An interpretation of **E-HA**$^\omega$ inside **HA**$^\omega$

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The plan

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- Introduce System T
- Introduce $\mathbf{HA}^{\omega}$ and a proof system capturing this theory
- Finally, in the framework of $\mathbf{HA}^{\omega}$, we will give an interpretation of extensional equality[1] on all sorts in a language that only contains equality on the sort $\mathbf{N}$.

Goal: Fully describe a (synctactical) translation between two proof systems where the source is wider than the target system.

[1] two functions are extensionally equal if they have the same graph

System T

What is System T ?

System T is

1. a programming language than can describe (some constructive) "functionals" (i.e functions over natural numbers, functions over functions over natural numbers...)
2. expressed as a typed lambda calculus
3. a theoretical tool invented by Gödel (in 1958) to give a computational interpretation of arithmetic with his so-called *Dialectica interpretation*.

Syntax of System T

System T is obtained by extending simple type lambda calculus with a based type $\mathbf{N}$ and native constructors to use it :

Sorts	σ, τ	$::=$	$\mathbf{N} \mid \sigma \to \tau$	($\to$ right associative)
Terms	t, u	$::=$	$x^\sigma \mid \lambda x^\sigma.t \mid tu$ $\mid 0 \mid \mathrm{S}\, t \mid \mathrm{Rec}^\sigma\, t\, u\, v$	(tu left associative)
Signatures	Δ	$::=$	$\emptyset \mid \Delta, x^\sigma$	(technicality)

Examples

$$\mathrm{add} \equiv \lambda x^{\mathbf{N}} \lambda y^{\mathbf{N}}.\mathrm{Rec}^{\mathbf{N}}\, x\, (\lambda h^{\mathbf{N}} \lambda m^{\mathbf{N}}.\mathrm{S}\, h)\, y \quad : \quad \mathbf{N} \to \mathbf{N} \to \mathbf{N}$$

$$\mathrm{app}_{\sigma,\tau} \equiv \lambda f^{\sigma \to \tau} \lambda x^\sigma.f\, x \quad : \quad (\sigma \to \tau) \to \sigma \to \tau$$

I may omit sort annotation when writing terms.

Computation in System T

We consider the following rules (β-reduction and ι-reduction) on terms

$$\begin{array}{rcl} (\lambda x.t)\,u & \succ & t[x ::= u] \\ \mathrm{Rec}\,t\;u\,0 & \succ & t \\ \mathrm{Rec}\,t\;u\,(\mathrm{S}\,v) & \succ & u\,(\mathrm{Rec}\,t\,u\,v)\,v \end{array}$$

from which we generate reduction

$$t \quad \twoheadrightarrow \quad u_{\text{ see footnote}^1}$$

and congruence

$$t \quad \cong \quad u_{\text{ see footnote}^2}.$$

[1]least compatible, reflexive and transitive relation containing $\succ$
[2]least equivalence relation containing $\twoheadrightarrow$

An example of reduction

$$
\begin{array}{rcl}
\mathrm{add}\,(\mathrm{S}\,(\mathrm{S}\,0))\,(\mathrm{S}\,0) & \equiv & (\lambda x^{\mathbf{N}} \lambda y^{\mathbf{N}}.\mathrm{Rec}^{\mathbf{N}}\, x\, (\lambda h^{\mathbf{N}} \lambda m^{\mathbf{N}}.\mathrm{S}\, h)\, y)\, (\mathrm{S}\,(\mathrm{S}\,0))\,(\mathrm{S}\,0) \\
& \twoheadrightarrow & \lambda y^{\mathbf{N}}.\mathrm{Rec}^{\mathbf{N}}\,(\mathrm{S}\,(\mathrm{S}\,0))\,(\lambda h^{\mathbf{N}} \lambda m^{\mathbf{N}}.\mathrm{S}\, h)\, y)\,(\mathrm{S}\,0) \\
& \twoheadrightarrow & \mathrm{Rec}^{\mathbf{N}}\,(\mathrm{S}\,(\mathrm{S}\,0))\,(\lambda h^{\mathbf{N}} \lambda m^{\mathbf{N}}.\mathrm{S}\, h)\,(\mathrm{S}\,0) \\
& \twoheadrightarrow & (\lambda h^{\mathbf{N}} \lambda m^{\mathbf{N}}.\mathrm{S}\, h)\,(\mathrm{Rec}^{\mathbf{N}}(\mathrm{S}\,(\mathrm{S}\,0))\,(\lambda h^{\mathbf{N}} \lambda m^{\mathbf{N}}.\mathrm{S}\, h)\,0)\,0 \\
& \twoheadrightarrow & (\lambda h^{\mathbf{N}} \lambda m^{\mathbf{N}}.\mathrm{S}\, h)\,(\mathrm{S}\,(\mathrm{S}\,0))\,0 \\
& \twoheadrightarrow & (\mathrm{S}\,(\mathrm{S}\,(\mathrm{S}\,0)))
\end{array}
$$

Derivation in System T

$$\dfrac{}{\Delta_1, x^\sigma, \Delta_2 \vdash_{\mathrm{T}} x^\sigma : \sigma}\ \text{(ax)}$$

$$\dfrac{\Delta, x^\sigma \vdash_{\mathrm{T}} t : \tau}{\Delta \vdash_{\mathrm{T}} \lambda x^\sigma . t : \sigma \to \tau}\ (\lambda\text{-intro}) \qquad \dfrac{\Delta \vdash_{\mathrm{T}} t : \sigma \to \tau \quad \Delta \vdash_{\mathrm{T}} u : \sigma}{\Delta \vdash_{\mathrm{T}} tu : \tau}\ \text{(app)}$$

$$\dfrac{}{\Delta \vdash_{\mathrm{T}} 0 : \mathbf{N}}\ (0) \qquad \dfrac{\Delta \vdash_{\mathrm{T}} t : \mathbf{N}}{\Delta \vdash_{\mathrm{T}} \mathrm{S}\, t : \mathbf{N}}\ (\mathrm{S})$$

$$\dfrac{\Delta \vdash_{\mathrm{T}} t : \sigma \quad \Delta \vdash_{\mathrm{T}} u : \sigma \to \mathbf{N} \to \sigma \quad \Delta \vdash_{\mathrm{T}} v : \mathbf{N}}{\Delta \vdash_{\mathrm{T}} \mathrm{Rec}^\sigma\ t\, u\, v : \sigma}\ (\mathrm{Rec})$$

Example of derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{f^{\sigma \to \tau}, x^{\sigma} \vdash f : \sigma \to \tau}\,(\text{ax}) \qquad \dfrac{}{f^{\sigma \to \tau}, x^{\sigma} \vdash x : \sigma}\,(\text{ax})}{f^{\sigma \to \tau}, x^{\sigma} \vdash f\, x : \tau f^{\sigma \to \tau}, x^{\sigma} \vdash f\, x : \tau}\,(\text{app})}{f^{\sigma \to \tau} \vdash \lambda x^{\sigma}.f\, x f^{\sigma \to \tau} \vdash \lambda x^{\sigma}.f\, x : \sigma \to \tau}\,(\lambda\text{-intro})}{\vdash_{\mathrm{T}} \mathrm{app}_{\sigma,\tau} \lambda f^{\sigma \to \tau} \lambda x^{\sigma}.f\, x \lambda f^{\sigma \to \tau} \lambda x^{\sigma}.f\, x : (\sigma \to \tau) \to \sigma \to \tau}\,(\lambda\text{-intro})
$$

Metatheoretical results

1. Canonicity : closed normal terms[1] of type **N** are of the form $\mathrm{S}^n\,0$, closed normal terms of type $\sigma \to \tau$ are of the form $\lambda x^\sigma.t$.
2. Strong normalisation : terms of System T are strongly normalisable.
3. Representable functions : a function $f : \mathbf{N} \to \mathbf{N}$ can be (extensionally) expressed as a term of System T if and only if it is a recursive function provably total in **PA**(Peano Arithmetic).

Finally, a generalized version of the weakening rule is admissible for this system :

$$\text{if } \Delta \subseteq \Delta' \text{ and } \Delta \vdash_{\mathrm{T}} t : \sigma \quad \text{then} \quad \Delta' \vdash_{\mathrm{T}} t : \sigma.$$

for a suitable definition of $\Delta \subseteq \Delta'$

[1] closed = without free variable (λ is the unique binder here)
normal = without possible reduction

Finite type arithmetic $\mathbf{HA}^\omega$

What is $\mathbf{HA}^\omega$?

$\mathbf{HA}^\omega$ is

1. a first order many sorted (intuitionistic) theory
2. a formal theory to speak about functionals
3. a conservative extension of Heyting Arithmetic (a.k.a the intuitionistic fragment of Peano Arithmetic).

Formulas of $\mathbf{HA}^\omega$

Formulas of $\mathbf{HA}^\omega$ are generated as follow :

$$\begin{array}{rcl}\Phi, \Psi & ::= & t = u \mid \bot \\ & & \mid \Phi \Rightarrow \Psi \mid \Phi \wedge \Psi \\ & & \mid \forall x^\sigma.\Phi\end{array}$$

where equality is restricted to the sort $\mathbf{N}$.

The following notation will be used :

$$\neg\Phi \quad \equiv \quad \Phi \Rightarrow \bot.$$

Axioms and rules of $\mathbf{HA}^\omega$

The axioms and rules of $\mathbf{HA}^\omega$ are the universal closures of :

1. the axioms and rules for many-sorted intuitionistic predicate logic
2. the axioms of equality (roughly an axiomatization of $\simeq$) [1]
3. for all formulas Φ that does not contain the variables $x^{\mathbf{N}}$ and $y^{\mathbf{N}}$:
$$x = y \Rightarrow \Phi[z := x] \Rightarrow \Phi[z := y]$$
(equal terms should satisfy the same properties)
4. $\neg(\mathrm{S}\, x) = 0$ (successors are different from 0)
5. for all formulas Φ :
$$\Phi[z := 0] \Rightarrow (\forall z^{\mathbf{N}}\, \Phi[z := x] \Rightarrow \Phi[z := \mathrm{S}\, x]) \Rightarrow \forall x^{\mathbf{N}}\, \Phi[z := x]$$
(a scheme of induction over natural numbers)

[1] because equality is restricted to the sort **N**, one needs to fully apply terms to axiomatize $\cong$.

Provability in $\mathbf{HA}^\omega$

We say that a formula Φ is provable in $\mathbf{HA}^\omega$ and we write

$$\vdash_{\mathbf{HA}^\omega} \Phi$$

if one can deduce Φ from the axioms and rules of $\mathbf{HA}^\omega$ in his favorite proof system (ex. hilbert system, natural deduction...).

We formalize this notion in few minutes :)

Models of $\mathbf{HA}^\omega$

A model of $\mathbf{HA}^\omega$ is given by the data of :

1. a family of sets $\{M_\sigma\}_\sigma$ indexed by the sorts of System T
2. an interpretation of all the constructions of System T inside these sets
3. such that this structure satisfies the axioms of $\mathbf{HA}^\omega$.

One can formally define what it means for a model to satisfy a closed formula (with parameters) Φ.

We will use the following theorem :

Theorem (Soundness)

If $\vdash_{\mathbf{HA}^\omega} \Phi$ then all models of $\mathbf{HA}^\omega$ satisfy Φ.

First example : the set theoretic model

The set theoretic model $\mathbf{M}$ is defined by

$$\begin{array}{rcl} \mathbf{M}_{\mathbf{N}} & \equiv & \mathbb{N} \\ \mathbf{M}_{\sigma \to \tau} & \equiv & \mathbf{M}_{\tau}^{\mathbf{M}_{\sigma}} \end{array}$$

where terms of System T are interpreted as functions.

Second example :**HRO**

The model of Hereditary Recursive Operations **HRO** defined by

$$\begin{array}{rcl} \mathbf{HRO_N} & \equiv & \mathbb{N} \\ \mathbf{HRO}_{\sigma\to\tau} & \equiv & \{e \in \mathbb{N} \mid \forall n \in \mathbf{HRO}_\sigma \ \{e\}(n) \downarrow\in \mathbf{HRO}_\tau\} \end{array}$$

where $\{e\}(n) \downarrow\in E$ means that the computation of the function of index e terminates on the input n and that the result of this computation is in E.

In this model, terms of System T are interpreted as indexes of the recursive functions they denote.

From an axiomatized theory to a type system

We now formulate the theory $\mathbf{HA}^\omega$ as a type system $\lambda\mathbf{HA}^\omega$ (without computation).

My motivation here is that I will work in a purely syntactic fashion and I'd rather manipulate proof terms (à la Church) than manipulating proof trees.

The sequents of $\lambda\mathbf{HA}^\omega$ will be of the form

$$\Delta; \Gamma \vdash M : \Phi$$

where

1. Δ is a signature of System T
2. Γ is a context of $\lambda\mathbf{HA}^\omega$ (to be defined next slide)
3. M is a proof term of $\lambda\mathbf{HA}^\omega$ (to be defined next slide)
4. Φ is a formula of $\mathbf{HA}^\omega$ (with one more predicate over the sort $\mathbf{N}$ for a technical detail).

The type system $\lambda\mathbf{HA}^\omega$

Formulas, proof terms and contexts of $\lambda\mathbf{HA}^\omega$ are generated by the following grammar :

$$\begin{array}{llcl}
\text{Formulas} & \Phi, \Psi & ::= & t = u \mid \bot \mid \mathrm{null}(t) \\
 & & & \mid \Phi \Rightarrow \Psi \mid \Phi \wedge \Psi \\
 & & & \mid \forall x^\sigma.\Phi \\
\text{Proof terms} & M, N & ::= & \xi \mid \mathrm{refl}\ t \mid \mathrm{peel}(t, u, M, \hat{x}.\Phi, N) \\
 & & & \mid \mathrm{efq}(M, \Phi) \\
 & & & \mid \lambda\xi.M \mid M\ N \\
 & & & \mid (M, N) \mid M.1 \mid M.2 \\
 & & & \mid \lambda x^\sigma.M \mid M\ t \\
 & & & \mid \mathrm{Ind}(\hat{x}.\Phi, M, N, t) \\
\text{Contexts} & \Gamma & ::= & \emptyset \mid \Gamma, \xi : \Phi
\end{array}$$

(again, equality is restricted to the sort $\mathbf{N}$.)

Derivation in $\lambda\mathbf{HA}^\omega$

$$\frac{(\Delta;\Gamma)\ \mathbf{wf}}{\Delta;\Gamma \vdash \xi : \Phi}\ (\xi : \phi \in \Gamma) \qquad \frac{\Delta;\Gamma \vdash M : \bot}{\Delta;\Gamma \vdash \mathrm{efq}(M,\Phi) : \Phi}\ (\mathbf{FV}(\Phi) \subseteq \Delta) \qquad \frac{\Delta;\Gamma \vdash M : \Phi}{\Delta;\Gamma \vdash M : \Psi}\ (\Phi \simeq \Psi)$$

$$\frac{\Delta;\Gamma, \xi : \Phi \vdash M : \Psi}{\Delta;\Gamma \vdash \lambda\xi.M : \Phi \Rightarrow \Psi} \qquad \frac{\Delta;\Gamma \vdash M : \Phi \Rightarrow \Psi \quad \Delta;\Gamma \vdash N : \Phi}{\Delta;\Gamma \vdash M\,N : \Psi}$$

$$\frac{\Delta;\Gamma \vdash M_1 : \Phi_1 \quad \Delta;\Gamma \vdash M_2 : \Phi_2}{\Delta;\Gamma \vdash (M_1, M_2) : \Phi_1 \wedge \Phi_2} \qquad \frac{\Delta;\Gamma \vdash M : \Phi_1 \wedge \Phi_2}{\Delta;\Gamma \vdash M.i : \Phi_i}\ (i = 1, 2)$$

$$\frac{\Delta, x^\sigma;\Gamma \vdash M : \Phi}{\Delta;\Gamma \vdash \lambda x^\sigma.M : \forall x^\sigma.\Phi}\ (x^\sigma \notin \mathbf{FV}(\Gamma)) \qquad \frac{\Delta;\Gamma \vdash M : \forall x^\sigma.\Phi \quad \Delta \vdash_{\mathrm{T}} t : \sigma}{\Delta;\Gamma \vdash M\,t : \Phi[x^\sigma := t]}$$

$$\frac{(\Delta;\Gamma)\ \mathbf{wf} \quad \Delta \vdash_{\mathrm{T}} t : \mathbf{N}}{\Delta;\Gamma \vdash \mathrm{refl}\,t : t = t} \qquad \frac{\Delta;\Gamma \vdash M : t = u \quad \Delta;\Gamma \vdash N : \Phi[x^{\mathbf{N}} := t]}{\Delta;\Gamma \vdash \mathrm{peel}(t, u, M, \hat{x}.\Phi, N) : \Phi[x^{\mathbf{N}} := u]}$$

$$\frac{\Delta;\Gamma \vdash M : \Phi[x^{\mathbf{N}} := 0] \quad \Delta;\Gamma \vdash N : \forall x^{\mathbf{N}}.(\Phi \Rightarrow \Phi[x^{\mathbf{N}} := \mathrm{S}\,x^{\mathbf{N}}]) \quad \Delta \vdash_{\mathrm{T}} t : \mathbf{N}}{\Delta;\Gamma \vdash \mathrm{Ind}(\hat{x}.\Phi, M, N, t) : \Phi[x := t]}$$

Theorem

$\vdash_{\mathbf{HA}^\omega} \Phi$ *if and only if there exists some proof term* M *such that* $\vdash M : \Phi$.

Final technical details

The congruence relation

$$\Phi \simeq \Psi$$

between formulas is generated from the reduction rules of System T and the two extra rules :

$$\begin{array}{rcl} \mathrm{null}(0) & \succ & \bot \Rightarrow \bot \\ \mathrm{null}(\mathrm{S}\,x) & \succ & \bot. \end{array}$$

The predicate $\mathrm{null}(x)$ is used to prove that successors are different from 0.

A pair of a signature and a context $(\Delta;\Gamma)$ is well formed if the free first order variables of Γ are contained in Δ, i.e

$$(\Delta;\Gamma)\ \mathbf{wf} \quad \equiv \quad \mathbf{FV}(\Gamma) \quad \subseteq \quad \Delta.$$

Some facts satisfied by $\lambda\mathbf{HA}^{\omega}$:

1. a generalization of the weakening lemma
2. if $\Delta;\Gamma \vdash M : \Phi$ then $\mathbf{FV}(\Phi) \subseteq \Delta$.

A case study : extension of equality

Some motivation

We work in a many sorted theory where the equality is only defined on the base sort **N**.

We would like to extend it (in an extensional fashion [1]) to all sorts.

Hence, for all sort σ we have to find an equivalence relation to serve as equality (i.e we want to quotient objects of sort σ by this relation).

Two objects in the same equivalence class should satisfy the same formulas, i.e they should be indiscernibles[2].

Two extensionally equal functions should be in the same equivalence class.

Finally, we want to express this extension in a purely syntactic fashion.

So let's study families of binary relations indexed by the sorts of System T!

[1] two functions are extensionally equal if they have the same graph

[2] in particular, it means that the relation should be compatible with all constructions of System T.

Two potential candidates ?

Let $\{=^{\mathrm{ext}}_{\sigma}\}_{\sigma}$ and $\{=^{\mathrm{pm}}_{\sigma}\}_{\sigma}$ be two families of binary relations indexed by the sorts of System T and defined in $\lambda\mathbf{HA}^{\omega}$ as follow :

$$\begin{array}{rcl} x^{\mathbf{N}} =^{\mathrm{ext}}_{\mathbf{N}} y^{\mathbf{N}} & \equiv & x = y \\ f^{\sigma\to\tau} =^{\mathrm{ext}}_{\sigma\to\tau} g^{\sigma\to\tau} & \equiv & \forall x \; f\,x =^{\mathrm{ext}}_{\tau} g\,x \end{array}$$

$$\begin{array}{rcl} x^{\mathbf{N}} =^{\mathrm{pm}}_{\mathbf{N}} y^{\mathbf{N}} & \equiv & x = y \\ f^{\sigma\to\tau} =^{\mathrm{pm}}_{\sigma\to\tau} g^{\sigma\to\tau} & \equiv & \forall x, y \; x =^{\mathrm{pm}}_{\sigma} y \Rightarrow f\,x =^{\mathrm{pm}}_{\tau} g\,y. \end{array}$$

Note that

1. The relation $=^{\mathrm{ext}}$ is obtained from equality by extending it to higher sorts in an extensional fashion (two functions are in $=^{\mathrm{ext}}$ if they are extensionally equal).
2. The relation $=^{\mathrm{pm}}$ is obtained from equality by extending it to higher sorts in a parametric fashion (in the sense of binary parametricity).

$=^{\mathrm{ext}}$ is a good candidate ?

Recall

$$\begin{array}{rcl} x^{\mathbf{N}} =^{\mathrm{ext}}_{\mathbf{N}} y^{\mathbf{N}} & \equiv & x = y \\ f^{\sigma\to\tau} =^{\mathrm{ext}}_{\sigma\to\tau} g^{\sigma\to\tau} & \equiv & \forall x\ f\,x =^{\mathrm{ext}}_{\tau} g\,x \end{array}$$

One can prove inside $\lambda\mathbf{HA}^{\omega}$ that for all sort σ, $=_{\sigma}^{=^{\mathrm{ext}}}$ is an equivalence relation[1].

But, one won't be able to prove that it is compatible, for instance :

$$\not\vdash \forall f^{(\mathbf{N}\to\mathbf{N})\to\mathbf{N}} \forall x^{\mathbf{N}\to\mathbf{N}} \forall y^{\mathbf{N}\to\mathbf{N}}\ x =^{\mathrm{ext}}_{\mathbf{N}\to\mathbf{N}} y \Rightarrow f\,x =^{\mathrm{ext}}_{\mathbf{N}} f\,y.$$

[1] note that you should first define what is an equivalence relation inside $\lambda\mathbf{HA}^{\omega}$

$=^{\mathrm{ext}}$ is not compatible

Proof.

By the contraposition of the Soundess theorem, it is sufficient to find one counter model.
We now work in **HRO**.
Let

$$\mathrm{quote} \in \mathbf{HRO}_{(\mathbf{N}\to\mathbf{N})\to\mathbf{N}}$$

be an index for the identity function and

$$p, q \in \mathbf{HRO}_{\mathbf{N}\to\mathbf{N}}$$

two distinct indexes for the same total unary function.
Note that

$$\begin{array}{lcl} \mathbf{HRO} & \vDash & p =^{\mathrm{ext}}_{\mathbf{N}\to\mathbf{N}} q \\ \mathbf{HRO} & \vDash & \{\mathrm{quote}\}(p) \neq^{\mathrm{Ext}}_{\mathbf{N}} \{\mathrm{quote}\}(q). \end{array}$$

Consequently, **HRO** is a counter model. □

$$\begin{array}{rcl} \mathbf{HRO}_{\mathbf{N}} & \equiv & \mathbb{N} \\ \mathbf{HRO}_{\sigma\to\tau} & \equiv & \{e \in \mathbb{N} \mid \forall n \in \mathbf{HRO}_\sigma\ \{e\}(n) \downarrow \in \mathbf{HRO}_\tau\} \end{array}$$

It shouldn't be a surprise : **HRO** is not an extensional model of (**N**-)$\mathbf{HA}^\omega$. We should restrict out attention to individuals that are "in a way" extensional (in particular, a term as quote should "not exist" in an "extensional world").

$=^{\mathrm{pm}}$ is a good candidate ?

One can prove that for all sort σ, $=^{\mathrm{pm}}_\sigma$ is symmetric and transitive. Formally, defining

$$\begin{array}{rcl} \mathbf{Sym}_{\mathcal{R}} & \equiv & \forall x^\sigma, y^\sigma \; x\mathcal{R}y \Rightarrow y\mathcal{R}x \\ \mathbf{Trans}_{\mathcal{R}} & \equiv & \forall x^\sigma, y^\sigma, z^\sigma \; x\mathcal{R}y \Rightarrow y\mathcal{R}z \Rightarrow x\mathcal{R}z \end{array}$$

one can exhibit proof terms

$$\begin{array}{rcll} \vdash & \mathrm{sym}^{\mathrm{pm}}_\sigma & : & \mathbf{Sym}_{=^{\mathrm{pm}}_\sigma} \\ \vdash & \mathrm{trans}^{\mathrm{pm}}_\sigma & : & \mathbf{Trans}_{=^{\mathrm{pm}}_\sigma} \end{array}$$

as follow

$$\begin{array}{rcl} \mathrm{sym}^{\mathrm{pm}}_{\mathbf{N}} & \equiv & \lambda x, y.\lambda\xi.\mathrm{peel}(x, y, \xi, \hat{z}.(z = x), \mathrm{refl}\, x) \\ \mathrm{sym}^{\mathrm{pm}}_{\sigma\to\tau} & \equiv & \lambda f, g.\lambda\xi.\lambda x, y.\lambda\eta.\mathrm{sym}^{\mathrm{pm}}_\tau (f\, y)(g\, x)(\xi\, y\, x\, (\mathrm{sym}^{\mathrm{pm}}_\sigma\, x\, y\, \eta)) \\ \mathrm{trans}^{\mathrm{pm}}_{\mathbf{N}} & \equiv & \lambda x, y, z.\lambda\xi, \eta.\mathrm{peel}(y, z, \eta, \hat{w}.x = w, \xi) \\ \mathrm{trans}^{\mathrm{pm}}_{\sigma\to\tau} & \equiv & \lambda f, g, h.\lambda\xi, \eta.\lambda x, y.\lambda\chi.\mathrm{trans}_\tau (f\, x)(g\, y)(h\, y)(\xi\, x\, y\, \chi) \\ & & \qquad (\eta\, y\, y(\mathrm{trans}_\sigma\, y\, x\, y(\mathrm{sym}^{\mathrm{pm}}_\sigma x\, y\, \chi)\chi)) \end{array}$$

$=^{\mathrm{pm}}$ is not reflexive...

Recall

$$\begin{array}{rcl} x^{\mathbf{N}} =^{\mathrm{pm}}_{\mathbf{N}} y^{\mathbf{N}} & \equiv & x = y \\ f^{\sigma\to\tau} =^{\mathrm{pm}}_{\sigma\to\tau} g^{\sigma\to\tau} & \equiv & \forall x, y\ x =^{\mathrm{pm}}_{\sigma} y \Rightarrow f\,x =^{\mathrm{pm}}_{\tau} g\,y. \end{array}$$

In **HRO**, one can show

$$\mathrm{quote} \neq^{\mathrm{pm}}_{(\mathbf{N}\to\mathbf{N})\to\mathbf{N}} \mathrm{quote}.$$

We conclude :

$$\nvdash \forall x^{(\mathbf{N}\to\mathbf{N})\to\mathbf{N}}\ x =^{\mathrm{pm}}_{(\mathbf{N}\to\mathbf{N})\to\mathbf{N}} x.$$

Is that bad, doctor ?

In logic, we are used to work with partial equivalence relation (**PER**).

Ok, it is not reflexive, but we can at least (try to) show that all terms of System T are in its domain[1].

[1] the domain of a binary relation $\mathcal{R}$ is all the individuals satisfying the formula $\mathbf{Dom}_{\mathcal{R}}(x) \equiv x\mathcal{R}x$.

Are all closed terms of System T in $\mathbf{Dom}_{=^{\mathrm{pm}}}$?

Formally, for all closed term

$$\vdash_{\mathrm{T}} t : \sigma$$

of System T, we want to build a proof

$$\vdash t^{\mathrm{pm}} : t =_{\sigma}^{\mathrm{pm}} t$$

of $\lambda\mathbf{HA}^{\omega}$.

With this goal in mind, we design a translation from System T to $\lambda\mathbf{HA}^{\omega}$

$$(\Delta \vdash_{\mathrm{T}} t : \sigma)^{\mathrm{pm}} \quad \rightsquigarrow \quad \Delta^1, \Delta^2; \Delta^{\mathrm{pm}} \vdash t^{\mathrm{pm}} : t^1 =_{\sigma}^{\mathrm{pm}} t^2.$$

where $\Delta^i, \Delta^{\mathrm{pm}}, t^{\mathrm{pm}}, t^i$ are yet to be defined.

Intuitions : Δ^1, Δ^2 are two disjoint copies of the context and Δ^{pm} says that $\Delta^1 =^{\mathrm{pm}} \Delta^2$.

Definition of Δ^i and t^i

Fixing $i = 1, 2$:
Declarations of variables in signatures are duplicated.

$$\begin{array}{rcl} \emptyset^i & \equiv & \emptyset \\ (\Delta, x^\sigma)^i & \equiv & \Delta^i, (x^i)^\sigma \end{array}$$

where x^i are fresh distinct variables (and $x^1 \neq x^2$).

Terms of System T are duplicated :

$$t^i \quad \equiv \quad t[\Delta := \Delta^i]$$

Note that

$$\Delta \vdash_{\mathrm{T}} t : \sigma \quad \text{implies} \quad \Delta^1, \Delta^2 \vdash_{\mathrm{T}} t^i : \sigma$$

and that

$$t^i \quad =_\alpha \quad t$$

if t is closed.

Definitions of Δ^{pm} and t^{pm}

Signatures of System T are translated into contexts of $\lambda\mathbf{HA}^\omega$:

$$\begin{array}{rcl} \emptyset^{\mathrm{pm}} & \equiv & \emptyset \\ (\Delta, x^\sigma)^{\mathrm{pm}} & \equiv & \Delta^{\mathrm{pm}}, x^{\mathrm{pm}} : x^1 =_\sigma^{\mathrm{pm}} x^2. \end{array}$$

Terms of System T are translated into proof terms of $\lambda\mathbf{HA}^\omega$:

$$\begin{array}{rcl} (x)^{\mathrm{pm}} & \equiv & x^{\mathrm{pm}} \\ (\lambda x^\sigma.t)^{\mathrm{pm}} & \equiv & \lambda x^1, x^2.\lambda x^{\mathrm{pm}}.t^{\mathrm{pm}} : \ \forall x^1 \forall x^2\ x^1 =_\sigma^{\mathrm{pm}} x^2 \Rightarrow (\lambda x^1.t^1)\, x^1 =^{\mathrm{pm}} (\lambda x^2.t^2)\, x^2 \\ (t\,u)^{\mathrm{pm}} & \equiv & t^{\mathrm{pm}}\, u^1\, u^2\, u^{\mathrm{pm}} : \ t^1 u^1 =^{\mathrm{pm}} t^2 u^2 \\ 0^{\mathrm{pm}} & \equiv & \mathrm{refl}\, 0 \\ (\mathrm{S}\, t)^{\mathrm{pm}} & \equiv & \mathrm{peel}\big(t^1, t^2, t^{\mathrm{pm}}, \hat{x}.(\mathrm{S}\, t^1 = \mathrm{S}\, x), \mathrm{refl}\,(\mathrm{S}\, t^1)\big) \\ (\mathrm{Rec}\, t\, u\, v)^{\mathrm{pm}} & \equiv & \mathrm{Ind}\big(\hat{x}.(\forall y^{\mathbf{N}}\ x = y \Rightarrow \mathrm{Rec}^\sigma\ t^1\, u^1\, x =_\sigma^{\mathrm{pm}} \mathrm{Rec}^\sigma\ t^2\, u^2\, y), \\ & & \quad \lambda y.\lambda\xi.\mathrm{peel}(0, y, \xi, \hat{z}.(t^1 =_\sigma^{\mathrm{pm}} \mathrm{Rec}^\sigma\ t^2\, u^2\, z), t^{\mathrm{pm}}), \\ & & \quad \lambda x.\lambda\eta.\lambda y.\lambda\xi.\mathrm{peel}(\mathrm{S}\, x, y, \xi, \hat{z}.(u^1(\mathrm{Rec}\, t^1\, u^1\, x)x =_\sigma^{\mathrm{pm}} (\mathrm{Rec}\, t^2\, u^2\, z)), \\ & & \quad u^{\mathrm{pm}}(\mathrm{Rec}\, t^1\, u^1\, x)(\mathrm{Rec}\, t^2\, u^2\, x)(\eta\, x\,(\mathrm{refl}\, x))\, x\, x\,(\mathrm{refl}\, x)), v^1\big) v^2\, v^{\mathrm{pm}} \end{array}$$

Soundness of the translation

Theorem

If

$$\Delta \quad \vdash_T \quad t : \sigma$$

then

$$\Delta^1, \Delta^2; \Delta^{\mathrm{pm}} \quad \vdash \quad t^{\mathrm{pm}} : t^1 =_\sigma^{\mathrm{pm}} t^2.$$

In particular

$$\vdash \quad t^{\mathrm{pm}} : t =_\sigma^{\mathrm{pm}} t.$$

for all closed terms of sort σ.

We showed that all closed terms of System T are in the domain of $=^{\mathrm{pm}}$ (and we will do more) !

An intuition on the translation

System T		$\lambda\mathbf{HA}^\omega$
t	$\rightsquigarrow$	$t^1 \xrightarrow{t^{\mathrm{pm}}} t^2$

Now what ?

Remember that for all closed terms of System T

$$\vdash t^{\mathrm{pm}} : t =^{\mathrm{pm}} t.$$

Our goal is to use $=^{\mathrm{pm}}_{\sigma}$ as the new equality predicate. It should at least be reflexive. But,

$$\forall x^{\sigma} x =^{\mathrm{pm}}_{\sigma} x.$$

cannot be proven.

How to fix our reflexivity issue ?

We want to design a translation of formulas

$$\Phi \mapsto \Phi^{\mathrm{pm}}$$

which is sound, interprets equality as the predicate $=^{\mathrm{pm}}$ and such that

$$(\forall x^{\sigma} x = x)^{\mathrm{pm}}$$

is provable.
The first idea is to restrict universal quantification to the domain of $=^{\mathrm{pm}}_{\sigma}$:

$$(\forall x^{\sigma} \Phi)^{\mathrm{pm}} \quad \equiv \quad \forall x^{\sigma}\ x =^{\mathrm{pm}}_{\sigma} x \Rightarrow \Phi^{\mathrm{pm}}$$

Unfortunately, it won't work as easily. The problem is that this translation won't be sound! Specifically, the induction hypothesis won't be strong enough (when trying to prove that open terms are in the domain of $=^{\mathrm{pm}}$ in the rule of $\forall$-elimination)...

The solution : internal parametricity

Taking the intuition of the first translation, we will interpret equalities and quantifications as follow :

$$\begin{array}{rcl} (t =_\sigma u)^{\mathrm{pm}} & \equiv & t^1 =_\sigma^{\mathrm{pm}} u^2 \\ (\forall x^\sigma \Phi)^{\mathrm{pm}} & \equiv & \forall x^1 \forall x^2 \; x^1 =_\sigma^{\mathrm{pm}} x^2 \Rightarrow \Phi^{\mathrm{pm}} \end{array}$$

It will allow us to define an interpretation of the system λ**E-HA**$^\omega$ (to be defined[1]) in λ**HA**$^\omega$.

This translation will give

1. a syntactic proof of relative consistency between **E-HA**$^\omega$ and **HA**$^\omega$ that can be formalized in a very weak framework
2. an interpretation of extensional equality (at all level) in a system that barely has equality on the sort **N**.

[1] basically contains λ**HA**$^\omega$ plus an extensional equality relation on all sorts

The proof system λE-HA$^\omega$

λ**E-HA**$^\omega$ is obtained from λ**HA**$^\omega$ by extending equality in an extensional way to all higher sorts, i.e by adding

1. atomic formulas $t =_\sigma u$ for all sort σ,
2. proof terms $(\mathrm{refl}_\sigma\ t)$, $\mathrm{peel}_\sigma(t, u, M, \hat{x}.\Phi, N)$ and $\mathrm{ext}_{\sigma,\tau}(M)$,
3. typing rules for the added proof terms :

$$\frac{(\Delta;\Gamma)\ \mathbf{wf} \quad \Delta \vdash_{\mathrm{T}} t : \sigma}{\Delta;\Gamma \vdash_{\mathrm{e}} \mathrm{refl}_\sigma\ t : t =_\sigma t} \qquad \frac{\Delta;\Gamma \vdash_{\mathrm{e}} M : t =_\sigma u \quad \Delta;\Gamma \vdash_{\mathrm{e}} N : \Phi[x^\sigma := t]}{\Delta;\Gamma \vdash_{\mathrm{e}} \mathrm{peel}_\sigma(t, u, M, \hat{x}.\Phi, N) : \Phi[x^\sigma := u]}$$

$$\frac{\Delta;\Gamma \vdash_{\mathrm{e}} M : \forall x^\sigma f\, x =_\tau g\, x}{\Delta;\Gamma \vdash_{\mathrm{e}} \mathrm{ext}_{\sigma,\tau}(M) : f =_{\sigma\to\tau} g}$$

The symbol

$$\vdash_{\mathrm{e}}$$

will be used to denote sequents (and provability) in λ**E-HA**$^\omega$.

A translation from λ**E-HA**$^\omega$ to λ**HA**$^\omega$

Our next goal is to define a translation

$$(\Delta;\Gamma \vdash_e M : \Phi)^{\mathrm{pm}} \quad \rightsquigarrow \quad \Delta^1, \Delta^2; \Delta^{\mathrm{pm}}, \Gamma^{\mathrm{pm}} \vdash M^{\mathrm{pm}} : \Phi^{\mathrm{pm}}.$$

where

1. $\Delta^i, t^i, \Delta^{\mathrm{pm}}, t^{\mathrm{pm}}$ are already defined
2. $\Phi \mapsto \Phi^{\mathrm{pm}}$ is a translation from formulas of λ**E-HA**$^\omega$ (containing the symbol $=_\sigma$ for all sorts σ) to formulas of λ**HA**$^\omega$ (only containing $=_{\mathbf{N}}$)
3. $\Gamma \mapsto \Gamma^{\mathrm{pm}}$ is a translation from contexts of λ**E-HA**$^\omega$ to contexts of λ**HA**$^\omega$
4. $M \mapsto M^{\mathrm{pm}}$ is a translation from proof terms of λ**E-HA**$^\omega$ (containing in particular proof terms peel_σ for all sorts σ) to proof terms of λ**HA**$^\omega$ (only containing $\mathrm{peel}_{\mathbf{N}}$).

$\Phi \mapsto \Phi^{\mathrm{pm}}$ and $\Gamma \mapsto \Gamma^{\mathrm{pm}}$

Formulas of λ**E-HA**$^{\omega}$ are translated into formulas of λ**HA**$^{\omega}$:

$$
\begin{array}{rcl}
(t =_{\sigma} u)^{\mathrm{pm}} & \equiv & t^1 =_{\sigma}^{\mathrm{pm}} u^2 \\
\bot^{\mathrm{pm}} & \equiv & \bot \\
(\Phi \Rightarrow \Psi)^{\mathrm{pm}} & \equiv & \Phi^{\mathrm{pm}} \Rightarrow \Psi^{\mathrm{pm}} \\
(\Phi \wedge \Psi)^{\mathrm{pm}} & \equiv & \Phi^{\mathrm{pm}} \wedge \Psi^{\mathrm{pm}} \\
(\forall x^{\sigma} \Phi)^{\mathrm{pm}} & \equiv & \forall x^1 \forall x^2 \; x^1 =_{\sigma}^{\mathrm{pm}} x^2 \Rightarrow \Phi^{\mathrm{pm}}.
\end{array}
$$

Contexts of λ**E-HA**$^{\omega}$ are translated into contexts of λ**HA**$^{\omega}$:

$$
\begin{array}{rcl}
\emptyset^{\mathrm{pm}} & \equiv & \emptyset \\
(\Gamma, \xi : \Phi)^{\mathrm{pm}} & \equiv & \Gamma^{\mathrm{pm}}, \xi : \Phi^{\mathrm{pm}}.
\end{array}
$$

The translation $M \mapsto M^{\mathrm{pm}}$

$$
\begin{array}{rcl}
(\xi)^{\mathrm{pm}} & \equiv & \xi \\
(\lambda\xi.M)^{\mathrm{pm}} & \equiv & \lambda\xi.M^{\mathrm{pm}} \\
(M\,N)^{\mathrm{pm}} & \equiv & M^{\mathrm{pm}}N^{\mathrm{pm}} \\
(M,N)^{\mathrm{pm}} & \equiv & (M^{\mathrm{pm}},N^{\mathrm{pm}}) \\
(M.i)^{\mathrm{pm}} & \equiv & M^{\mathrm{pm}}.i \\
(\lambda x.M)^{\mathrm{pm}} & \equiv & \lambda x^1,x^2\lambda x^{\mathrm{pm}}.M^{\mathrm{pm}} \\
(M\,t)^{\mathrm{pm}} & \equiv & M^{\mathrm{pm}}t^1t^2t^{\mathrm{pm}} \\
(\mathrm{efq}(M,\Phi))^{\mathrm{pm}} & \equiv & \mathrm{efq}(M^{\mathrm{pm}},\Phi^{\mathrm{pm}}) \\
(\mathrm{refl}_\sigma\ t)^{\mathrm{pm}} & \equiv & t^{\mathrm{pm}}\ :\ t^1 =^{\mathrm{pm}}_\sigma t^2 \\
\big(\mathrm{peel}_\sigma(t,u,M,\hat{x}.\Phi,N)\big)^{\mathrm{pm}} & \equiv & \mathrm{postpone} \\
\big(\mathrm{ext}_{\sigma,\tau}(M)\big)^{\mathrm{pm}} & \equiv & M^{\mathrm{pm}} \\
\big(\mathrm{Ind}(\hat{x}.\Phi,M,N,t)\big)^{\mathrm{pm}} & \equiv & \mathrm{Ind}(\hat{x}.\forall y\ x=y \Rightarrow \Phi^{\mathrm{pm}}[x_1:=x][x_2:=y], \\
& & \quad \lambda y\lambda\xi.\mathrm{peel}(0,y,\xi,\hat{z}.\Phi^{\mathrm{pm}}[x^1:=0][x_2:=z],M^{\mathrm{pm}}), \\
& & \quad \lambda x\lambda\eta\lambda y\xi.\mathrm{peel}(\mathrm{S}\,x,y,\xi,\hat{z}.\Phi^{\mathrm{pm}}[x^1:=\mathrm{S}\,x][x^2:=z], \\
& & \qquad N^{\mathrm{pm}}x\,x\,(\mathrm{refl}\,x)(\eta\,x\,(\mathrm{refl}\,x)), \\
& & \quad t^1)\ t^2t^{\mathrm{pm}}
\end{array}
$$

The case of peel

Recall that peel is the proof term allowing us to prove that equal terms satisfy the same properties.

But why is it called peel?

If your equality is a compatible equivalence relation, you can prove by "peeling" (simply by external induction on) your formulas that indeed two equal terms satisfy the same formulas!

This is what we will do with the interpretation of $=_\sigma$, i.e

$$=_\sigma^{\mathrm{pm}}.$$

Peeling the formulas

We first construct a family of terms $\mathrm{Elim}_{x^{\hat{\sigma}}.\Phi}$ satisfying that if

$$\mathbf{FV}(\Phi) \quad \subseteq \quad \Delta$$

then

$$\Delta^1, \Delta^2; \Delta^{\mathrm{pm}} \quad \vdash \quad \mathrm{Elim}_{x^{\hat{\sigma}}.\Phi} : \forall x^1 x^2 y^1 y^2 \\ x^1 =^{\mathrm{pm}}_{\sigma} y^1 \Rightarrow x^2 =^{\mathrm{pm}}_{\sigma} y^2 \Rightarrow \Phi^{\mathrm{pm}} \Rightarrow \Phi^{\mathrm{pm}}[x^1 := y^1][x^2 := y^2].$$

It is done by induction on the syntax of formulas :

$$\begin{array}{rcl}
\mathrm{Elim}_{\hat{x}.t=_{\sigma}u} & \equiv & \lambda x^1, x^2, y^1, y^2 \lambda \xi^1, \xi^2, \xi.\mathrm{trans}^{\mathrm{pm}}\ t^1[x^1 := y^1]\ t^1\ u^2[x^2 := y^2] \\
 & & \quad (\mathrm{Elim}^1_{\hat{x}.t}\ y^1\ x^1\ (\mathrm{sym}^{\mathrm{pm}}\ x^1\ y^1\ \xi^1)) \\
 & & \quad (\mathrm{trans}^{\mathrm{pm}}\ t^1\ u^2\ u^2[x^2 := y^2]\xi(\mathrm{Elim}^2_{\hat{x}.u}\ x^2\ y^2\ \xi^2)) \\
\mathrm{Elim}_{\hat{x}.\bot} & \equiv & \lambda x^1, x^2, y^1, y^2 \lambda \xi^1, \xi^2, \xi.\xi \\
\mathrm{Elim}_{\hat{x}.(\Phi \Rightarrow \Psi)} & \equiv & \lambda x^1, x^2, y^1, y^2 \lambda \xi^1, \xi^2, \xi.\lambda \eta.\mathrm{Elim}^+_{\hat{x}.\Psi}(\xi\ (\mathrm{Elim}^-_{\hat{x}.\Phi} \eta)) \\
\mathrm{Elim}_{\hat{x}.(\Phi \wedge \Psi)} & \equiv & \lambda x^1, x^2, y^1, y^2 \lambda \xi^1, \xi^2, \xi.(\mathrm{Elim}^+_{\hat{x}.\Phi}\xi.1, \mathrm{Elim}^+_{\hat{x}.\Psi}\xi.2) \\
\mathrm{Elim}_{\hat{x}.(\forall z \Phi)} & \equiv & \lambda x^1, x^2, y^1, y^2 \lambda \xi^1, \xi^2, \xi.\lambda z^1, z^2 \lambda z^{\mathrm{pm}}.\mathrm{Elim}_{\hat{x}.\Phi}(\xi\ z\ z^1\ z^{\mathrm{pm}})
\end{array}$$

where

$$\begin{array}{rcl}
\mathrm{Elim}^+_{\hat{x}.\Phi} & \equiv & \mathrm{Elim}_{\hat{x}.\Phi}\ x^1\ x^2\ y^1\ y^2\ \xi^1\ \xi^2 \\
\mathrm{Elim}^-_{\hat{x}.\Phi} & \equiv & \mathrm{Elim}_{\hat{x}.\Phi}\ y^1\ y^2\ x^1\ x^2\ (\mathrm{sym}^{\mathrm{pm}}\ x^1\ y^1\ \xi^1)(\mathrm{sym}^{\mathrm{pm}}\ x^2\ y^2\ \xi^2) \\
\mathrm{Elim}^i_{z^{\hat{\sigma}}.t} & : & \forall z^1, z^2.z^1 =^{\mathrm{pm}}_{\sigma} z^2 \Rightarrow t^i[z^i = z^1] =^{\mathrm{pm}}_{\tau} t^i[z^i = z^2].
\end{array}$$

Defining peel

We can now set

$$\left(\mathrm{peel}_\sigma(t, u, M, \hat{x}.\Phi, N)\right)^{\mathrm{pm}} \quad \equiv \quad \mathrm{Elim}_{\hat{x}.\Phi}\, t^1 t^2 u^1 u^2 (\mathrm{trans}^{\mathrm{pm}}\, t^1 u^2 u^1 M^{\mathrm{pm}} (\mathrm{sym}^{\mathrm{pm}}\, u^1 u^2 u^{\mathrm{pm}})) \\ (\mathrm{trans}^{\mathrm{pm}}\, t^2 t^1 u^2 (\mathrm{sym}^{\mathrm{pm}}\, t^1 t^2 t^{\mathrm{pm}}) M^{\mathrm{pm}}) N^{\mathrm{pm}}$$

Soundess of the translation

...and we state the

Theorem

If

$$\Delta;\Gamma \quad \vdash_{\mathrm{e}} \quad M : \Phi$$

then

$$\Delta^1, \Delta^2; \Delta^{\mathrm{pm}}, \Gamma^{\mathrm{pm}} \quad \vdash \quad M^{\mathrm{pm}} : \Phi^{\mathrm{pm}}.$$

An intuition on the translation

λ**E-HA**$^\omega$		λ**HA**$^\omega$
t	$\rightsquigarrow$	$\begin{array}{c} t^1 \\ \Big\downarrow{\scriptstyle t^{\mathrm{pm}}} \\ t^2 \end{array}$
$t = u$	$\rightsquigarrow$	$\begin{array}{ccc} t^1 & \xrightarrow{(t=u)^{\mathrm{pm}}} & u^1 \\ \Big\downarrow{\scriptstyle t^{\mathrm{pm}}} & & \Big\downarrow{\scriptstyle u^{\mathrm{pm}}} \\ t^2 & & u^2 \end{array}$

Characterizing the image of the translation

How do we know that we have fully characterized the translation $(\_)^{\mathrm{pm}}$? To be more specific, we proved that

$$\vdash_e \Phi \quad \text{implies} \quad \vdash_e \Phi^{\mathrm{pm}}.$$

But we also would like to know that

$$\vdash \Phi^{\mathrm{pm}} \quad \text{implies} \quad \vdash_e \Phi.$$

It is done in three steps

1. first we show that in λ**E-HA**$^\omega$, $=^{\mathrm{pm}}_\sigma$ collapses to $=_\sigma$ at all level of sorts σ
2. then, using this fact, we will be able to show that in λ**E-HA**$^\omega$, for closed formulas Φ, we have :

$$\vdash_e \Phi \Leftrightarrow \Phi^{\mathrm{pm}}$$

3. therefore, if Φ^{pm} is provable in λ**HA**$^\omega$, then it is also provable in λ**E-HA**$^\omega$ and consequently Φ is also provable in λ**E-HA**$^\omega$.

$=^{\mathrm{pm}}$ collapses to $=$ in λ**E-HA**$^\omega$

For every sorts σ, we construct a proof term

$$\vdash_e \mathrm{Collaps}_\sigma : \forall x^\sigma \forall y^\sigma \; x =_\sigma y \Leftrightarrow x =^{\mathrm{pm}}_\sigma y$$

by external induction on the sorts of System T :

$$\begin{array}{rcl} \mathrm{Collaps}_{\mathbf{N}} & \equiv & \lambda x, y.\big(\lambda\xi.\xi, \lambda\xi.\xi\big) \\ \mathrm{Collaps}_{\sigma\to\tau} & \equiv & \lambda f, g.(\lambda\xi.\lambda x, y.\lambda\eta.\mathrm{Collaps}_\tau.1\,(f\,x)\,(g\,y)\,\mathrm{app}_{\sigma,\tau}(\xi, x, y, \mathrm{Collaps}_\sigma.2\,x\,y\,\eta), \\ & & \quad \lambda\xi.\mathrm{ext}_{\sigma,\tau}(\lambda z.\mathrm{Collaps}_\tau.2\,(f\,z)\,(g\,z)(\xi\,z\,z\,(\mathrm{Collaps}_\sigma.1\,z\,z_,(\mathrm{refl}\,z))))\big) \end{array}$$

Characterizing the image of the translation (1)

We exhibit a family of proof terms Equiv^i_Φ for $i = 1, 2$ satisfying for any formula Φ and any signatures Δ containing the free variables of Φ

$$\begin{array}{lll} \Delta^1, \Delta^2; \Delta^{\mathrm{pm}} & \vdash_e & \mathrm{Equiv}^1_\Phi : \Phi^1 \Rightarrow \Phi^{\mathrm{pm}} \\ \Delta^1, \Delta^2; \Delta^{\mathrm{pm}} & \vdash_e & \mathrm{Equiv}^2_\Phi : \Phi^{\mathrm{pm}} \Rightarrow \Phi^1 \end{array}$$

as follow :

$$\begin{array}{lll} \mathrm{Equiv}^1_{t=_\sigma u} & \equiv & \lambda\xi.\mathrm{trans}_\sigma\ t^1\ u^1\ u^2(\mathrm{Collaps}_\sigma.1\ t^1\ u^1\ \xi)\ u^{\mathrm{pm}} \\ \mathrm{Equiv}^2_{t=_\sigma u} & \equiv & \lambda\xi.\mathrm{Collaps}_\sigma.2\ t^1\ u^1\ (\mathrm{trans}_\sigma\ t^1\ u^2\ u^1\ \xi\ (\mathrm{sym}^{\mathrm{pm}}\ u^1\ u^2\ u^{\mathrm{pm}})) \end{array}$$

$$\begin{array}{lll} \mathrm{Equiv}^1_{\Phi\Rightarrow\Psi} & \equiv & \lambda\xi,\eta.\mathrm{Equiv}^1_\Psi(\xi\ (\mathrm{Equiv}^2_\Phi\ \eta)) \\ \mathrm{Equiv}^2_{\Phi\Rightarrow\Psi} & \equiv & \lambda\xi,\eta.\mathrm{Equiv}^2_\Psi(\xi\ (\mathrm{Equiv}^1_\Phi\ \eta)) \end{array}$$

$$\begin{array}{lll} \mathrm{Equiv}^1_{\forall x^\sigma\ \Phi} & \equiv & \lambda\xi.\lambda x^1, x^2.\lambda x^{\mathrm{pm}}.\mathrm{Equiv}^1_\Phi\ (\xi\ x^1) \\ \mathrm{Equiv}^2_{\forall x^\sigma\ \Phi} & \equiv & \lambda\xi.\lambda x^1.\mathrm{Equiv}^2_\Phi[x_2 := x_1][x^{\mathrm{pm}} := (\mathrm{Collaps}_\sigma.1\ x^1\ x^1\ (\mathrm{refl}_\sigma\ x^1))] \\ & & \quad (\xi\ x^1\ x^1\ (\mathrm{Collaps}_\sigma.1\ x^1\ x^1\ (\mathrm{refl}_\sigma\ x^1))) \end{array}$$

other cases are left as an exercise :)

Characterizing the image of the translation (2)

Finally, for closed formula Φ :

$$\vdash_e (\mathrm{Equiv}^1_\Phi, \mathrm{Equiv}^2_\Phi) : \Phi \Leftrightarrow \Phi^{\mathrm{pm}}.$$

and

$$\vdash \Phi^{\mathrm{pm}} \quad \text{implies} \quad \vdash_e \Phi.$$

Therefore, we can state our last theorem

Theorem

For every closed formula Φ

$$\vdash \Phi^{\mathrm{pm}} \quad \textbf{if and only if} \quad \vdash_e \Phi.$$

Conclusion

We designed a translation from λ**E-HA**$^\omega$ to λ**HA**$^\omega$ using techniques reminiscent of parametricity, giving an interpretation of extensional equality on all sorts in a language that contains only equality on the sort **N**.

In fact, this interpretation was not new and was already used by R. Gandy. on 1956 (see On The Axiom of Extensionality -Part I. The Journal of Symbolic Logic, Vol. 21, 1956.).

Ideas of future work include

1. analysis of the computational content of this translation
2. generalizing the base type **N** to find out if only a **PER** is sufficient.
3. generalization to other proof systems.