The implicative tripos

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Different notions of models

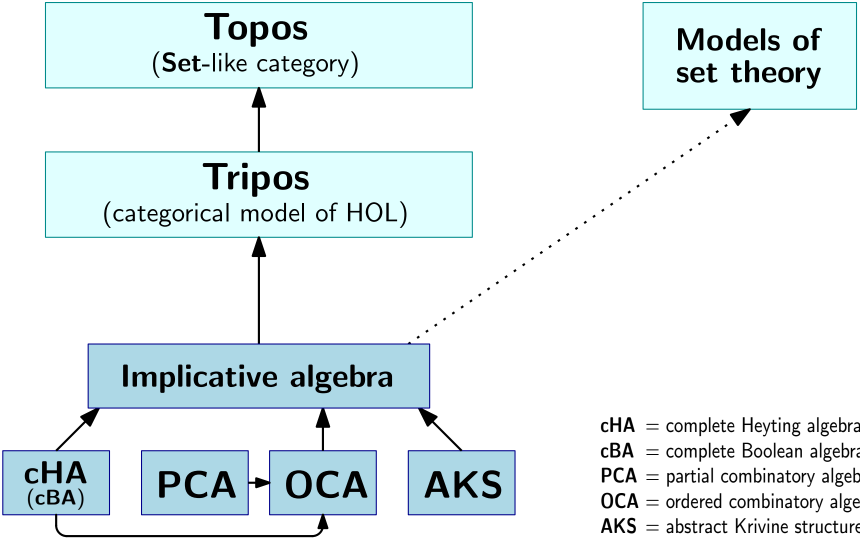
- **Tarski models:** $[\![\phi]\!] \in \{0;1\}$
  - Interprets classical provability (correctness/completeness)
- **Intuitionistic realizability:** $[\![\phi]\!] \in \mathfrak{P}(\Lambda)$ [Kleene'45]
  - Interprets intuitionistic proofs
  - Independence results in intuitionistic theories
  - Definitely incompatible with classical logic
- **Cohen forcing:** $[\![\phi]\!] \in \mathfrak{P}(C)$ [Cohen'63]
  - Independence results, in classical theories
    (Negation of continuum hypothesis, Solovay's axiom, etc.)
- **Boolean-valued models:** $[\![\phi]\!] \in \mathcal{B}$ [Scott, Solovay, Vopěnka]
- **Classical realizability:** $[\![\phi]\!] \in \mathfrak{P}(\Lambda_c)$ [Krivine'94, '01, '03, '09–]
  - Interprets classical proofs
  - Generalizes Tarski models... and forcing!

The categorical tradition of realizability

- **Categorical logic** [Lawvere, Tierney '70]
  - Hyperdoctrines = models of 1st order theories
    (Slogan: $\exists/\forall$ are left/right adjoints!)
  - Modern definition of the notion of topos
    (generalizes Grothendieck's definition)
- **Categorical realizability** [Hyland, Johnstone, Pitts '80]
  - Major input from forcing and Boolean-valued models [Scott]
  - Effective topos [Hyland]
  - Notion of tripos and tripos-to-topos construction [Pitts]
  - Generalization to partial combinatory algebras (PCAs)

    ... but incompatible with classical logic
- **What about classical realizability?**

The categorical problem



cHA = complete Heyting algebra
cBA = complete Boolean algebra
PCA = partial combinatory algebra
OCA = ordered combinatory algebra
AKS = abstract Krivine structure

Plan

Plan

Heyting algebras

Definition (Heyting algebra)

A Heyting algebra is a poset $(H, \leq)$ such that:

1. H has extremal elements
$$\bot \;:=\; \min(H) \qquad\qquad \top \;:=\; \max(H)$$
2. Any two elements $x, y \in H$ have a meet and a join:
$$x \wedge y \;:=\; \inf\{x, y\} \qquad\qquad x \vee y \;:=\; \sup\{x, y\}$$
3. Any two elements $x, y \in H$ have a relative pseudo-complement
$$x \to y \;:=\; \max\{z \in H : (z \wedge x) \leq y\}$$
which is characterized by the adjunction
$$z \leq (x \to y) \quad\Leftrightarrow\quad (z \wedge x) \leq y \qquad\qquad \text{(for all } z \in H\text{)}$$

In other words, a Heyting algebra is a bounded lattice with an operation of relative pseudo-complement (a.k.a. Heyting's implication)

Some remarks

- In a Heyting algebra $(H, \leq)$, the ordering $x \leq y$ is characterized from each of the three operations $\wedge$, $\vee$ and $\rightarrow$ by:

$$\begin{array}{rcl} x \leq y & \Leftrightarrow & x \wedge y = x \\ & \Leftrightarrow & x \vee y = y \\ & \Leftrightarrow & (x \rightarrow y) = \top \end{array}$$

- **Soundness:** All the intuitionistic equivalences hold in any Heyting algebra. In particular, the two distributivity laws:

$$\begin{array}{rcl} x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z) \end{array}$$

  are valid, so that every HA is also a (bounded) distributive lattice

- **Completeness:** The set of propositional formulas, quotiented by intuitionistic equivalence, is itself a Heyting algebra

  $\Rightarrow$ Lindenbaum algebra

Negation

- In a Heyting algebra, negation is defined by $\neg x \;:=\; (x \to \bot)$
  This operation is (in general) not involutive: $\neg\neg x \;\neq\; x$
- However, we still have the following (in)equalities:

$$x \;\leq\; \neg\neg x \qquad\qquad \neg x \vee y \;\leq\; x \to y \;\leq\; \neg y \to \neg x$$

$$\neg x \wedge \neg y \;=\; \neg(x \vee y) \qquad\qquad \neg x \vee \neg y \;\leq\; \neg(x \wedge y)$$

But the converse inequalities do not hold in general

Proposition (Boolean algebras)

In a Heyting algebra $(H, \leq)$, the following are equivalent:

1. $\neg\neg x \;=\; x$ for all $x \in H$ (negation is involutive)
2. $(((x \to y) \to x) \to x) \;=\; \top$ for all $x, y \in H$ (Peirce's law)
3. $(H, \leq)$ is a Boolean algebra

- In a Boolean algebra, all the former inequalities become equalities

Morphisms of Heyting algebras

Definition (Morphism of Heyting algebras)

A morphism of Heyting algebras is a function $F : H \to H'$ such that

$$\begin{array}{rclcrcl} F(x \wedge y) & = & F(x) \wedge F(y) & \qquad & F(\top) & = & \top \\ F(x \vee y) & = & F(x) \vee F(y) & & F(\bot) & = & \bot \\ F(x \to y) & = & F(x) \to F(y) & & & & \end{array}$$

for all $x, y \in H$

- In other words, a morphism of Heyting algebras is a morphism of bounded lattices that also preserves Heyting's implication
- Such a function is necessarily monotonic: $x \leq y \;\Rightarrow\; F(x) \leq F(y)$

The category of Heyting algebras

- The category of Heyting algebras (notation: $\mathbf{HA}$) is the category whose objects are the Heyting algebras and whose arrows are the morphisms of Heyting algebras
- $\mathbf{HA}$ is a (non-full) sub-category of $\mathbf{Pos}$ (the category of posets)
- Note that:
  1. An arrow is an isomorphism in $\mathbf{HA}$ iff it is an isomorphism in $\mathbf{Pos}$
  2. Any injective morphism of HAs is also an embedding in $\mathbf{Pos}$:
     $$x \leq y \quad \Leftrightarrow \quad F(x) \leq F(y)$$
  3. Any bijective morphism of HAs is also an isomorphism
- The category of Boolean algebras (notation: $\mathbf{BA}$) is the full sub-category of $\mathbf{HA}$ whose objects are the Boolean algebras (Notion of morphism is the very same in $\mathbf{BA}$ and $\mathbf{HA}$)

Galois connections

- A Galois connection between two posets A and B is a pair of functions $F : A \to B$ and $G : B \to A$ such that:
$$F(x) \leq_B y \quad \Leftrightarrow \quad x \leq_A G(y) \qquad \text{(for all } x \in A,\ y \in B)$$
- In this situation (notation: $F \dashv G$), we observe that:
  1. $F : A \to B$ and $G : B \to A$ are necessarily monotonic
  2. $F : A \to B$ is uniquely determined by $G : B \to A$:
  $$F(x) \;=\; \min\{y \in B : x \leq_A G(y)\} \qquad \text{(for all } x \in A)$$
  F is called the left adjoint of G, and written $F = G_L$
  3. $G : B \to A$ is uniquely determined by $F : A \to B$:
  $$G(y) \;=\; \max\{x \in A : F(x) \leq_B y\} \qquad \text{(for all } y \in B)$$
  G is called the right adjoint of F, and written $G = F_R$

Adjunction in $\mathbf{HA}$

- In what follows, we shall work mainly with arrows $F \in \mathbf{HA}(H, H')$ having both adjoints, written $F_L, F_R : H' \to H$.
- When they exist, F_L and F_R are unique and monotonic, but in general, they are not morphisms of HAs (only arrows in $\mathbf{Pos}$)

Proposition (Functoriality)

1. If $F \in \mathbf{HA}(H, H')$ and $G \in \mathbf{HA}(H', H'')$ have left adjoints, then:
$$(G \circ F)_L \ = \ F_L \circ G_L \qquad (\in \mathbf{Pos}(H'', H))$$
2. If $F \in \mathbf{HA}(H, H')$ and $G \in \mathbf{HA}(H', H'')$ have right adjoints, then:
$$(G \circ F)_R \ = \ F_R \circ G_R \qquad (\in \mathbf{Pos}(H'', H))$$
3. If $F \in \mathbf{HA}(H, H')$ is an isomorphism, then:
$$F_L \ = \ F_R \ = \ F^{-1} \qquad (\in \mathbf{Pos}(H', H))$$

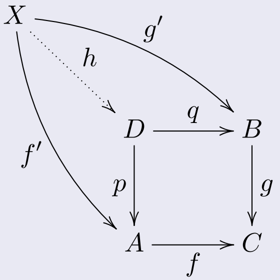
In particular: $\mathrm{id}_L = \mathrm{id}_R = \mathrm{id}$

Pullbacks (1/3)

Definition (Pullback)

In a category $\mathscr{C}$, a pullback of two arrows $A \xrightarrow{f} C \xleftarrow{g} B$ is an object D equipped with arrows $A \xleftarrow[p]{} D \xrightarrow[q]{} B$ such that:

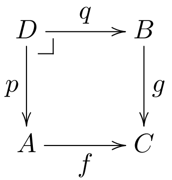
1. $f \circ p = g \circ q$
2. For each object X with arrows $A \xleftarrow{f'} X \xrightarrow{g'} B$ such that $f \circ f' = g \circ g'$, there is a unique arrow $h : X \to D$ such that:



$$\begin{cases} f' = p \circ h \\ g' = q \circ h \end{cases}$$

Pullbacks

- The pullback (D, p, q) of two arrows $A \xrightarrow{f} C \xleftarrow{g} B$, when it exists, is unique up to unique isomorphism



It is written $D = A \times_C B$ (and indicated with a "right angle" sign)

- When $C = 1$ is the terminal object of $\mathscr{C}$, the pullback of A and B amounts to the binary product:

$$\begin{array}{rcl} A \times_1 B & = & A \times B \\ p & = & \pi_{A,B} \quad : \quad A \times B \to A \\ q & = & \pi'_{A,B} \quad : \quad A \times B \to B \end{array}$$

Pullbacks (3/3)

- In the category **Set**, the pullback of two arrows $A \xrightarrow{f} C \xleftarrow{g} B$ always exists; it is the fibered product:

$$
\begin{array}{rcl}
A \times_C B & = & \{(x,y) \in A \times B \mid f(x) = g(y)\} \\
p & = & ((x,y) \mapsto x) \quad : \quad A \times_C B \to A \\
q & = & ((x,y) \mapsto y) \quad : \quad A \times_C B \to B
\end{array}
$$

- Pullbacks are constructed similarly in the categories **Pos** (posets), **HA** (Heyting algebras), **Top** (topological spaces), **Mon** (monoids), **Grp** (groups), **Ring** (rings) and R-**Mod** (R-modules)

- **A useful pullback:** The square

$$
\begin{array}{ccc}
A \times C & \xrightarrow{\pi_{A,C}} & A \\
{\scriptstyle f \times \mathrm{id}_C}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
B \times C & \xrightarrow[\pi_{B,C}]{} & B
\end{array}
$$

is always a pullback (provided $A \times C$ and $B \times C$ exist)

Set-based triposes

Definition (Set-based tripos)

A (**Set**-based) tripos is a contravariant functor $\mathbf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ such that:

(1) For each map $f : I \to J$ (in **Set**), the corresponding map $\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I$ (in **HA**) has left & right adjoints $\exists f, \forall f : \mathbf{P}I \to \mathbf{P}J$ (in **Pos**)

(2) Beck-Chevalley condition: Each pullback square in **Set** (on the l.h.s.) induces the following two commutative squares in **Pos** (on the r.h.s.):

$$\begin{array}{ccc} I & \xrightarrow{f_1} & I_1 \\ {\scriptstyle f_2}\downarrow & \lrcorner & \downarrow{\scriptstyle g_1} \\ I_2 & \xrightarrow[g_2]{} & J \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathbf{P}I & \xrightarrow{\exists f_1} & \mathbf{P}I_1 \\ {\scriptstyle \mathbf{P}f_2}\uparrow & & \uparrow{\scriptstyle \mathbf{P}g_1} \\ \mathbf{P}I_2 & \xrightarrow[\exists g_2]{} & \mathbf{P}J \end{array} \qquad \begin{array}{ccc} \mathbf{P}I & \xrightarrow{\forall f_1} & \mathbf{P}I_1 \\ {\scriptstyle \mathbf{P}f_2}\uparrow & & \uparrow{\scriptstyle \mathbf{P}g_1} \\ \mathbf{P}I_2 & \xrightarrow[\forall g_2]{} & \mathbf{P}J \end{array}$$

(3) The functor $\mathbf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ has a generic predicate $\mathrm{tr}_\Sigma \in \mathbf{P}\Sigma$ (for some set Σ), i.e. such that for all sets I, the following map is surjective:

$$\begin{array}{rcl} \Sigma^I & \to & \mathbf{P}I \\ \sigma & \mapsto & \mathbf{P}\sigma(\mathrm{tr}_\Sigma) \end{array}$$

On the definitions of the notion of tripos

- The above definition is the initial definition of triposes, such as introduced in Hyland, Johnstone, Pitts: *Tripos theory* (1980)

- Pitts' PhD *The Theory of Triposes* (1981) generalizes the notion of tripos in essentially two directions:

  1. The category **Set** is replaced by an arbitrary Cartesian category $\mathscr{C}$ (intuitively: a category of 'contexts'), and the generic predicate is replaced by a more general membership predicate[1]

  2. The Beck-Chevalley condition is only required for certain pullback squares (the projection squares), and may not hold for all

- However, all forcing/realizability/implicative triposes are triposes in the sense of the initial definition (i.e. **Set**-based triposes); therefore we shall only consider these

[1] Due to the fact that the Cartesian category $\mathscr{C}$ is not necessarily closed. But when $\mathscr{C}$ is a ccc, the existence of the generic predicate is sufficient

Triposes: some intuitions (1/7)

Intuitively, a ($\mathbf{Set}$-based) tripos is a model of intuitionistic HOL, where higher-order types are modeled by sets. In this framework:

- The (contravariant) functor $\mathbf{P} : \mathbf{Set}^{\text{op}} \to \mathbf{HA}$ associates to each set I a particular Heyting algebra $\mathbf{P}I$ of predicates over I
  - Each predicate $p \in \mathbf{P}I$ can be viewed as an abstract formula $p(x)$ depending on a variable $x : I$. Intuitively:

    $$\begin{array}{lll} p \leq q & \text{means:} & (\forall x : I)\,(p(x) \Rightarrow q(x)) \\ p = q & \text{means:} & (\forall x : I)\,(p(x) \Leftrightarrow q(x)) \end{array}$$

    (So that in this description, the ordering $\leq$ represents inclusion whereas equality represent extensional equality of predicates)
  - $\mathbf{P}I$ is a Heyting algebra, which means that predicates $p, q \in \mathbf{P}I$ can be assembled using the constructions

    $$\bot, \quad \top, \quad p \wedge q, \quad p \vee q, \quad p \to q$$

    The axioms of Heyting algebras express that all the deduction rules of intuitionistic propositional calculus are valid

Triposes: some intuitions (2/7)

- The correspondence $I \mapsto \mathbf{P}I$ is functorial: each map $f : I \to J$ (in **Set**) induces a substitution map $\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I$ (in **HA**):
  - Given $p \in \mathbf{P}J$, the predicate $\mathbf{P}f(p) \in \mathbf{P}I$ represents the pre-image of p by f: $\mathbf{P}f(p) \equiv$ "$p \circ f$" or "$f^{-1}(p)$"
  - Or, if we see p as a formula $p(y)$ (in the context $y : J$) then $\mathbf{P}f(p)$ is the formula $p(y)[y := f(x)]$ (in the context $x : I$)
- The fact that $\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I$ is a morphism of HAs expresses that substitution commutes with all connectives:

$$\begin{array}{rcl} (p(y) \wedge q(y))[y := f(x)] & \equiv & p(f(x)) \wedge q(f(x)) \\ (p(y) \vee q(y))[y := f(x)] & \equiv & p(f(x)) \vee q(f(x)) \\ (p(y) \to q(y))[y := f(x)] & \equiv & p(f(x)) \to q(f(x)) \end{array}$$

- Identities $\mathbf{P}\,\mathrm{id}_X = \mathrm{id}_{\mathbf{P}(X)}$ and $\mathbf{P}(g \circ f) = \mathbf{P}f \circ \mathbf{P}g$ express that the operation of substitution (or pre-image) is contravariant

Triposes: some intuitions (3/7)

- Axiom (1) expresses that for each map $f : I \to J$ (in $\mathbf{Set}$), the map $\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I$ (in $\mathbf{HA}$) has left/right adjoints $\exists f, \forall f : \mathbf{P}I \to \mathbf{P}J$ (in $\mathbf{Pos}$), representing $\exists/\forall$-quantifications along $f : I \to J$:
  - Given $p \in \mathbf{P}I$:

$$\begin{array}{lll} \exists f(p) & \text{means:} & (\exists x : I)(f(x) = y \land p(x)) \\ \forall f(p) & \text{means:} & (\forall x : I)(f(x) = y \Rightarrow p(x)) \end{array} \qquad \text{(in context } y : J\text{)}$$

  - Given $p \in \mathbf{P}I$ and $q \in \mathbf{P}J$, the adjunctions

$$\begin{array}{lll} \exists f(p) \le q & \text{iff} & p \le \mathbf{P}f(q) \\ q \le \forall f(p) & \text{iff} & \mathbf{P}f(q) \le p \end{array}$$

    represent the logical equivalences

$$\begin{array}{lcl} (\forall y : J)[(\exists x : I)(f(x) = y \land p(x)) \;\Rightarrow\; q(y)] & \Leftrightarrow & (\forall x : I)[p(x) \;\Rightarrow\; q(f(x))] \\ (\forall y : J)[q(y) \;\Rightarrow\; (\forall x : I)(f(x) = y \Rightarrow p(x))] & \Leftrightarrow & (\forall x : I)[q(f(x)) \;\Rightarrow\; p(x)] \end{array}$$

- **Beware!** Adjoints $\exists f, \forall f : \mathbf{P}I \to \mathbf{P}J$ are only monotonic; they are not morphisms of HAs in general. (**Intuition:** $\forall/\exists$ do not commute with $\Rightarrow$)

Triposes: some intuitions (4/7)

- In the particular case where $f := \pi_{I,K} : I \times K \to I$ is the first projection, the left/right adjoints $\exists\pi_{I,K}, \forall\pi_{I,K} : \mathbf{P}(I \times K) \to \mathbf{P}I$ represent pure quantifications over the variable $z : K$
  - Given $p \in \mathbf{P}(I \times K)$:

$$\begin{array}{lll} \exists\pi_{I,K}(p) & \text{means:} & (\exists z : K)\, p(x,z) \\ \forall\pi_{I,K}(p) & \text{means:} & (\forall z : K)\, p(x,z) \end{array} \qquad \text{(in context } x : I\text{)}$$

  - Given $p \in \mathbf{P}(I \times K)$ and $q \in \mathbf{P}(I)$, the adjunctions

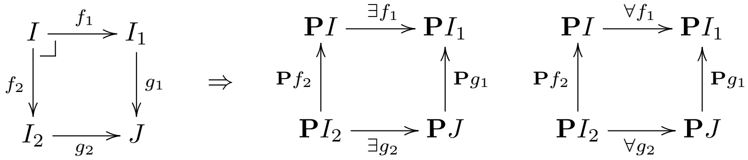
$$\begin{array}{lcl} \exists\pi_{I,K}(p) \le q & \Leftrightarrow & p \le \mathbf{P}\pi_{I,K}(q) \\ q \le \forall\pi_{I,K}(p) & \Leftrightarrow & \mathbf{P}\pi_{I,K}(q) \le p \end{array}$$

    represent the logical equivalences:

$$\begin{array}{lcl} (\forall x : I)[(\exists z : K)\, p(x,z) \Rightarrow q(x)] & \Leftrightarrow & (\forall x : I,\ z : K)[p(x,z) \Rightarrow q(x)] \\ (\forall x : I)[q(x) \Rightarrow (\forall z : K)\, p(x,z)] & \Leftrightarrow & (\forall x : I,\ z : K)[q(x) \Rightarrow p(x,z)] \end{array}$$

Triposes: some intuitions (5/7)

- Axiom (2) (Beck-Chevalley condition) expresses that each pullback in **Set** (l.h.s.) induces two commutative squares in **Pos** (r.h.s.):



$\mathbf{P}g_1 \circ \exists g_2 = \exists f_1 \circ \mathbf{P}f_2$ $\qquad$ $\mathbf{P}g_1 \circ \forall g_2 = \forall f_1 \circ \mathbf{P}f_2$

- Both commutation properties (r.h.s.) are actually equivalent up to the symmetry w.r.t. the diagonal (by exchanging indices 1 and 2 in the initial pullback square):

$$\mathbf{P}g_1 \circ \exists g_2 = \exists f_1 \circ \mathbf{P}f_2 \qquad \text{iff} \qquad \mathbf{P}g_2 \circ \forall g_1 = \forall f_2 \circ \mathbf{P}f_1$$

So that in order to prove the Beck-Chevalley condition, we only need to check that all $\exists$-diagrams commute, or that all $\forall$-diagrams commute

Triposes: some intuitions (6/7)

- When considering the pullback

$$\begin{array}{ccc} I \times K & \xrightarrow{\pi_{I,K}} & I \\ {\scriptstyle f \times \mathrm{id}_K} \downarrow & \lrcorner & \downarrow {\scriptstyle f} \\ J \times K & \xrightarrow[\pi_{J,K}]{} & J \end{array}$$

the corresponding Beck-Chevalley condition (2)

$$\begin{array}{ccc} \mathbf{P}(I \times K) & \xrightarrow{\exists \pi_{I,K}} & \mathbf{P}I \\ {\scriptstyle \mathbf{P}(f \times \mathrm{id}_K)} \uparrow & & \uparrow {\scriptstyle \mathbf{P}f} \\ \mathbf{P}(J \times K) & \xrightarrow[\exists \pi_{J,K}]{} & \mathbf{P}J \end{array} \qquad \begin{array}{ccc} \mathbf{P}(I \times K) & \xrightarrow{\forall \pi_{I,K}} & \mathbf{P}I \\ {\scriptstyle \mathbf{P}(f \times \mathrm{id}_K)} \uparrow & & \uparrow {\scriptstyle \mathbf{P}f} \\ \mathbf{P}(J \times K) & \xrightarrow[\forall \pi_{J,K}]{} & \mathbf{P}J \end{array}$$

$$\mathbf{P}f \circ \exists \pi_{J,K} = \exists \pi_{I,K} \circ \mathbf{P}(f \times \mathrm{id}_K) \qquad \mathbf{P}f \circ \forall \pi_{J,K} = \forall \pi_{I,K} \circ \mathbf{P}(f \times \mathrm{id}_K)$$

expresses the behavior of substitution w.r.t. pure quantifications:

$$\begin{array}{rcl} \big((\exists z : K)\, p(y,z)\big)[y := f(x)] & \equiv & (\exists z : K)\big(p(y,z)[y := f(x), z := z]\big) \\ \big((\forall z : K)\, p(y,z)\big)[y := f(x)] & \equiv & (\forall z : K)\big(p(y,z)[y := f(x), z := z]\big) \end{array}$$

Triposes: some intuitions (7/7)

- Axiom (3) assumes the existence of a set Σ of propositions equipped with a generic predicate $\mathsf{tr} \in \mathbf{P}\Sigma$, that allows us to turn any functional proposition into a predicate via the map

$$\begin{array}{rcl} \Sigma^I & \to & \mathbf{P}I \\ f & \mapsto & \mathbf{P}f(\mathsf{tr}) \end{array} \qquad (I \in \mathbf{Set})$$

We assume that the above map is surjective, so that each predicate $p \in \mathbf{P}I$ is represented by (at least) a functional proposition $f \in \Sigma^I$

- **Remark:** The generic predicate $\mathsf{tr} \in \mathbf{P}\Sigma$ is never unique. Indeed:
  (1) Given a generic predicate $\mathsf{tr} \in \mathbf{P}\Sigma$ and a surjection $h : \Sigma' \to \Sigma$, we can always construct another generic predicate $\mathsf{tr}' \in \mathbf{P}\Sigma'$, letting $\mathsf{tr}' := \mathbf{P}h(\mathsf{tr})$ (using AC)
  (2) If $\mathsf{tr} \in \mathbf{P}\Sigma$ and $\mathsf{tr}' \in \mathbf{P}\Sigma'$ are two generic predicates of the same tripos $\mathbf{P}$, then there are always two conversion maps $h : \Sigma' \to \Sigma$ and $h' : \Sigma \to \Sigma'$ such that $\mathsf{tr}' = \mathbf{P}h(\mathsf{tr})$ and $\mathsf{tr} = \mathbf{P}h'(\mathsf{tr})$

Set-based triposes (recall)

Definition (Set-based tripos)

A (**Set**-based) tripos is a contravariant functor $\mathbf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ such that:

(1) For each map $f : I \to J$ (in $\mathbf{Set}$), the corresponding map $\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I$ (in $\mathbf{HA}$) has left & right adjoints $\exists f, \forall f : \mathbf{P}I \to \mathbf{P}J$ (in $\mathbf{Pos}$)

(2) Beck-Chevalley condition: Each pullback square in $\mathbf{Set}$ (on the l.h.s.) induces the following two commutative squares in $\mathbf{Pos}$ (on the r.h.s.):

$$\begin{array}{ccc} I & \xrightarrow{f_1} & I_1 \\ {\scriptstyle f_2}\downarrow & \lrcorner & \downarrow{\scriptstyle g_1} \\ I_2 & \xrightarrow[g_2]{} & J \end{array} \quad\Rightarrow\quad \begin{array}{ccc} \mathbf{P}I & \xrightarrow{\exists f_1} & \mathbf{P}I_1 \\ {\scriptstyle \mathbf{P}f_2}\uparrow & & \uparrow{\scriptstyle \mathbf{P}g_1} \\ \mathbf{P}I_2 & \xrightarrow[\exists g_2]{} & \mathbf{P}J \end{array} \qquad \begin{array}{ccc} \mathbf{P}I & \xrightarrow{\forall f_1} & \mathbf{P}I_1 \\ {\scriptstyle \mathbf{P}f_2}\uparrow & & \uparrow{\scriptstyle \mathbf{P}g_1} \\ \mathbf{P}I_2 & \xrightarrow[\forall g_2]{} & \mathbf{P}J \end{array}$$

(3) The functor $\mathbf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ has a generic predicate $\mathrm{tr}_\Sigma \in \mathbf{P}\Sigma$ (for some set Σ), i.e. such that for all sets I, the following map is surjective:

$$\begin{array}{rcl} \Sigma^I & \to & \mathbf{P}I \\ \sigma & \mapsto & \mathbf{P}\sigma(\mathrm{tr}_\Sigma) \end{array}$$

Isomorphism of (Set-based) triposes

Definition (Isomorphism of triposes)

Two triposes $\mathbf{P}, \mathbf{P}' : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ are isomorphic when there is a natural isomorphism $\phi : \mathbf{P} \Rightarrow \mathbf{P}'$, i.e. a family of isos $\phi_I : \mathbf{P}I \to \mathbf{P}'I$ $(I \in \mathbf{Set})$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{P}I & \xrightarrow[\sim]{\phi_I} & \mathbf{P}'I \\ {\scriptstyle \mathbf{P}f}\uparrow & & \uparrow{\scriptstyle \mathbf{P}'f} \\ \mathbf{P}J & \xrightarrow[\phi_J]{\sim} & \mathbf{P}'J \end{array} \qquad \text{for all maps} \qquad \begin{array}{c} I \\ {\scriptstyle f}\downarrow \\ J \end{array}$$

- The notion of iso can be taken indifferently in $\mathbf{HA}$ or in $\mathbf{Pos}$, since a map $\phi_I : \mathbf{P}I \to \mathbf{P}'I$ is an iso in $\mathbf{HA}$ iff it is an iso in $\mathbf{Pos}$
- There is no need to take care about generic predicates!

  Reason: A natural iso will automatically map any generic predicate of $\mathbf{P}$ to a generic predicate of $\mathbf{P}'$

Plan

Example 1: The powerset tripos

Theorem (Powerset tripos)

The functor $\mathfrak{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{HA}$ defined by:

$$\mathfrak{P}I \;:=\; \mathfrak{P}(I) \quad \text{(powerset)} \qquad \text{(for each set } I\text{)}$$

$$\mathfrak{P}f \;:\; \mathfrak{P}J \to \mathfrak{P}I \;:=\; (p \mapsto f^{-1}(p)) \qquad \text{(for each map } f : I \to J\text{)}$$

is a ($\mathbf{Set}$-based) tripos

- For each $f : I \to J$, the adjoints $\exists f, \forall f : \mathfrak{P}I \to \mathfrak{P}J$ are given by:

$$\begin{aligned} \exists f(p) &:= \{y \in J \mid \exists x \in f^{-1}(y),\ x \in p\} \\ &= \{y \in J \mid f^{-1}(y) \cap p \ \text{inhabited}\} = f(p) \\ \forall f(p) &:= \{y \in J \mid \forall x \in f^{-1}(y),\ x \in p\} \\ &= \{y \in J \mid f^{-1}(y) \subseteq p\} \end{aligned} \qquad (p \in \mathfrak{P}(I))$$

- Generic predicate:

$$\Sigma \;:=\; \mathfrak{P}(\{\bullet\}) \quad (\cong_{\mathsf{LK}} \{0,1\})$$

$$\mathsf{tr}_\Sigma \;:=\; \{\{\bullet\}\} \;\in\; \mathfrak{P}(\Sigma)$$

Example 2: Localic triposes

Theorem (Localic tripos)

Given a complete Heyting algebra H (also known as a locale), the functor $\mathbf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ defined by:

$$\mathbf{P}I := H^I \qquad \text{(for each set } I\text{)}$$

$$\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I := (p \mapsto p \circ f) \qquad \text{(for each map } f : I \to J\text{)}$$

is a ($\mathbf{Set}$-based) tripos, called a localic tripos, or a forcing tripos

- For each $f : I \to J$, the adjoints $\exists f, \forall f : \mathbf{P}I \to \mathbf{P}J$ are given by:

$$\exists f(p) := \Big(j \in J \mapsto \bigvee_{i \in f^{-1}(j)} p_i\Big)$$
$$\forall f(p) := \Big(j \in J \mapsto \bigwedge_{i \in f^{-1}(j)} p_i\Big) \qquad (p \in H^I)$$

- Generic predicate: $\Sigma := H, \quad \mathrm{tr}_\Sigma := \mathrm{id}_H \in \mathbf{P}\Sigma \ (= H^H)$
- When $H = \mathfrak{P}(\{\bullet\})$ $(\cong_{\mathsf{LK}} \{0,1\})$, we get the powerset tripos $\mathfrak{P}$

Example 3: Intuitionistic realizability triposes (1/3)

Recall that:

Definition (Partial combinatory algebra)

A partial combinatory algebra (PCA) is partial applicative structure $(P, \cdot)$ containing two elements $\mathbf{K}, \mathbf{S} \in P$ such that for all $x, y, z \in P$:

$$\mathbf{K} \cdot x \cdot y \downarrow= x$$

$$\mathbf{S} \cdot x \cdot y \downarrow$$

$$\mathbf{S} \cdot x \cdot y \cdot z \downarrow= (x \cdot z) \cdot (y \cdot z) \qquad \text{(whenever the rhs is defined)}$$

Examples:

- $P := \Lambda/\beta\eta$ equipped with application is a (total) CA
- $P := \mathbb{N}$ equipped with Kleene application is a PCA

Example 3: Intuitionistic realizability triposes (2/3)

Given a partial combinatory algebra $(P, \cdot, \mathbf{K}, \mathbf{S})$:

- For each set I, we endow $\mathfrak{P}(P)^I$ with the relation $\vdash_I$ defined by:

$$(X_i)_{i\in I} \ \vdash_I \ (Y_i)_{i\in I} \qquad \text{iff} \qquad \bigcap_{i\in I}(X_i \to Y_i) \ \text{ is inhabited}$$

(writing $X_i \to Y_i \ := \ \{p \in P \mid \forall q \in X_i,\ p \cdot q \downarrow\in Y_i\}$ for Kleene's implication)

Proposition

For each set I, the pair $\big(\mathfrak{P}(P)^I, \vdash_I\big)$ is a pre-Heyting algebra

- In the pre-Heyting algebra $\big(\mathfrak{P}(P)^I, \vdash_I\big)$:

$$\begin{aligned}
(X_i)_{i\in I} \wedge (Y_i)_{i\in I} &= \big(\{\langle p,q\rangle \mid p\in X_i,\ q\in Y_i\}\big)_{i\in I}\\
(X_i)_{i\in I} \vee (Y_i)_{i\in I} &= \big(\{\langle \bar{0},p\rangle \mid p\in X_i\} \cup \{\langle \bar{1},q\rangle \mid q\in Y_i\}\big)_{i\in I}\\
(X_i)_{i\in I} \to (Y_i)_{i\in I} &= \big(X_i \to Y_i\big)_{i\in I}
\end{aligned}$$

where $\langle p,q\rangle := \mathbf{BC}(\mathbf{CI})pq$, ...

Example 3: Intuitionistic realizability triposes (3/3)

Theorem (Intuitionistic realizability tripos)

Given a partial combinatory algebra $(P, \cdot, \mathbf{K}, \mathbf{S})$, the functor $\mathbf{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{HA}$ defined by:

$$\mathbf{P}I := \mathbf{Pos}(\mathfrak{P}(P)^I, \vdash_I) \quad (= \text{poset reflection}) \qquad (\text{for each set } I)$$

$$\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I := ([(X_j)_{j\in J}] \mapsto [(X_{f(i)})_{i\in I}]) \qquad (\text{for each } f : I \to J)$$

is a ($\mathbf{Set}$-based) tripos, called an intuitionistic realizability tripos

- For each $f : I \to J$, the adjoints $\exists f, \forall f : \mathbf{P}I \to \mathbf{P}J$ are given by:

$$\exists f([(X_i)_{i\in I}]) := \left[\left(\bigcup_{i\in f^{-1}(j)} X_i\right)_{j\in J}\right]$$

$$\forall f([(X_i)_{i\in I}]) := \left[\left(\bigcap_{i\in f^{-1}(j)} X_i\right)_{j\in J}\right]$$

- Generic predicate: $\Sigma := \mathfrak{P}(P)$, $\mathsf{tr}_\Sigma := [\mathsf{id}_{\mathfrak{P}(P)}] \in \mathbf{P}\Sigma$
- When $P = \{\bullet\}$ (+ trivial app.), we get the powerset tripos $\mathfrak{P}$ again

From triposes to toposes

Triposes are mainly useful to construct toposes[2] (= $\mathbf{Set}$-like categories), via the tripos-to-topos construction:

$$\mathbf{P} \quad \mapsto \quad \underbrace{\mathbf{Set}[\mathbf{P}]}_{\text{topos induced by } \mathbf{P}}$$

So that:

- Triposes induced by HAs yield localic (= forcing) toposes
- Triposes induced by PCAs yield intuitionistic realizability toposes.
  Example: The effective topos [Hyland '82]
- Triposes induced by AKSs yield classical realizability toposes [Streicher '13]
- ... and triposes induced by imp. algebras yield implicative toposes (to be studied soon :-)

[2] Or topoi, following the Greek etymology

What is a topos?

Formally, an (elementary) topos is a category $\mathscr{T}$ with all finite limits, exponentials and a subobject classifier

Intuitively, a topos is a category $\mathscr{T}$ in which:

- Objects behave as types, whereas morphisms behave as functions
- Objects (= types) are closed under products $A \times B$, sums $A + B$, exponentials B^A (or $A \to B$), comprehension types $\{x : A \mid \phi(x)\}$ and quotient types $A/{\sim}$
- Each topos $\mathscr{T}$ has a subobject classifier Ω (= type of propositions) that induces the internal logic of $\mathscr{T}$ (at least intuitionistic)
- Most toposes have a natural numbers object $\mathbb{N}$ (including all $\mathbf{Set}[\mathbf{P}]$)

Topos = $\mathbf{Set}$-like category in which one can formalize all elementary mathematics (at least intuitionistically)

Plan

Implicative algebras (recall)

Definition (Implicative structures & algebras)

1. An implicative structure is a complete lattice $(\mathscr{A}, \preccurlyeq)$ equipped with a binary operation $(\to) : \mathscr{A}^2 \to \mathscr{A}$ such that:

   (1) If $a' \preccurlyeq a$ and $b \preccurlyeq b'$, then $(a \to b) \preccurlyeq (a' \to b')$

   (2) For all $a \in \mathscr{A}$ and $B \subseteq \mathscr{A}$, we have: $a \to \bigcurlywedge_{b \in B} b \;=\; \bigcurlywedge_{b \in B} (a \to b)$

2. A separator of $(\mathscr{A}, \preccurlyeq, \to)$ is a subset $S \subseteq \mathscr{A}$ such that:

   (1) If $a \in S$ and $a \preccurlyeq a'$, then $a' \in S$

   (2) $\bigcurlywedge_{a,b \in \mathscr{A}} (a \to b \to c) \;\; (= \mathbf{K}^{\mathscr{A}}) \;\in\; S$ and
   $\bigcurlywedge_{a,b,c \in \mathscr{A}} ((a \to b \to c) \to (a \to b) \to a \to c) \;\; (= \mathbf{S}^{\mathscr{A}}) \;\in\; S$

   (3) If $(a \to b) \in S$ and $a \in S$, then $b \in S$

3. An implicative algebra is an implicative structure $(\mathscr{A}, \preccurlyeq, \to)$ together with a separator $S \subseteq \mathscr{A}$

Product of a family of implicative structures (1/2)

Given a family of implicative structures $(\mathscr{A}_i)_{i\in I} = (\mathscr{A}_i, \preccurlyeq_i, \to_i)_{i\in I}$

- The product $\mathscr{A} = \prod_{i\in I}\mathscr{A}_i$ of the family $(\mathscr{A}_i)_{i\in I} = (\mathscr{A}_i, \preccurlyeq_i, \to_i)_{i\in I}$ is clearly an implicative structure, where:

$$(a_i)_{i\in I} \preccurlyeq (b_i)_{i\in I} \quad :\equiv \quad \forall i\in I,\ a_i \preccurlyeq_i b_i \qquad \text{(product ordering)}$$

$$(a_i)_{i\in I} \to (b_i)_{i\in I} \quad := \quad (a_i \to_i b_i)_{i\in I} \qquad \text{(componentwise)}$$

Proposition (Properties of the product implicative structure $\prod_{i\in I}\mathscr{A}_i$)

In the product $\mathscr{A} = \prod_{i\in I}\mathscr{A}_i$, we have:

1. $ab \; = \; (a_ib_i)_{i\in I}$ for all $a, b\in\mathscr{A}$
2. $(t)^{\mathscr{A}} \; = \; \big((t)^{\mathscr{A}_i}\big)_{i\in I}$ for all closed λ-terms t
3. $\mathbf{S}^{\mathscr{A}} = \big(\mathbf{S}^{\mathscr{A}_i}\big)_{i\in I}$ $\quad\mathbf{K}^{\mathscr{A}} = \big(\mathbf{K}^{\mathscr{A}_i}\big)_{i\in I}$ $\quad\text{cc}^{\mathscr{A}} = \big(\text{cc}^{\mathscr{A}_i}\big)_{i\in I}$ etc.
4. $a\times b = (a_i\times b_i)_{i\in I},\quad a+b = (a_i+b_i)_{i\in I}$ for all $a, b\in\mathscr{A}$

Product of a family of implicative structures (2/2)

Given a family of implicative structures $(\mathscr{A}_i)_{i\in I} = (\mathscr{A}_i, \preccurlyeq_i, \to_i)_{i\in I}$

- The product $S = \prod_{i\in I} S_i$ of a family of separators $(S_i \subseteq \mathscr{A}_i)_{i\in I}$ is clearly a separator of the product $\mathscr{A} = \prod_{i\in I} \mathscr{A}_i$
- Moreover: $a \vdash_S b$ iff $\forall i \in I,\ a_i \vdash_{S_i} b_i$ (for all $a, b \in \mathscr{A}$)

Proposition (Factorization of the quotient)

$$\mathscr{A}/S \;=\; \Big(\prod_{i\in I}\mathscr{A}_i\Big) \Big/ \Big(\prod_{i\in I} S_i\Big) \;\cong\; \prod_{i\in I}(\mathscr{A}_i/S_i) \qquad \text{(iso. in } \mathbf{HA}\text{)}$$

- **Beware!** We only have the inclusions

$$S^0(\mathscr{A}) \;\subseteq\; \prod_{i\in I} S^0(\mathscr{A}_i) \qquad \text{(intuitionistic core)}$$

$$S^0_K(\mathscr{A}) \;\subseteq\; \prod_{i\in I} S^0_K(\mathscr{A}_i) \qquad \text{(classical core)}$$

Power of an implicative structure

Given an implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ and a set I, we write

$$\mathscr{A}^I \ := \ (\mathscr{A}^I, \preccurlyeq^I, \to^I) \ := \ \prod_{i \in I} \mathscr{A} \qquad \text{(power implicative structure)}$$

Each separator $S \subseteq \mathscr{A}$ induces two separators in $\mathscr{A}^I$:

- The power separator $\quad S^I \ := \ \prod_{i \in I} S \ \subseteq \ \mathscr{A}^I$,

  for which we have: $\qquad \mathscr{A}^I / S^I \ \cong \ (\mathscr{A}/S)^I$

- The uniform power separator $\quad S[I] \ \subseteq \ S^I \ \subseteq \ \mathscr{A}^I \quad$ defined by:

  $$S[I] \ := \ \{(a_i)_{i \in I} \in \mathscr{A}^I \ \mid \ (\exists s \in S)(\forall i \in I)\, s \preccurlyeq a_i\} \ = \ \uparrow\delta(S)$$

  where $\uparrow\delta(S)$ is the upwards closure (in $\mathscr{A}^I$) of the image of S through the canonical map $\delta : \mathscr{A} \to \mathscr{A}^I$ defined by $\delta(a) := (i \mapsto a) \in \mathscr{A}^I$ for all $a \in \mathscr{A}$

- In general, the inclusion $\quad S[I] \ \subseteq \ S^I \quad$ is strict!

Properties of the uniform power separator

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure, and I a set.

Each separator $S \subseteq \mathscr{A}$ induces a uniform power separator $S[I] \subseteq \mathscr{A}^I$

Proposition (Entailment w.r.t. $S[I]$)

For all families $a = (a_i)_{i \in I},\ b = (b_i)_{i \in I} \in \mathscr{A}^I$, we have:

$$a \vdash_{S[I]} b \quad \Leftrightarrow \quad (a \to b) \in S[I] \quad \Leftrightarrow \quad \bigcurlywedge_{i \in I} (a_i \to b_i) \in S$$

$$a \dashv\vdash_{S[I]} b \quad \Leftrightarrow \quad (a \leftrightarrow b) \in S[I] \quad \Leftrightarrow \quad \bigcurlywedge_{i \in I} (a_i \leftrightarrow b_i) \in S$$

Recall that $a \leftrightarrow b \ := \ (a \to b) \times (b \to a)$ (in any implicative structure)

We can also notice that:

- $S^0(\mathscr{A}^I) \ = \ S^0(\mathscr{A})[I] \ \subseteq \ \big(S^0(\mathscr{A})\big)^I$ (intuitionistic core of $\mathscr{A}^I$)
- $S^0_K(\mathscr{A}^I) \ = \ S^0_K(\mathscr{A})[I] \ \subseteq \ \big(S^0_K(\mathscr{A})\big)^I$ (classical core of $\mathscr{A}^I$)

Tripos associated to an implicative algebra (1/2)

Let $(\mathscr{A}, S) = (\mathscr{A}, \preccurlyeq, \to, S)$ be an implicative algebra

For each set I, we let $\mathbf{P}I \ := \ \mathscr{A}^I/S[I]$

- The poset $(\mathbf{P}I, \leq_{S[I]})$ is a Heyting algebra, where:

$$\begin{aligned} [a] \to [b] &= [(a_i \to b_i)_{i\in I}] & & \\ [a] \wedge [b] &= [(a_i \times b_i)_{i\in I}] & \top &= [(\top)_{i\in I}] \\ [a] \vee [b] &= [(a_i + b_i)_{i\in I}] & \bot &= [(\bot)_{i\in I}] \end{aligned}$$

- The correspondence $I \mapsto \mathbf{P}I$ is functorial:
  - Each $f : I \to J$ induces a substitution map $\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I$:

$$\mathbf{P}f([(a_j)_{j\in J}]) \ := \ [(a_{f(i)})_{i\in I}] \ \in \ \mathbf{P}I$$

  - The map $\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I$ is a morphism of Heyting algebras
  - $\mathbf{P}\,\mathrm{id}_I = \mathrm{id}_{\mathbf{P}(I)}$ and $\mathbf{P}(g \circ f) = \mathbf{P}f \circ \mathbf{P}g$ (contravariance)

Therefore: $\mathbf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ is a (contravariant) functor

Tripos associated to an implicative algebra (2/2)

Theorem (Associated tripos)

The functor $\mathbf{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{HA}$ induced by $(\mathscr{A}, S)$ is a tripos

- Each substitution map $\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I$ has both left and right adjoints $\exists f, \forall f : \mathbf{P}I \to \mathbf{P}J$:

$$\exists f\big([(a_i)_{i\in I}]\big) \;:=\; \Big[\Big(\exists_{i\in f^{-1}(j)}\, a_i\Big)_{j\in J}\Big] \;\in\; \mathbf{P}J$$

$$\forall f\big([(a_i)_{i\in I}]\big) \;:=\; \Big[\Big(\forall_{i\in f^{-1}(j)}\, a_i\Big)_{j\in J}\Big] \;\in\; \mathbf{P}J$$

(+ satisfies the Beck-Chevalley condition)

- There is a propositional object $\Sigma \in \mathbf{Set}$ together with a generic predicate $\mathsf{tr} \in \mathbf{P}\Sigma$:

$$\Sigma \;:=\; \mathscr{A} \qquad\qquad \mathsf{tr} \;:=\; [\mathsf{id}_{\mathscr{A}}] \;\in\; \mathbf{P}\Sigma$$

To sum up...

- The above construction encompasses many well-known tripos constructions:
  - Forcing triposes, which correspond to the case where $(\mathscr{A}, \preccurlyeq, \to)$ is a complete Heyting/Boolean algebra, and $S = \{\top\}$ (i.e. no quotient)
  - Triposes induced by (total) combinatory algebras... (int. realizability)
    ... and even by partial combinatory algebras, via some completion trick
  - Triposes induced by abstract Krivine structures (class. realizability)
- As for any tripos, each implicative tripos can be turned into a topos via the standard tripos-to-topos construction
- **Question:** What do implicative triposes bring new w.r.t.
  - Forcing triposes (intuitionistic or classical)?
  - Intuitionistic realizability triposes?
  - Classical realizability triposes?

Plan

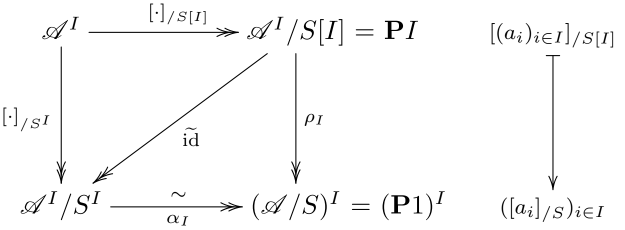
The fundamental diagram (1/3)

Given an implicative algebra $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to, S)$ and a set I, the separator $S \subseteq \mathscr{A}$ induces two separators in $\mathscr{A}^I$:

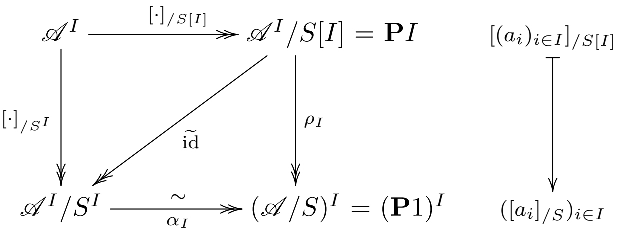
- The power separator $\quad S^I \subseteq \mathscr{A}^I$
- The uniform power separator $\quad S[I] \subseteq S^I \subseteq \mathscr{A}^I \quad$ defined by:

$$S[I] := \{(a_i)_{i\in I} \in \mathscr{A}^I \ :\ (\exists s \in S)(\forall i \in I)\, s \preccurlyeq a_i\}$$

We thus get the following (commutative) diagram: (in $\mathbf{Set}/\mathbf{HA}$)



The fundamental diagram (2/3)



Proposition

The following are equivalent:

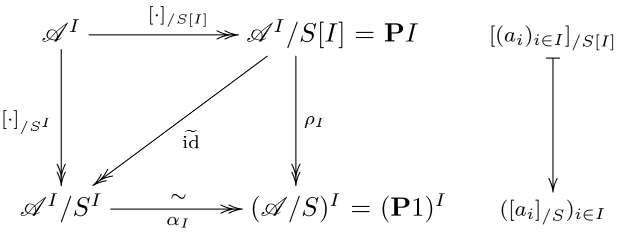
(1) The map $\rho_I : (\mathscr{A}^I/S[I]) \to (\mathscr{A}/S)^I$ is injective

(2) The map $\rho_I : (\mathscr{A}^I/S[I]) \to (\mathscr{A}/S)^I$ is an isomorphism (of HAs)

(3) $S[I] = S^I$

(4) The separator $S \subseteq \mathscr{A}$ is closed under all I-indexed meets.

The fundamental diagram (3/3)



Proof.

- Recall that in $\mathbf{HA}$, a morphism is an iso if and only if it is bijective. Since ρ is surjective and α_I is an iso, it is clear that:

$$(1)\ \ \rho \text{ injective} \quad\Leftrightarrow\quad (2)\ \ \rho \text{ iso.} \quad\Leftrightarrow\quad \widetilde{\mathrm{id}} \text{ iso.} \quad\Leftrightarrow\quad (3)\ \ S[I] = S^I$$

- $(3) \Rightarrow (4)$ Let $(a_i)_{i\in I} \in S^I$. Since $S^I = S[I]$ (by (3)), there is $s \in S$ such that $s \preccurlyeq a_i$ for all $i \in I$. Hence $s \preccurlyeq \curlywedge_{i\in I} a_i \in S$.

- $(4) \Rightarrow (3)$ Let $(a_i)_{i\in I} \in S^I$. By (4), we have that $s := \curlywedge_{i\in I} a_i \in S$. Since $s \preccurlyeq a_i$ for all $i \in I$, we get $(a_i)_{i\in I} \in S[I]$. Therefore: $S^I = S[I]$. □

Forcing triposes (recall)

Proposition and definition (Forcing triposes)

Given a complete Heyting (or Boolean) algebra H:

1. The functor $\mathbf{P} := H^{(-)} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ is a tripos
2. For all $I, J \in \mathbf{Set}$, $f : I \to J$:
   - $\mathbf{P}I := H^I$ is a complete HA
   - $\mathbf{P}f : \mathbf{P}J \to \mathbf{P}I$ is a morphism of complete HAs
3. $\Sigma := H$ and $\mathrm{tr} := \mathrm{id}_H$ (generic predicate)

Such a tripos is called a forcing tripos

- Forcing triposes are the ones underlying Kripke (or Cohen) forcing
- Each forcing tripos (induced by H) can be seen as an implicative tripos, constructed from the implicative algebra

$$(\mathscr{A}, \preccurlyeq, \to, S) := (H, \leq_H, \to_H, \{\top_H\})$$

Isomorphism of triposes

Definition (Isomorphism of triposes)

Two triposes $\mathbf{P}, \mathbf{P}' : \mathbf{Set}^{\mathsf{op}} \to \mathbf{HA}$ are isomorphic when there is a natural isomorphism $\beta : \mathbf{P} \Rightarrow \mathbf{P}'$ (in the category $\mathbf{HA}$):

$$\begin{array}{ccc} I & \qquad & \mathbf{P}I \xrightarrow[\sim]{\beta_I} \mathbf{P}'I \\ {\scriptstyle f}\downarrow & & {\scriptstyle \mathbf{P}f}\uparrow \qquad\qquad \uparrow{\scriptstyle \mathbf{P}'(f)} \\ J & & \mathbf{P}J \xrightarrow[\beta_J]{\sim} \mathbf{P}'J \end{array}$$

- We have seen that each Heyting tripos is isomorphic to a particular implicative tripos, taking $(\mathscr{A}, \preccurlyeq, \to, S) := (H, \leq_H, \to_H, \{\top\})$
- But more generally, what are the implicative triposes that are isomorphic to a forcing tripos?

Characterizing forcing triposes (1/4)

Theorem

Let $\mathbf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ be the tripos induced by an implicative algebra $(\mathscr{A}, \preccurlyeq, \to, S)$. Then the following are equivalent:

(1) The tripos $\mathbf{P}$ is isomorphic to a forcing tripos

(2) The separator $S \subseteq \mathscr{A}$ is a principal filter of $\mathscr{A}$

(3) The separator $S \subseteq \mathscr{A}$ is finitely generated and $\pitchfork^{\mathscr{A}} \in S$

Remark: These conditions do not imply that $(\mathscr{A}, \preccurlyeq, \to)$ is a Heyting algebra!
Counter-example: Krivine realizability with an instruction $\pitchfork$ (in the separator)

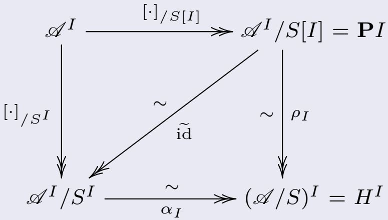
Proof.

- We have already seen that (3) $\Leftrightarrow$ (2), in a previous talk
- So it remains to prove that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) (...)

Characterizing forcing triposes (2/4)

Proof (continued).

- (2) ⇒ (1) When $S \subseteq \mathscr{A}$ is a principal filter of $\mathscr{A}$, we have seen that $H := \mathscr{A}/S$ is a complete Heyting algebra. Moreover, since S is closed under arbitrary meets, we have $S[I] = S^I$ for all sets I. Therefore the arrow ρ_I of the fundamental diagram



is an isomorphism of (complete) Heyting algebras for all sets I. It is also clearly natural in I, hence we can take $\beta_I := \rho_I$. (...)

Characterizing forcing triposes (3/4)

Proof (continued).

- (1) ⇒ (2) Assume that there is a natural isomorphism $\beta_I : \mathbf{P}I \tilde{\to} H^I$ (in I) for some complete Heyting algebra H. In particular, we have $\beta_1 : \mathbf{P}1 \tilde{\to} H^1 = H$, so that $\mathscr{A}/S = \mathbf{P}1 \cong H$ is a complete HA.

  Now, fix a set I, and write $c_i := \{0 \mapsto i\} : 1 \to I$ for each $i \in I$.

  Via the two (contravariant) functors $\mathbf{P}, H^{(-)} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$, we easily check that the arrow $c_i : 1 \to I$ is mapped to:

  $$\mathbf{P}c_i = \rho_i : \mathscr{A}^I/S[I] \to \mathscr{A}/S$$

  and

  $$H^{(c_i)} = \pi_i : H^I \to H$$

  where:

  - ρ_i is the ith component of the surjection $\rho_I : \mathscr{A}^I/S[I] \twoheadrightarrow (\mathscr{A}/S)^I$ of the fundamental diagram, given by: $\rho_i([a]_{/S[I]}) = [a_i]_{/S}$
  - π_i is the ith projection from H^I to H (...)

Characterizing forcing triposes (4/4)

Proof (continued).

- (1) ⇒ (2) (continued) We get the following commutative diagrams:

$$
\begin{array}{c}
1 \\
\Big\downarrow{\scriptstyle c_i} \\
I
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{A}/S & \xrightarrow[\sim]{\beta_1} & H \\
{\scriptstyle \mathbf{P}c_i=\rho_i}\Big\uparrow & & \Big\uparrow{\scriptstyle \pi_i=H^{(c_i)}} \\
\mathscr{A}^I/S[I] & \xrightarrow[\beta_I]{\sim} & H^I
\end{array}
\qquad
\begin{array}{ccc}
(\mathscr{A}/S)^I & \xrightarrow[\sim]{\beta_1^I} & H^I \\
{\scriptstyle \rho_I=\langle\rho_i\rangle_{i\in I}}\Big\uparrow & & \sim\Big\uparrow{\scriptstyle \mathrm{id}=\langle\pi_i\rangle_{i\in I}} \\
\mathscr{A}^I/S[I] & \xrightarrow[\beta_I]{\sim} & H^I
\end{array}
$$

  - 1st commutative square (for $i \in I$) comes from the naturality of β
  - 2nd commutative square is deduced from the first one by glueing the arrows ρ_i and π_i for all indices $i \in I$

From the 2nd commutative square, it is clear that $\rho_I : \mathscr{A}^I/S[I] \to (\mathscr{A}/S)^I$ is an isomorphism for all sets I. Therefore, the separator $S \subseteq \mathscr{A}$ is closed under arbitrary meets, which means that it is a principal filter □

Classical realizability triposes (1/2)

Definition (Abstract Krivine structure)

An abstract Krivine structure (AKS) $\mathscr{A}$ is given by:

- 2 sets $\mathbf{\Lambda}$ ($\mathscr{A}$-terms), $\mathbf{\Pi}$ ($\mathscr{A}$-stacks)
- 3 functions $(@) : \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{\Lambda}$, $(\cdot) : \mathbf{\Lambda} \times \mathbf{\Pi} \to \mathbf{\Pi}$, $(\mathbf{k}_{\_}) : \mathbf{\Pi} \to \mathbf{\Lambda}$
- 3 combinators $\mathbf{S}, \mathbf{K}, \mathbf{cc} \in \mathbf{\Lambda}$
- A subset $\mathrm{PL} \subseteq \mathbf{\Lambda}$ (of proof-like $\mathscr{A}$-terms) that contains the combinators $\mathbf{S}, \mathbf{K}, \mathbf{cc}$ and that is closed under application $(@)$.
- A binary relation $\perp\!\!\!\perp \subseteq \mathbf{\Lambda} \times \mathbf{\Pi}$ (the pole) such that:

$$\begin{array}{rcl}
t \star u \cdot \pi \in \perp\!\!\!\perp & \text{implies} & tu \star \pi \in \perp\!\!\!\perp \\
t \star \pi \in \perp\!\!\!\perp & \text{implies} & \mathbf{K} \star t \cdot u \cdot \pi \in \perp\!\!\!\perp \\
tv(uv) \star \pi \in \perp\!\!\!\perp & \text{implies} & \mathbf{S} \star t \cdot u \cdot v \cdot \pi \in \perp\!\!\!\perp \\
t \star \mathbf{k}_\pi \cdot \pi \in \perp\!\!\!\perp & \text{implies} & \mathbf{cc} \star t \cdot \pi \in \perp\!\!\!\perp \\
t \star \pi \in \perp\!\!\!\perp & \text{implies} & \mathbf{k}_\pi \star t \cdot \pi' \in \perp\!\!\!\perp
\end{array}$$

Classical realizability triposes (2/2)

- Each abstract Krivine structure $\mathcal{K} = (\mathbf{\Lambda}, \mathbf{\Pi}, \ldots, \mathsf{PL}, \perp\!\!\!\perp)$ induces a classical implicative algebra $\mathscr{A}_{\mathcal{K}} = (\mathscr{A}_{\mathcal{K}}, \preccurlyeq_{\mathcal{K}}, \to_{\mathcal{K}}, S_{\mathcal{K}})$ defined by:

$$\begin{array}{rcll} \mathscr{A}_{\mathcal{K}} & := & \mathfrak{P}(\mathbf{\Pi}) & \\ a \preccurlyeq_{\mathcal{K}} b & :\equiv & a \supseteq b & \\ a \to_{\mathcal{K}} b & := & a^{\perp\!\!\!\perp} \cdot b & \text{(Krivine's implication)} \\ S_{\mathcal{K}} & := & \{a \in \mathscr{A} \mid a^{\perp\!\!\!\perp} \cap \mathsf{PL} \neq \varnothing\} & \end{array}$$

- **Remark:** $\mathscr{A}_{\mathcal{K}}$ consistent iff $\bot_{\mathscr{A}_{\mathcal{K}}}\ (= \mathbf{\Pi}) \notin S_{\mathcal{K}}$
  iff $\mathbf{\Pi}^{\perp} \cap \mathsf{PL} = \varnothing$
  iff $\mathcal{K}$ consistent (as an AKS)

- The classical implicative algebra $\mathscr{A}_{\mathcal{K}} = (\mathscr{A}_{\mathcal{K}}, \preccurlyeq_{\mathcal{K}}, \to_{\mathcal{K}}, S_{\mathcal{K}})$ in turn induces a classical implicative tripos $\mathbf{P}_{\mathscr{A}_{\mathcal{K}}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$, called the classical realizability tripos induced by $\mathcal{K}$ [Streicher '13]

- Are all classical implicative triposes of this form?

Universality of AKS (1/3)

Theorem (Universality of AKS)

Each classical implicative tripos $\mathbf{P}_{\mathscr{A}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ is isomorphic to some classical realizability tripos $\mathbf{P}_{\mathscr{A}_{\mathcal{K}}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ (for some AKS $\mathcal{K}$)

Classical implicative algebras have thus the same expressiveness as Krivine realizability

The theorem follows from the following lemma:

Lemma (Reduction of implicative algebras)

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq_{\mathscr{A}}, \to_{\mathscr{A}}, S_{\mathscr{A}})$ and $\mathscr{B} = (\mathscr{B}, \preccurlyeq_{\mathscr{B}}, \to_{\mathscr{B}}, S_{\mathscr{B}})$ be two implicative algebras. If there exists a surjective map $\psi : \mathscr{B} \to \mathscr{A}$ (a reduction from $\mathscr{B}$ onto $\mathscr{A}$) such that

(1) $\psi\left(\bigwedge_{i\in I} b_i\right) = \bigwedge_{i\in I} \psi(b_i)$ (for all $I \in \mathbf{Set}$ and $b \in \mathscr{B}^I$)

(2) $\psi(b \to_{\mathscr{B}} b') = \psi(b) \to_{\mathscr{A}} \psi(b')$ (for all $b, b' \in \mathscr{B}$)

(3) $b \in S_{\mathscr{B}}$ iff $\psi(b) \in S_{\mathscr{A}}$ (for all $b \in \mathscr{B}$)

then the corresponding triposes $\mathbf{P}_{\mathscr{A}}, \mathbf{P}_{\mathscr{B}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ are isomorphic

Universality of AKS (2/3)

Proof of the Lemma

For each set I, write $\psi^I : \mathscr{B}^I \to \mathscr{A}^I$ the map defined by $\psi^I(b) = \psi \circ b$ for all $b \in \mathscr{B}^I$. Given points $b, b' \in \mathscr{B}^I$, we observe that:

$$\begin{array}{rll} b \vdash_{S_{\mathscr{B}}[I]} b' & \text{iff} & \bigwedge_{i\in I}(b_i \to_{\mathscr{B}} b'_i) \in S_{\mathscr{B}} \\ & \text{iff} & \psi\big(\bigwedge_{i\in I}(b_i \to_{\mathscr{B}} b'_i)\big) \in S_{\mathscr{A}} \\ & \text{iff} & \bigwedge_{i\in I}(\psi(b_i) \to_{\mathscr{A}} \psi(b'_i)) \in S_{\mathscr{A}} \\ & \text{iff} & \psi^I(b) \vdash_{S_{\mathscr{A}}[I]} \psi^I(b') \end{array}$$

From this, we deduce that:

(1) The map $\psi^I : \mathscr{B}^I \to \mathscr{A}^I$ is compatible with the preorders $\vdash_{S_{\mathscr{B}}[I]}$ (on $\mathscr{B}^I$) and $\vdash_{S_{\mathscr{A}}[I]}$ (on $\mathscr{A}^I$), and thus factors into a monotonic map $\tilde{\psi}_I : \mathbf{P}_{\mathscr{B}} I \to \mathbf{P}_{\mathscr{A}} I$ through the quotients $\mathbf{P}_{\mathscr{B}} I = \mathscr{B}^I / S_{\mathscr{B}}[I]$ and $\mathbf{P}_{\mathscr{A}} I = \mathscr{A}^I / S_{\mathscr{A}}[I]$

(2) The monotonic map $\tilde{\psi}_I : \mathbf{P}_{\mathscr{B}} I \to \mathbf{P}_{\mathscr{A}} I$ is an embedding of partial orderings, in the sense that $p \leq p'$ iff $\tilde{\psi}_I(p) \leq \tilde{\psi}_I(p')$ for all $p, p' \in \mathbf{P}_{\mathscr{B}} I$

Moreover, since $\psi : \mathscr{B} \to \mathscr{A}$ is onto, the maps $\psi^I : \mathscr{B}^I \to \mathscr{A}^I$ and $\tilde{\psi}_I : \mathbf{P}_{\mathscr{B}} I \to \mathbf{P}_{\mathscr{A}} I$ are onto as well. Therefore $\tilde{\psi}_I : \mathbf{P}_{\mathscr{B}} I \to \mathbf{P}_{\mathscr{A}} I$ is an isomorphism in $\mathbf{Pos}$, and thus an isomorphism in $\mathbf{HA}$. The naturality of $\tilde{\psi}_I : \mathbf{P}_{\mathscr{B}} I \to \mathbf{P}_{\mathscr{A}} I$ (in I) follows from the naturality of $\psi^I : \mathscr{A}^I \to \mathscr{B}^I$ (in I), which is obvious by construction □

Universality of AKS (3/3)

Proof of the Theorem

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to, S)$ be a classical implicative algebra. We consider the AKS $\mathcal{K} = (\Lambda, \Pi, @, \cdot, \mathtt{k}_{\_}, \mathtt{K}, \mathtt{S}, \mathtt{cc}, \mathrm{PL}, \perp\!\!\!\perp)$ defined by

- $\mathbf{\Lambda} = \mathbf{\Pi} := \mathscr{A}$
- $a@b := ab, \quad a \cdot b := a \to b \quad$ and $\quad \mathbf{k}_a := a \to \bot$ (for all $a, b \in \mathscr{A}$)
- $\mathbf{K} := \mathbf{K}^{\mathscr{A}}, \quad \mathbf{S} := \mathbf{S}^{\mathscr{A}} \quad$ and $\quad \mathbf{cc} := \mathbf{cc}^{\mathscr{A}}$
- $\mathrm{PL} := S \quad$ and $\quad \perp\!\!\!\perp := (\preccurlyeq_{\mathscr{A}}) = \{(a,b) \in \mathscr{A}^2 : a \preccurlyeq b\}$

Clearly, the above structure $\mathcal{K}$ is an AKS, in which for each set $\beta \subseteq \mathbf{\Pi}$ $(= \mathscr{A})$, we have $\beta^{\perp\!\!\!\perp} = \{a \in \mathscr{A} : \forall b \in \beta,\ a \preccurlyeq b\} = \downarrow\{\curlywedge\beta\} \subseteq \mathbf{\Lambda}$ $(= \mathscr{A})$

Now, the AKS $\mathcal{K}$ induces the implicative algebra $\mathscr{B} = (\mathscr{B}, \preccurlyeq_{\mathscr{B}}, \to_{\mathscr{B}}, S_{\mathscr{B}})$ defined by:

- $\mathscr{B} := \mathfrak{P}(\mathbf{\Pi}) = \mathfrak{P}(\mathscr{A})$
- $\beta \preccurlyeq_{\mathscr{B}} \beta' \ :\Leftrightarrow\ \beta \supseteq \beta'$ (for all $\beta, \beta' \in \mathscr{B}$)
- $\beta \to_{\mathscr{B}} \beta' := \beta^{\perp\!\!\!\perp} \cdot \beta' = \{a \to a' : a \preccurlyeq \curlywedge\beta,\ a' \in \beta'\}$ (for all $\beta, \beta' \in \mathscr{B}$)
- $S_{\mathscr{B}} := \{\beta \in \mathscr{B} : \beta^{\perp\!\!\!\perp} \cap \mathrm{PL} \neq \varnothing\} = \{\beta \in \mathfrak{P}(\mathscr{A}) : \curlywedge\beta \in S_{\mathscr{A}}\}$

We now define $\psi : \mathscr{B} \to \mathscr{A}$ by $\psi(\beta) = \curlywedge\beta$ for all $\beta \in \mathscr{B}$ $(= \mathfrak{P}(\mathscr{A}))$. We easily check that $\psi : \mathscr{B} \to \mathscr{A}$ is a reduction from $\mathscr{B}$ onto $\mathscr{A}$. So that from the previous Lemma, the triposes induced by the implicative algebras $\mathscr{A}$ and $\mathscr{B}$ are isomorphic □

Plan

Conclusion

- Each implicative algebra $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to, S)$ induces a $(\mathbf{Set}$-based$)$ tripos $\mathbf{P}_{\mathscr{A}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ (that can later be turned into a topos)
- This construction encompasses all the tripos known so far, namely:
  - All localic triposes (induced by complete HAs)
  - All realizability triposes induced by (total) combinatory algebras

    (The construction extends to the realizability triposes induced by PCAs, using some completion trick not shown here)
  - All classical realizability triposes induced by AKSs
- In this structure: forcing = non deterministic realizability
- Classical implicative structures have the very same expressiveness as abstract Krivine structures (with a much lighter machinery)
- **Question:** Are all triposes implicative (up to isomorphism)?