent from well-known tricks of computer architecture (protection rings, virtualization, monitoring...)
- New insights in logic:
    - Logical meaning of explicit environments
    - Logical meaning of a particular side effect
    - Backtrack defines the limit between the stack and the memory

Related and future work

- How this computation model works in practical cases of forcing?

  $\rightsquigarrow$ Need to take into account the generic set G

  - Particular case when $C[p]$ is a data type:
    Lionel Rieg: Herbrand theorem by forcing (PhD thesis) [2014]
  - Variations on the same theme, in a linear setting:
    Aloïs Brunel: *The Monitoring Power of Forcing* (PhD) [2014]
  - Formalization of the generic set in PAω^+ (general case):
    Pierre Pradic (Master 2 thesis) [2015]

- Does the same methodology apply to other logical translations?
  - Pierre-Marie Pédrot: *A Materialist Dialectica* (PhD) [2015]

- Use this methodology the other way around!
  - Deduce new logical translations from computation models
    borrowed to computer architecture, operating systems...

- Towards an integration of side effects into the CH correspondence?