Implicative algebras: a new foundation for forcing and realizability

Alexandre Miquel

UNIVERSIDAD DE LA REPUBLICA URUGUAY

FACULTAD DE INGENIERIA

EQUIPO DE LÓGICA · UDELAR

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EN: **implicative** *adj.*
1. LOGIC: relative to implication

ES: **implicativo** *adj.* (*f.* **implicativa**)
1. LÓGICA: relativo a la implicación

FR: **implicatif** *adj.* (*f.* **implicative**)
1. LOGIQUE: relatif à l'implication

DE: **implikativ** *adj.*
1. LOGIK: bezüglich der Implikation

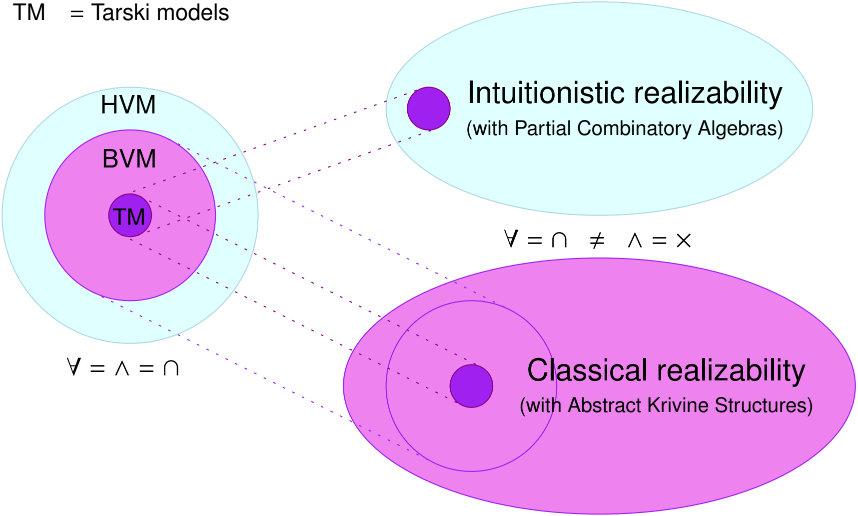
...

Different notions of models (1/2)

- **Tarski models:** $[\![\phi]\!] \in \{0;1\}$
  - Interprets classical provability (correctness/completeness)
- **Intuitionistic realizability:** $[\![\phi]\!] \in \mathfrak{P}(\Lambda)$ [Kleene '45]
  - Interprets intuitionistic proofs
  - Independence results in intuitionistic theories
  - Definitely incompatible with classical logic
- **Cohen forcing:** $[\![\phi]\!] \in \mathfrak{P}(C)$ [Cohen '63]
  - Independence results, in classical theories
    (Negation of continuum hypothesis, Solovay's axiom, etc.)
- **Boolean-valued models:** $[\![\phi]\!] \in \mathcal{B}$ [Scott, Solovay, Vopěnka]
- **Classical realizability:** $[\![\phi]\!] \in \mathfrak{P}(\Lambda_c)$ [Krivine '94, '01, '03, '09–]
  - Interprets classical proofs
  - Generalizes Tarski models... and forcing!

Different notions of models (2/2)

HVM = Heyting-valued models ≈ Kripke forcing
BVM = Boolean-valued models ≈ Cohen forcing
TM = Tarski models



The algebraic structures underlying forcing

Definition (Heyting/Boolean algebras)

1. A Heyting algebra is a bounded lattice $(H, \leq)$ that has relative pseudo-complements

$$a \to b \;:=\; \max\{c \in H : (c \wedge a) \leq b\} \qquad \text{(Heyting's implication)}$$

for all $a, b \in H$, so that we get the adjunction:

$$(c \wedge a) \leq b \quad \Leftrightarrow \quad c \leq (a \to b) \qquad \text{(Heyting's adjunction)}$$

2. A Boolean algebra is a Heyting algebra in which the operation of negation $\neg a \;:=\; (a \to \bot)$ is involutive:

$$\neg\neg a \;\; (= (a \to \bot) \to \bot) \;=\; a \qquad (a \in H)$$

3. A Heyting/Boolean algebra is complete when the underlying lattice is

- Complete Heyting algebras $\Rightarrow$ Kripke (i.e. intuitionistic) forcing
- Complete Boolean algebras $\Rightarrow$ Cohen (i.e. classical) forcing

The algebraic structures underlying realizability (1/2)

Definition (Partial combinatory algebras)

1. A partial applicative structure (PAS) is a set P together with a partial operation $(\cdot) : P \times P \rightharpoonup P$ called application
2. A partial combinatory algebra (PCA) is a PAS $(P, \cdot)$ containing two elements $\mathbf{K}, \mathbf{S} \in P$ such that for all $x, y, z \in P$:

$$\begin{aligned} \mathbf{K} \cdot x \cdot y &\downarrow= x \\ \mathbf{S} \cdot x \cdot y &\downarrow \\ \mathbf{S} \cdot x \cdot y \cdot z &\downarrow= (x \cdot z) \cdot (y \cdot z) \end{aligned}$$

(whenever the rhs is defined)

3. A combinatory algebra (CA) is a PCA whose application is total

Examples:

- $P := \Lambda/\beta\eta$ equipped with application is a (total) CA
- $P := \mathbb{N}$ equipped with Kleene application is a PCA

The algebraic structures underlying realizability (2/2)

Definition (Abstract Krivine structure)

An Abstract Krivine structure (AKS) $\mathscr{A}$ is given by:

- 2 sets $\mathbf{\Lambda}$ ($\mathscr{A}$-terms), $\mathbf{\Pi}$ ($\mathscr{A}$-stacks)
- 3 functions $(@) : \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{\Lambda}$, $(\cdot) : \mathbf{\Lambda} \times \mathbf{\Pi} \to \mathbf{\Pi}$, $(\mathbf{k}_{\_}) : \mathbf{\Pi} \to \mathbf{\Lambda}$
- 3 combinators $\mathbf{S}, \mathbf{K}, \mathbf{cc} \in \mathbf{\Lambda}$
- A subset $\mathrm{PL} \subseteq \mathbf{\Lambda}$ (of proof-like $\mathscr{A}$-terms) that contains the combinators $\mathbf{S}, \mathbf{K}, \mathbf{cc}$ and that is closed under application $(@)$.
- A binary relation $\perp\!\!\!\perp \subseteq \mathbf{\Lambda} \times \mathbf{\Pi}$ (the pole) such that:

$$
\begin{array}{rcl}
t \star u \cdot \pi \in \perp\!\!\!\perp & \text{implies} & tu \star \pi \in \perp\!\!\!\perp \\
t \star \pi \in \perp\!\!\!\perp & \text{implies} & \mathbf{K} \star t \cdot u \cdot \pi \in \perp\!\!\!\perp \\
tv(uv) \star \pi \in \perp\!\!\!\perp & \text{implies} & \mathbf{S} \star t \cdot u \cdot v \cdot \pi \in \perp\!\!\!\perp \\
t \star \mathbf{k}_\pi \cdot \pi \in \perp\!\!\!\perp & \text{implies} & \mathbf{cc} \star t \cdot \pi \in \perp\!\!\!\perp \\
t \star \pi \in \perp\!\!\!\perp & \text{implies} & \mathbf{k}_\pi \star t \cdot \pi' \in \perp\!\!\!\perp
\end{array}
$$

Unifying all kinds of models

Aim: Define an algebraic structure to encompass:

- Complete Heyting Algebras (for Heyting-valued models, Kripke forcing)
- Complete Boolean Algebras (for Boolean-valued models, Cohen forcing)
- Partial Combinatory Algebras (for Intuitionistic realizability)
- Ordered Combinatory Algebras (for Intuitionistic realizability)
- Abstract Krivine Structures (for Classical realizability)

Implicative algebras can be used to construct:

- Categorical models (triposes, toposes)
- Models of (intuitionistic/classical) set theory

Underlying ideas are reminiscent from earlier work of

- Ruyer '07, Streicher '13 (and many others!)

Plan

Plan

Implicative structures

Definition (Implicative structure)

An implicative structure is a triple $(\mathscr{A}, \preccurlyeq, \to)$ where

(1) $(\mathscr{A}, \preccurlyeq)$ is a complete (meet semi-)lattice

(2) $(\to) : \mathscr{A}^2 \to \mathscr{A}$ is a binary operation such that:

(2a) $a' \preccurlyeq a,\ \ b \preccurlyeq b'$ entails $(a \to b) \preccurlyeq (a' \to b')$ $\quad (a, a', b, b' \in \mathscr{A})$

(2b) $\displaystyle\bigwedge_{b \in B}(a \to b) \ =\ a \to \bigwedge_{b \in B} b$ $\quad$ (for all $B \subseteq \mathscr{A}$)

- Write $\bot$ (resp. $\top$) the smallest (resp. largest) element of $\mathscr{A}$
- When $B = \varnothing$, axiom (2b) gives: $(a \to \top) = \top$ $\quad (a \in \mathscr{A})$

Examples of implicative structures

- Complete Heyting algebras $(\mathscr{A}, \preccurlyeq)$, where $\to$ is defined by:

$$a \to b \;:=\; \max\{c \in \mathscr{A} \;:\; (c \curlywedge a) \preccurlyeq b\} \qquad \text{(Heyting's implication)}$$

$+$ complete Boolean algebras (as a particular case of Heyting algebras)

- Given a (total) combinatory algebra $(P, \cdot, \mathbf{K}, \mathbf{S})$, we let:
  - $\mathscr{A} \;:=\; \mathfrak{P}(P)$
  - $a \preccurlyeq b \;:=\; a \subseteq b$
  - $a \to b \;:=\; \{z \in P : \forall x \in a,\; z \cdot x \in b\}$ (Kleene's implication)

Note: if we do the same with a partial combinatory algebra (PCA), we only get a quasi-implicative structure (cf next slide)

$+$ similar construction for ordered combinatory algebras (OCA)

- Given an abstract Krivine structure $(\boldsymbol{\Lambda}, \boldsymbol{\Pi}, \ldots, \mathrm{PL}, \perp\!\!\!\perp)$, we let:
  - $\mathscr{A} \;:=\; \mathfrak{P}(\boldsymbol{\Pi})$
  - $a \preccurlyeq b \;:=\; a \supseteq b$
  - $a \to b \;:=\; a^{\perp\!\!\!\perp} \cdot b$ (Krivine's implication)

Relaxing the definition

In some situations, it is desirable to have $(a \to \top) \neq \top$

Definition (Quasi-implicative structure)

Same definition as for an implicative structure, but axiom

(2b) $$\bigwedge_{b \in B}(a \to b) \;=\; a \to \bigwedge_{b \in B} b \qquad (\text{if } B \neq \varnothing)$$

only required for the non-empty subsets $B \subseteq \mathscr{A}$

Examples:

- Each partial combinatory algebra $(P, \cdot, \mathbf{K}, \mathbf{S})$ more generally induces a quasi-implicative structure: $(\mathfrak{P}(P), \subseteq, \to)$

  This structure is an implicative structure iff application $\cdot$ is total

- Usual notions of reducibility candidates (Tait, Girard, Parigot, etc.) induce quasi-implicative structures (built from the λ-calculus)

Viewing truth values as (generalized) realizers (1/2)

- The Curry-Howard correspondence:

  Syntax: Proof = Program : Formula = Type

  Semantics: Realizer $\in$ Truth value

- But in most semantics, we can associate to every realizer t its principal type $[t]$, i.e. the smallest truth value containing t:

$$t \ : \ A \ \text{(typing)} \qquad \text{iff} \qquad [t] \subseteq A \ \text{(subtyping)}$$

- Identifying t with $[t]$, we get the inclusion:

$$\text{Realizers} \quad \subseteq \quad \text{Truth values}$$

- Moreover, we shall see that application and abstraction can be lifted at the level of truth values. Therefore:

$$\text{Truth values} \quad = \quad \text{Generalized realizers}$$

Viewing truth values as (generalized) realizers (2/2)

- Fundamental ideas underlying implicative structures:
  1. Operations on λ-terms can be lifted to truth values
  2. Truth values can be used as generalized realizers
  3. Realizers and truth values live in the same world!

$$\text{Proof} = \text{Program} = \text{Type} = \text{Formula}$$

(The ultimate Curry-Howard identification)

- In an implicative structure, the relation $a \preccurlyeq b$ may read:
  - a is a subtype of b (viewing a and b as truth values)
  - a has type b (viewing a as a realizer, b as a truth value)
  - a is more defined than b (viewing a and b as realizers)

- In particular:

ordering of sybtyping $\preccurlyeq$ $\equiv$ reverse Scott ordering $\sqsupseteq$

Encoding application & abstraction

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

Definition (Application & Abstraction)

Given $a, b \in \mathscr{A}$ and a function $f : \mathscr{A} \to \mathscr{A}$, we let:

$$ab \;:=\; \bigcurlywedge\{c \in \mathscr{A} \;:\; a \preccurlyeq (b \to c)\} \qquad \text{(application)}$$

$$\boldsymbol{\lambda} f \;:=\; \bigcurlywedge_{a \in \mathscr{A}} (a \to f(a)) \qquad \text{(abstraction)}$$

- **Properties:**
  1. If $a \preccurlyeq a'$ and $b \preccurlyeq b'$, then $ab \preccurlyeq a'b'$ (Monotonicity)
  2. If $f \preccurlyeq g$ (pointwise), then $\boldsymbol{\lambda} f \preccurlyeq \boldsymbol{\lambda} g$ (Monotonicity)
  3. $(\boldsymbol{\lambda} f)a \;\preccurlyeq\; f(a)$ (β-reduction)
  4. $a \;\preccurlyeq\; \boldsymbol{\lambda}(x \mapsto ax)$ (η-expansion)
  5. $ab \preccurlyeq c$ iff $a \preccurlyeq (b \to c)$ (Adjunction)

Encoding the λ-calculus

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \rightarrow)$ be an implicative structure

- To each closed λ-term t with parameters (i.e. constants) in $\mathscr{A}$, we associate a truth value $t^{\mathscr{A}} \in \mathscr{A}$:

$$\begin{array}{rcl} a^{\mathscr{A}} & := & a \\ (\lambda x\,.\,t)^{\mathscr{A}} & := & \boldsymbol{\lambda}(a \mapsto (t[x := a])^{\mathscr{A}}) \\ (tu)^{\mathscr{A}} & := & t^{\mathscr{A}} u^{\mathscr{A}} \end{array}$$

- **Properties:**
  - β-rule: If $t \twoheadrightarrow_{\beta} t'$, then $(t)^{\mathscr{A}} \preccurlyeq (t')^{\mathscr{A}}$
  - η-rule: If $t \twoheadrightarrow_{\eta} t'$, then $(t)^{\mathscr{A}} \succcurlyeq (t')^{\mathscr{A}}$

- **Remarks:**
  - This is *not* a denotational model of the λ-calculus!
  - The map $t^{\mathscr{A}}$ is not injective in general

Semantic typing (1/2)

Elements of $\mathscr{A}$ can be used as semantic types for λ-terms:

- **Types:** $a \in \mathscr{A}$

  Terms: λ-terms with parameters in $\mathscr{A}$

  Contexts: $\Gamma \equiv x_1 : a_1, \dots, x_n : a_n \qquad (a_1, \dots, a_n \in A)$

  Judgment: $\Gamma \vdash t : a$

- **Remark:** Each context $\Gamma \equiv x_1 : a_1, \dots, x_n : a_n$ can also be used as a substitution: $\Gamma \equiv x_1 := a_1, \dots, x_n := a_n$

- The validity of a judgment is defined directly (i.e. semantically); not from a set of inference rules:

Definition (Semantic validity)

$$\Gamma \vdash t : a \quad :\equiv \quad FV(t) \subseteq \text{dom}(\Gamma) \;\text{ and }\; (t[\Gamma])^{\mathscr{A}} \preccurlyeq a$$

Semantic typing (2/2)

Definition (Semantic validity)

$$\Gamma \vdash t : a \qquad :\equiv \qquad FV(t) \subseteq \mathsf{dom}(\Gamma) \;\text{ and }\; (t[\Gamma])^{\mathscr{A}} \preccurlyeq a$$

Proposition

The following semantic typing rules are valid:

$$\frac{}{\Gamma \vdash x : a}\ ((x{:}a) \in \Gamma) \qquad \frac{}{\Gamma \vdash a : a} \qquad \frac{}{\Gamma \vdash t : \top}\ (FV(t) \subseteq \mathsf{dom}(\Gamma))$$

$$\frac{\Gamma, x : a \vdash t : b}{\Gamma \vdash \lambda x\,.\,t : a \to b} \qquad \frac{\Gamma \vdash t : a \to b \quad \Gamma \vdash u : a}{\Gamma \vdash tu : b}$$

$$\frac{\Gamma \vdash t : a_i \quad (\text{for all } i \in I)}{\Gamma \vdash t : \bigcurlywedge_{i \in I} a_i} \qquad \frac{\Gamma \vdash t : a}{\Gamma \vdash t : a'}\ (a \preccurlyeq a') \qquad \frac{\Gamma \vdash t : a}{\Gamma' \vdash t : a}\ (\Gamma' \preccurlyeq \Gamma)$$

Note: $\Gamma' \preccurlyeq \Gamma$ means: $\Gamma'(x) \preccurlyeq \Gamma(x)$ for all $x \in \mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\Gamma')$.

Remarkable identities (1/2)

- Recall that in (Curry-style) system F, we have:

$$\begin{array}{lcll} \mathbf{I} & := & \lambda x\,.\,x & : \forall\alpha\,(\alpha\to\alpha) \\ \mathbf{K} & := & \lambda xy\,.\,x & : \forall\alpha,\beta\,(\alpha\to\beta\to\alpha) \\ \mathbf{S} & := & \lambda xyz\,.\,xz(yz) & : \forall\alpha,\beta,\gamma\,((\alpha\to\beta\to\gamma)\to(\alpha\to\beta)\to\alpha\to\gamma) \end{array}$$

Proposition

In any implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ we have:

$$\begin{array}{lcll} \mathbf{I}^{\mathscr{A}} & := & (\lambda x\,.\,x)^{\mathscr{A}} & = \bigcurlywedge_{a\in\mathscr{A}}(a\to a) \\ \mathbf{K}^{\mathscr{A}} & := & (\lambda xy\,.\,x)^{\mathscr{A}} & = \bigcurlywedge_{a,b\in\mathscr{A}}(a\to b\to a) \\ \mathbf{S}^{\mathscr{A}} & := & (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} & = \bigcurlywedge_{a,b,c\in\mathscr{A}}((a\to b\to c)\to(a\to b)\to a\to c) \end{array}$$

Remarkable identities (2/2)

- The same property holds for:

$$\begin{array}{lcll} \mathbf{C} & := & \lambda xyz\,.\,xzy & : \ \forall\alpha,\beta,\gamma\,((\alpha\to\beta\to\gamma)\to\beta\to\alpha\to\gamma) \\ \mathbf{W} & := & \lambda xy\,.\,xyy & : \ \forall\alpha,\beta\,((\alpha\to\alpha\to\beta)\to\alpha\to\beta) \end{array}$$

but not for

$$\mathbf{II} \ := \ (\lambda x\,.\,x)(\lambda x\,.\,x) \ : \ \forall\alpha\,(\alpha\to\alpha)$$

(Thanks to a remark of Étienne Miquey)

- By analogy, we let:

$$\text{\ae}^{\mathscr{A}} \ := \ \bigwedge_{a,b\in\mathscr{A}}(((a\to b)\to a)\to a) \qquad \text{(Peirce's law)}$$

$$= \ \bigwedge_{a\in\mathscr{A}}((\neg a\to a)\to a) \qquad \text{(where } \neg a := (a\to\bot)\text{)}$$

From this, we extend the encoding of the λ-calculus to all λ-terms enriched with the constant æ (= proof-like λ_c-terms)

Particular case: $\mathscr{A}$ is a complete Heyting algebra (1/2)

Complete Heyting algebras are the particular implicative structures $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ where $\to$ is defined from the ordering $\preccurlyeq$ by

$$a \to b \ := \ \max\{c \in \mathscr{A} \ : \ (c \curlywedge a) \preccurlyeq b\}$$

Recall: Complete Heyting (or Boolean) algebras are the structures underlying forcing (in the sense of Kripke or Cohen)

Proposition

When $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ is a complete Heyting algebra:

1. For all $a, b \in \mathscr{A}$: $ab = a \curlywedge b$ (application = binary meet)
2. For all λ-terms t with free variables $x_1, \ldots, x_k$ $(k \geq 0)$ and for all $a_1, \ldots, a_k \in \mathscr{A}$, we have:
$$(t[x_1 := a_1, \ldots, x_k := a_k])^{\mathscr{A}} \ \succcurlyeq \ a_1 \curlywedge \cdots \curlywedge a_k$$
3. In particular, when t is closed: $(t)^{\mathscr{A}} = \top$
4. $\mathscr{A}$ is a (complete) Boolean algebra iff $\text{cc}^{\mathscr{A}} = \top$

Particular case: $\mathscr{A}$ is a complete Heyting algebra (2/2)

Proof.

1. For all $c \in \mathscr{A}$, we have: $ab \preccurlyeq c \quad\Leftrightarrow\quad a \preccurlyeq (b \to c) \quad\Leftrightarrow\quad a \curlywedge b \preccurlyeq c$, hence $ab = a \curlywedge b$.

2. We prove that $(t[\vec{x} := \vec{a}])^{\mathscr{A}} \succcurlyeq a_1 \curlywedge \cdots \curlywedge a_k$ by induction on t

   - $t \equiv x$ (variable). Obvious.
   - $t \equiv t_1 t_2$ (application). Obvious from point 1.
   - $t \equiv \lambda x_0 \,.\, t_0$ (abstraction). In this case, we have:

$$\begin{aligned}(t[\vec{x} := \vec{a}])^{\mathscr{A}} &= \bigcurlywedge_{a_0} (a_0 \to (t_0[x_0 := a_0, \vec{x} := \vec{a}])^{\mathscr{A}}) \\ &\succcurlyeq \bigcurlywedge_{a_0} (a_0 \to a_0 \curlywedge a_1 \curlywedge \cdots \curlywedge a_k) \qquad \text{(by IH)} \\ &\succcurlyeq a_1 \curlywedge \cdots \curlywedge a_k\end{aligned}$$

   using the relation $b \preccurlyeq (a \to a \curlywedge b)$ of Heyting Algebras.

3. In particular, when t is closed, we get: $(t)^{\mathscr{A}} \succcurlyeq \top$

4. $(\mathscr{A}, \preccurlyeq)$ Boolean algebra iff $\text{æ}^{\mathscr{A}} = \top$: Obvious. □

Logical strength of an implicative structure

- **Warning!** We may have $(t)^{\mathscr{A}} = \bot$ for some closed λ-term t.

  Intuitively, this means that the corresponding term is inconsistent in (the logic represented by) the implicative structure $\mathscr{A}$

- We say that the implicative structure $\mathscr{A}$ is:
  - intuitionistically consistent when $(t)^{\mathscr{A}} \neq \bot$ for all closed λ-terms
  - classically consistent when $(t)^{\mathscr{A}} \neq \bot$ for all closed λ-terms with cc

- **Examples:**
  - Every non-degenerated complete Heyting algebra is int. consistent
  - Every non-degenerated complete Boolean algebra is class. consistent
  - Every implicative structure induced by a total combinatory algebra is intuitionistically consistent
  - Every implicative structure induced by an AKS whose pole $\perp\!\!\!\perp$ is coherent (cf [Krivine'12]) is classically consistent

Two trivial examples...

Trivial example 1:

- Given a complete lattice $(\mathscr{A}, \preccurlyeq)$, we let
$$a \to b \ := \ b \qquad \text{(for all } a, b \in \mathscr{A}\text{)}$$
Clearly, $(\mathscr{A}, \preccurlyeq, \to)$ is an implicative structure
- In this structure, we have: $\mathbf{I}^{\mathscr{A}} \ := \ \bigcurlywedge_{a \in \mathscr{A}} (a \to a) \ = \ \bigcurlywedge_{a \in \mathscr{A}} a \ = \ \bot \ \ (!)$

Trivial example 2:

- Given a complete lattice $(\mathscr{A}, \preccurlyeq)$, we let
$$a \to b \ := \ \top \qquad \text{(for all } a, b \in \mathscr{A}\text{)}$$
Again, $(\mathscr{A}, \preccurlyeq, \to)$ is an implicative structure!
- In this structure, we have: $\mathbf{I}^{\mathscr{A}} \ := \ \bigcurlywedge_{a \in \mathscr{A}} (a \to a) \ = \ \top,$ but
$$(\mathbf{II})^{\mathscr{A}} \ := \ \top\top \ = \ \bigcurlywedge \{c \in \mathscr{A} : \top \preccurlyeq (\top \to c)\} \ = \ \bigcurlywedge \mathscr{A} \ = \ \bot \ \ (!)$$

… and a non trivial example (1/2)

(The following example is inspired from Girard's phase semantics for LL)

- Let $(M, \cdot, 1)$ be a commutative monoid. We let:
  - $\mathscr{A} \;:=\; \mathfrak{P}(M)$
  - $a \preccurlyeq b \;:=\; a \subseteq b$
  - $a \to b \;:=\; \{\gamma \in M : (\forall \alpha \in a)\, \gamma\alpha \in b\}$ (for all $a, b \in \mathscr{A}$)

  Clearly, $(\mathscr{A}, \preccurlyeq, \to)$ is an implicative structure
  (since the product $\cdot$ is a total operation)

- We easily check that for all $a, b \in \mathscr{A}$:

$$ab \;:=\; a \cdot b \;=\; \{\alpha\beta : \alpha \in a,\ \beta \in b\}$$

  Therefore:
  - $ab = ba$ (application is commutative)
  - $(ab)c = a(bc)$ (application is associative)
  - $aa \neq a$, in general (application is not idempotent)

… and a non trivial example (2/2)

Proposition

1. In the implicative structure $(\mathscr{A}, \preccurlyeq, \to) = (\mathfrak{P}(M), \subseteq, \to)$:

$$\begin{array}{rcllcl} \mathbf{I}^{\mathscr{A}} & := & (\lambda x\,.\,x)^{\mathscr{A}} & = \{1\} & \neq & \bot \\ \mathbf{C}^{\mathscr{A}} & := & (\lambda xyz\,.\,xzy)^{\mathscr{A}} & = \{1\} & \neq & \bot \\ \mathbf{B}^{\mathscr{A}} & := & (\lambda xyz\,.\,x(yz))^{\mathscr{A}} & = \{1\} & \neq & \bot \end{array}$$

2. Moreover, if we assume that $\alpha^2 \neq \alpha$ for some $\alpha \in M$, then:

$$\begin{array}{rcllcl} \mathbf{K}^{\mathscr{A}} & := & (\lambda xy\,.\,x)^{\mathscr{A}} & = \varnothing & = & \bot \\ \mathbf{W}^{\mathscr{A}} & := & (\lambda xy\,.\,xyy)^{\mathscr{A}} & = \varnothing & = & \bot \\ \mathbf{S}^{\mathscr{A}} & := & (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} & = \varnothing & = & \bot \end{array}$$

More generally, for each closed λ-term t, we (should) have:

$$(t)^{\mathscr{A}} = \begin{cases} \{1\} & \text{if } t \text{ is linear} \\ \varnothing & \text{otherwise} \end{cases} \qquad \text{(to be checked)}$$

Plan

Separators

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

Definition (Separator)

A separator of $\mathscr{A}$ is a subset $S \subseteq \mathscr{A}$ such that:

(1) If $a \in S$ and $a \preccurlyeq b$, then $b \in S$ (upwards closed)

(2) $\mathbf{K}^{\mathscr{A}} = (\lambda xy\,.\,x)^{\mathscr{A}} \in S$ and $\mathbf{S}^{\mathscr{A}} = (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} \in S$

(3) If $(a \to b) \in S$ and $a \in S$, then $b \in S$ (modus ponens)

We say that S is consistent (resp. classical) when $\bot \notin S$ (resp. $œ^{\mathscr{A}} \in S$)

Remarks:

- Under (1), axiom (3) is equivalent to:

  (3′) If $a, b \in S$, then $ab \in S$ (closure under application)

- In a complete Heyting algebra: separator = filter
- But in general, separators are not closed under binary meets

λ-terms and separators

Intuition: Separator $S \subseteq \mathscr{A}$ = criterion of truth (in $\mathscr{A}$)

- All separators are closed under the operations of the λ-calculus:

Proposition

Given a separator $S \subseteq \mathscr{A}$:

1. For all λ-terms t with free variables $x_1, \ldots, x_k$ and for all $a_1, \ldots, a_k \in S$, we have: $(t[x_1 := a_1, \ldots, x_k := a_k])^{\mathscr{A}} \in S$
2. For all closed λ-terms t: $(t)^{\mathscr{A}} \in S$

- Alternative formulation:

Given a closed λ-term t with parameters in S:

$$\vdash t : a \qquad \text{implies} \qquad a \in S$$

If a has a "proof" t (possibly using "axioms" $\in S$), then a is true ($\in S$)

Intuitionistic and classical cores

Definition (intuitionistic & classical cores)

Given an implicative algebra we write:

- $S_J^0(\mathscr{A})$ the smallest separator of $\mathscr{A}$ (intuitionistic core)
- $S_K^0(\mathscr{A})$ the smallest classical separator of $\mathscr{A}$ (classical core)

We easily check that:

$$\begin{aligned} S_J^0(\mathscr{A}) &= \uparrow\{(t)^{\mathscr{A}} \ : \ t \text{ closed } \lambda\text{-term}\} \\ S_K^0(\mathscr{A}) &= \uparrow\{(t)^{\mathscr{A}} \ : \ t \text{ closed } \lambda\text{-term with } \text{œ}\} \end{aligned}$$

writing $\uparrow B$ the upwards closure of a subset $B \subseteq \mathscr{A}$

Proposition

An implicative algebra $\mathscr{A}$ is intuitionistically (resp. classically) consistent if and only if $\bot \notin S_J^0(\mathscr{A})$ (resp. $\bot \notin S_K^0(\mathscr{A})$)

Encoding conjunction and disjunction

In any implicative structure, conjunction and disjunction are defined by:

$$a \times b \quad := \quad \bigwedge_{c \in \mathscr{A}} ((a \to b \to c) \to c) \qquad \text{(conjunction)}$$

$$a + b \quad := \quad \bigwedge_{c \in \mathscr{A}} ((a \to c) \to (b \to c) \to c) \qquad \text{(disjunction)}$$

Proposition

The following semantic typing rules are valid:

$$\frac{\Gamma \vdash t : a \quad \Gamma \vdash u : b}{\Gamma \vdash \lambda z \,.\, z\, t\, u : a \times b} \qquad \frac{\Gamma \vdash t : a \times b}{\Gamma \vdash t\, (\lambda xy \,.\, x) : a} \qquad \frac{\Gamma \vdash t : a \times b}{\Gamma \vdash t\, (\lambda xy \,.\, y) : b}$$

$$\frac{\Gamma \vdash t : a}{\Gamma \vdash \lambda zw \,.\, z\, t : a + b} \qquad \frac{\Gamma \vdash t : b}{\Gamma \vdash \lambda zw \,.\, w\, t : a + b}$$

$$\frac{\Gamma \vdash t : a + b \quad \Gamma, x : a \vdash u : c \quad \Gamma, y : b \vdash v : c}{\Gamma \vdash t\, (\lambda x \,.\, u)\, (\lambda y \,.\, v) : c}$$

Moreover, we have: $(\lambda z \,.\, z\, a\, b)^{\mathscr{A}} = \langle a, b \rangle^{\mathscr{A}} = a \times b$ (pairing = conjunction)

Encoding quantifiers

Given a family $(a_i)_{i \in I}$, we let:

$$\bigforall_{i \in I} a_i \quad := \quad \bigcurlywedge_{i \in I} a_i$$

$$\bigexists_{i \in I} a_i \quad := \quad \bigcurlywedge_{c \in \mathscr{A}} \Big(\bigcurlywedge_{i \in I} (a_i \to c) \;\to c \Big)$$

Proposition

The following semantic typing rules are valid:

$$\frac{\Gamma \vdash t : a_i \quad (\text{for all } i \in I)}{\Gamma \vdash t : \forall_{i \in I} a_i} \qquad \frac{\Gamma \vdash t : \forall_{i \in I} a_i}{\Gamma \vdash t : a_{i_0}} \; (i_0 \in I)$$

$$\frac{\Gamma \vdash t : a_{i_0}}{\Gamma \vdash \lambda z \,.\, z\, t : \exists_{i \in I} a_i} \; (i_0 \in I) \qquad \frac{\Gamma \vdash t : \exists_{i \in I} a_i \qquad \Gamma, x : a_i \vdash u : c \quad (\text{for all } i \in I)}{\Gamma \vdash t\,(\lambda x \,.\, u) : c}$$

Note: The simpler encoding $\exists_{i \in I} a_i \; := \; \curlyvee_{i \in I} a_i$ does not work in classical realizability

A note on existential quantification

- The interpretation of $\forall$ and $\exists$ is asymmetric:

$$\forall(a_i)_{i\in I} \;:=\; \bigcurlywedge_{i\in I} a_i \qquad \exists(a_i)_{i\in I} \;:=\; \bigcurlywedge_{c\in\mathscr{A}}\Big(\bigcurlywedge_{i\in I}(a_i\to c)\;\to c\Big)$$

Why not taking $\exists(a_i)_{i\in I} \;:=\; \bigcurlyvee_{i\in I} a_i$?

- **Reason:** The latter interpretation "$\exists = \curlyvee$" fails to interpret the elimination rule of $\exists$. In general:

$$\forall(a_i\to b)_{i\in I}\to \curlyvee(a_i)_{i\in I}\to b \quad\notin\quad S^0_J(\mathscr{A})$$

(There are counter-examples with Krivine realizability)

- However, the interpretation "$\exists = \curlyvee$" works when:
  - $\mathscr{A}$ is a complete Heyting/Boolean algebra
  - $\mathscr{A} = (\mathfrak{P}(A), \subseteq, \to)$ is the implicative structure induced by a (total) combinatory algebra $(P, \cdot\,, \mathbf{K}, \mathbf{S})$

Interpreting first-order logic

- Formulas of first-order logic are interpreted by:

$$
\begin{aligned}
[\![\phi \Rightarrow \psi]\!] &= [\![\phi]\!] \to [\![\psi]\!] \\
[\![\neg\phi]\!] &= [\![\phi]\!] \to \bot \\
[\![\phi \wedge \psi]\!] &= \bigwedge_{a \in \mathscr{A}} \big(([\![\phi]\!] \to [\![\psi]\!] \to a) \to a\big) \\
[\![\phi \vee \psi]\!] &= \bigwedge_{a \in \mathscr{A}} \big(([\![\phi]\!] \to a) \to ([\![\psi]\!] \to a) \to a\big) \\
[\![\forall x\, \phi(x)]\!] &= \bigwedge_{v \in M} [\![\phi(v)]\!] \\
[\![\exists x\, \phi(x)]\!] &= \bigwedge_{a \in \mathscr{A}} \Big(\bigwedge_{v \in M} ([\![\phi(v)]\!] \to a) \ \to \ a \Big)
\end{aligned}
$$

Theorem (Soundness)

If $\vdash_{\mathsf{LJ}} \phi$ (resp. $\vdash_{\mathsf{LK}} \phi$), then $[\![\phi]\!] \in S_J^0(\mathscr{A})$ (resp. $[\![\phi]\!] \in S_K^0(\mathscr{A})$)

Implicative algebras

Definition (Implicative algebra)

An implicative algebra is a quadruple $(\mathscr{A}, \preccurlyeq, \to, S)$ where

- $(\mathscr{A}, \preccurlyeq, \to)$ is an implicative structure (= algebra of truth values)
- $S \subseteq \mathscr{A}$ is a separator (+ criterion of truth)

An implicative algebra $(\mathscr{A}, \preccurlyeq, \to, S)$ is

- consistent when $\bot \notin S$
- classical when $\text{Œ}^{\mathscr{A}} \in S$

- The separator $S \subseteq \mathscr{A}$ induces a preorder of entailment:

$$a \vdash_S b \quad :\equiv \quad (a \to b) \in S \qquad \text{(for all } a, b \in \mathscr{A}\text{)}$$

- The poset reflection of $(\mathscr{A}, \vdash_S)$ is written $\mathscr{A}/S$:

$$[a] \leq_S [b] \quad \text{iff} \quad a \vdash_S b \qquad \text{(for all } a, b \in \mathscr{A}\text{)}$$

The induced Heyting algebra

Proposition

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to, S)$ be an implicative algebra

1. The quotient poset $H = (\mathscr{A}/S, \leq_S)$ is a Heyting algebra, where:

$$
\begin{aligned}
[a] \to_H [b] &= [a \to b] \\
[a] \wedge_H [b] &= [a \times b] \qquad & [a] \vee_H [b] &= [a + b] \\
\bot_H &= [\bot] \qquad & \top_H &= [\top] = S
\end{aligned}
$$

2. When $\mathscr{A}$ is classical (i.e. $\text{œ}^{\mathscr{A}} \in S$), this poset is a Boolean algebra

The poset $H = (\mathscr{A}/S, \leq_S)$ is called the Heyting algebra induced by $\mathscr{A}$

Remarks:

- The Heyting algebra H is in general not complete
- **Beware!** The ordering $\leq_S$ on H comes from $\vdash_S$ (entailment), and not from $\preccurlyeq$ (subtyping). However, we have: $a \preccurlyeq b \;\Rightarrow\; [a] \leq_S [b]$.

Ultraseparators (1/2)

Although separators are *not* filters (w.r.t. the order $\preccurlyeq$), they can be manipulated similarly to filters. For instance:

- We call an ultraseparator any separator $S \subseteq \mathscr{A}$ that is consistent and maximal (w.r.t. inclusion) among consistent separators
- By Zorn's lemma, we easily check that any consistent separator can be extended into an ultraseparator

Trivial Boolean algebra

$S \subseteq \mathscr{A}$ is an ultraseparator if and only if the induced Heyting algebra $(\mathscr{A}/S, \leq_S)$ is the trivial Boolean algebra:

$$S \subseteq \mathscr{A} \text{ ultraseparator} \qquad \text{iff} \qquad (\mathscr{A}/S, \leq_S) \;\approx\; \mathbf{2}$$

Remark: Works even when the ultraseparator $S \subseteq \mathscr{A}$ is not classical!

Ultraseparators (2/2)

Remark: There are non-classical ultraseparators!

Typical example is given by intuitionistic realizability:

- Let $(\mathscr{A}, \preccurlyeq, \to)$ be the implicative structure induced by a total combinatory algebra $(P, \cdot, \mathbf{K}, \mathbf{S})$:
  - $\mathscr{A} \; := \; \mathfrak{P}(P)$ (sets of combinators)
  - $a \preccurlyeq b \; := \; a \subseteq b$ (inclusion)
  - $a \to b \; := \; \{z \in P : \forall x \in a, \; z \cdot x \in b\}$ (Kleene's implication)
- Let $S = \mathfrak{P}(P) \setminus \{\varnothing\} = \mathscr{A} \setminus \{\bot\}$. We easily check that S is a consistent separator, obviously maximal. Hence: $\mathscr{A}/S \approx \mathbf{2}$.
- Identity $\mathscr{A}/S \approx \mathbf{2}$ reflects the fact that in intuitionistic realizability, we have either $\Vdash \phi$ or $\Vdash \neg\phi$ for each closed formula ϕ.
- On the other hand, we have: $\text{œ}^{\mathscr{A}} = \bigwedge_{a} ((\neg a \to a) \to a) = \varnothing$

  (Indeed, from a realizer $t \in \text{œ}^{\mathscr{A}}$, we would easily solve the halting problem)

Plan

Separators and filters

- In the theory of implicative algebras, separators play the same role as filters in the theory of Heyting algebras.

  However, separators $S \subseteq \mathscr{A}$ are in general *not* filters:

$$\begin{array}{rcl} a, b \in S & \Rightarrow & ab \in S \\ a, b \in S & \Rightarrow & a \times b \in S \\ a, b \in S & \not\Rightarrow & a \curlywedge b \in S \end{array}$$

- On the other hand, in the particular case where $\mathscr{A}$ is (derived from) a complete Heyting algebra, we have: separator = filter

- We shall now study in the general case the situations where a separator happens to be also a filter

Non deterministic choice

- Given an implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$, we let:

$$\pitchfork^{\mathscr{A}} := \bigcurlywedge_{a,b} (a \to b \to a \curlywedge b)$$

(non deterministic choice)

We shall also use the symbol $\pitchfork$ (non-deterministic choice operator) as an extra constant of the λ-calculus (like cc), that is interpreted by $\pitchfork^{\mathscr{A}}$

- In Krivine's λ_c-calculus, universal realizers of the "type" $\pitchfork^{\mathscr{A}}$ are the instructions $\pitchfork$ with the non-deterministic evaluation rule:

$$\pitchfork \star u \cdot v \cdot \pi \succ \begin{cases} u \star \pi \\ v \star \pi \end{cases}$$

[Guillermo & M., 2014]

"Attention à l'instruction fork qui a des effets dévastateurs!"
J.-L. Krivine, 12/03/2012

Non deterministic choice and parallel 'or'

- Let $\mathsf{Nat}^{\mathscr{A}}(n) \;:=\; \bigcurlywedge_{a\in\mathscr{A}^{\mathbb{N}}}\Big(a(0) \to \bigcurlywedge_{p\in\mathbb{N}}(a(p) \to a(p+1)) \to a(n)\Big)$

Fact

1. $\pitchfork^{\mathscr{A}} \;=\; (\lambda xy\,.\,x)^{\mathscr{A}} \curlywedge (\lambda xy\,.\,y)^{\mathscr{A}}$ $\qquad$ $(\mathtt{tt} \curlywedge \mathtt{ff})$
2. $\pitchfork^{\mathscr{A}} \;\dashv\vdash_S\; \bigcurlywedge_{n\in\mathbb{N}} \mathsf{Nat}^{\mathscr{A}}(n)$ $\qquad$ (in any separator $S \subseteq \mathscr{A}$)

- Non deterministic choice is related to the parallel 'or'

$$\text{p-or}^{\mathscr{A}} \;:=\; (\bot \to \top \to \bot) \curlywedge (\top \to \bot \to \bot) \qquad \text{(parallel 'or')}$$

Fact

1. $\pitchfork^{\mathscr{A}} \;\preccurlyeq\; \text{p-or}^{\mathscr{A}}$
2. $\pitchfork^{\mathscr{A}} \;\dashv\vdash_S\; \text{p-or}^{\mathscr{A}}$ $\qquad$ (in any classical separator $S \subseteq \mathscr{A}$)

Non deterministic choice, parallel 'or' and filters

- Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure
- It is clear that a separator $S \subseteq \mathscr{A}$ is a filter if and only if it is closed under binary meets: $a, b \in S \;\Rightarrow\; a \curlywedge b \in S$ (for all $a, b \in \mathscr{A}$)

Proposition (Characterizing filters)

1. A separator $S \subseteq \mathscr{A}$ is a filter if and only if: $\pitchfork^{\mathscr{A}} \in S$
2. A classical separator $S \subseteq \mathscr{A}$ is a filter if and only if: p-or$^{\mathscr{A}} \in S$

Proof.

1. $(\Rightarrow)$ In any separator $S \subseteq \mathscr{A}$, we have $(\lambda xy\,.\,x)^{\mathscr{A}}, (\lambda xy\,.\,y)^{\mathscr{A}} \in S$. So that when S is a filter, we get $\pitchfork^{\mathscr{A}} = (\lambda xy\,.\,x)^{\mathscr{A}} \curlywedge (\lambda xy\,.\,y)^{\mathscr{A}} \in S$.

   $(\Leftarrow)$ If $\pitchfork^{\mathscr{A}} \in S$, then $(a \to b \to a \curlywedge b) \in S$ for all $a, b \in \mathscr{A}$. So that if $a, b \in S$, we get $a \curlywedge b$ (applying the modus ponens twice in S).
2. Obvious from item 1, since: $\pitchfork^{\mathscr{A}} \in S$ iff p-or$^{\mathscr{A}} \in S$. □

Generating separators

- Given any subset $X \subseteq \mathscr{A}$, we write:
  - $\mathrm{App}(X)$ the applicative algebra generated by X, i.e. the smallest subset of $\mathscr{A}$ containing X and closed under application
  - $\uparrow X$ the upwards closure of X in $\mathscr{A}$ (w.r.t. $\preccurlyeq$)

Lemma (Separator generated by a subset of $\mathscr{A}$)

For all $X \subseteq \mathscr{A}$, the subset $\uparrow\mathrm{App}(X \cup \{\mathbf{K}^{\mathscr{A}}, \mathbf{S}^{\mathscr{A}}\}) \subseteq \mathscr{A}$ is the smallest separator of $\mathscr{A}$ containing X as a subset

- A separator $S \subseteq \mathscr{A}$ is finitely generated when it is of the form

$$S = \uparrow\mathrm{App}(X) \qquad \text{for some finite subset } X \subseteq \mathscr{A}$$

- We observe that both separators $S_J^0(\mathscr{A}) \subseteq \mathscr{A}$ (intuitionistic core) and $S_K^0(\mathscr{A}) \subseteq \mathscr{A}$ (classical core) are finitely generated

Finitely generated separators and principal filters (1/4)

Theorem

Given a separator $S \subseteq \mathscr{A}$, the following are equivalent:

1. S is finitely generated and $⋔^{\mathscr{A}} \in S$
2. S is a principal filter: $S = \uparrow\{\Theta\}$ for some $\Theta \in S$

   (Θ is called the universal proof of S)
3. The induced Heyting algebra $H := (\mathscr{A}/S, \leq_S)$ is complete, and the surjection $[\cdot] : \mathscr{A} \to H$ commutes with infinitary meets:

$$\Big[\bigcurlywedge_{i \in I} a_i\Big] = \bigwedge_{i \in I} [a_i]$$

In model theoretic terms, this situation corresponds to a collapse of (intuitionistic/classical) realizability into (Kripke/Cohen) forcing!

Finitely generated separators and principal filters (2/4)

Proof.

- S finitely generated + $\pitchfork^{\mathscr{A}} \in \mathscr{S} \quad \Rightarrow \quad S$ principal filter

Suppose that $S = \uparrow\mathrm{App}(\{g_1, g_2, \ldots, g_n\})$ and $\pitchfork^{\mathscr{A}} \in S$. From the latter, S is a filter, so that for all $k \geq 1$, we have more generally:

$$\begin{aligned}\pitchfork_k^{\mathscr{A}} \;&:=\; \bigcurlywedge_{a_1,\ldots,a_k} (a_1 \to \cdots \to a_k \to a_1 \curlywedge \cdots \curlywedge a_k) \\ &=\; \bigcurlywedge_{i=1..k} (\lambda x_1 \cdots x_k \,.\, x_i)^{\mathscr{A}} \;\in\; S\end{aligned}$$

We let: $\Theta \;:=\; (\mathbf{Y}\,(\lambda r\,.\,\pitchfork_{n+1}^{\mathscr{A}}\, g_1 \,\cdots\, g_n\,(r\,r)))^{\mathscr{A}} \;\in\; S$, where $\mathbf{Y} \equiv (\lambda yf\,.\,f(yyf))(\lambda yf\,.\,f(yyf))$ is Turing's fixpoint combinator.

By construction we have $\Theta \;\preccurlyeq\; \pitchfork_{n+1}^{\mathscr{A}}\, g_1 \,\cdots\, g_n\,(\Theta\,\Theta)$, hence:

$$\Theta \preccurlyeq g_1, \quad \ldots, \quad \Theta \preccurlyeq g_n \quad \text{and} \quad \Theta \preccurlyeq \Theta\,\Theta$$

By induction, we get $\Theta \preccurlyeq a$ for all $a \in \mathrm{App}(g_1, \ldots, g_n)$, and thus $\Theta \preccurlyeq a$ for all $a \in S$. Therefore: $\Theta = \min(S)$ and $S = \uparrow\{\Theta\}$. (...)

Finitely generated separators and principal filters (3/4)

Proof (continued).

- S principal filter $\Rightarrow$ H complete + commutation property

  Suppose that $S = \uparrow\{\Theta\}$, and let $[a_i]_{i\in I} \in H^I$ be a family of elements of H, defined from a family of representatives $(a_i)_{i\in I} \in \mathscr{A}^I$. Since $\left(\bigcurlywedge_{i\in I} a_i\right) \preccurlyeq a_i$ for all $i \in I$, $\left[\bigcurlywedge_{i\in I} a_i\right]$ is a lower bound of the family $[a_i]_{i\in I}$ in H.

  Conversely, if $[b]$ is a lower bound of the family $[a_i]_{i\in I}$ in H, we have $(b \to a_i) \in S$ for all $i \in I$. And since $S = \uparrow\{\Theta\}$, we get $\Theta \preccurlyeq (b \to a_i)$ for all $i \in I$, so that:

  $$\Theta \;\preccurlyeq\; \bigcurlywedge_{i\in I}(b \to a_i) \;=\; b \to \bigcurlywedge_{i\in I} a_i\,.$$

  Hence $[b] \leq_S \left[\bigcurlywedge_{i\in I} a_i\right]$. Therefore, $\left[\bigcurlywedge_{i\in I} a_i\right]$ is the g.l.b. of the family $[a_i]_{i\in I}$, hence the commutation property $\left[\bigcurlywedge_{i\in I} a_i\right] = \bigwedge_{i\in I}[a_i]$. (...)

Finitely generated separators and principal filters (4/4)

Proof (continued).

- H complete + commut. property $\Rightarrow$ S finitely generated + $\pitchfork^{\mathscr{A}} \in S$

  Suppose that $H = \mathscr{A}/S$ is complete and that the surjection $[\,\cdot\,] : \mathscr{A} \to H$ commutes with infinitary meets. Let $\Theta = \curlywedge S$. From the commutation property, we have:

$$[\Theta] \;=\; \Big[\curlywedge_{a\in S} a\Big] \;=\; \bigwedge_{a\in S}[a] \;=\; \bigwedge_{a\in S}\top_H \;=\; \top_H,$$

  hence $\Theta \in S$, so that $\Theta = \min(S)$ and $S = \uparrow\{\Theta\}$. Therefore the separator S is a (principal) filter, hence we have $\pitchfork^{\mathscr{A}} \in S$.

  S is also finitely generated, by the unique generator Θ. □

Uniform existential quantification

- We say that an implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ has uniform existential quantification when for all $(a_i)_{i \in I} \in \mathscr{A}^I$ and $b \in \mathscr{A}$:

$$(*) \qquad \bigcurlywedge_{i \in I} (a_i \to b) \;=\; \Big(\bigcurlyvee_{i \in I} a_i\Big) \to b$$

- This equality (that corresponds to $\exists$-elim) holds in:
  - all complete Heyting/Boolean algebras
  - all the implicative algebras induced by total combinatory algebras $(P, \cdot, \mathbf{K}, \mathbf{S})$ (intuitionistic realizability)
- When $(*)$ holds, we can let: $\displaystyle\exists_{i \in I} a_i \;:=\; \bigcurlyvee_{i \in I} a_i$

Proposition

If $\mathscr{A}$ has uniform existential quantifications, then:

1. p-or $:= (\bot \to \top \to \bot) \curlywedge (\top \to \bot \to \bot) \;=\; \top$
2. All classical separators $S \subseteq \mathscr{A}$ are filters

Morality: Uniform $\exists/\forall$ (both) are incompatible with classical realizability

Plan

Conclusion

We introduced implicative algebras, a simple algebraic structure that is common to forcing and realizability (intuitionistic & classical)

- Relies on the fundamental idea that truth values can be manipulated as generalized realizers (via the operations of the λ-calculus)

Proof = Program = Type = Formula

- Criterion of truth given by a separator (generalizing filters)
- Implicative algebras can be used to construct:
  - Models of 1st-order logic (implicative models)
  - Categorical models of higher-order logic: implicative triposes/toposes
  - Models of (I)ZF set theory
- In this structure: forcing = non deterministic realizability
- **Remark:** One can show that classical implicative algebras have the same expressiveness as abstract Krivine structures (but with a lighter machinery)