Orbifold Topological Quantum Field Theories in Dimension 2

Ana González[*], Ernesto Lupercio[†], Carlos Segovia[‡] and Bernardo Uribe[§]

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Contents

[*]Instituto de Matemática y Estadística "Prof. Ing. Rafael Laguardia", Facultad de Ingeniería Julio Herrera y Reissig 565 CP11300. Montevideo, Uruguay.

[†]Centro de Investigaciones y Estudios Avanzados, Av. Instituto Politécnico Nacional # 2508, Col. San Pedro Zacatenco, México

[‡]Mathematisches Institut, Ruprecht-Karls-Universität Heidelberg, Im Neuenheimer Feld 288, 69120, Heidelberg, Germany

[§]Departamento de Matemáticas, Universidad de los Andes, Carrera 1 N. 18A-10, Bogotá, Colombia

1 Introduction

In this book we explore 2-dimensional topological quantum field theories (TQFTs) and certain algebraic structures that model them. Such algebraic structures are called *Frobenius* structures for they all are generalizations of the concept of Frobenius algebra and they are relatively simple to define and study from a mathematical perspective, while at the same time they preserve a lot of the formal properties of quantum field theories that appear in string theory. Let us start by describing the simplests of cases.

A *Frobenius algebra* over a field $\mathbb{k}$ is a (non-necessarily commutative) associative algebra A, together with a non-degenerate trace $\theta : A \longrightarrow \mathbb{k}$. In other words, we have that $\langle x|y\rangle = \theta(xy) = \theta(yx)$ is a non-degenerate bilinear form. They have been studied since the 1930's – specially in representation theory – for their very nice duality properties [BN37], [Nak39], and [Nak41]. In recent times the surprising connection found to topological quantum field theories has made them subject of renewed interest. In this book most Frobenius algebras will be (graded) commutative (or super commutative).

Every finite group G provides us with the basic example, the center of the group algebra $A = \mathbb{C}[G]$ with the trace $\theta\left(\sum \alpha_g g\right) = \frac{a_1}{|G|}$ is a Frobenius algebra.

A second important example is the Poincaré algebra associated to every compact closed manifold M, provided by its cohomology algebra $A = H^*(M)$ with trace

$$\theta(w) = \int_M w\,,$$

for $w \in H^*(M)$.

Poincaré duality is equivalent to the assertion that this is a Frobenius algebra.

In topology this fact manifests in many ways, for instance, in the existence of an intersection product in homology that becomes a coproduct in cohomology. The coproduct Δ is the composition of the Poincaré duality isomorphism $D : H_*(M) \xrightarrow{\cong} H^*(M)$ with the dual map for the ordinary cup product $\mu : H^*(M) \otimes H^*(M) \longrightarrow H^*(M)$

$$\begin{tikzcd} A \otimes A & A \arrow[l, "\Delta"'] \\ A^* \otimes A^* \arrow[u, "D\otimes D"] & A^* \arrow[l, "\mu^*"] \arrow[u, "D"'] \end{tikzcd}$$

recall here that we are working over a field $\mathbb{k}$.

If we consider the case of a non-compact manifold M, its cohomology algebra is no longer a Frobenius algebra, but we may ask ourselves what structure remains. In this way we arrive at the following definition:

Definition 1.1. A *nearly Frobenius algebra* A is an algebra together with a commutative coassociative comultiplication $\Delta : A \longrightarrow A \otimes A$ such that Δ is an A-bimodule morphism.

What this means explicitly is that whenever $\Delta(b) = \sum_i b_i \otimes b_i'$ we have in turn that the following equation holds: $\Delta(ab) = \sum_i (a \cdot b_i) \otimes c_i$. Also if $\Delta(a) = \sum_i a_i \otimes a_i'$ then $\Delta(ab) = \sum_i a_i \otimes (a_i' \cdot b)$. We write these identities more compactly as follows:

$$\Delta(ab) = a\Delta(b) = \Delta(a)b, \tag{1}$$

and we call these equations the Abrams' condition.

Clearly every Frobenius algebra is also a nearly Frobenius algebra. The first important algebraic result [Abr96] is that a nearly Frobenius algebra is a Frobenius algebra if and only if Δ admits a co-unit (or trace). In chapter 3 of this book we look at Frobenius algebras and nearly Frobenius algebras from the point of view of an algebraist, and we prove this result. In particular we classify semi-simple nearly Frobenius algebras.

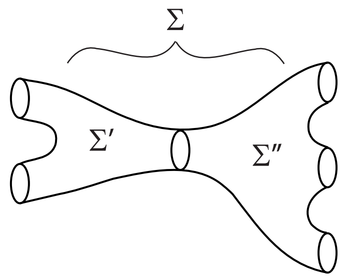
For ordinary Frobenius algebras there is a striking folk theorem stating that it is the same to have a Frobenius algebra as it is to have a $(1+1)$-dimensional topological quantum field theory (TQFT). A TQFT [Ati88] is a rule that assigns to every topological (real 2-dimensional) oriented surface Σ whose boundary is divided (according to orientation) into n incoming circles and m-outgoing circles, a linear map:

$$Z_\Sigma : A^{\otimes n} \longrightarrow A^{\otimes m}$$

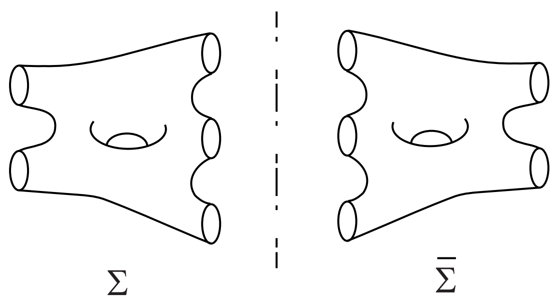
where A is a fixed (finite dimensional) vector space –the space of states of the theory. We agree to define $A^{\otimes 0} = \Bbbk$. As these maps Z_Σ run over all surfaces Σ, they must satisfy certain compatibility conditions, the most important of which states if we cut up a surface Σ into two smaller surfaces Σ' and Σ'' in such a manner that the intersection $\Sigma' \cap \Sigma''$ is the same as the outgoing circles of Σ' and, in turn, equal to the incoming circles of Σ''; we must have that the operator Z_Σ is the composition of the operators $Z_{\Sigma'}$ and $Z_{\Sigma''}$, as in the picture.

We will also request that reflecting a picture Σ in a mirror (changing the orientation) to obtain $\overline{\Sigma}$, change the operator Z_Σ by dualizing it:

$$Z_{\bar{\Sigma}} = Z_\Sigma^*$$

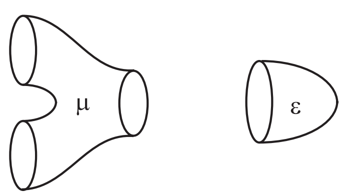


$$Z_\Sigma = Z_{\Sigma''} \circ Z_{\Sigma'}$$



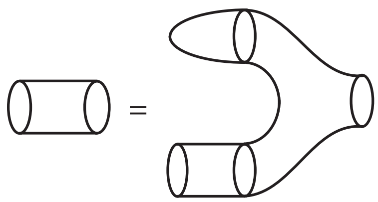
Finally, and without loss of generality, we will assume that the cylinder

corresponds to the identity operator, namely $Z_{S^1 \times I} = \mathrm{id}_A$. The structure of a TQFT on A automatically endows A with the structure of a Frobenius algebra, where we have the product as the operator induced by the pair of pants and the trace as the operator induced by the right sided cap:

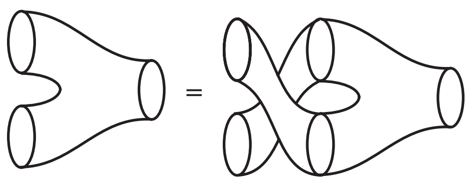


$$A \otimes A \xrightarrow{\mu} A \qquad\qquad A \xrightarrow{\varepsilon} \Bbbk$$

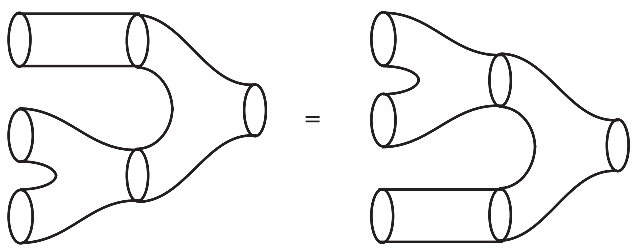
It is a fun exercise to show that the following pictures imply commutativity, associativity, identity and non-degeneracy of the trace.



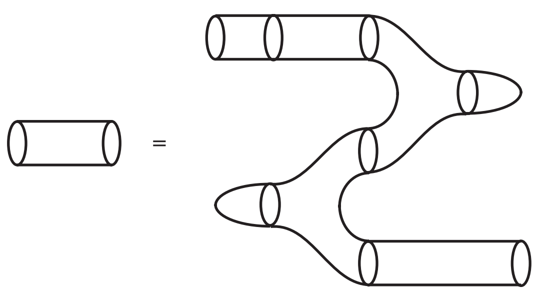
unit axiom



commutativity



associativity



non-degeneracy

If, instead of using $(1+1)$-dimensional surfaces interpolating between, we use $(n+1)$-dimensional manifolds interpolating between n-dimensional manifolds we arrive at the concept of $(n+1)$-dimensional TQFT.

Whenever we consider a closed $(n+1)$-manifold Σ (with empty boundary), we define the number $\Psi_\Sigma \in \Bbbk = A^{\otimes 0}$ by observing that the linear map $Z_\Sigma \colon \Bbbk \to \Bbbk$ if of the form:

$$Z_\Sigma(\nu) = \Psi_\Sigma x.$$

The number $\Psi_\Sigma \in \Bbbk = A^{\otimes 0}$ is called that *partition function* of the theory (evaluated at Σ) and it is a topological invariant of Σ.

We motivate this abstract definition by the formalism of Feynman path integrals in the first two sections of chapter 2. Sections 2.1, 2.2 and 5.1 in this book are heuristic rather than mathematical, but the results of the rest of this work does not depend logically on what appears there. They are included for the matter of exposition.

In physics partition functions are thought of as containing the whole information of a quantum field theory; however, there are more general topological quantum field theories which do not have a partition function. We define a *positive boundary topological quantum field theory* (TQFT+) just as an ordinary TQFT except that Z_Σ is possibly non-defined whenever the outgoing boundary of Σ is empty. Clearly every TQFT is a TQFT+ but not conversely. If a Z is a TQFT+ it may not have a partition function. In chapter 2 of this book, we prove that *it is the same to have a TQFT+ as it is to have a nearly Frobenius algebra.* This fact generalizes the folk theorem that we alluded at before. A TQFT+ is to be thought as a TQFT on a *non-compact* background.

Example 1.1. There is a very beautiful example of a $(n+1)$-dimensional TQFT due to Dijkgraaf and Witten [DW90, Seg99, CV]. This is a $(n+1)$-dimensional TQFT $(A^G, \Psi^G, Z^G)_{n+1}$ associated to a finite group G. In this model we have:

- $\mathcal{F}(Y) = [Y, BG] = \mathrm{Bun}_G(Y)$, where $\mathrm{Bun}_G(Y)$ is the set isomorphism classes of G-principal bundles on Y. This is called the space of *fields* of this theory. In chapter 2, we will explain in more generality what we mean by a field.

- $A^G(Y) = \mathrm{Maps}(\mathrm{Bun}_G(Y), \mathbb{C})$. Here we remark that $\mathrm{Bun}_G(Y) \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_1(Y), G)/\sim$, this last bijection being induced by the holonomy of the bundle. The symbol $\sim$ denotes conjugation.

- For a boundaryless Y we have $Z^G(Y) = |\mathrm{Hom}(\pi_1(Y); G)|/|G|$.

- If $\partial Y = Z$ has no output boundary then for each $P \in \mathrm{Bun}_G(Z)$ we have:

$$\Psi_Y(P) = \sum_{Q \in \mathrm{Bun}_G(Y),\ Q|Z = P} \frac{1}{|\mathrm{Aut}(Q)|} \in \mathbb{C}$$

Segal has shown that when the dimension of the model is $1+1$, then we have:

- The Frobenius algebra (A_G, θ_G) associated to $(A^G, \Psi^G, Z^G)_{1+1}$ is isomorphic to the center of the group algebra $\mathbb{C}[G]$, with trace

$$\theta_G\left(\sum_g \lambda_g g\right) = \frac{1}{|G|}\lambda_1.$$

- For a boundaryless genus g Riemann surface Σ we have:

$$\Psi_\Sigma = Z(\Sigma) = |G|^{2g-2}\sum_V \frac{1}{(\dim V)^{2g-2}}$$

  where g is the genus of Σ and V runs through irreducible representations of G.

Example 1.2. Consider any *compact*, closed manifold M, its cohomology $A_M := H^*(M)$ is a Frobenius algebra (because Poincaré duality holds). From this we conclude that there is a TQFT Z_M associated to M. We will return later on to the construction of this theory from a space of fields. If we consider a non-compact manifold M, then A_M is a nearly Frobenius algebra, and we can construct from this a positive boundary TQFT.

The most fruitful method to produce examples of TQFTs in dimension 2 is to consider moduli spaces $\mathcal{M}$ of maps from the surface Σ to a background manifold X. integrating over the (virtual) fundamental class of the space $\mathcal{M}$ (also called obstruction class) one gets rid of the dependance on the map and obtains a *bona fide* TQFT. Orbifolds thus play a dual role in this book, for often the moduli space $\mathcal{M}$ is naturally an orbifold, but also we are motivated by the physics to consider background spaces X that are orbifolds and their virtual fundamental classes. These are the subjects covered in chapters 5 (virtual fundamental classes) and 7 (orbifolds).

An orbifold $\mathcal{X}$ is a space X together with an structure that is very much like that of a manifold, only that instead of locally looking like $\mathbb{R}^n$, orbifolds locally look like $\mathbb{R}^n/G$, where G is a *finite* subgroup of $\mathrm{GL}_n(\mathbb{R})$. In this book we will often deal with orientable orbifolds so that we will further have $G \subset \mathrm{SL}_n(\mathbb{R})$ for every local group G. We will write $\mathcal{X}|_V \cong [U/G]$ to indicate that $U \to U/G \cong V$ is a local chart of $\mathcal{X}$, where $U \cong \mathbb{R}^n$ and $G \subset \mathrm{SL}_n(\mathbb{R})$ is finite. An orbifold could be of the form $\mathcal{X} = [M/G]$ for M a smooth manifold and G a finite group acting by diffeomorphisms of M, and in this case we follow convention and call it a *global quotient* orbifold [Moe02]. We will very often think about the case $\mathcal{X} = [M/G]$. In chapter 7 we deal with the formalism of orbifolds in a more leisurely manner.

Topological quantum field theories in dimension 2 on compact backgrounds [Ati88] associated to orbifolds appeared first in physics [DHVW86, DHVW85]. In the last decade such theories have become the object of intense study in mathematics, specially since the seminal paper of Chen and Ruan [CR04a]. Chen-Ruan cohomology is the state space for a TQFT with closed strings on a compact background [JKK07]. Chen-Ruan cohomology and its variants are covered in chapter 11.

So far we have been dealing only with closed strings while physics teaches us that introducing open strings with boundary conditions (branes) is a very fruitful approach. A mathematical axiomatization of general 2-dimensional topological theories with open strings and branes on *compact* backgrounds was put forward by Moore and Segal in [MS]. In this book we introduce a generalization to the case of non compact backgrounds. Both formalisms are dealt with in detail in chapter 4. These structures are indistinctly called *(nearly) Calabi-Yau categories* or *(nearly) Frobenius structures*, although we favour the latter name. From an algebraist point of view they are categorical generalizations of Frobenius algebras. The orbifold or equivariant formalism where we include the action of a global finite group G is studied carefully in chapter 8.

The remaining chapters deal with important examples of nearly Frobenius structures.

There is a TQFT+ whose state space is the homology $\mathsf{H}_*(\mathcal{L}\mathsf{M})$ of the free loop space $\mathcal{L}\mathsf{M}$ of any smooth manifold M. This theory was first introduced in [CS] and studied from the point of view of obstruction classes in [CJ02]. Cohen and Godin [CG04] proved that this theory is a TQFT+ (nearly Frobenius algebra). It cannot be made into a TQFT even when M is compact. We give a new proof of their result. We introduce string topology in chapter 6 and prove that this nearly Frobenius algebra (TQFT+) can be extended to a full nearly Frobenius structure. This is closely related to the result of Blumberg, Cohen and Teleman [BCT09].

Chapter 9 deals with the generalization of string topology from a background manifold M to a background orbifold $\mathcal{X}$. We prove in this chapter that orbifold string topology [LUX08] admits the structure of a nearly G-Frobenius structure.

While, on a manifold we have that the natural circle action (rotating the loops) of the circle S^1 on $\mathcal{L}\mathsf{M}$ has as its fixed points $(\mathcal{L}\mathsf{M})^{\mathsf{S}^1}$, the situation is slightly different for an orbifold $\mathcal{X}$ where we have that $\mathcal{X} \subset \mathrm{I}(\mathcal{X}) = (\mathsf{L}\mathcal{X})^{\mathsf{S}^1}$. (the localization principle of theorem 7.37), where $\mathrm{I}(\mathcal{X})$ is the inertia orbifold of $\mathcal{X}$, also called the space of ghost loops of $\mathcal{X}$. We can define a new TQFT+ by considering a sort of string topology of ghost loops. We call such a theory *virtual orbifold cohomology*

[LUX07]. We prove in chapter 10 that virtual orbifold homology is the state space of a full nearly G-Frobenius structure.

In chapter 11 we prove that when $\mathcal{X}$ is hyperkahler the virtual orbifold cohomology of the ghost loop orbifold $I(\mathcal{X})$ is isomorphic to the Chen-Ruan theory of $\mathcal{X}$ providing a link between different theories.

Discrete torsion is a beautiful degree of freedom for orbifold theories. In chapter 12 we motivate gerbes from the point of view of electromagnetic theory; then we see discrete torsion as a particular case of a gerbe over a orbifold. Then we show that discrete torsion provides a universal example of a G-Frobenius algebra, and by tensoring Frobenius algebras, we can twist any G-Frobenius algebra by discrete torsion providing an algebraic approach to this degree of freedom.

In chapter 12 everything comes together in the study of a beautiful example. The (naive) symmetric product of a space X is defined often as the *topological space* $M^n/\mathfrak{S}_n := M \times \cdots \times M/\mathfrak{S}_n$. In this book we consider instead the orbifold $\mathcal{X} := [M^n/\mathfrak{S}_n] := [M \times \cdots \times M/\mathfrak{S}_n]$. The final chapter of this book is a study of the whole theory for the orbifold $\mathcal{X} = [M^n/\mathfrak{S}_n]$.

The prerequisites for this book are minimal, all the necessary background should be covered in the graduate courses in geometry, topology, and algebra for first year graduate students. The book tries hard to be self-contained, and the distinct chapters can be read mostly independently. We include seven appendices of standard material to help the novice. We tried to make the book amenable to physicists and we hope that it may serve as a bridge between researchers in physics and mathematics.

We would like to thank all the mathematicians that influenced this work through conversations and correspondence, in particular, Alejandro Adem, Ralph Cohen, Dan Freed, Hugo García-Compean, Tommaso de Fernex, Samuel Gitler, André Haefliger, Mariana Hain, Nigel Hitchin, Tyler Jarvis, Maxim Kontsevich, Ieke Moerdijk, Jack Morava, Jacob Mostovoy, Thomas Nevins, Mainak Poddar, Yongbin Ruan, Graeme Segal, Dennis Sullivan, Constantin Teleman, Ed Witten and Miguel Xicoténcatl were very influential in our approach to these questions.

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We dedicate this book to Anika, Ayelén, Leonardo, Karla and Luisa. May their generation overcome the challenges we bequeath them.

2 Classifying 2-dimensional TQFTs

2.1 Motivation

Let M be $\mathbb{R}^3$ the 3-dimensional euclidean space.

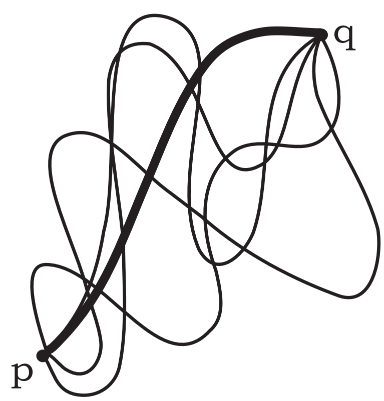
Given two points in M, say p and q, we would like to compute the probability that a particle which starts in p lands in q after certain amount of time T. The answer is, of course, zero, but we can nevertheless still ask what the probability is that the particle will be at a distance less than ϵ from q.

Feynman gave a remarkable formula for the probability [Fey06]. Say that ϕ is the initial probability distribution for the position of the particle at $t = 0$ (meaning that $\int_U |\phi_0|$ is the probability that the particle is in U at $t = 0$). Then the probability distribution $|\phi_T|$ for the position at $t = T$ is given by the *path integral*:

$$\phi_T(q) = \int_{\mathcal{P}_q} \phi_0(\gamma(0))e^{-i\hbar S(\gamma)}\mathcal{D}\gamma \tag{2}$$

where

$$\mathcal{P}_q = \{\gamma\colon [0,T] \to M | \gamma(T) = q\} \subset \mathrm{Maps}([0,T], M)$$



and

$$S(\gamma) = \frac{1}{2}\int_0^T |\gamma'(t)|^2 dt.$$

In the picture we stress the classical (Euler-Lagrange) path minimizing S.

Moreover, if we think of $|\phi_t\rangle = \phi(q,t)$ as a one parameter family of vectors (*kets*) in $\mathcal{H} = \mathrm{Maps}(M, \mathbb{C})$ (usually thought of as a Hilbert space) then we have that the main result of Feynman in this case is that ϕ *satisfies the Schrödinger equation.*

We can try to extract the formal structure behind formula 2 as follows:

Consider P_T to be a compact 1-dimensional manifold with boundary (namely $P = [0, T]$). We define the *fields* on a 1-manifold Y to be

$$\mathcal{F}(Y) = \mathrm{Maps}(Y, M),$$

the *moduli space* of all maps from Y to M. We will return to the subject of moduli spaces below. Moduli spaces are often orbifolds. In any case we will divide the boundary of Y into two portions that we will call the *incoming* and *outgoing* boundaries

$$\partial Y = \partial_0 Y \coprod \partial_1 Y.$$

As part of the structure we need an *action* map:

$$S_Y \colon \mathcal{F}(Y) \to \mathbb{R}$$

which in our case could be given by:

$$S(\gamma) = \frac{1}{2}\int_Y |\gamma'|^2.$$

We have the following properties:

i) We have restriction maps (forming a *correspondence*)

$$\begin{tikzcd} & \mathcal{F}(Y) \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ \mathcal{F}(\partial_0 Y) & & \mathcal{F}(\partial_1 Y) \end{tikzcd}$$

ii) Whenever we have $Y = Y' \cup Y''$ where $Y' \cap Y'' = \partial_1 Y' = \partial_0 Y''$

then

$$S_Y(\gamma) = S'_Y(\gamma|_{Y'}) + S''_Y(\gamma|_{Y''})$$

iii) We have the following pull-back diagram. The fact that this diagram is cartesian implies that we do have a 1-parameter action on $\mathcal{H} = \mathrm{Maps}(M, \mathbb{C}) =$

$\mathrm{Maps}(\mathcal{F}(\bullet), \mathbb{C})$:

$$\begin{tikzcd}
& & \mathcal{F}(Y) \arrow[dl, "\pi''"'] \arrow[dr, "\pi'"] & & \\
& \mathcal{F}(Y') \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1'"] & & \mathcal{F}(Y'') \arrow[dl, "\pi_0''"'] \arrow[dr, "\pi_1"] & \\
\mathcal{F}(\partial_0 Y) & & \mathcal{F}(Y' \cap Y'') & & \mathcal{F}(\partial_1 Y'')
\end{tikzcd}$$

iv) The initial ket[1] $|\phi_0\rangle$ evolves along Y according to the formula

$$|\phi_T\rangle = (\pi_T)_!(\pi_0^*(|\phi_0\rangle) \cdot e^{-i\hbar S})). \tag{3}$$

We will call this *the pull-push evolution formula.* It is the fundamental formula for all that follows and requires some clarification.

- $|\phi_0\rangle \in \mathcal{H}$ can be seen as an element in $\mathrm{Maps}(\mathcal{F}(\partial_0 Y), \mathbb{C})$ for $\partial_0 Y = \bullet$ a point and hence $\mathcal{F}(\partial_0 Y) = \mathcal{F}(\bullet) = \mathrm{Maps}(\bullet, M) \cong M$.
- $\pi_0^*(|\phi_0\rangle)$ is an element in $\mathrm{Maps}(\mathcal{F}(Y), \mathbb{C})$. In fact when we evaluate at $\gamma \in \mathcal{F}(Y)$, we get $(\pi_0^*(|\phi_0\rangle))(\gamma) = \phi_0(\gamma(\partial_0 Y)) = \phi_0(\gamma(0))$.
- $(\pi_1)_!\colon \mathrm{Maps}(\mathcal{F}(Y), \mathbb{C}) \to \mathrm{Maps}(\mathcal{F}(\bullet), \mathbb{C})$ is the map that integrates over the fiber of $\pi_1\colon \mathcal{F}(Y) \to \mathcal{F}(\bullet)$ (which in this example is the path space $\mathcal{P}_q$ and therefore it is given by a path integral). Namely:

$$((\pi_1)_!(\Phi))(q) = \int_{\mathcal{P}_q} \Phi(\gamma)\mathcal{D}\gamma$$

- You may want to think of the exponential term as a sort of Chern class for a line bundle over $\mathcal{F}(Y)$. It causes the integral to become oscillatory, and when $\hbar$ approaches 0, stationary phase approximation makes the probability that the particle travels the classical (Euler-Lagrange) path approach to 1. Feynman designed it with this specific purpose [Fey06].
- Formula 3 is in fact exactly equivalent to formula 2.

[1]The word *ket* comes from bracket. So for a given vector space $\mathcal{H}$ elements $|\phi\rangle$ in $\mathcal{H}$ itself are called kets and elements $\langle T|$; in the dual space $\mathcal{H}^*$ are called bras. The numerical evaluation $\langle T|\phi\rangle$ ends up being a bracket. The joke is Dirac's.

The algebraic abstract structure that we will extract from this is the following. Define

$$\mathcal{H}_{\mathsf{Y}} \colon = \mathrm{Maps}(\mathcal{F}(\mathsf{Y}), \mathbb{C})$$

then we have

a) We will write $\mathcal{H}$ for $\mathcal{H}(\bullet)$. To every 0-dimensional manifold we have associated a vector space $\mathcal{H}$.

b) To every 1-dimensional manifold (say of length T) we have associated a linear operator

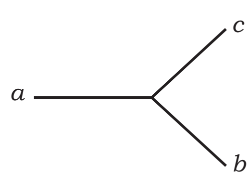
$$\mathsf{Z}_{\mathsf{T}} \colon \mathcal{H} \to \mathcal{H}$$

$$\mathsf{Z}_{\mathsf{T}}(\phi_0) = \phi_{\mathsf{T}}.$$

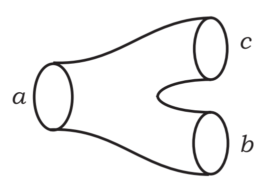
c) Whenever we glue 1-manifolds, we compose the corresponding linear operators. Namely we have homomorphism from $\mathbb{R}$ to $\mathrm{GL}(\mathcal{H})$.

The field theory just described is not topological, for the operators *depend* on the length T of the 1-manifold. In a topological theory the operators are independent on the geometry of the 1-manifold and only depend on their topology (hence we only have two operators: the one associated to *the* interval, and the number associated to the circle).

Here we should mention that in string theory we usually start by assuming that, rather than point particles interacting at singular points, we consider extended strings as in the following picture:



Particle interaction



String interaction

In the picture we have a particle a scattering in to a pair of particles b and c, and the corresponding situation with a string scattering. You should think of this picture as living inside the ambient space time M. Notice that the string interaction has *no singularity.*

Traditionally one thinks of M as a smooth manifold, for example, in general relativity. Later on we will think instead that the ambient space-time is an orbifold $\mathcal{X}$. While a (parameterized) string on a manifold can be modeled by an element of the free loop space:

$$\gamma \in \mathcal{L}M = \mathrm{Map}(S^1, M),$$

namely a piecewise smooth map form the circle to M; in an orbifold the definition of a loop is more intricate, we will come on this issue later. For now let us see what the basic formal structure is for string interactions.

2.2 Feynman's Path Integral Heuristics.

The purpose of this section is to provide motivation to the definition of a topological quantum field theory (TFT) in geometry and topology. The subject has a long and very interesting history in physics before it entered the mathematician's language, where it was incepted primarily though the influence of E. Witten [Wit89]. It was him who proved that the concept was very fruitful to study a host of mathematical phenomena in geometry and topology, specifically giving remarkable applications to knot theory.

Let us start by describing briefly what is usually meant by a quantum field theory in physics. We start by a *space-time* M which is a given smooth manifold of dimension $n+1$. We are also given for every manifold M (with boundary) a space of fields $\mathcal{F}(M)$. For every $x \in M$, we have (complex valued) local observables of the form $O_x \colon \mathcal{F}(M) \longrightarrow \mathbb{C}$, so that $O_x(\phi) \in \mathbb{C}$ for every field $\phi \in \mathcal{F}(M)$. The notation $O_x(\phi)$ is meant to signify that its value depends on ϕ_x, the germ of ϕ around x. The most important part of the structure is a probability measure μ on $\mathcal{F}(M)$ called the *Feynman measure.* All the physics of a quantum system is then contained in the expectation values $\langle O_x \rangle$, and the correlation values $\langle O^{(1)}_{x_1} O^{(2)}_{x_2} O^{(3)}_{x_3} \cdots O^{(k)}_{x_k} \rangle$.

In a great majority of examples we have that

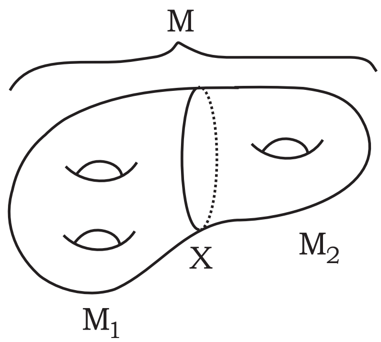
$$\mu = e^{-iS(\phi)} \mathcal{D}\phi,$$

where the *action* $S \colon \mathcal{F}(M) \to \mathbb{R}$ if of the form

$$S(\phi) = \int_M \mathcal{L}(\phi, D\phi) dx,$$

where $\mathcal{L}\colon TM \to \mathbb{R}$ is called the *Lagrangian* of the theory.

Following Atiyah [Ati88] and Segal [Seg88a],[Seg99] we will extract an algebraic gadget out of this picture. To do this, notice that whenever we cut up a manifold M into two submanifolds M_1 and M_2 with common boundary X as in the picture:



We can use the fact that $S(\phi) = S(\phi_1) + S(\phi_2)$ where ϕ_i is the restriction of ϕ to M_i, and roughly write:

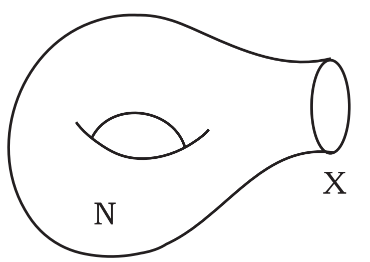
$$Z_M = \int_{\mathcal{F}(M)} e^{-iS(\phi)}\mathcal{D}\phi = \int_{\mathcal{F}(X)} Z_1(\psi)Z_2(\psi)\mathcal{D}\psi,$$

where

$$Z_i(\psi) = \int_{\phi_i \in \mathcal{F}(M_i),\ \phi_i|_X = \psi} e^{-iS(\phi_i)}\mathcal{D}\phi_i.$$

Let us denote by $H_X := \mathrm{Maps}(\mathcal{F}(X), \mathbb{C})$. Clearly H_X has the structure of a vector space, and we have that since $Z_i\colon \mathcal{F}(X) \to \mathbb{C}$, then $Z_i = Z_{M_i} \in H_X$ for a $n+1$ dimensional manifold M_i with boundary X. In other words, whenever a $n+1$ dimensional manifold N has as its boundary a n dimensional manifold X we set:

$$Z_N(\psi) = \int_{\phi_i \in \mathcal{F}(N),\ \phi_i|_X = \psi} e^{-iS(\phi_i)}\mathcal{D}\phi_i.$$



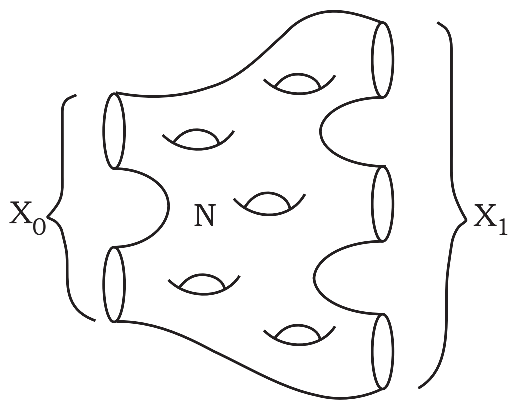
Obtaining in this manner a vector:

$$Z_N \in H_X.$$

.

In this way a quantum field theory (of dimension $n+1$) provides us with an assignment $X \mapsto H_X$ of a vector space for every n-dimensional manifold, and a vector $N \mapsto Z_N$ whenever a $n+1$ dimensional manifold has boundary $\partial N = X$.

We can do a little better. Consider a scattering process. Suppose now that we think of the manifold as having an *initial* boundary $\partial_0 N = X_0$ and a *final* boundary $\partial_1 = X_1$:



Let $H_{X_i} := \mathrm{Maps}(\mathcal{F}(X_i), \mathbb{C})$. Then we can write a linear operator of the form:

$$Z_N \colon H_{X_0} \longrightarrow H_{X_1},$$

by the formula:

$$(Z_N(\Psi))(\psi_1) = \int_{\mathcal{F}(X_0)} K(\psi_1, \psi_0)\Psi(\psi_0)\mathcal{D}\psi_0,$$

where the kernel K is given by

$$K(\phi_1, \phi_2) = \int_{\phi \in \mathcal{F}(N),\ \phi|_{X_i} = \psi_i} e^{-iS(\phi)}\mathcal{D}\phi.$$

We should also note that (formally at least) since $H_X := \mathrm{Maps}(\mathcal{F}(X), \mathbb{C})$, then we have that:

$$H_{X_1 \coprod X_2} := \mathrm{Maps}(\mathcal{F}(X_1 \coprod X_2), \mathbb{C}) = \mathrm{Maps}(\mathcal{F}(X_1), \mathbb{C}) \times \mathrm{Maps}(\mathcal{F}(X_2), \mathbb{C}) = H_{X_1} \times H_{X_2}.$$

If, as in the picture above, X_0 (resp. X_1) can be written as the disjoint union of its connected components $X_{01} \coprod X_{02}$ (resp. $X_{11} \coprod X_{12} \coprod X_{13}$,) then the map

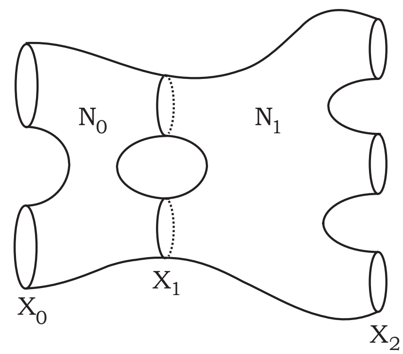
$$Z_N \colon H_{X_{01}} \times H_{X_{02}} \longrightarrow H_{X_{11}} \times H_{X_{12}} \times H_{X_{13}},$$

is actually a map

$$Z_N \colon H_{X_{01}} \otimes H_{X_{02}} \longrightarrow H_{X_{11}} \otimes H_{X_{12}} \otimes H_{X_{13}},$$

for the required multilinearity conditions are easy to verify.

Also easy to verify is that, whenever we glue two cobordisms $N = N_0 \cup N_1$ as depicted below:



we have that

$$Z_N = Z_{N_1} \circ Z_{N_0}.$$

What is quite surprising at first is that for many examples, roughly speaking, the assignments:

$$X \mapsto H_X, \quad N \mapsto Z_N,$$

for all X and for all N, contain *all* the information of the field theory, namely we can recover all correlations from those mappings. For topological field theories and 2-dimensional conformal field theories, this is the case. This is great news for mathematicians since the purported measure on the space of fields $\mathcal{F}(M)$ often does not exists. Nevertheless the assignments do exist and provide a mathematical definition for the field theories in question.

When the assignment $N \mapsto Z_N$ depends on the metric of N we refer to the theory as an *Euclidean field theory*, when it depends only on the conformal structure we call it a *conformal field theory*, and when it only depends on the topology of N we call it a *topological field theory*. In the last case the correlations will be independent of the metric.

2.3 Topological Field Theories in Dimension 1+1

Michael Atiyah in [Ati88] and [Ati90] defined *nD-Topological Field Theory* (nD-TFT) Z^A, using the following data:

1. A vector space $Z^A(\Sigma)$ associated to each $(n-1)$-dimensional closed manifold Σ.

2. A vector $Z^A(M) \in Z^A(\partial M)$ associated to each oriented n-dimensional manifold M with boundary ∂M.

3. An isomorphism $Z(f) : Z(\Sigma_1) \to Z(\Sigma_2)$, where $f : \Sigma_1 \to \Sigma_2$ is an orientation preserving diffeomorphism.

This data is subject to the following axioms:

(i) Z^A is *functorial* with respect to orientation-preserving diffeomorphisms of Σ and M.

(ii) Z^A is *involutory*, i.e. $Z^A(\Sigma^*) = Z^A(\Sigma)^*$ where Σ^* is Σ with opposite orientation and $Z^A(\Sigma)^*$ is the dual vector space of $Z^A(\Sigma)$.

(iii) Z^A is *multiplicative*

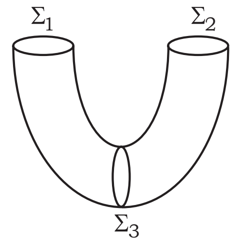
$$Z^A(\Sigma_1 \sqcup \Sigma_2) = Z^A(\Sigma_1) \otimes Z^A(\Sigma_2).$$

(iv) $Z^A(\emptyset) = \Bbbk$, where $\emptyset$ is interpreted as the empty $(n-1)$-dimensional closed manifold.

(v) $Z^A(\emptyset) = 1$, where $\emptyset$ is interpreted as the empty n-dimensional manifold which interpolates between two empty $(n-1)$-dimensional closed manifolds.

(vi) If $f : \Sigma_1 \to \Sigma_2$ is an orientation-preserving diffeomorphism, then $Z(f) : Z(\Sigma_1) \to Z(\Sigma_2)$ is an isomorphism.

These axioms are meant to be understood as follows: The functoriality axiom means that an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma'$ induces an isomorphism $Z^A(f) : Z^A(\Sigma) \to Z^A(\Sigma')$ and that $Z^A(gf) = Z^A(g)\, Z^A(f)$ for $g : \Sigma' \to \Sigma''$. Also if f extends to an orientation-preserving diffeomorphism $M \to M'$, with $\partial M = \Sigma$ and $\partial M' = \Sigma'$, then $Z^A(f)$ takes the element $Z^A(M)$ to $Z^A(M')$. The multiplicative axiom is clear. Moreover if $\partial M_1 = \Sigma_1 \sqcup \Sigma_3^*$, $\partial M_2 = \Sigma_3 \sqcup \Sigma_2$ and $M = M_1 \sqcup_{\Sigma_3} M_2$ is the manifold obtained by gluing together the common Σ_3-component:



Then we require:

$$Z^A(M) = \langle Z^A(M_1), Z^A(M_2) \rangle$$

where $\langle\, , \rangle$ denotes the natural pairing coming from the duality map,

$$Z^A(\Sigma_1) \otimes Z^A(\Sigma_3)^* \otimes Z^A(\Sigma_3) \otimes Z^A(\Sigma_2) \to Z^A(\Sigma_1) \otimes Z^A(\Sigma_2)$$

defined by $a \otimes \varphi \otimes b \otimes c \longmapsto \varphi(b) a \otimes c$. This is a very powerful axiom which implies that $Z^A(M)$ can be computed (in many different ways) by "cutting M in half" along Σ_3.

2.4 Categorical Definition of a TQFT

The first step is to define an appropiate category of cobordisms that permits us to give a functorial definition of a nD-TFT.

Definition 2.1. Let Σ_0 and Σ_1 two compact, connected, oriented $(n-1)$-manifolds, we say that they are *cobordant* if there is a n-manifold M, with boundary $\Sigma_1^* \sqcup \Sigma_2$; in this case we say that M is a n-*cobordism* of Σ_1 to Σ_2.

If we fix a positive integer n, we can construct a category $n\widetilde{Cob}$ where the objects are the closed smooth $(n-1)$-dimensional manifolds, and the morphisms are the oriented smooth n-dimensional manifolds (n-cobordism). We need to address whether the composition of two cobordisms of the same dimension is a smooth manifold. Up to a smoothing process this can be arranged (see [Koc04]). Let be $nCob' = n\widetilde{Cob}/\sim$ where $\sim$ is equivalence by diffeomorphisms. Let Σ be a closed submanifold of M of codimension 1. We assume that both are oriented. At a point $x \in \Sigma$, let $[v_1, \ldots, v_{n-1}]$ be a positive basis for $T_x\Sigma$. A vector $w \in T_xM$ is called a *positive normal* if $[v_1, \ldots, v_{n-1}, w]$ is a positive basis for T_xM. Now suppose Σ is a connected component of the boundary of M with an specific orientation; then it makes sense to ask whether the positive normal w points inward or it points outward as compared to M. Locally the situation is the following, a vector in $\mathbb{R}^n$ either points inward or outward with respect to the half-space $\mathbb{H}^n$ ($\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0)\}$). If a positive normal points inward we call Σ an *in-boundary*, and if it points outward we call it an *out-boundary*. To see that this makes sense we have to check that this does not depend on the choice of positive normal (or the choice of the point $x \in \Sigma$). If some positive normal points inward, it is easy to verify that every other positive normal at any other point $y \in \Sigma$ points inward as well. This follows from the fact that the normal bundle is a trivial line bundle on Σ. This in turn is a consequence of the assumption that both M and Σ are orientable (see Hirsch [Hir95], theorem 4.4.2). Thus the boundary of a manifold M is the union of various in-boundaries and out-boundaries. The in-boundary of M may be empty, and the out-boundary may also be empty. Note that if we reverse the orientation of both M and its boundary Σ, then the notion of what is in-boundary or out-boundary remains the same. We will denote by nCob the category $n\mathsf{Cob}'$ where every object is given an orientation (therefore any cobordism has a direction).

For the next definition we will assume that the reader is familiar with the concept of monoidal category; if this is not the case, we refer the reader to Appendix 16.

Definition 2.2. An n-*dimensional topological field theory* is a symmetric monoidal functor Z^C, from (nCob,$\sqcup$,$\emptyset$,T) to (Vect$_\Bbbk$,$\otimes$,$\Bbbk$,σ).

In all that follows we will further assume that the topological cylinder $\Sigma_0 := S^1 \times [0,1]$ seen as a cobordisms between a circle and itself gets assigned the identity map by the functor, to wit $\mathrm{Z}^C(\Sigma_0) = \mathrm{id}$.

Proposition 2.3. *Atiyah's definition and the categorical definition of a TFT coincide.*

Proof. Suppose Z^A is a TFT in the sense of Atiyah, then for M an oriented n-dimensional manifold, the next isomorphism gives the correspondence

$$\begin{array}{cccc} \Psi & \mathrm{Z}^A(\Sigma_1)^* \otimes \mathrm{Z}^A(\Sigma_2) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{Z}^A(\Sigma_1), \mathrm{Z}^A(\Sigma_2)) \\ & \mathrm{Z}^A(M) & \longmapsto & \mathrm{Z}^C(M) \end{array} \tag{4}$$

where $\partial M = \Sigma_1^* \sqcup \Sigma_2$. Set $\mathrm{Z}^C(M) := \mathrm{Z}^A(M)$; if we identify the image of the idempotent element $\mathrm{Z}^A(\Sigma \times \mathrm{I})$ with the identity $1_{\mathrm{Z}^A(\Sigma)}$, then we get a functor $\mathrm{Z}^C : \mathrm{nCob} \to \mathrm{Vect}_\Bbbk$. This functor is well defined by the *functorial* and *multiplicative* axioms. Moreover, the monoidal structure is given by $\sqcup \to \otimes$ and it is symmetrical since $Z^C(T_{\Sigma,\Sigma'}) = \sigma_{Z^C(\Sigma),Z^C(\Sigma')}$.

Conversely, given a symmetrical monoidal functor $Z^C : \mathrm{nCob} \to \mathrm{Vect}_{\Bbbk}$, if Σ is a closed $(n-1)$-dimensional smooth manifold, set $Z^A(\Sigma) := Z^C(\Sigma)$. For M a n-dimensional oriented smooth manifold we take

$$Z^A(M) = Z^C(M')(1) \in Z^C(\Sigma_{\mathrm{In}})^* \otimes Z^C(\Sigma_{\mathrm{Out}}),$$

where M' is M reversing the orientation to the in-boundary. By hypothesis, we have $Z^C(\emptyset) = \Bbbk$. Moreover, the functor Z^C is multiplicative and it is independent of the cut by the correspondence 64. As consequence, the axioms (iii) and (iv) are satisfied. Clearly $Z^A(\emptyset) = \widehat{1} \otimes 1$. Axiom (v) follows from $\Psi(Z^A(\emptyset)) = \Psi(\widehat{1} \otimes 1) = \Bbbk$. Axiom (i) is satisfied because Z^C factors through differential homotopy classes. Axiom (ii) is proposition 2.5.

♣

Corollary 2.4. *For a Topological Field Theory* Z *of any dimension and* Σ *an object in* nCob*, the image of* Σ *under* Z *is a finite dimensional vector space.*

Proof. Let

$$\langle\ ,\ \rangle_{\Sigma} : Z(\Sigma) \otimes Z(\Sigma^*) \longrightarrow \Bbbk$$

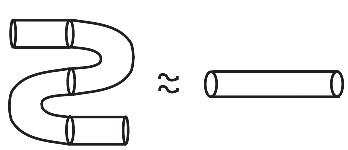
and

$$\theta_{\Sigma} : \Bbbk \longrightarrow Z(\Sigma^*) \otimes Z(\Sigma)$$

the maps associated to  and  respectively. Since Z is a TFT, then the next diagram:

$$\begin{tikzcd}
Z(\Sigma) \arrow[d, "\simeq"'] & (Z(\Sigma) \otimes Z(\Sigma)^*) \otimes Z(\Sigma) \arrow[r, "\langle,\rangle_{\Sigma} \otimes \mathrm{id}_{Z(\Sigma)}"] & \Bbbk \otimes Z(\Sigma) \arrow[d, "\simeq"] \\
Z(\Sigma) \otimes \Bbbk \arrow[r, "1_{Z(\Sigma)} \otimes \theta_{\Sigma}"] & Z(\Sigma) \otimes (Z(\Sigma^*) \otimes Z(\Sigma)) \arrow[u, "\simeq"'] & Z(\Sigma)
\end{tikzcd}$$

is the identity map. Graphically:



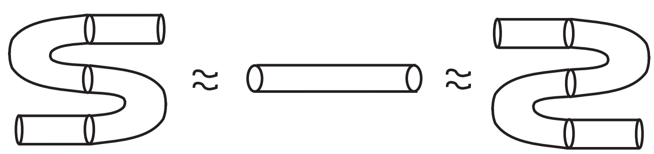
then we have $(\langle\, ,\, \rangle_\Sigma \otimes 1_{Z(\Sigma)}) \circ (1_{Z(\Sigma)} \otimes \theta_\Sigma) = 1_{Z(\Sigma)}$. For $\theta_\Sigma(1) = \sum v_j \otimes w_j$ and $a \in Z(\Sigma)$ then we have:

$$\begin{aligned} a \xrightarrow{\sim} a \otimes 1 &= (\langle\, ,\, \rangle_\Sigma \otimes 1_{Z(\Sigma)}) \circ (1_{Z(\Sigma)} \otimes \theta_\Sigma)(a \otimes 1) \\ &= (\langle\, ,\, \rangle_\Sigma \otimes 1_{Z(\Sigma)})(\sum a \otimes v_j \otimes w_j) \\ &= \sum \langle a, v_j \rangle_\Sigma \otimes w_j \xrightarrow{\sim} \sum \langle a, v_j \rangle_\Sigma w_j. \end{aligned}$$

Therefore $a = \sum \langle a, v_j \rangle_\Sigma w_j$, and consequently $\{w_j\}$ generates $Z(\Sigma)$, but since $\mathbb{k}$ is at least a division ring, we can extract a basis from the generating set. Now since every division ring has the property of invariance of dimension then $Z(\Sigma)$ is finitely generated with $n = \mathsf{rank}(A) \leq |\{w_j\}|$.

♣

The simplicity of the definition may be misleading: it is remarkable how much information a TFT encodes. For example, the fact that the theory only depends on the topology implies that to the cobordisms



we associate the same linear transformation, which is the identity. This equivalences are called the *zig-zag identities*. This simple fact implies that for any n-dimensional TFT the vector space associated to every object of $n\mathsf{Cob}$ inherits the structure of a *Frobenius algebra*.

Proposition 2.5. *Let* Z *be an* n*-dimensional TFT, and* Σ *an* n*-dimensional oriented closed smooth manifold, then* $Z(\Sigma)$ *is equipped with a nondegenerate pairing and* $Z(\Sigma^*) \simeq Z(\Sigma)^*$.

Proof. Similarly to 2.4 we have that the next diagrams:

$$\begin{tikzcd} Z(\Sigma) \arrow[d, "\simeq"'] & (Z(\Sigma) \otimes Z(\Sigma^*)) \otimes Z(\Sigma) \arrow[r, "\langle , \rangle_\Sigma \otimes 1_{Z(\Sigma)}"] & \mathbb{k} \otimes Z(\Sigma) \arrow[d, "\simeq"] \\ Z(\Sigma) \otimes \mathbb{k} \arrow[r, "1_{Z(\Sigma)} \otimes \theta_\Sigma"] & Z(\Sigma) \otimes (Z(\Sigma^*) \otimes Z(\Sigma)) \arrow[u, "\simeq"'] & Z(\Sigma) \end{tikzcd}$$

and

$$\begin{tikzcd}
\Bbbk \otimes Z(\Sigma^*) \arrow[r, "\theta_\Sigma \otimes 1_{Z(\Sigma^*)}"] & (Z(\Sigma^*) \otimes Z(\Sigma)) \otimes Z(\Sigma^*) \arrow[d, "\simeq"] & Z(\Sigma^*) \\
Z(\Sigma^*) \arrow[u, "\simeq"] & Z(\Sigma^*) \otimes (Z(\Sigma) \otimes Z(\Sigma^*)) \arrow[r, "1_{Z(\Sigma^*)} \otimes \langle , \rangle_\Sigma"] & Z(\Sigma^*) \otimes \Bbbk \arrow[u, "\simeq"']
\end{tikzcd}$$

are the identity maps of $Z(\Sigma)$ and $Z(\Sigma^*)$ respectively, i.e.

$$1_{Z(\Sigma)} = (\langle \, , \, \rangle_\Sigma \otimes 1_{Z(\Sigma)}) \circ (1_{Z(\Sigma)} \otimes \theta_\Sigma)$$

and

$$1_{Z(\Sigma)} = (1_{Z(\Sigma^*)} \otimes \langle \, , \, \rangle_\Sigma) \circ (\theta_\Sigma \otimes 1_{Z(\Sigma^*)})$$

An easy algebraic exercise proves that $\langle \, , \, \rangle_\Sigma$ is a nondegenerate pairing and that the map

$$\begin{array}{rccc} \lambda_{\text{left}} : & Z(\Sigma^*) & \longrightarrow & Z(\Sigma)^* \\ & y & \longmapsto & \langle x, y \rangle_\Sigma \end{array}$$

is an isomorphism (for we can use that $Z(\Sigma)$ and $Z(\Sigma^*)$ are finitely generated).

♣

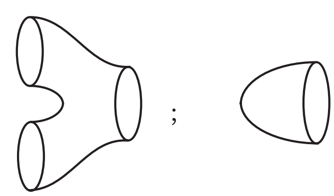
2.5 (1+1)-Dimensional TQFTs as Frobenius Algebras

Theorem 2.6. *There is a canonical equivalence of categories*

$$\text{2D-TFT}_{\Bbbk} \simeq \text{cFA}_{\Bbbk}$$

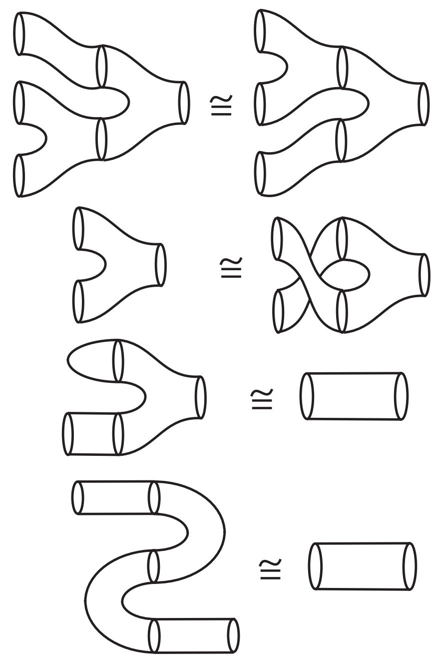
where $\text{cFA}_{\Bbbk}$ *is the category of commutative Frobenius algebras.*

Proof. We only sketch the proof. We closely follow Moore-Segal [MS] for this. It is easy to see that a 2D-TFT determines a Frobenius algebra. This is the vector space A associated to the circle. The next cobordisms induce a product $\mu : A \otimes A \to A$ and a unid $u : \Bbbk \to A$.

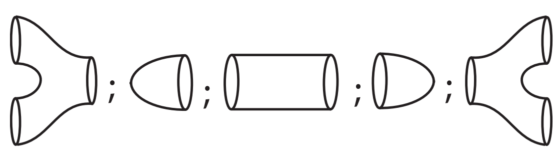


μ:A⊗A⟶A $\quad$ u:$\Bbbk$⟶A

The next pictures imply respectively the properties of associativity, commutativity, unit and non-degeneracy:



We need to prove that when we have a commutative Frobenius algebra we can assign a well defined functor from $2\,\mathrm{Cob}$ to $\mathrm{Vect}_{\Bbbk}$, for this first we note that the category is generated under composition and disjoint unions by the next five elementary cobordisms:



For this fix a 2-dimensional cobordism Σ. It is not hard to associate a linear operator to a pair consisting of a cobordism together with a decomposition on the previous five elementary building blocks. The problem is to show that the operator is independent of the chosen decomposition.

The basic idea of the proof is analogous to the proof of the Poincaré-Hopf theorem, where one embeds the discrete space of triangulations on the continuous space of vector fields on a manifold and moving around in the space of vector fields one proves that the Euler characteristic does not depend on the triangulation. Now we will embed the discrete space of possible decompositions of Σ into the continuous space of Morse functions on Σ.

Given a Morse function $f\colon \Sigma \to \mathbb{R}$ on a 2-dimensional cobordism (with the boundaries attaining constant values corresponding to the max and the min of the function f, and all critical points of Morse type and taking different values) we must associate a decomposition of Σ. This is easily achieved by cutting up sigma along $f^{-1}(t)$ for one choice of t between any two consecutive critical values of f.

Moreover every decomposition in elementary cobordisms can be achieved by a Morse function of this sort. The construction of a well defined functor is possible because there is a path in the space of Morse functions that joins any pair of Morse functions associated to a specific cobordism. According to Cerf's theory [Cer70], two Morse functions can always be connected by a good path in which every element is a Morse function except for a finite set which belongs to one of the two following cases:

1. The function has one degenerate critical point where in local coordinates (x, y) it has the form $\pm x^2 + y^3$.

2. Only two critical values of Morse type coincide.

It is understood that in any of the two cases the remaining critical values are different (for the case 1, they are even different to the degenerate critical points) and of Morse type. The invariance of the operator associated to Σ in the first case is implied by the unit and counit axioms; for the second case we must use the identity for the Euler number:

$$\chi = \sum (-1)^{\lambda} c_{\lambda}$$

with c_{λ} the number of critical points of index λ of its Morse function. Since every elementary cobordism has at most a critical point of index 0, 1 or 2; then for the case $\chi = 2$ the cobordism corresponding to the two critical values has Euler number -2, 0 or 2. When $\chi = 0$ or 2 the only relevant possibilities are the cylinder and the sphere while for $\chi = -2$ it is just a torus with two holes or the sphere with four holes. In the case $(1, 1, 1)$ (one entry, genus one and one exit) there is nothing to check, because, though a torus with two holes can be cut into two pair of pants by many different isotopy classes of cuts, there is only one possible composite cobordism, and we have only one possible composite map:

$$A \to A \otimes A \to A.$$

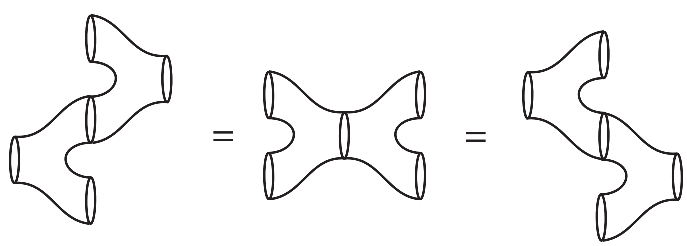
Note that the coproduct is just

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\lambda"'] & A \otimes A \\ A^* \arrow[r, "m^*"'] & A^* \otimes A^* \arrow[u, "\lambda^{-1}\otimes\lambda^{-1}"'] \end{tikzcd}$$

where λ is the corresponding Frobenius isomorphism between A and its dual. For a commutative algebra is easy to prove that

$$\Delta(a) = \sum a e_i \otimes e_i^{\#} = \sum e_i \otimes e_i^{\#} a$$

with $\{e_i\}$ a basis for $\mathcal{A}$ and $\#$ denotes the dual. For the sphere with four holes when we have $(3,0,1)$ and $(1,0,3)$ these cases are covered by the associativity of the product and coassociative of the coproduct respectively. Finally for $(2,0,2)$ it is enough to prove that it is well defined for all the possible pants decomposition; it is known that for a compact surface (m,g,n) (meaning m input circles, genus g and n output circles,) every pair-of-pants decomposition has $3g-3+m+n$ simple closed curves which cut the surface in $2g-2+m+n$ pairs of pants, hence for this case we have only a curve dividing in two pair of pants and then the only possibilities are:



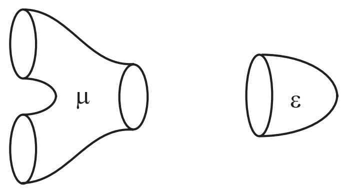
but this is clearly Abrams' condition 1 from the introduction.

♣

Notice that to have a full proof of the theorem we need the result stating that an almost Frobenius algebra with a count is exactly the same as a Frobenius algebra. We will prove this result in the next chapter.

2.6 The Case of Positive Boundaries

We define the category $n\widetilde{\mathrm{Cob}}^+$ by considering its object to be oriented *non-empty* closed smooth $(n-1)$-dimensional manifolds, and the morphisms are the oriented smooth n-dimensional manifolds(n-cobordism). Notice we do not allow the empty manifold to be the in-bounday nor the out-boundary. We always have components on both sides so in the following picture the first cobordism m is allowed while the second ε is *forbidden*:



Let be $n\mathrm{Cob}^+ = n\widetilde{\mathrm{Cob}}^+/\sim$ where $\sim$ is the relation of diffeomorphism equivalence.

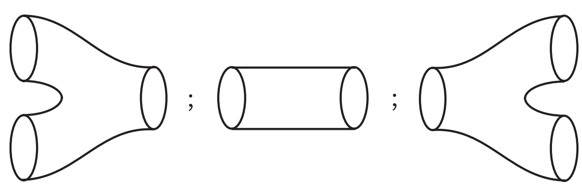
Definition 2.7. An n-*dimensional positive boundary topological field theory* (TQFT+) is a symmetric monoidal functor Z^C, from $(n\mathrm{Cob}^+,\sqcup,\emptyset,T)$ to $(\mathrm{Vect}_{\mathbb{k}},\otimes,\mathbb{k},\sigma)$.

Theorem 2.8. *There is a one-to-one correspondence between nearly Frobenius algebras and (1+1)-dimensional positive boundary topological quantum field theories.*

Proof. We want to show that it is the same to have a positive boundary 2-dim TQFT (where every connected component of the 2-dim surfaces that represent the morphisms in Cob_2^+ always have non-empty input and output boundaries) as it is to have a nearly Frobenius algebra (without units). To do this, we must consider a surface with $m > 0$ incoming boundary circles, $n > 0$ outgoing boundary circles, and genus g which we denote by $\Sigma_{m,g,n}$. We want to define the maps:

$$\Psi_{\Sigma_{m,g,n}} : A^{\otimes m} \longrightarrow A^{\otimes n}$$

and to do so we decompose the surface into three types of elementary pieces, namely pieces that look like pair of pants, cylinders, and inverted pair of pants:



To do this, it is enough to consider a perfect Morse function over $\Sigma_{m,g,n}$, that is to say a function $f : \Sigma_{m,g,n} \longrightarrow \mathbb{R}$ with isolated critical points $x_1, ..., x_k \in \Sigma$ and a strict inequality $f(x_1) < f(x_2) < ... < f(x_k)$. Moreover, we can request that $f^{-1}(0) = \delta_{in}\Sigma$ and $f^{-1}(1) = \delta_{out}\Sigma$. Let us pick real numbers

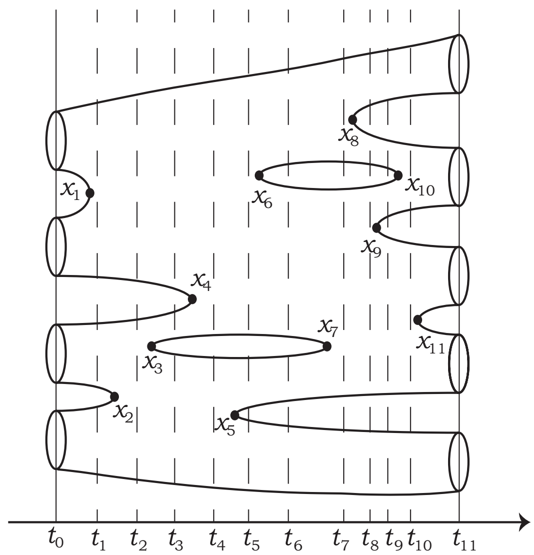
$$0 = t_0 < f(x_1) < t_1 < f(x_2) < t_2 < ... < f(x_k) < t_k.$$

As all of the t_k are regular values we have that $f^{-1}(t_j)$ is a union of circles, also $f^{-1}[t_j, t_{j+1}]$ has only one critical point and so $f^{-1}[t_j, t_{j+1}]$ is the disjoint union of elementary pieces. Therefore the function f, together with the choice of $t_1, ..., t_k$, gives a decomposition of $\Sigma_{m,g,n}$ that (by the way) is clearly independent of the choice of $t_1, ..., t_k$.

For a given decomposition of $\Sigma_{m,g,n}$, a convenient f realizing one such decomposition can always be obtained from a particular embedding $J_f : \Sigma_{m,g,n} \hookrightarrow \mathbb{R}^3$, in such a manner that

$$f = \pi_1 \circ J_f : \Sigma_{m,g,n} \overset{J_f}{\hookrightarrow} \mathbb{R}^3 \overset{\pi_1}{\hookrightarrow} \mathbb{R}.$$

For example consider the following decomposition of $\Sigma_{4,2,5}$ obtained by an embedding J_f:

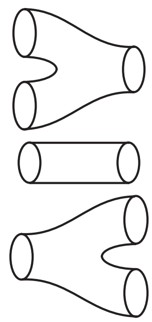


So let us assume that we are given a decomposition of $\Sigma_{m,g,n}$ in elementary pieces. We can associate a linear mapping

$$Z_{m,g,n} : A^{\otimes m} \longrightarrow A^{\otimes n}$$

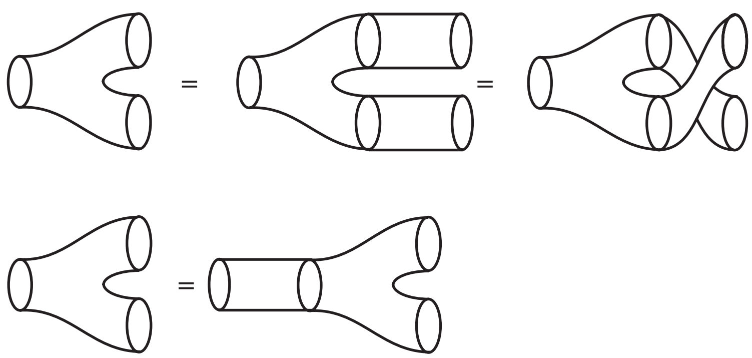
using only the given decomposition, and the structure of nearly Frobenius algebra (A, μ, Δ) over A. Recall that the multiplication is a map $\mu : A \longrightarrow A \otimes A$ and the comultiplication is map $\Delta : A \longrightarrow A \otimes A$.

Using μ and Δ and the decomposition of $\Sigma_{m,g,n}$ we can construct $Z_{m,g,n}$ by associating to every elementary piece a map as follows



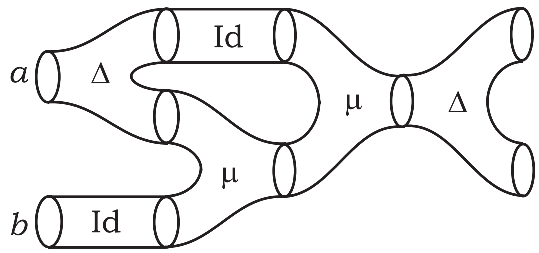
$$\longmapsto \quad \mu : A \otimes A \longrightarrow A$$
$$\longmapsto \quad \mathrm{id}_A : A \longrightarrow A$$
$$\longmapsto \quad \Delta : A \longrightarrow A \otimes A$$

At this moment it is relevant to point at that we can identify two decomposition of $\Sigma_{m,g,n}$ that differ only by insertion or deletion of cylinders, for example nothing is gained or lost by the following insertions:



The five subdivision of $\Sigma_{1,0,2}$ get the same map $\Delta : A \longrightarrow A \otimes A$ associated to them. It is at this point that we use both the commutativity of μ and the co-commutativity of Δ in order to be able to uncross the cylinders in the case that they are braided. Therefore from now on we will identify two subdivisions if they differ by the insertion or deletion of cylinders, even when they are braided (in which case we are allowed to unbraid them).

To construct the map $Z_{m,g,n}$ we use the symmetric monoidal structure $\otimes$ and the composition. For example to the following decomposition of $\Sigma_{2,1,2}$:

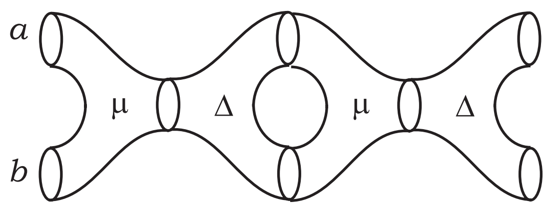


We associate the map $Z_{2,1,2} : A^{\otimes 2} \longrightarrow A^{\otimes 2}$ given as follows. Write $\Delta(a) = \sum_i \xi_i \otimes \eta_j$, then consider the composition

$$Z_{2,1,2} : a \otimes b \xmapsto{\Delta\otimes 1} \Delta(a) \otimes b \xmapsto{1\otimes\mu} \Delta(a)b = \sum_i \xi_i \otimes (\eta_i b)$$

$$\xmapsto{\mu} \sum_i \xi_i(\eta_i b) \xmapsto{\Delta} \Delta\left(\sum_i \xi_i \eta_i b\right)$$

Notice that we could decompose $\Sigma_{2,1,2}$ in a different manner:



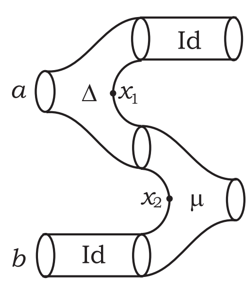
getting in turn:

$$Z'_{2,1,2} : a \otimes b \xmapsto{\mu} ab \xmapsto{\Delta} \Delta(ab) \xmapsto{\mu} \mu(\Delta(ab)) \xmapsto{\Delta} \Delta(\mu(\Delta(ab)))$$

Namely $Z'_{2,1,2}(a \otimes b) = \Delta(\mu(\Delta(ab)))$.
Now the Frobenius identity:

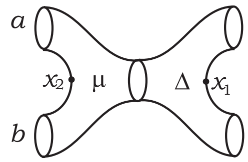
$$\Delta(a)b = \Delta(ab).$$

Tells us that we can always exchange a portion of the decomposition that looks like



$a \otimes b \longmapsto \Delta(a)b$

for one that looks like:



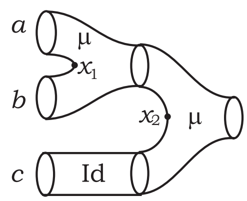
$a \otimes b \longmapsto \Delta(ab)$

Geometrically this has the effect of exchanging the critical points x_1 and x_2 permuting them.

Algebraically this shows that $Z_{2,1,2} = Z'_{2,1,2}$ for we have

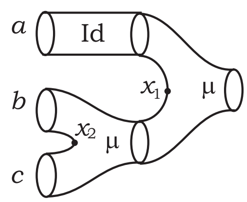
$$\begin{aligned} Z'_{2,1,2} &\stackrel{\text{def}}{=} \Delta(\mu(\Delta(ab))) = \\ &= \Delta(\mu(\Delta(a)b))) = \Delta(\mu(\sum \xi_i \otimes (\eta_i b))) = \\ &= \Delta(\sum \xi_i \eta_i b) = Z_{2,1,2}(a \otimes b) \end{aligned}$$

The associativity of the product μ allows one to exchange two left-handed saddle points of f as follows:



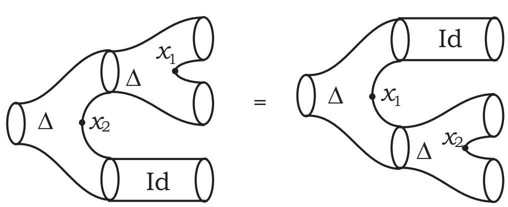
$(ab)c$

is the same as



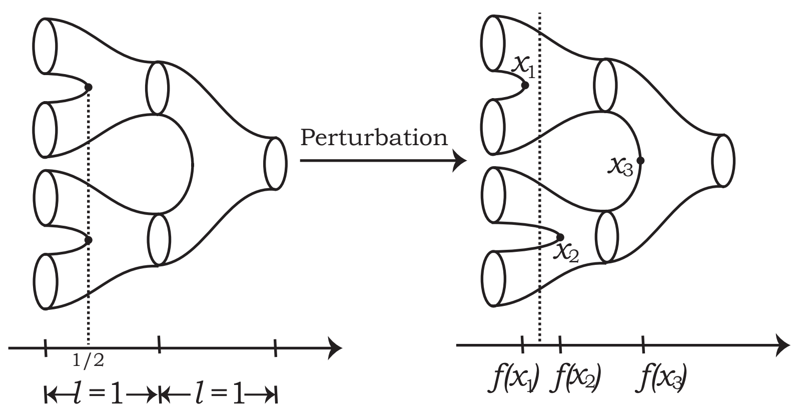
$a(bc)$

Correspondingly, the coassociativity of Δ also permits the exchange of critical points but the right handed ones:



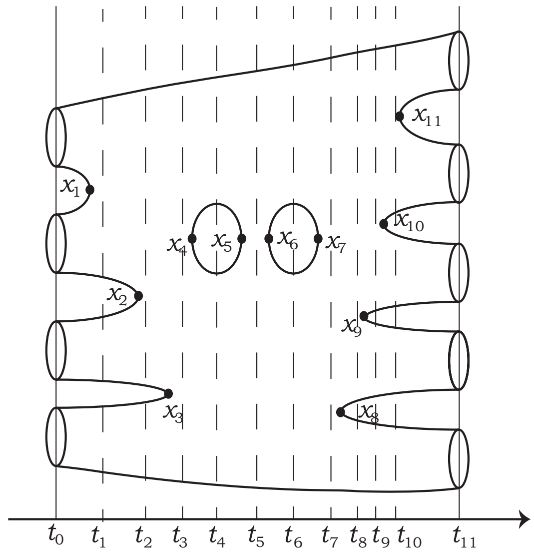
We want to show more generally that given $\Sigma_{m,g,n}$, and a pair of pants decomposition (and cylinders) the map $Z_{m,g,n}$ defined by the above procedure **does not** depend on the chosen decomposition.

Let us start by pointing out that any pair of pants decomposition can be realized by an $f = \pi_1 \circ J$ for an embedding $J : \Sigma_{m,g,n} \hookrightarrow \mathbb{R}^3$ depending on the decomposition. Given the combinatorial data of the decomposition we obtain the embedding by assembling back $\Sigma_{m,g,n}$ on top of a flat wall out of elementary pieces that look like straight pairs of pants and straight cylinders. All the pair of pants (left sided and right sided) we use are of some length $l = 1$ but their height h is allowed to change arbitrarily, we can stretch a pair of pants horizontally. Cylinders are all of the same length $l = 1$. Starting by the combinatorial data of the decomposition we assemble a 3-dimensional model of the surface $J : \Sigma_{m,g,n} \hookrightarrow \mathbb{R}^3$ by using the pieces we described, adding as many cylinders as necessary. At this point we may need to slightly perturb the embedding to make sure that the Morse function $f = \pi_1 \circ J : \Sigma_{m,g,n} \longrightarrow \mathbb{R}$ satisfies $f(x_i) \neq f(x_j)$ for every pair of critical points x_i, x_j. The picture below depicts an example of such perturbation:



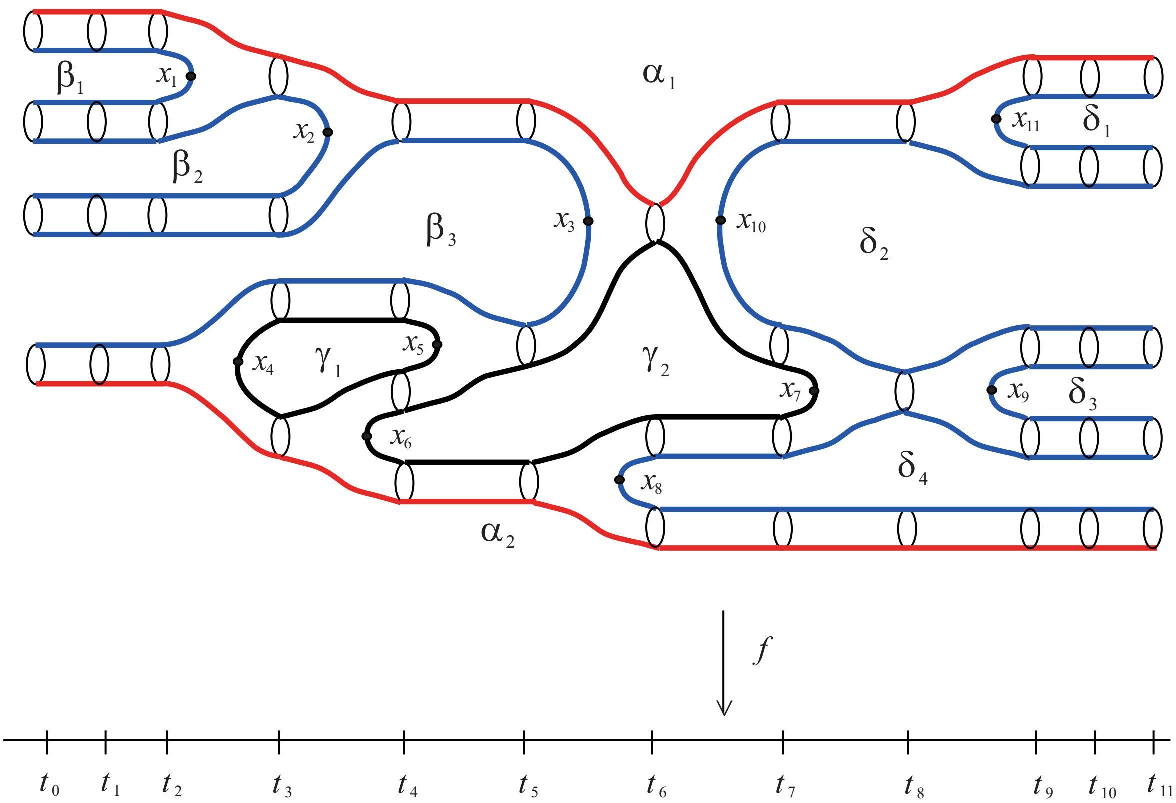
$$f(x_1) < f(x_2) < f(x_3)$$

The previous procedure produces the embedding $J : \Sigma_{m,g,n} \hookrightarrow \mathbb{R}^3$ that induces the given decomposition (up to cylinders) by making a cut in between every two consecutive critical values, $f(x_i) < t_i f(x_{i+1})$. We need the extra cylinders to construct the embedding J, but the linear mapping $Z_{m,g,n} : A^{\otimes m} \longrightarrow A^{\otimes n}$ is unaffected by them. We want to show that $Z_{m,g,n}$ is independent of the initial decomposition. To do this let us introduce the normal embedding $J^{Normal} : Z_{m,g,n} \hookrightarrow \mathbb{R}^3$. In the picture below we depict the normal embedding $J : \Sigma_{4,2,5} \hookrightarrow \mathbb{R}^3$,



$$f(x_1) < f(x_2) < f(x_3) < f(x_4) < f(x_5) < f(x_6) < f(x_7) < f(x_8) < f(x_9) < f(x_{10}) < f(x_{11})$$

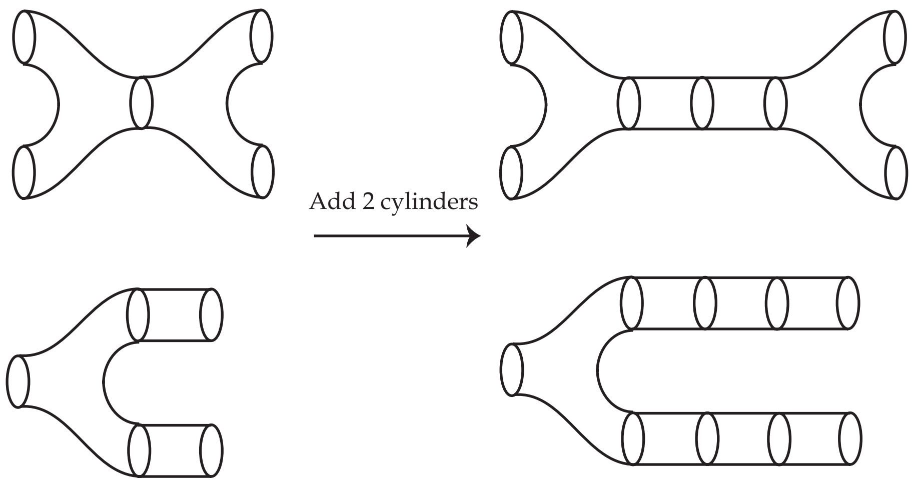
The dotted lines in the figure indicate the values at which we cut the surface. The induced pair of pants decomposition of $\Sigma_{m,g,n}$ is called the normal decomposition, and the corresponding map is $Z^{\mathrm{Normal}}_{m,g,n} : A^{\otimes m} \longrightarrow A^{\otimes n}$. All we need to show for an arbitrary decomposition with associated linear map $Z : A^{\otimes m} \longrightarrow A^{\otimes n}$ is that $Z^{\mathrm{Normal}}_{m,g,n}$, and this will imply the independence on the decomposition. Let us start with an arbitrary decomposition of $\Sigma_{m,g,n}$; and construct as above its associated embedding $J : \Sigma_{m,g,n} \hookrightarrow \mathbb{R}^3$ as below:



Notice that using the fact that the Euler characteristic satisfies:

- $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$
- $\chi(S^1) = \chi(S^1 \times I) = 0$
- $\chi(\Sigma_{2,0,1}) = -1$

We can conclude that $-\chi(\Sigma_{m,g,n})$ is the number of pairs of pants in any decomposition, and this in turn is equal to the number of critical points $x_1, ..., x_k$ for any $f = \pi_1 \circ J$; and therefore we can conclude that the curves $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3, \delta_4$ are in one to one correspondence with the corresponding curves for the normal embeding J^{Normal}. To finish the proof all we need to do is to pull one by one every one of the curves $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3, \delta_4$ into normal form. To facilitate this we first may need to add as many cylinders as necessary at any given cut point t_k as in the picture below



Notice that $s = m - 1$, $r = n - 1$ and g are topological invariants.
As we pull the curves into normal form one at a time, and each curve has at most one critical point we got a finite sequence of decompositions, and a finite sequence of linear mappings

$$Z_0 = Z, Z_1, Z_2, ..., Z_N = Z^{\text{Normal}}_{m,g,n}.$$

To show that $Z_i = Z_{i+1}$ notice that the corresponding decompositions (or rather embeddings) differ by the crossing of two critical points of $f = \pi_1 \circ J_t$ (where t is the time parameter for the time dependent embedding as we pull the curves into normal form). But then $Z_i = Z_{i+1}$ is ensured by the Frobenius equation, the associativity of μ and the coassociativity of Δ. This concludes the proof.

♣

We should point out here that the most general version of this type of classification theory is the Baez-Dolan cobordism hypothesis proved by Jacob Lurie [BD95, Lur09].

3 Frobenius Algebras

3.1 Frobenius Algebras

We start with the classical result of F. G. Frobenius from 1903 ([Fro03]) characterizing the finite dimensional algebras over fields for which the left and right regular representations are equivalent, which are called Frobenius algebras. We present the basic characterizations of Frobenius algebras, which was established in 1937-1941 by R. Brauer, C. Nesbitt and T. Nakayama.

Fix a field $\mathbb{k}$ of characteristic zero. A unital $\mathbb{k}$-*algebra* is a $\mathbb{k}$-vector space A together with two $\mathbb{k}$-linear maps

$$m : A \otimes A \to A \quad \text{and} \quad u : \mathbb{k} \to A$$

called *multiplication* and *unit* such that m is associative and u is the unit ($u(1) = 1_A$).

Let A be a finite dimensional $\mathbb{k}$-algebra, $\{a_1, a_2, \ldots, a_n\}$ is a basis of the $\mathbb{k}$-vector space A, and $\alpha_{ijk} \in \mathbb{k}$, $i, j, k \in \{1, 2, \ldots, n\}$ are the associated structure constants, that is,

$$a_j a_k = \sum_{i=1}^{n} \alpha_{ijk} a_i$$

for all $j, k \in \{1, 2, \ldots, n\}$.

We consider the matrices

$$L(a_j) = \big(L(a_j)_{ik}\big) = \big(\alpha_{ijk}\big)_{ik} \in M_n(\mathbb{k}), j \in \big\{1, \ldots, n\big\},$$

$$R(a_l) = \big(R(a_l)_{ik}\big) = \big(\alpha_{ikl}\big)_{ik} \in M_n(\mathbb{k}), l \in \big\{1, \ldots, n\big\},$$

which determine $\mathbb{k}$-linear maps $L : A \longrightarrow M_n(\mathbb{k})$ and $R : A \longrightarrow M_n(\mathbb{k})$, respectively. We denote also by $R^t : A \longrightarrow M_n(\mathbb{k})$ the $\mathbb{k}$-linear map such that $R^t(a) = R(a)^t$, the transpose of the matrix $R(a)$, for any $a \in A$. The maps

$$L : A \longrightarrow M_n(\mathbb{k}) \quad \text{and} \quad R^t : A \longrightarrow M_n(\mathbb{k})$$

are representations of the algebra A over $\mathbb{k}$, called by Frobenius the *first (left) regular representation* and the *second (right) regular representation* of A over $\mathbb{k}$, respectively.

Definition 3.1. A finite dimensional $\Bbbk$-algebra A over a field $\Bbbk$ is said to be a *Frobenius algebra* if the first (left) regular representation L and the second (right) regular representation R^t of A over $\Bbbk$ are equivalent (for a chosen basis of A over $\Bbbk$).

For a finite dimensional $\Bbbk$-algebra A over a field $\Bbbk$, a $\Bbbk$-bilinear form $\langle\ ,\ \rangle : A \otimes A \to \Bbbk$ is said to be *associative* if $\langle ab, c\rangle = \langle a, bc\rangle$ for all elements $a, b, c \in A$. Moreover, a $\Bbbk$-bilinear form $\langle\ ,\ \rangle : A \otimes A \to \Bbbk$ is said to be *non-degenerate*, if for every nonzero element $a \in A$, the linear forms $\langle a,\ \rangle, \langle\ , a\rangle : A \longrightarrow \Bbbk$ are nonzero.

We present theorems of R. Brauer, C. Nesbitt and T. Nakayama from [BN37], [Nak39], and [Nak41] which give a criteria for a finite dimensional $\Bbbk$-algebra A to be a Frobenius algebra, and are independent of the choice of a basis of A.

Theorem 3.2. *Let A be a finite dimensional $\Bbbk$-algebra over a field $\Bbbk$. The following conditions are equivalent.*

(i) *A is a Frobenius algebra.*

(ii) *There exists a non-degenerate associative $\Bbbk$-bilinear form $\langle\ ,\ \rangle : A \otimes A \to \Bbbk$.*

(iii) *There exists a $\Bbbk$-linear form $\varepsilon : A \to \Bbbk$ such that* $\ker(\varepsilon)$ *does not contain a nonzero right ideal of A.*

(iv) *There exists an isomorphism $\lambda : A \to A^*$ of right A-modules, where the dual space A^* is an A-module with the action $(f \leftharpoonup a)(b) = f(ab)$, for all $b \in A$.*

Proof. To show the equivalence (i)⇔(ii) we observe that a matrix $P = \big(p_{ij}\big)_{ij}$ determines the $\Bbbk$-bilinear form $\langle\ ,\ \rangle_P : A \otimes A \to \Bbbk$ such that $\langle a_i, a_j\rangle_P = p_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$. Conversely, every $\Bbbk$-bilinear form $\langle\ ,\ \rangle : A \otimes A \to \Bbbk$ is of the form $\langle\ ,\ \rangle = \langle\ ,\ \rangle_P$, where $P = \big(p_{ij}\big)_{ij} \in M_n(\Bbbk)$ with $p_{ij} = \langle a_i, a_j\rangle$ for $i, j \in \{1, 1, \ldots, n\}$.

Let $P = \big(p_{ij}\big)_{ij}$ be a matrix from $M_n(\Bbbk)$. We claim that the following equivalences hold:

(1) The form $\langle\ ,\ \rangle_P$ is associative if and only if $PL(a) = R^t(a)P$ for all $a \in A$.

(2) The form $\langle\ ,\ \rangle_P$ is non-degenerate if and only if the matrix P is invertible.

The equivalences (1) and (2) show that the conditions (i) and (ii) are equivalent.

(ii)⇒(iii) Let $\langle\ ,\ \rangle : A \otimes A \to \Bbbk$ be a non-degenerate associative $\Bbbk$-bilinear form. Define the $\Bbbk$-linear form $\varepsilon : A \to \Bbbk$ by

$$\varepsilon(a) = \langle a, 1_A\rangle = \langle 1_A, a\rangle \quad \text{for } a \in A.$$

(iii)⇒(ii) Let $\varepsilon : A \to \Bbbk$ be a $\Bbbk$-linear form such that $\varepsilon(I) \neq 0$ for any nonzero right ideal of A. Define the $\Bbbk$-bilinear form $\langle\ ,\ \rangle : A \otimes A \to \Bbbk$ by

$$\langle a, b\rangle = \varepsilon(ab) \quad \text{for all} \quad a, b \in A$$

(iii)⇒(iv) We define the $\Bbbk$-linear map $\lambda : A \to A^*$ such that $\lambda(a)(b) = \varepsilon(ab)$ for $a, b \in A$.

(iv)⇒(iii) Assume $\lambda : A \to A^*$ is an isomorphism of right A-modules. Define the $\Bbbk$-linear map $\varepsilon : A \to \Bbbk$ by $\varepsilon = \lambda(1) \in A^*$.

♣

Lowell Abrams in [Abr96] and Aaron D. Lauda in [Lau05] gave two additional characterizations of Frobenius algebras. They assumed that the algebra A is commutative to prove these results. We prove, in the next theorem, the same results in the general possibly non-commutative case (see [Hai06]).

Theorem 3.3. *A finite dimensional $\Bbbk$-algebra A over a field $\Bbbk$ (possibly non-commutative) is a Frobenius algebra if and only if it satisfies one of the next conditions*

(1) There are linear maps $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \Bbbk$ such that (A, Δ, ε) is a coalgebra and Δ satisfies the Frobenius identities. Explicitly, the following diagrams commute:

- *The coalgebra axioms*

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta\otimes 1"] \\ A \otimes A \arrow[r, "1\otimes\Delta"'] & A \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes \Bbbk & A \otimes A \arrow[l, "1\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes 1"] & \Bbbk \otimes A \\ & A \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

If we note $\Delta(x) = \sum x_1 \otimes x_2$, then for $x \in A$ the coalgebra axioms are given by the next relations

$$(\Delta \otimes 1)(\Delta(x)) = \sum x_{11} \otimes x_{12} \otimes x_2 = \sum x_1 \otimes x_{21} \otimes x_{22} = (1 \otimes \Delta)(\Delta(x))$$

$$(1 \otimes \varepsilon)(\Delta(x)) = \sum x_1 \varepsilon(x_2) = x = \sum \varepsilon(x_1) x_2 = (\varepsilon \otimes 1)(\Delta(x)).$$

- *The Frobenius identities*

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "1\otimes\Delta"'] & A \arrow[d, "\Delta"] \\ A \otimes A \otimes A \arrow[r, "m\otimes 1"'] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "\Delta\otimes 1"'] & A \arrow[d, "\Delta"] \\ A \otimes A \otimes A \arrow[r, "1\otimes m"'] & A \otimes A \end{tikzcd}$$

i.e. $\sum xy_1 \otimes y_2 = \sum (xy)_1 \otimes (xy)_2 = \sum x_1 \otimes x_2 y$, *for* $x, y \in A$.

(2) *There exists a co-pairing* $\theta : \Bbbk \to A \otimes A$ *and a linear map* $\varepsilon : A \to \Bbbk$ *such that the following diagrams commute:*

$$\begin{tikzcd} A \arrow[r, "1\otimes\theta"] \arrow[d, "\theta\otimes 1"'] \arrow[dr, "\Delta"] & A \otimes A \otimes A \arrow[d, "m\otimes 1"] \\ A \otimes A \otimes A \arrow[r, "1\otimes m"'] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "\theta"] \arrow[d, "\theta"'] \arrow[dr, "u"] & A \otimes A \arrow[d, "\varepsilon\otimes 1"] \\ A \otimes A \arrow[r, "1\otimes\varepsilon"'] & A \end{tikzcd}$$

Let $x \in A$, *if we denote* $\theta(1) = \sum \xi_i \otimes \xi^j$ *then the Lauda condition is the following:*

$$\sum x\xi_1 \otimes \xi_2 = \sum x_1 \otimes x_2 = \sum \xi_1 \otimes \xi_2 x,$$

and

$$\sum \varepsilon(\xi_1)\xi_2 = 1_A = \sum \xi_1 \varepsilon(\xi_2).$$

Proof. (1) ($\Leftarrow$) We suppose that $(A, m, u, \Delta, \varepsilon)$ is an algebra-coalgebra where Δ satisfies the Frobenius identities. We define the linear map $\lambda : A \to A^*$ as $\lambda = \varepsilon \leftharpoonup$, where $\lambda(a)(b) = \big(\varepsilon \leftharpoonup a\big)(b) = \varepsilon(ab)$. To prove that λ is an isomorphism we will prove that $1_A \leftharpoonup : A^* \to A$ is the inverse function, where $1_A \leftharpoonup f = \sum f(1_1)1_2$ with $\Delta\big(1_A\big) = \sum 1_1 \otimes 1_2$,

$$(1_A \leftharpoonup) \circ (\varepsilon \leftharpoonup)(x) = 1_A \leftharpoonup (\varepsilon \leftharpoonup x) = \sum \varepsilon(x 1_1) 1_2 = x 1_A = x,$$

the last identification is because Δ satisfies the Frobenius identities, and the other identity

$$\big((\varepsilon \leftharpoonup) \circ (1_A \leftharpoonup)(f)\big)(x) = \sum f(1_1)\varepsilon(1_2 x) = f\left(\sum 1_1 \varepsilon(1_2 x)\right) = f(x), \text{ for all } x \in A,$$

as before, the last identification is because Δ satisfies the Frobenius identities.

The last step is to prove that λ is a morphism of right A-modules, i.e. the commutativity of the following diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "\varepsilon\leftharpoonup\otimes 1"'] & A \arrow[d, "\varepsilon\leftharpoonup"] \\ A^* \otimes A \arrow[r, "\leftharpoonup"] & A \otimes A^* \end{tikzcd}$$

$$\big((\varepsilon \leftharpoonup) \circ m(x \otimes y)\big)(z) = \big(\varepsilon \leftharpoonup (xy)\big)(z) = \varepsilon\big((xy)z\big)$$

$$\big((\varepsilon \leftharpoonup x) \leftharpoonup y\big)(z) = (\varepsilon \leftharpoonup x)(yz) = \varepsilon\big(x(yz)\big)$$

and the commutativity is due to the associativity of the product.

($\Rightarrow$) Now we suppose that $\big(A, m, u, \varepsilon \leftharpoonup\big)$ is a Frobenius algebra, $\{e_1, \ldots, e_n\}$ a basis of A and $\{e_i^*\}$ the dual basis. We define the coproduct $\Delta : A \to A \otimes A$ by

$$\Delta(x) = \sum_i x e_i \otimes \big(\varepsilon \leftharpoonup\big)^{-1}\big(e_i^*\big)$$

It is easy to prove that this coproduct is given by

$$\begin{tikzcd} A \arrow[r, "\Delta^{op}"] \arrow[d, "\varepsilon\leftharpoonup"'] & A \otimes A \\ A^* \arrow[r, "m^*"] & A^* \otimes A^* \arrow[u, "(\varepsilon\leftharpoonup)^{-1} \otimes (\varepsilon\leftharpoonup)^{-1}"'] \end{tikzcd}$$

- *Coassociativity:* $(\Delta \otimes 1)\Delta(x) = \sum_{i,j} x e_i e_j \otimes (\varepsilon \leftharpoonup)^{-1}\big(e_j^*\big) \otimes (\varepsilon \leftharpoonup)^{-1}\big(e_i^*\big)$, $(1 \otimes \Delta)\Delta(x) = \sum_{i,j} x e_j \otimes (\varepsilon \leftharpoonup)^{-1}\big(e_j^*\big)e_i \otimes (\varepsilon \leftharpoonup)^{-1}\big(e_i^*\big)$.
  Applying the isomorphism $1 \otimes (\varepsilon \leftharpoonup) \otimes (\varepsilon \leftharpoonup)$ we need to prove

  $$\sum_{i,j} x e_i e_j \otimes e_j^* \otimes e_i^* = \sum_{i,j} x e_j \otimes e_j^* \leftharpoonup e_i \otimes e_i^*, \tag{5}$$

  where $(\varepsilon \leftharpoonup)\big((\varepsilon \leftharpoonup)^{-1}\big(e_j^*\big)e_i\big) = e_j^* \leftharpoonup e_i$. If we prove that $\sum_{i,j} xze_j \otimes e_j^* = \sum_{i,j} xe_j \otimes e_j^* \leftharpoonup z$, for all $z \in A$ we deduce 64.

  $$\left(\sum_{i,j} xze_j \otimes e_j^*\right)(w) = \sum xze_j e_j^*(w) = (xz)\left(\sum e_j e_j^*(w)\right) = (xz)w.$$

  $$\left(\sum_{i,j} xe_j \otimes e_j^* \leftharpoonup z\right)(w) = \sum xe_j\big(e_j^* \leftharpoonup z\big)(w) = x\left(\sum e_j e_j^*(zw)\right) = x(zw).$$

- *Counit axiom:* $\big(\varepsilon \otimes 1\big)\Delta(x) = \sum \varepsilon\big(xe_i\big)(\varepsilon \leftharpoonup)^{-1}\big(e_i^*\big) = (\varepsilon \leftharpoonup)^{-1}\left((\sum \varepsilon\big(xe_i\big)e_i^*\right)$
  $= (\varepsilon \leftharpoonup)^{-1}(\varepsilon \leftharpoonup x) = x.$
  $\big(1 \otimes \varepsilon\big)\Delta(x) = \sum xe_i\varepsilon\left((\varepsilon \leftharpoonup)^{-1}\big(e_i^*\big)\right) = \sum xe_i(\varepsilon \leftharpoonup)(\varepsilon \leftharpoonup)^{-1}\big(e_i^*\big)\big(1_A\big) = x\sum e_i e_i^*\big(1_A\big) = x1_A = x$

- *Frobenius identities:* $\Delta(xy) = \sum xye_i \otimes (\varepsilon \leftharpoonup)^{-1}\big(e_i^*\big) = (x \otimes 1)\Delta(y)$.
  On the other hand, $\Delta(x)(1 \otimes y) = \sum xe_i \otimes (\varepsilon \leftharpoonup)^{-1}\big(e_i^*\big)y = \sum xe_i \otimes (\varepsilon \leftharpoonup)^{-1}\big(e_i^* \leftharpoonup y\big)$.
  Applying the isomorphism $1 \otimes (\varepsilon \leftharpoonup)$ we need to prove that

$$\sum xye_i \otimes e_i^* = \sum xe_i \otimes \big(e_i^* \leftharpoonup y\big).$$

  That is true because

$$\Big(\sum xye_i \otimes e_i^*\Big)(z) = (xy)\Big(\sum e_i e_i^*(z)\Big) = (xy)z = x(yz) = x\Big(\sum e_i e_i^*(yz)\Big)$$

$$= x \sum e_i\big(e_i^* \leftharpoonup y\big)(z) = \Big(\sum xe_i \otimes \big(e_i^* \leftharpoonup y\big)\Big)(z), \quad \text{for all } z \in A.$$

(2) It is easy to see that this condition is equivalent to condition (1). Given the coproduct $\Delta : A \to A \otimes A$ we define $\theta : \Bbbk \to A \otimes A$ by $\theta = \Delta \circ u$. We deduce the commutativity of the diagrams using the Frobenius identities. If we consider the co-pairing $\theta : \Bbbk \to A \otimes A$ we define $\Delta : A \to A \otimes A$ as follows

$$\Delta = (1 \otimes m) \circ (\theta \otimes 1) = (m \otimes 1) \circ (1 \otimes \theta).$$

♣

Yet another characterization of Frobenius algebras is given by a relation between the coalgebra structure and an isomorphism by the dual algebra and the coalgebra, as the next result shows:

Theorem 3.4. *A is a Frobenius algebra if and only if A is a coalgebra and there exists an isomorphism $\phi : A^* \to A$ of right A^*-modules, where A is an A^*-module with the action $\leftharpoonup : A^* \otimes A \to A$ given by $f \leftharpoonup x = \sum f(x_1)x_2$.*

Proof. ($\Rightarrow$) Let $\big(A, m, u, \Delta, \varepsilon\big)$ be a Frobenius algebra, then $\big(A, \Delta, \varepsilon\big)$ is a coalgebra. We define $\phi = 1_A \leftharpoonup : A^* \to A$, note that $1_A \leftharpoonup$ is an isomorphism and the inverse function is $\varepsilon \leftharpoonup : A \to A^*$. We only need to prove that $1_A \leftharpoonup$ is a morphism of right A^*-modules, that is, the next diagram commute

$$\begin{tikzcd}
A^* \otimes A^* \arrow[r, "\Delta^*"] \arrow[d, "1_A \leftharpoonup \otimes 1"'] & A^* \arrow[d, "1_A \leftharpoonup"] \\
A \otimes A^* \arrow[r, "\leftharpoonup"] & A
\end{tikzcd}$$

$$1_A \leftharpoonup \Delta^*\big(f\otimes g\big) = \sum \Delta^*\big(f\otimes g\big)\big(1_1\big)1_2 = \sum f\big(1_1\big)g\big(1_2\big)1_3,$$

$$\big(1_A \leftharpoonup f\big) \leftharpoonup g = \sum f\big(1_1\big)1_2 \leftharpoonup g = \sum f\big(1_1\big)g\big(1_2\big)1_3,$$

then $1_A \leftharpoonup$ is a morphism of right A^*-modules.

($\Leftarrow$) Let be $\big(A,\Delta,\varepsilon\big)$ a coalgebra and $\phi : A^* \to A$ an isomorphism of right A^*-modules. First, we observe that the morphisms $\phi : A^* \to A$ of right A^*-modules are of the form $\phi = t \leftharpoonup$, for some $t \in A$. To prove this, we only need to check that for $t = \phi(\varepsilon)$ we have $\phi = t \leftharpoonup$. We know that the diagram

$$\begin{tikzcd}
A^*\otimes A^* \arrow[r, "\Delta^*"] \arrow[d, "\phi\otimes 1"'] & A^* \arrow[d, "\phi"] \\
A\otimes A^* \arrow[r, "\leftharpoonup"] & A
\end{tikzcd}$$

commutes, then $\phi\left(\Delta^*\big(\varepsilon\otimes f\big)\right) = \phi(\varepsilon) \leftharpoonup f = t \leftharpoonup f$, and $\Delta^*\big(\varepsilon\otimes f\big)(x) = \sum \varepsilon\big(x_1\big)f\big(x_2\big) = f\left(\sum \varepsilon\big(x_1\big)x_2\right) = f(x)$. Therefore $\Delta^*\big(\varepsilon\otimes f\big) = f$ and $\phi(f) = t \leftharpoonup f$, for all $f \in A^*$.

Then we suppose then $\phi = t \leftharpoonup$ and we define $m(x\otimes y) = xy = \sum (t \leftharpoonup)^{-1}(x)\big(y_1\big)y_2$, for all $x, y \in A$. It is easy to prove that this product satisfies

$$\begin{tikzcd}
A\otimes A \arrow[r, "m^{op}"] \arrow[d, "(t\leftharpoonup)^{-1}\otimes(t\leftharpoonup)^{-1}"'] & A \\
A^*\otimes A^* \arrow[r, "\Delta^*"] & A^* \arrow[u, "(t\leftharpoonup)^{-1}"']
\end{tikzcd}$$

where m^{op} is the opposite product. Note that, as $t \leftharpoonup$ is an isomorphism of right A^*-modules, the inverse function $(t \leftharpoonup)^{-1}$ satisfies that

$$\begin{tikzcd}
A^*\otimes A^* \arrow[r, "\Delta^*"] & A^* \\
A\otimes A^* \arrow[r, "\leftharpoonup"] \arrow[u, "(t\leftharpoonup)^{-1}\otimes 1"] & A \arrow[u, "(t\leftharpoonup)^{-1}"']
\end{tikzcd} \tag{6}$$

commutes.

- *Unit axiom:* We will prove that $t = t \leftharpoonup \varepsilon$ is the unit.

$$xt = \sum (t \leftharpoonup)^{-1}(x)\big(t_1\big)t_2 = t \leftharpoonup (t \leftharpoonup)^{-1}(x) = \big((t \leftharpoonup)\circ(t \leftharpoonup)^{-1}\big)(x) = x$$

$$tx = \sum (t \leftharpoonup)^{-1}(t)\big(x_1\big)x_2 = \sum \varepsilon(x_1)x_2 = x,$$

- *Frobenius identities:* $(x \otimes 1)\Delta(y) = \Delta(xy) = \Delta(x)(1 \otimes y)$.

$$\Delta(xy) = \sum (t \leftharpoonup)^{-1}(x)\big(y_1\big)y_2 \otimes y_3,$$

$$(x \otimes 1)\Delta(y) = \sum xy_1 \otimes y_2 = \sum (t \leftharpoonup)^{-1}(x)\big(y_1\big)y_2 \otimes y_3,$$

$$\Delta(x)(1 \otimes y) = \sum x_i \otimes x_2 y = \sum x_1 \otimes (t \leftharpoonup)^{-1}\big(x_2\big)\big(y_1\big)y_2.$$

Then $\Delta(xy) = (x \otimes 1)\Delta(y)$.
Note that, if we prove that $\sum (t \leftharpoonup)^{-1}\big(x_2\big)(y)x_1 = \sum (t \leftharpoonup)^{-1}(x)\big(y_1\big)y_2$, then we conclude that $\Delta(xy) = \Delta(x)(1 \otimes y)$.

As 6 commutes we have that $\big((t \leftharpoonup)^{-1}(x \leftharpoonup f)\big)(y) = \Delta^*\big((t \leftharpoonup)^{-1}(x) \otimes f\big)(y)$, for all $y \in A^*$. Then

$$\begin{aligned} f\Big(\sum (t \leftharpoonup)^{-1}\big(x_2\big)(y)x_1\Big) &= \sum f(x_1)(t \leftharpoonup)^{-1}\big(x_2\big)(y) \\ &= (t \leftharpoonup)^{-1}\Big(\sum f(x_1)x_2\Big)(y) \\ &= \Big((t \leftharpoonup)^{-1}(x \leftharpoonup f)\Big)(y) \\ &= \Delta^*\big((t \leftharpoonup)^{-1}(x) \otimes f\big)(y) \\ &= \sum (t \leftharpoonup)^{-1}(x)(y_1)f(y_2) \\ &= f\Big(\sum (t \leftharpoonup)^{-1}(x)\big(y_1\big)y_2\Big) \end{aligned}$$

for all $f \in A^*$, and $\sum (t \leftharpoonup)^{-1}\big(x_2\big)(y)x_1 = \sum (t \leftharpoonup)^{-1}(x)\big(y_1\big)y_2$.

- *Associativity:* $(xy)z = \sum (t \leftharpoonup)^{-1}(x)(y_1)(t \leftharpoonup)^{-1}(y_2)(z_1)z_2 = z \leftharpoonup (t \leftharpoonup)^{-1}\big(y \leftharpoonup (t \leftharpoonup)^{-1}(x)\big)$, using that 6 commutes we have

$$\begin{aligned} (xy)z &= z \leftharpoonup \Delta^*\Big((t \leftharpoonup)^{-1}(y) \otimes (t \leftharpoonup)^{-1}(x)\Big) \\ &= \sum \Delta^*\Big((t \leftharpoonup)^{-1}(y) \otimes (t \leftharpoonup)^{-1}(x)\Big)(z_1)z_2 \\ &= \sum (t \leftharpoonup)^{-1}(y)(z_1)(t \leftharpoonup)^{-1}(x)(z_2)z_3 \\ &= x(yz). \end{aligned}$$

♣

Definition 3.5. A *Frobenius algebra homomorphism* $\phi : \big(A, \varepsilon\big) \longrightarrow \big(A', \varepsilon'\big)$ between two Frobenius algebras is an algebra homomorphism which is at the same time a coalgebra homomorphism. In particular it preserves the Frobenius form, in the sense that $\varepsilon = \phi \circ \varepsilon'$.

Let $\mathrm{FA}_{\Bbbk}$ denotes the category of Frobenius algebras, and let $\mathrm{cFA}_{\Bbbk}$ denotes the full subcategory of all commutative Frobenius algebras.

Lemma 3.6. *If a $\Bbbk$-algebra homomorphism ϕ, between two Frobenius algebras $\left(A, \varepsilon\right)$ and $\left(A', \varepsilon'\right)$, is compatible with the forms in the sense that the diagram*

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\varepsilon"'] & & A' \arrow[dl, "\varepsilon'"] \\ & \Bbbk & \end{tikzcd}$$

commutes, then ϕ is injective.

Proof. The kernel of ϕ is an ideal and it is clearly contained in $\ker(\varepsilon)$. But $\ker(\varepsilon)$ contains no nontrivial ideals, so $\ker(\phi) = 0$ and thus ϕ is injective.

♣

Lemma 3.7. *A Frobenius algebra homomorphism $\phi : A \to A'$ is always invertible. In other words, the category $\mathrm{FA}_{\Bbbk}$ is a groupoid and similarly is $\mathrm{cFA}_{\Bbbk}$.*

Proof. Since ϕ is comultiplicative and respects the counits ε and ε' (as well as the units η and η'), the dual map $\phi^* : A'^* \to A^*$ is multiplicative and respects the units and counits. But by the preceding lemma an immediate consequence is that ϕ^* is injective. Since A is a finite-dimensional vector space, then ϕ is surjective. We already know it is injective, hence it is invertible.

♣

Remark 3.8. Notice that for a fixed algebra structure on A, the set of all compatible Frobenius algebra structures on A is determined by one, that is, two traces on A will differ only by an invertible element of the algebra A.

Example 3.1. Let $A = \Bbbk$, and $\varepsilon : A \to \Bbbk$ be the identity map of $\Bbbk$. Clearly there are no ideals in the kernel of this map, so we have a Frobenius algebra.

Example 3.2. The field of complex numbers $\mathbb{C}$ is a Frobenius algebra over $\mathbb{R}$: an obvious Frobenius form is taking the real part

$$\begin{aligned} \mathbb{C} &\to \mathbb{R} \\ a + ib &\mapsto a. \end{aligned}$$

Example 3.3. Let A be a skew-field (also called division algebra) of finite dimension over $\mathbb{k}$. Similarly as for a field, a skew-field has no nontrivial left ideals (or right ideals), any nonzero linear form $A \to \mathbb{k}$ will make A into a Frobenius algebra over $\mathbb{k}$, for example the quaternions $\mathbb{H}$ form a Frobenius algebra over $\mathbb{R}$.

Example 3.4. Let A be the space $\mathrm{Mat}_n(\mathbb{k})$ of all $n \times n$ matrices over $\mathbb{k}$, this is a Frobenius algebra with the usual trace map

$$\begin{aligned} \mathrm{Tr} : \mathrm{Mat}_n(\mathbb{k}) &\to \quad \mathbb{k} \\ (a_{ij}) \qquad &\mapsto \sum_i a_{ii} \end{aligned}$$

To see that the bilinear pairing resulting from Tr is nondegenerate, take the linear basis of $\mathrm{Mat}_n(\mathbb{k})$ consisting of E_{ij} with only one nonzero entry $e_{ij} = 1$. Clearly E_{ji} is the dual basis element to E_{ij} under this pairing. Note that this is a symmetric Frobenius algebra since two matrix products AB and BA have the same trace. If we twist the Frobenius form by multiplication with a noncentral invertible matrix we obtain a nonsymmetric Frobenius algebra.

For example consider $\mathrm{Mat}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ with the usual trace map

$$\begin{array}{rccc} \mathrm{Tr} : & \mathrm{Map}_2(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & a + d \end{array}$$

We can twist and then take as the Frobenius form the composition

$$\begin{array}{ccccc} \mathrm{Mat}_2(\mathbb{R}) & \longrightarrow & \mathrm{Mat}_2(\mathbb{R}) & \xrightarrow{\mathrm{Tr}} & \mathbb{R} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \longmapsto & b + c \end{array},$$

the composition is not central, for if we take $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ then $AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$.

Example 3.5. Let $G = \{e, g_1, \ldots, g_n\}$ be a finite group, the *group algebra* $A := \mathbb{C}[G]$ is defined as the set of formal linear combinations $\sum_{i=0}^{n} c_i g_i$, where $c_i \in \mathbb{C}$, with multiplication given by the multiplication of G. It can be made into a Frobenius algebra by taking the Frobenius form to be the functional

$$\begin{array}{rccll} \varepsilon: & \mathbb{C}G & \longrightarrow & \mathbb{C} & \\ & e & \longmapsto & 1 & \\ & g_i & \longmapsto & 0 & \text{for } i \neq 0. \end{array}$$

Indeed, the corresponding pairing $g \otimes h \mapsto \varepsilon(gh)$ is nondegenerate since $g \otimes h \mapsto 1$ if and only if $h = g^{-1}$.

Example 3.6. Assume the group field is $\Bbbk = \mathbb{C}$. Let G be a finite group of order n. A *class function* on G is a function $G \to \mathbb{C}$ which is constant on each conjugacy class; the class functions form a ring denoted $A := R(G)$. In particular, the characters (traces of representations) are class functions, and in fact every class function is a linear combination of characters. There is a bilinear pairing on $R(G)$ defined by

$$\langle \phi, \psi \rangle := \frac{1}{n} \sum_{t \in G} \phi(t) \psi\big(t^{-1}\big).$$

The characters form an orthonormal basis of $R(G)$with respect to this bilinear pairing, so in particular the pairing is nondegenerate and provides a Frobenius algebra structure on $R(G)$.

Example 3.7. Let M be a compact, closed, connected, oriented manifold of finite dimension n. Let us consider the singular cohomology of M and write $A := H^*(M)$. We can define a counit map $\varepsilon : H^*(M) \to \Bbbk$ by

$$\varepsilon(\varphi) = \varphi\big([M]\big) = \int_M \varphi,$$

where $[M]$ is the fundamental class of M in homology. This map induces the pairing

$$\langle \, , \rangle : H^*(M) \otimes H^*(M) \to \Bbbk$$

defined by $\langle \varphi, \psi \rangle = \varepsilon\big(\varphi \smile \psi\big) = \big(\varphi \smile \psi\big)\big([M]\big) = \varphi\big([M] \frown \psi\big)$. Remember that we have the next isomorphism induced by Poincaré duality

$$\Phi : H^{n-k}(M) \xrightarrow{h} \mathrm{Hom}_{\Bbbk}(H_{n-k}(M), \Bbbk) \xrightarrow{D^*} \mathrm{Hom}_{\Bbbk}(H^k(M), \Bbbk)$$

where h is the map induced by the evaluation of cochains on chains, and D^* is the dual of Poincaré duality. Then $\Phi(\varphi)(\psi) = \varphi\big([M] \frown \psi\big)$, this proves that the pairing is nondegenerate.

3.2 Nearly Frobenius Algebras

In this section we focus on one of the central objects of study in this book. That is the structure of *nearly Frobenius algebra*.

Definition 3.9. An algebra A with product $m : A \otimes A \to A$ is a *nearly Frobenius algebra* if there exists a coproduct $\Delta : A \to A \otimes A$ that makes the following diagrams commutative:

1. The coalgebra axioms

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta\otimes 1"] \\ A \otimes A \arrow[r, "1\otimes\Delta"'] & A \otimes A \otimes A \end{tikzcd}$$

If we note $\Delta(x) = \sum x_1 \otimes x_2$, then for $x \in \mathcal{A}$ the coalgebra axioms are given by the next relations

$$\begin{aligned} (\Delta \otimes 1)\big(\Delta(x)\big) &= \sum x_{11} \otimes x_{12} \otimes x_2 = \sum x_1 \otimes x_{21} \otimes x_{22} = (1 \otimes \Delta)\big(\Delta(x)\big), \\ (1 \otimes \varepsilon)\big(\Delta(x)\big) &= \sum x_1 \varepsilon(x_2) = x = \sum \varepsilon(x_1) x_2 = (\varepsilon \otimes 1)\big(\Delta(x)\big). \end{aligned}$$

2. The Frobenius identities

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "\Delta\otimes 1"'] & A \arrow[d, "\Delta"] \\ A \otimes A \otimes A \arrow[r, "1\otimes m"'] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "1\otimes\Delta"'] & A \arrow[d, "\Delta"] \\ A \otimes A \otimes A \arrow[r, "m\otimes 1"'] & A \otimes A \end{tikzcd}$$

i.e. $\sum xy_1 \otimes y_2 = \sum (xy)_1 \otimes (xy)_2 = \sum x_1 \otimes x_2 y$, for all $x, y \in \mathcal{A}$.

Lemma 3.10. *Let A be a $\Bbbk$-algebra and $\Delta : A \to A \otimes A$ a $\Bbbk$-linear map such that*

$$\Delta(x) = (x \otimes 1)\Delta(1) = \Delta(1)(1 \otimes x)$$

for all $x \in A$. Then Δ is an A-bimodule morphism.

Proof. The linear map Δ is an A-bimodule morphism if $\Delta \circ m = (1 \otimes m) \circ (\Delta \otimes 1)$ and $\Delta \circ m = (m \otimes 1) \circ (1 \otimes \Delta)$. Let $x, y \in A$ then

$$(1 \otimes m) \circ (\Delta \otimes 1)(x \otimes y) = \Delta(x)(1 \otimes y) = \Delta(1)(1 \otimes x)(1 \otimes y) = \Delta(1)(1 \otimes xy) = \Delta(xy)$$

$$(m \otimes 1) \circ (1 \otimes \Delta)(x \otimes y) = (x \otimes 1)\Delta(y) = (x \otimes 1)(y \otimes 1)\Delta(1) = (xy \otimes 1)\Delta(1) = \Delta(xy)$$

♣

Example 3.8. Every Frobenius algebra is also a nearly Frobenius algebra, but a nearly Frobenius algebra is a Frobenius algebra only when we have a compatible counit.

Example 3.9. Let A be the truncated polynomial algebra in one variable $\Bbbk[x]/x^{n+1}$. We will determine all the nearly-Frobenius structures on A.

We consider the canonical basis $B = \{1, x, \dots, x^n\}$ of A. Then the general expression of a $\Bbbk$-linear map $\Delta : A \to A \otimes A$ in the value 1 is

$$\Delta(1) = \sum_{i,j=1}^{n} a_{ij} x^i \otimes x^j.$$

This map is an A-bimodule morphism if

$$\Delta(x^k) = (x^k \otimes 1)\Delta(1) = \Delta(1)(1 \otimes x^k), \quad \forall\, k \in \{0, \dots, n\}. \tag{7}$$

The equation (7) when $k = 1$ is

$$\sum_{i,j=1}^{n} a_{ij} x^{i+1} \otimes x^j = \sum_{ij,=1}^{n} a_{ij} x^i \otimes x^{j+1}.$$

This happens if $a_{0j-1} = 0$, $j = 1, \dots, n$; $a_{i-10} = 0$, $i = 1, \dots, n$ and $a_{ij-1} = a_{i-1j}$ in other case. Then

$$\Delta(1) = \sum_{k=0}^{n} a_{kn} \left(\sum_{i+j=n+k} x^i \otimes x^j \right)$$

We denote $a_k = a_{kn}$. Applying the lemma 3.10 we need to prove that $\Delta(x^k) = \big(x^k \otimes 1\big)\Delta(1) = \Delta(1)\big(1 \otimes x^k\big)$ to conclude that Δ is an A-bimodule morphism.

$$\begin{aligned}
\Delta(1)\big(1 \otimes x^l\big) &= \sum_{k=0}^{n} a_k \left(\sum_{i+j=n+k} x^i \otimes x^j \right) \big(1 \otimes x^l\big) = \sum_{k=0}^{n} a_k \left(\sum_{i+j=n+k} x^i \otimes x^{j+l} \right) \\
&= \sum_{k=0}^{n} a_k \left(\sum_{i+m=n+k+l} x^i \otimes x^m \right) = \sum_{k=0}^{n} a_k \left(\sum_{r+m=n+k} x^{r+l} \otimes x^m \right) \\
&= \big(x^l \otimes 1\big) \sum_{k=0}^{n} a_k \left(\sum_{r+m=n+k} x^r \otimes x^m \right) = \big(x^l \otimes 1\big)\Delta(1).
\end{aligned}$$

Finally, we need to check the coassociativity axiom: Let $x^l \in \mathcal{A}$ with $l \geq 0$.

$$\begin{aligned}
(\Delta \otimes 1)(\Delta(x^l)) &= (\Delta \otimes 1)\left(\sum_{k=0}^{n} a_k \left(\sum_{i+j=n+k+l} x^i \otimes x^j\right)\right) = \sum_{k=0}^{n} a_k \left(\sum_{i+j=n+k+l} \Delta(x^i) \otimes x^j\right) \\
&= \sum_{k,m=0}^{n} a_k a_m \left(\sum_{i+j=n+k+l} \sum_{r+s=n+m+i} x^r \otimes x^s \otimes x^j\right) \\
&= \sum_{k,m=0}^{n} a_k a_m \left(\sum_{r+s+j=2n+m+k+l} x^r \otimes x^s \otimes x^j\right) \\
(1 \otimes \Delta)(\Delta(x^l)) &= (1 \otimes \Delta)\left(\sum_{k=0}^{n} a_k \left(\sum_{i+j=n+k+l} x^i \otimes x^j\right)\right) = \sum_{k=0}^{n} a_k \left(\sum_{i+j=n+k+l} x^i \otimes \Delta(x^j)\right) \\
&= \sum_{k,m=0}^{n} a_k a_m \left(\sum_{i+j=n+k+l} \sum_{r+s=n+m+j} x^i \otimes x^r \otimes x^s\right) \\
&= \sum_{k,m=0}^{n} a_k a_m \left(\sum_{r+s+j=2n+m+k+l} x^r \otimes x^s \otimes x^j\right).
\end{aligned}$$

Then the pair (A, Δ) is a nearly-Frobenius algebra. In particular we have that the coproduct Δ is a linear combination of the coproducts Δ_k defined by

$$\Delta_k(x^l) = \sum_{i+j=n+k+l} x^i \otimes x^j, \quad \text{for } k \in \{0, \dots, n\}$$

that is $\Delta = \sum_{k=0}^{n} a_k \Delta_k$ where $a_k \in \Bbbk$ for all $k \in \{1, \dots, n\}$.

Note that Δ_0 is the Frobenis coproduct of A where the trace map $\varepsilon : A \to \mathbb{C}$ is given by $\varepsilon(x^i) = \delta_{i,n}$. The other coproducts, Δ_k $k \neq 0$, do not come from a Frobenius algebra structure. That is, it does not exist a trace map $\varepsilon : A \to A \otimes A$ such that $(A, \Delta_k, \varepsilon)$ is a Frobenius algebra for $k = 1, \dots, n$.

$$m(\varepsilon \otimes 1)(\Delta_k(x^l)) = \sum_{i+j=n+k+l} \varepsilon(x^i) x^j,$$

with $j > l$, so $m(\varepsilon \otimes 1)\Delta_k(x^l) \neq x^l$.

Example 3.10. Let A be the algebra $\mathbb{C}[[x, x^{-1}]]$ of formal Laurent series. Consider the coproducts given by:

$$\Delta_j(x^i) = \sum_{k+l=i+j} x^k \otimes x^l.$$

These coproducts define nearly Frobenius structures that do not come from a Frobenius structure.

Example 3.11. The Poincaré algebra $A := H^*(M)$ of a non-compact manifold M is a nearly Frobenius algebra. Consider the diagram:

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & M \times M \arrow[d, "1\times\Delta"] \\ M \times M \arrow[r, "\Delta\times 1"'] & M \times M \times M \end{tikzcd}$$

Using transversality we have that:

$$(\Delta \times 1)^*(1 \times \Delta)^! = \Delta^!\Delta^*,$$

where $\Delta^* : H^*(M) \otimes H^*(M) = H^*(M \times M) \to H^*(M)$ is the map induced by the diagonal map in cohomology, and $\Delta^! : H^*(M) \to H^*(M) \otimes H^*(M)$ is the Gysin map of the diagonal map. Therefore

$$\big(\Delta^* \otimes 1\big)\big(1 \otimes \Delta^!\big) = \Delta^!\Delta^*.$$

Then $H^*(M)$ is an algebra with a coproduct which is a module homomorphism. For non-compact manifolds we can not assume the existence of a fundamental class in homology, so we can not integrate and we do not have a trace in cohomology.

An interesting family of examples of nearly-Frobenius algebras is the produced by quivers. In [AGL], Artenstein, Lanzilotta and the first author studied the nearly-Frobenius structures that admit these objects. We describe briefly these results (see [ASS06]).

Definition 3.11. A *quiver* $Q = \big(Q_0, Q_1, s, t\big)$ is a quadruple consisting of two sets: Q_0 (whose elements are called *points*, or *vertices*) and Q_1 (whose elements are called *arrows*), and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its *source* $s(\alpha) \in Q_0$ and its *target* $t(\alpha) \in Q_0$, respectively.

An arrow $\alpha \in Q_1$ of source $a = s(\alpha)$ and target $b = t(\alpha)$ is usually denoted by $\alpha : a \to b$. A quiver $Q = \big(Q_0, Q_1, s, t\big)$ is usually denoted briefly by $Q = (Q_0, Q_1)$ or even simply by Q. Thus, a quiver is nothing but an oriented graph without any restriction as to the number of arrows between two points, to the existence of loops or oriented cycles.

Definition 3.12. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $a, b \in Q_0$. A *path* of *length* $l \geq 1$ with source a and target b (or, more briefly, from a to b) is a sequence

$$(a|\alpha_1, \alpha_2, \ldots, \alpha_l|b),$$

where $\alpha_k \in Q_1$ for all $1 \leq k \leq l$, and we have $s(\alpha_1) = a$, $t(\alpha_k) = s(\alpha_{k+1})$ for each $1 \leq k < l$, and finally $t(\alpha_l) = b$. Such a path is denoted briefly by $\alpha_1\alpha_2 \ldots \alpha_l$.

Definition 3.13. Let Q be a quiver. The *path algebra* $\Bbbk Q$ is the $\Bbbk$-algebra whose underlying $\Bbbk$-vector space has as its basis the set of all paths $(a|\alpha_1, \alpha_2, \ldots, \alpha_l|b)$ of length $l \geq 0$ in Q and such that the product of two basis vectors $(a|\alpha_1, \alpha_2, \ldots, \alpha_l|b)$ and $(c|\beta_1, \beta_2, \ldots, \beta_k|d)$ of $\Bbbk Q$ is defined by

$$(a|\alpha_1, \alpha_2, \ldots, \alpha_l|b)(c|\beta_1, \beta_2, \ldots, \beta_k|d) = \delta_{bc}(a|\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_k|d),$$

where δ_{bc} denotes the Kronecker delta. In other words, the product of two paths $\alpha_1 \ldots \alpha_l$ and $\beta_1 \ldots \beta_k$ is equal to zero if $t(\alpha_l) \neq s(\beta_1)$ and is equal to the composed path $\alpha_1 \ldots \alpha_l\beta_1 \ldots \beta_k$ if $t(\alpha_l) = s(\beta_1)$. The product of basis elements is then extended to arbitrary elements of $\Bbbk Q$ by distributivity.

Example 3.12. If Q is the following quiver:

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha_1"] & \underset{2}{\bullet} \arrow[r, "\alpha_2"] & \underset{3}{\bullet} \arrow[r, "\alpha_3"] & \underset{4}{\bullet} \arrow[r, dotted, no head] & \underset{n\text{-}1}{\bullet} \arrow[r, "\alpha_{n\text{-}1}"] & \underset{n}{\bullet} \end{tikzcd}$$

Then the path algebra $A = \Bbbk Q$

$$\Bbbk Q = \langle e_1, e_2, \ldots, e_n, \alpha_i \ldots \alpha_{i+j} : i = 1, \ldots, n, j \geq 0 \rangle.$$

admits a unique nearly-Frobenius structure, where the coproduct is defines as follows

$$\begin{aligned} \Delta(e_1) &= a\alpha_1 \ldots \alpha_{n-1} \otimes e_1, \\ \Delta(e_n) &= ae_n \otimes \alpha_1 \ldots \alpha_{n-1}, \\ \Delta(e_i) &= a\alpha_i \ldots \alpha_{n-1} \otimes \alpha_1 \ldots \alpha_{i-1}, \\ \Delta(\alpha_i \ldots \alpha_j) &= a\alpha_i \ldots \alpha_{n-1} \otimes \alpha_1 \ldots \alpha_j, \end{aligned}$$

where $a \in \Bbbk$.

Theorem 3.14. *Let $A = \Bbbk Q$ with Q a finite, connected quiver with no oriented cycles. Then A has a nearly-Frobenius structure if and only if $Q = A_n$ with all the arrows in Q having the same orientation.*

If we introduce relations in the quiver Q then the nearly-Frobenius structures over Q are very interesting.

Proposition 3.15. *The path algebra associated to the quiver*

$$Q_R: \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \cdots\cdots \underset{m}{\bullet} \xrightarrow{\alpha_m} \underset{0}{\bullet} \xrightarrow{\beta_1} \underset{m+1}{\bullet} \cdots\cdots \bullet \xrightarrow{\beta_n} \underset{m+n}{\bullet}$$

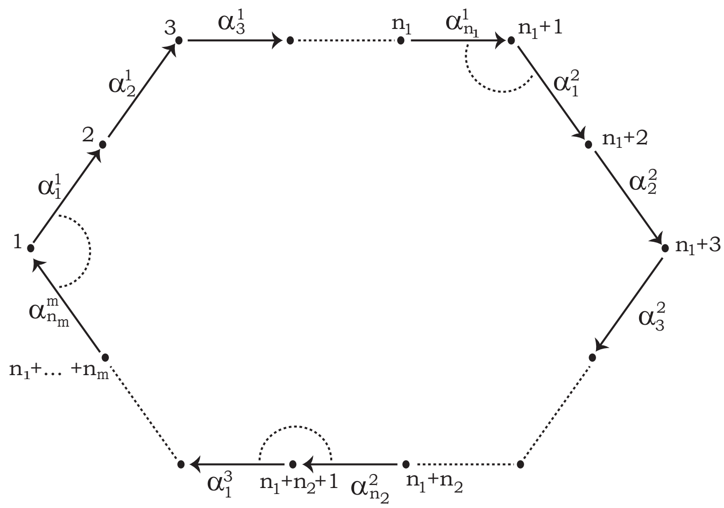
with the relation $\alpha_m\beta_1 = 0$, admits $mn+2$ independent nearly-Frobenius structures, these are

$$\begin{array}{rclrcl}
\Delta(e_1) & = & a\alpha_1 \dots \alpha_m \otimes e_1 & \Delta(e_{m+1}) & = & b\beta_2 \dots \beta_n \otimes \beta_1 \\
\vdots & & & \vdots & & \\
\Delta(e_i) & = & a\alpha_i \dots \alpha_m \otimes \alpha_1 \dots \alpha_{i-1} & \Delta(e_{m+i}) & = & b\beta_{i+1} \dots \beta_n \otimes \beta_1 \dots \beta_i \\
\vdots & & & \vdots & & \\
\Delta(e_m) & = & a\alpha_m \otimes \alpha_1 \dots \alpha_{m-1} & \Delta(e_{m+n}) & = & be_{m+n} \otimes \beta_1 \dots \beta_n \\
\Delta(\alpha_i \dots \alpha_j) & = & a\alpha_i \dots \alpha_m \otimes \alpha_1 \dots \alpha_j & \Delta(\beta_i \dots \beta_j) & = & b\beta_i \dots \beta_n \otimes \beta_1 \dots \beta_j
\end{array}$$

$$\Delta(e_0) = ae_0 \otimes \alpha_1 \dots \alpha_m + b\beta_1 \dots \beta_n \otimes e_0 + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\beta_1 \dots \beta_j \otimes \alpha_i \dots \alpha_m$$

where $a, b, c_{ij} \in \Bbbk$.

Theorem 3.16. *The path algebra A associated to the cyclic quiver Q*



with m *maximal paths of length* n_i, $i = 1, \dots, m$ *admits* R *nearly-Frobenius structures, where*

$$R = m + \sum_{i=1}^{m} n_i n_{i+1}$$

whit $n_{m+1} = m_1$.

The next results construct nearly-Frobenius structures in tensor algebras and quotient algebras.

Theorem 3.17. *If* (A, Δ_1) *and* (B, Δ_2) *are nearly-Frobenius algebras then* $(A \otimes B, \Delta)$ *is a nearly-Frobenius algebra where*

$$\Delta = (1 \otimes \tau \otimes 1) \circ (\Delta_1 \otimes \Delta_2), \quad \textit{with } \tau \textit{ is the transposition.}$$

Proof. The map Δ is coassociative because the external diagram is commutative since the internal diagrams commute:

$$\begin{tikzcd}
A \otimes B \arrow[r, "\Delta_1 \otimes \Delta_2"] \arrow[d, "\Delta_1 \otimes \Delta_2"'] & A \otimes A \otimes B \otimes B \arrow[r, "1 \otimes \tau \otimes 1"] \arrow[d, "\Delta_1 \otimes 1 \otimes \Delta_2 \otimes 1"] & (A \otimes B) \otimes (A \otimes B) \arrow[d, "\Delta_1 \otimes \Delta_2 \otimes 1 \otimes 1"] \\
A \otimes A \otimes B \otimes B \arrow[r, "1 \otimes \Delta_1 \otimes 1 \otimes \Delta_2"] \arrow[d, "1 \otimes \tau \otimes 1"'] & A \otimes A \otimes A \otimes B \otimes B \otimes B \arrow[r, "1 \otimes \tau \otimes 1"] \arrow[d, "1 \otimes \tau \otimes 1"] & A \otimes A \otimes B \otimes B \otimes A \otimes B \arrow[d, "1 \tau \otimes 1 \otimes 1 \otimes 1"] \\
A \otimes B \otimes B \otimes A \otimes B \arrow[r, "1 \otimes 1 \otimes 1 \otimes \Delta_1 \otimes \Delta_2"'] & A \otimes B \otimes A \otimes A \otimes B \otimes B \arrow[r, "1 \otimes 1 \otimes 1 \otimes 1 \otimes \tau \otimes 1"'] & A \otimes B \otimes A \otimes B \otimes A \otimes B
\end{tikzcd}$$

The linear map Δ satisfies the Frobenius identities because the next external diagram is commutative using that the internal diagrams commute:

$$\begin{tikzcd}
(A \otimes B) \otimes (A \otimes B) \arrow[r, "1 \otimes \tau \otimes 1"] \arrow[d, "\Delta_1 \otimes \Delta_2 \otimes 1"'] & A \otimes A \otimes B \otimes B \arrow[r, "m_1 \otimes m_2"] \arrow[d, "\Delta_1 \otimes 1 \otimes \Delta_2 \otimes 1"] & (A \otimes B) \arrow[d, "\Delta_1 \otimes \Delta_2"] \\
A \otimes A \otimes B \otimes B \otimes A \otimes B \arrow[r, "1 \otimes \tau \otimes 1"] \arrow[d, "1 \otimes \tau \otimes 1 \otimes 1"'] & A \otimes A \otimes A \otimes B \otimes B \otimes B \arrow[r, "1 \otimes m_1 \otimes 1 \otimes m_2"] \arrow[d, "1 \otimes \tau \otimes 1 \otimes 1"] & A \otimes A \otimes B \otimes B \arrow[d, "1 \tau \otimes 1"] \\
A \otimes B \otimes A \otimes B \otimes A \otimes B \arrow[r, "1 \otimes 1 \otimes \tau \otimes 1"'] & A \otimes B \otimes A \otimes A \otimes B \otimes B \arrow[r, "1 \otimes m_1 \otimes m_2"'] & A \otimes B \otimes A \otimes B
\end{tikzcd}$$

♣

Let be (A, Δ) a nearly-Frobenius algebra.

Definition 3.18. A linear subspace J in A is called a *nearly-Frobenius ideal* if

(a) J is an ideal of A and

(b) $\Delta(J) \subset J \otimes A + A \otimes J$.

Proposition 3.19. *Let be (A, Δ) a nearly-Frobenius algebra, J a nearly-Frobenius ideal and $p : A \to A/J$ the natural projection. Then A/J admits a unique nearly-Frobenius structure such that p is a coalgebra morphism.*

Proof. Since $(p \otimes p)\Delta(J) \subset (p \otimes p)\big(J \otimes A + A \otimes J\big) = 0$, by the universal property of the factor vector space it follows that there exists a unique morphism of vector spaces

$$\exists! \quad \overline{\Delta} : A/J \to A/J \otimes A/J$$

for which the diagram

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "\Delta"'] & A/J \arrow[d, "\overline{\Delta}\otimes\overline{\Delta}"] \\ A \arrow[r, "p\otimes p"'] & A/J \otimes A/J \end{tikzcd}$$

is commutative. This map is defined by $\overline{\Delta}(\overline{a}) = \sum \overline{a_1} \otimes \overline{a_2}$ where $\overline{a} = p(a)$, i.e. $\overline{\Delta} = (p \otimes p) \circ \Delta$.

The fact that $\big(\overline{\Delta} \otimes 1\big)\overline{\Delta}(\overline{a}) = \big(1 \otimes \overline{\Delta}\big)\overline{\Delta}(\overline{a}) = \sum \overline{a_1} \otimes \overline{a_2} \otimes \overline{a_3}$ follows immediately from the commutativity of the diagram.

The last step is to prove that the coproduct is a bimodule morphism:

$$\begin{tikzcd} A/J \otimes A/J \arrow[r, "\overline{m}"] \arrow[d, "\overline{\Delta}\otimes 1"'] & A/J \arrow[d, "\overline{\Delta}"] \\ A/J \otimes A/J \otimes A/J \arrow[r, "1\otimes\overline{m}"'] & A/J \otimes A/J \end{tikzcd} \qquad \begin{tikzcd} A/J \otimes A/J \arrow[r, "\overline{m}"] \arrow[d, "1\otimes\overline{\Delta}"'] & A/J \arrow[d, "\overline{\Delta}"] \\ A/J \otimes A/J \otimes A/J \arrow[r, "\overline{m}\otimes 1"'] & A/J \otimes A/J \end{tikzcd}$$

First note that $\Delta(a) = \sum a_1 \otimes a_2$, $\Delta(b) = \sum b_1 \otimes b_2$.

$\overline{\Delta}\overline{m}(\overline{a} \otimes \overline{b}) = \overline{\Delta}(p(ab)) = (p \otimes p)\Delta(ab) = (p \otimes p)\big((1 \otimes m)(\Delta \otimes 1)(a \otimes b)\big)$ $= (p \otimes p)\big(\sum a_1 \otimes a_2 b\big) = \sum \overline{a_1} \otimes \overline{a_2 b}$.

On the other hand

$(1 \otimes \overline{m})(\overline{\Delta} \otimes 1)(\overline{a} \otimes \overline{b}) = (1 \otimes \overline{m})\big(\sum \overline{a_1} \otimes \overline{a_2} \otimes \overline{b}\big) = \sum \overline{a_1} \otimes \overline{a_2 b}$. Then the first diagram is commutative.

♣

3.3 The Moduli Space of nearly Frobenius Structures of a Fixed Algebra A

Theorem 3.20. *Let A be a fixed $\mathbb{k}$-algebra. Then the set of nearly Frobenius coproducts of A making it into a nearly Frobenius algebra is a $\mathbb{k}$-vector space.*

Proof. Let be $B = \{e_i\}_{i \in I}$ a basis as $\mathbb{k}$-vector space of A and Δ_1 and Δ_2 two nearly Frobenius coproducts.

The product $e_i e_j$ can be expressed as $\sum_k c_{ij}^k e_k$, the value of the coproduct Δ_1 on 1 as $\sum_{i,j} d_{ij} e_i \otimes e_j$ and $\Delta_2(1) = \sum_{i,j} a_{ij} e_i \otimes e_j$.

Now, we consider the linear map $\Delta = \alpha\Delta_1 + \beta\Delta_2 : A \to A \otimes A$, with $\alpha, \beta \in \mathbb{k}$. First we prove that this map is an A-bimodule morphism.

$$\begin{aligned}(m \otimes 1)(1 \otimes \Delta) &= (m \otimes 1)\big(1 \otimes (\alpha\Delta_1 + \beta\Delta_2)\big)\\ &= \alpha(m \otimes 1)(1 \otimes \Delta_1) + \beta(m \otimes 1)(1 \otimes \Delta_2)\\ &= \alpha\Delta_1 m + \beta\Delta_2 m = \Delta m.\end{aligned}$$

In the same way $(1 \otimes m)(\Delta \otimes 1) = \Delta m$.

To prove the coassociativity, first we note that $\Delta_1(e_k) = \big(e_k \otimes 1\big)\Delta_1(1) = \Delta_1(1)\big(1 \otimes e_k\big)$ is equivalent to say that

$$\sum_l d_{ij} c_{jk}^l = \sum_l d_{jl} c_{kj}^i \tag{8}$$

and $\Delta_2(e_k) = \big(e_k \otimes 1\big)\Delta_2(1) = \Delta_2(1)\big(1 \otimes e_k\big)$ is equivalent to say that

$$\sum_l a_{ij} c_{jk}^l = \sum_l a_{jl} c_{kj}^i. \tag{9}$$

The coassociativity condition is

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$$

using the definition of Δ this is equivalent to say

$$\big((\Delta_2 \otimes 1)\Delta_1 - (1 \otimes \Delta_1)\Delta_2\big) + \big((\Delta_1 \otimes 1)\Delta_2 - (1 \otimes \Delta_2)\Delta_1\big) = 0.$$

Note that if we prove that $(\Delta_2 \otimes 1)\Delta_1(1) - (1 \otimes \Delta_1)\Delta_2(1) = 0$ then we can conclude that the map $(\Delta_2 \otimes 1)\Delta_1 - (1 \otimes \Delta_1)\Delta_2 = 0$. This is because

$$\begin{aligned}(\Delta_2 \otimes 1)\Delta_1(x) - (1 \otimes \Delta_1)\Delta_2(x) &= (\Delta_2 \otimes 1)\Delta_1(1)(1 \otimes x) - (1 \otimes \Delta_1)\Delta_2(1)(1 \otimes x) \\ &= \big((\Delta_2 \otimes 1)\Delta_1(1) - (1 \otimes \Delta_1)\Delta_2(1)\big)(1 \otimes x) = 0.\end{aligned}$$

$$(\Delta_2 \otimes 1)\Delta_1(1) = \sum_{j,k,l}\left(\sum_{i,l} d_{ij}a_{kl}c_{li}^{m}\right) e_k \otimes e_m \otimes e_j$$

and

$$(1 \otimes \Delta_1)\Delta_2(1) = \sum_{j,k,l}\left(\sum_{i,l} d_{ml}a_{ki}c_{li}^{j}\right) e_k \otimes e_m \otimes e_j$$

Using the equations 8 and 9 we see that

$$\sum_{i,l} d_{ml}a_{ki}c_{li}^{j} = \sum_{i,l} d_{ij}a_{kl}c_{li}^{m}.$$

If we change Δ_1 with Δ_2 we conclude that $(\Delta_1 \otimes 1)\Delta_2 - (1 \otimes \Delta_2)\Delta_1 = 0$.

♣

Definition 3.21. The Frobenius space associated to an algebra A is the vector of all the possible co-products Δ that make it into a nearly Frobenius algebra. Its dimension over $\Bbbk$ is called the *Frobenius* dimension of A.

3.4 Semisimple Algebras

In this section we study the particular case of semi-simple algebras.

Example 3.13. We consider a non-commutative field $\Bbbk$.
The linear map $\Delta : \Bbbk \to \Bbbk$ satisfies the Frobenius identities, then

$$\Delta(x) = \Delta(1)(1 \otimes x) = (x \otimes 1)\Delta(1), \quad \forall\, x \in \Bbbk.$$

If we define $\Delta(1) = a1 \otimes 1 = a \otimes 1$, with $a \in \Bbbk$ then

$$\Delta(x) = a \otimes x = ax \otimes 1 = xa \otimes 1 \Leftrightarrow ax = xa \Leftrightarrow a \in Z(\Bbbk).$$

Finally, we need to prove that this coproduct is coassociative:

$$\begin{aligned}(\Delta \otimes 1)\Delta(x) &= \Delta(1) \otimes ax &= a \otimes 1 \otimes ax &= a^2x \otimes 1 \otimes 1 \\ (1 \otimes \Delta)\Delta(x) &= ax \otimes \Delta(1) &= ax \otimes a \otimes 1 &= axa \otimes 1 \otimes 1\end{aligned}$$

As $a \in Z(\Bbbk)$ we have that $a^2x = axa$, then $(\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x)$, $\forall x \in \Bbbk$. Therefore the algebra $\Bbbk$ is a nearly-Frobenius algebra and we have as much nearly-Frobenius structures as elements in the center of $\Bbbk$.
Note that these structures come from Frobenius structures, the trace map $\varepsilon : \Bbbk \to \Bbbk$ is $\varepsilon(1) = 1$.

Example 3.14. Let be A the matrix algebra $M_{n\times n}(\Bbbk)$, with $\Bbbk$ a commutative field. We consider the canonical basis of A, $\{E_{ij} : i, j = 1, \dots, n\}$, where $E_{ij} = (e_{kl})_{kl}$ with

$$e_{kl} = \begin{cases} 1 & \text{if } k = i, l = j \\ 0 & \text{in other case} \end{cases}.$$

First note that $E_{ij}E_{kl} = \begin{cases} E_{il} & \text{if } j = k \\ 0 & \text{in other case} \end{cases}$. In particular $E_{ii}E_{ij} = E_{ij}$ and $E_{ij}E_{jj} = E_{ij}$, then

$$\Delta(E_{ij}) = \Delta(E_{ij})(1 \otimes E_{jj}) = (E_{ii} \otimes 1)\Delta(E_{ij})$$

and

$$\Delta(E_{ij}) = \Delta(E_{ii})(1 \otimes E_{ij}) = (E_{ij} \otimes 1)\Delta(E_{jj}).$$

The last equations imply that

$$\Delta(E_{ij}) = \sum_{k,l=1}^{n} a^{ij}_{kl} E_{ik} \otimes E_{lj} = \sum_{k,l=1}^{n} a^{ii}_{kl} E_{ik} \otimes E_{lj} = \sum_{k,l=1}^{n} a^{jj}_{kl} E_{ik} \otimes E_{lj},$$

then $a^{ij}_{kl} = a^{ii}_{kl} = a^{jj}_{kl}$, for all $k, l = 1 \dots, n$. As a consequence we have that

$$\Delta(E_{ij}) = \sum_{k,l=1}^{n} a_{kl} E_{ik} \otimes E_{lj}, \quad \forall\, i, j.$$

Finally we need to check that this coproduct is coassociative:

$$\begin{aligned}
(\Delta \otimes 1)\Delta(E_{ij}) &= \sum_{k,l=1}^{n} a_{kl}\Delta(E_{ik}) \otimes E_{lj} &&= \sum_{k,l=1}^{n}\sum_{r,s=1}^{n} a_{kl}a_{rs} E_{ir} \otimes E_{sk} \otimes E_{lj} \\
(1 \otimes \Delta)\Delta(E_{ij}) &= \sum_{r,s=1}^{n} a_{rs} E_{ir} \otimes \Delta(E_{sj}) &&= \sum_{r,s=1}^{n}\sum_{k,l=1}^{n} a_{rs}a_{kl} E_{ir} \otimes E_{sk} \otimes E_{lj}
\end{aligned}$$

As $\Bbbk$ is commutative we have that $(\Delta \otimes 1)\Delta(E_{ij}) = (1 \otimes \Delta)\Delta(E_{ij})$.

Note that $M_{n\times n}(\Bbbk)$ admits $n \times n$ independent coproducts, one for each a_{kl}, that is

$$\Delta(E_{ij}) = \sum_{k,l=1}^{n} a_{kl}\Delta_{kl}(E_{ij}), \text{ where } \Delta_{kl}(E_{ij}) = E_{ik} \otimes E_{lj}.$$

Example 3.15. Let G be a cyclic finite group. The group $\Bbbk[G]$ is a nearly-Frobenius algebra. A basis, as vector space, of $\Bbbk[G]$ is $\{g^i : i = 1, \dots, n\}$ where $|G| = n$. As before, if we determine the value of the coproduct in the unit of the group we have the value over all element of the algebra.

A general expression of $\Delta(1)$ is

$$\Delta(1) = \sum_{i,j=1}^{n} \alpha_{ij} g^i \otimes g^j.$$

Using that $\Delta(g^k) = \Delta(1)(1 \otimes g^k) = (g^k \otimes 1)\Delta(1)$ we have that

$$\sum_{i,j^1}^{n} \alpha_{ij} g^{k+i} \otimes g^j = \sum_{i,j=1}^{n} \alpha_{ij} g^i \otimes g^{j+k}.$$

then $\alpha_{i-kj} = \alpha_{ij-k}$, also $\alpha_{1j-1} = \alpha_{nj}$ and $\alpha_{in} = \alpha_{i-11}$. This permit us to express the coproduct as

$$\Delta(1) = \sum_{i=2}^{n} \alpha_i \left\{ \sum_{k=1}^{i-1} g^k \otimes g^{i-k} + \sum_{k=i}^{n} g^k \otimes g^{n+i-k} \right\}.$$

This implies that

$$\begin{aligned}
\Delta(g^k) &= \Delta(1)(1 \otimes g^k) \\
&= \sum_{i=2}^{n} \alpha_i \left\{ \sum_{k=1}^{i-1} g^{k+l} \otimes g^{i-k} + \sum_{k=i}^{n} g^{k+l} \otimes g^{n+i-k} \right\} \\
\Delta(g^k) &= (g^k \otimes 1)\Delta(1) \\
&= \sum_{i=2}^{n} \alpha_i \left\{ \sum_{k=1}^{i-1} g^{k} \otimes g^{i+l-k} + \sum_{k=i}^{n} g^{k} \otimes g^{n+i+l-k} \right\}.
\end{aligned}$$

These expressions of $\Delta(g^k)$ coincide by a simple change of variables. An a similar reasoning permit us to prove the coassociativity of the coproduct.

Theorem 3.22. *1. Let A be a $\Bbbk$-algebra. Then, A is a nearly-Frobenius algebra if and only if A^{op} is a nearly-Frobenius algebra.*

2. Let $A_1, \dots, A_n$ be $\Bbbk$-algebras and $A = A_1 \times A_2 \times \cdots \times A_n$. Then A is a nearly-Frobenius algebra if and only if $A_1, \dots, A_n$ are nearly-Frobenius algebras.

Proof. 1. The opposite algebra A^{op} of the algebra A is the algebra with the same set of elements and the same addition but with multiplication $*$ given by $a * b = ba$ for a and b in A.

We define the coproduct $\Delta^{op} : A^{op} \to A^{op} \otimes A^{op}$ as $\tau \circ \Delta$, where Δ is the coproduct in A and τ is the transposition. It is clear that Δ^{op} is coassociative because Δ is coassociative. We need to check that Δ^{op} is morphism of A^{op}-bimodule.

$$\Delta^{op}(a*b) = \Delta^{op}(ba) = \tau\big(\Delta(ba)\big) = \sum a_2 \otimes b a_1 = \sum a_2 \otimes a_1 * b = \big(1 \otimes *\big)\big(\Delta^{op}(a) \otimes b\big)$$

$$\Delta^{op}(a*b) = \Delta^{op}(ba) = \tau\big(\Delta(ba)\big) = \sum b_2 a \otimes b_1 = \sum a * b_2 \otimes b_1 = \big(* \otimes 1\big)\big(a \otimes \Delta^{op}(b)\big)$$

2. First, we suppose that $A_1, \dots, A_n$ are nearly-Frobenius algebras and $\Delta_1 \dots, \Delta_n$ are the associated coproducts.

Note that a direct product for a finite index is identical to the direct sum. We can suppose that $A = A_1 \oplus \cdots \oplus A_n$ in $\mathrm{Vect}_{\Bbbk}$ and let $q_i : A_i \to \bigoplus_{i=1}^{n} A_i$ be the canonical injections. Then there exists a unique morphism Δ in $\mathrm{Vect}_{\Bbbk}$ such that the diagram

$$\begin{tikzcd}
A_j \arrow[r, "q_j"] \arrow[d, "\Delta_j"'] & \bigoplus_{i=1}^{n} A_i \arrow[d, "\Delta"] \\
A_j \otimes A_j \arrow[r, "q_j \otimes q_j"'] & \big(\bigoplus_{i=1}^{n} A_i\big) \otimes \big(\bigoplus_{i=1}^{n} A_i\big)
\end{tikzcd}$$

commute.

The coassociativity is a consequence of the commutativity of the cube

$$
\begin{tikzcd}
& A \otimes A \otimes A & & A \otimes A \arrow[ll, "\Delta\otimes 1"'] \\
A_i \otimes A_i \otimes A_i \arrow[ur, "q_i\otimes q_i\otimes q_i"] & & A_i \otimes A_i \arrow[ll, "\Delta_i\otimes 1"'] \arrow[ur, "q_i\otimes q_i"] & \\
& A \otimes A \arrow[uu, "1\otimes\Delta"] & & A \arrow[ll, "\Delta"'] \arrow[uu, "\Delta"'] \\
A_i \otimes A_i \arrow[uu, "1\otimes\Delta_i"] \arrow[ur, "q_i\otimes q_i"'] & & A_i \arrow[ll, "\Delta_i"] \arrow[uu, "\Delta_i"] \arrow[ur, "q_i"'] &
\end{tikzcd}
$$

To prove the Frobenius identities, first we note that the diagram

$$
\begin{tikzcd}
A \arrow[r, "p_i"] \arrow[d, "\Delta"'] & A_i \arrow[d, "\Delta_i"] \\
A \otimes A \arrow[r, "p_i\otimes p_i"'] & A_i \otimes A_i
\end{tikzcd}
$$

commute, where $p_j : \bigoplus_{i=1}^{n} A_i \to A_j$ is the canonical projection. This implies that the next cube commute.

$$
\begin{tikzcd}
& A \otimes A \arrow[rr, "m"] \arrow[dl, "p_i\otimes p_i"'] \arrow[dd, "\Delta\otimes 1"' near start] & & A \arrow[dl, "p_i"] \arrow[dd, "\Delta"] \\
A_i \otimes A_i \arrow[rr, "m_i" near end] \arrow[dd, "\Delta_i\otimes 1"] & & A_i \arrow[dd, "\Delta_i"' near start] & \\
& A \otimes A \otimes A \arrow[rr, "1\otimes m" near start] \arrow[dl, "p_i\otimes p_i\otimes p_i"'] & & A \otimes A \arrow[dl, "p_i\otimes p_i"'] \\
A_i \otimes A_i \otimes A_i \arrow[rr, "1\otimes m_i"] & & A_i \otimes A_i &
\end{tikzcd}
$$

Then (A, Δ) is a nearly-Frobenius algebra.

Now, we suppose that $A = \bigoplus_{i=1}^{n} A_i$ is a nearly-Frobenius algebra and $\Delta : A \to A \otimes A$ is the associated coproduct.

We define the linear map $\Delta_i : A_i \to A_i \otimes A_i$ as the composition

$$A_i \xrightarrow{q_i} A \xrightarrow{\Delta} A \otimes A \xrightarrow{p_i \otimes p_i} A_i \otimes A_i$$

By the universal property of the coproduct $A = \bigoplus_{i=1}^{n} A_i$ we have that the next diagram is commutative

$$\begin{tikzcd} A_i \arrow[r, "q_i"] \arrow[d, "\Delta_i"'] & A \arrow[d, "\Delta"] \\ A_i \otimes A_i \arrow[r, "q_i \otimes q_i"'] & A \otimes A \end{tikzcd}$$

Then $\Delta(x_1, \ldots, x_n) = \sum_{i=1}^{n} (q_i \otimes q_i)\Delta_i(x_i)$. It can be checked immediately looking at this expression that (A_i, Δ_i) is a nearly-Frobenius algebra.

♣

Corollary 3.23. *If* $\mathsf{char}(\Bbbk)$ *does not divide the order of* G*, then* $\Bbbk[G]$ *is a nearly-Frobenius algebra.*

Proof. Applying the Maschke's theorem we have that $\Bbbk[G]$ is semisimple, then it is a product of simple algebras $M_{n_i \times n_i}(\Bbbk)$. Therefore, by the Theorem 3.22, we conclude that $\Bbbk[G]$ is a nearly-Frobenius algebra. Even more we can determine all the nearly-Frobenius structures that it admits.

♣

From what we have seen we conclude that in the case of semi-simple algebras the Frobenius space of A is a vector space of dimension equal to the dimension of A, and that it has a one dimensional subspace (minus the origin) of *bona fide* Frobenius structures.

4 (Non-compact) Calabi-Yau Categories

A *2-dimensional open-closed topological field theory* (2D O-C TFT) is a generalization of a 2D TFT. Now the category of cobordism is modified in the sense the boundary objects are compact, oriented, one-manifolds, X, together with a labeling of the components of the boundary, ∂X, by objects of a $\mathbb{C}$-linear category $\mathcal{B}$, see figure 1. You can think of such objects as labels, or colors. So now the boundary of a surface is coloured by objects of $\mathcal{B}$, and the color black. The morphisms generalize the usual notion of a cobordism between manifolds with boundary, but with the additional data of the labeling category $\mathcal{B}$. A cobordism Σ_{X_1,X_2} between two objects X_1 and X_2 is an oriented surface Σ, whose boundary is partitioned into three parts: the incoming boundary $\partial_{in}\Sigma$ which is identified with X_1, the outgoing boundary $\partial_{out}\Sigma$ which is identified with X_2, and the remaining part of the boundary is referred as the "free part" $\partial_{free}\Sigma$ whose path components are labeled by objects of $\mathcal{B}$. Note that $\partial_{free}\Sigma$ is a cobordism between ∂X_1 and ∂X_2, which preserves the labeling, see figure 2.

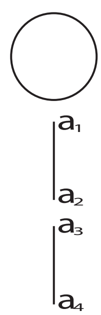


Figure 1: A one manifold with labels $a_i \in \mathrm{Obj}(\mathcal{B})$.

A monoidal functor from this category to the category of complex vector spaces will be called a (1+1)-dimensional *open-closed topological fiel theory.* We write A for the vector space associated to the standard circle S^1, and $\mathcal{O}_{ab} = \mathrm{Hom}(a, b)$ for the vector space associated to the interval $[0, 1]$, with ends labeled by $a, b \in \mathrm{Obj}(\mathcal{B})$.

You may imagine that such a bordism represents the evolution of closed and open strings in time, and that the labels are boundary conditions on the open strings.

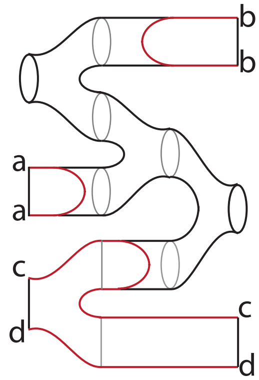


Figure 2: An open-closed cobordism.

4.1 Algebraic Structure of the Moore-Segal Formalism for Compact Backgrounds.

Recall that an ordinary closed string TFT is the same as a Frobenius algebra. Moore and Segal [MS] consider 2D O-C TFTs and prove that to have such a theory is the same as to have a *Calabi-Yau category* also called a *Frobenius structure* whose definition is as follows:

A *Frobenius structure* consists of the following algebraic data:

1. $(A, \cdot, \Delta_A, 1_A)$ is a commutative Frobenius algebra.
2. A $\mathbb{C}$-linear category $\mathcal{B}$, where $\mathcal{O}_{ab} = \mathrm{Hom}(a, b)$ for $a, b \in \mathcal{B}$.

2a. With associative linear maps η^b_{ac} and units u_a

$$\eta^b_{ac} : \mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}, \tag{10}$$

$$u_a : \mathbb{C} \to \mathcal{O}_{aa}. \tag{11}$$

2b. The spaces $\mathcal{O}_{aa}$ have nondegenerate traces

$$\Theta_a : \mathcal{O}_{aa} \to \mathbb{C}. \tag{12}$$

In particular, each $\mathcal{O}_{aa}$ is not necessarily a commutative Frobenius algebra.

2c. Moreover,

$$\begin{array}{ccccc} \mathcal{O}_{ab} \otimes \mathcal{O}_{ba} & \xrightarrow{\eta^b_{aa}} & \mathcal{O}_{aa} & \xrightarrow{\Theta_a} & \mathbb{C} \\ \mathcal{O}_{ba} \otimes \mathcal{O}_{ab} & \xrightarrow{\eta^a_{bb}} & \mathcal{O}_{bb} & \xrightarrow{\Theta_b} & \mathbb{C} \end{array} \tag{13}$$

are perfect pairings with

$$\Theta_a(\psi_1\psi_2) = \Theta_b(\psi_2\psi_1) \tag{14}$$

for $\psi_1 \in \mathcal{O}_{ab}$, and $\psi_2 \in \mathcal{O}_{ba}$.

3. There are linear maps

$$\iota_a : A \to \mathcal{O}_{aa}, \quad \iota^a : \mathcal{O}_{aa} \to A \tag{15}$$

such that

3a. ι_a is an algebra homomorphism

$$\iota_a(\phi_1\phi_2) = \iota_a(\phi_1)\iota_a(\phi_2), \tag{16}$$

3b. the identity is preserved

$$\iota_a(1_A) = 1_a. \tag{17}$$

3c. Moreover, ι_a is central in the sense that

$$\iota_a(\phi)\psi = \psi\iota_b(\phi), \tag{18}$$

for all $\phi \in A$ and $\psi \in \mathcal{O}_{ab}$.

3d. ι_a and ι^a are adjoint

$$\Theta_A(\iota^a(\psi)\phi) = \Theta_a(\psi\iota_a(\phi)).$$

3e. The "Cardy conditions". Define the map

$$\pi^a_b := \eta^a_{bb} \circ \tau \circ \Delta^b_{aa} : \mathcal{O}_{aa} \to \mathcal{O}_{bb},$$

where $\tau : \mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \to \mathcal{O}_{ba} \otimes \mathcal{O}_{ab}$ is the transposition map. We require the "Cardy condition":

$$\pi^a_b = \iota_b \circ \iota^a. \tag{19}$$

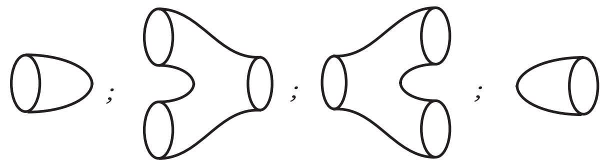


Figure 3: Four diagrams defining the Frobenius structure.

4.2 Topological Interpretation

For the case of a closed 2D TFT the Frobenius structure is provided by the diagrams in Fig. 3. The consistency conditions follow from Fig. 4. In the open case, entirely analogous considerations lead to the construction of a non necessarily commutative Frobenius algebra in the open sector. The basic data are summarized in Fig. 5. The fact that the traces are dual pairings follows from Fig. 6. The new ingredients in the open-closed theory are the open to closed and closed to open transitions. in 2D TFT these are the maps ι_a, ι^a. they are represented by Fig. 7. There are five new consistency conditions associated with the open-closed transitions. They are illustrated in Fig. 8 to Fig 13.

Theorem 4.1. *There is a one-to-one correspondence between (1+1)-dimensional Open-Closed Topological Field Theories and Frobenius structures.*

The proof of this theorem is a slightly more elaborate than that of the theorem relating commutative Frobenius algebras with TQFTs, but the basic ideas are all the same. The interested reader can find a full account of this proof in [MS].

4.3 Example: Representations of a Finite Group G

A simple example of an open-closed TFT is the associated to a finite group G. Where the category $\mathcal{B}$ is the category $\mathcal{R}ep(G)$ of finite representations of G. If $E \in \mathrm{Obj}\big(\mathcal{R}ep(G)\big)$ the trace $\theta_E : \mathcal{O}_{EE} \to \mathbb{C}$ takes $\psi : E \to E$ to $\frac{1}{|G|}\,\mathrm{tr}(\psi)$.
The algebra A is the center of the group algebra $\mathbb{C}[G]$ such that

$$\begin{aligned} \iota_E : Z\big(\mathbb{C}[G]\big) &\to \mathcal{O}_{EE}, \\ \sum_g \alpha_g g &\mapsto \sum_g \alpha_g \rho_g \end{aligned}$$

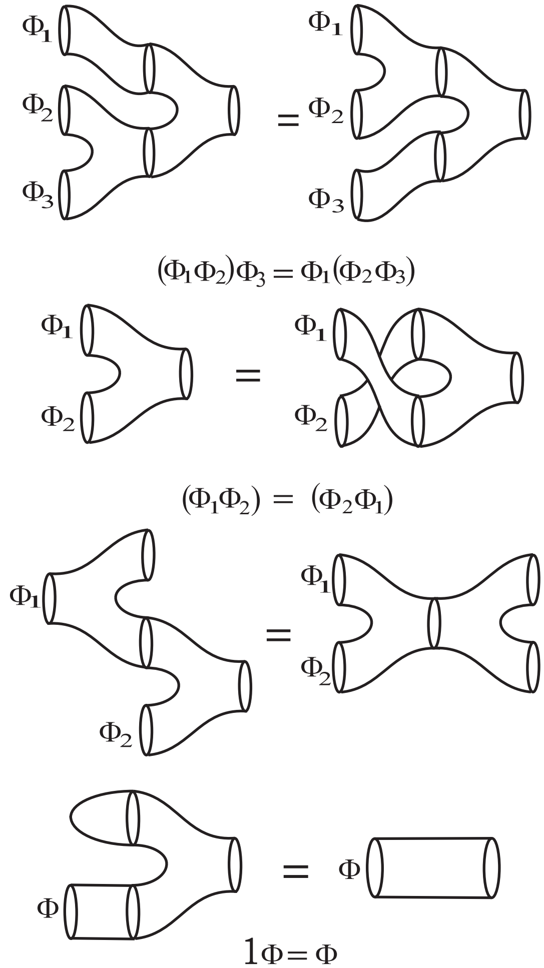


Figure 4: Associativity, commutativity, Abrams condition and unit constraints in the closed case.

$$\iota^{E} : \mathcal{O}_{EE} \to Z\big(\mathbb{C}[G]\big),$$
$$\psi : E \to E \mapsto \sum_{g} \mathrm{tr}\big(\psi g|_{E}\big) g$$

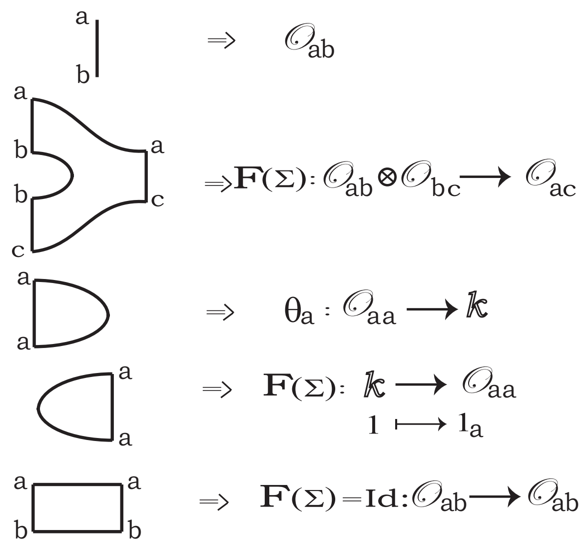


Figure 5: Basic data for the open theory.

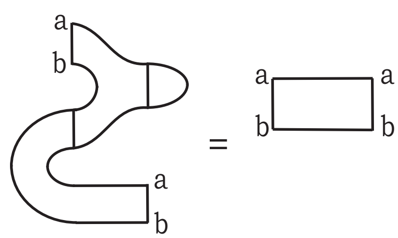


Figure 6: Assuming that the strip corresponds to the identity morphism we must have perfect pairings.

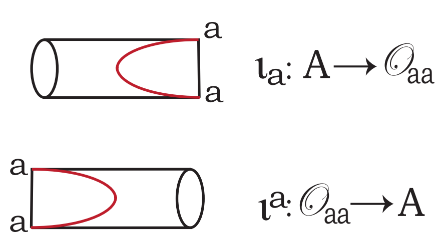


Figure 7: Two ways of representing open to closed and closed to open transitions.

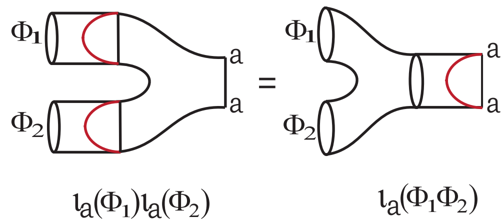


Figure 8: ι_a is a homomorphism.

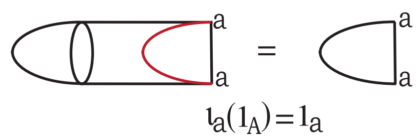


Figure 9: ι_a preserves the identity.

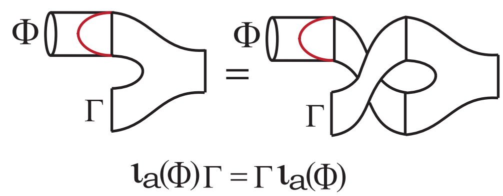


Figure 10: ι_a maps into the center of $\mathcal{O}_{aa}$.

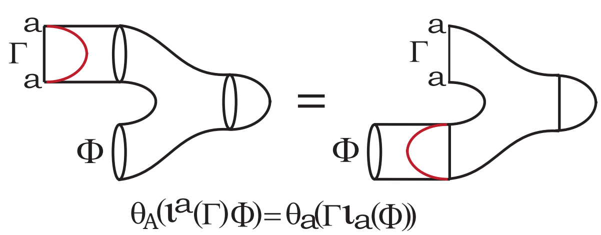


Figure 11: ι^a is the adjoint of ι_a.

and the trace

$$\theta_{Z(\mathbb{C}[G])} : Z(\mathbb{C}[G]) \to \mathbb{C}$$
$$\sum_g \alpha_g g \mapsto \frac{\alpha_1}{|G|}.$$

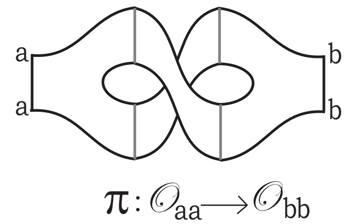


Figure 12: The double-twist diagram defines the map $\pi^a_b : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$.

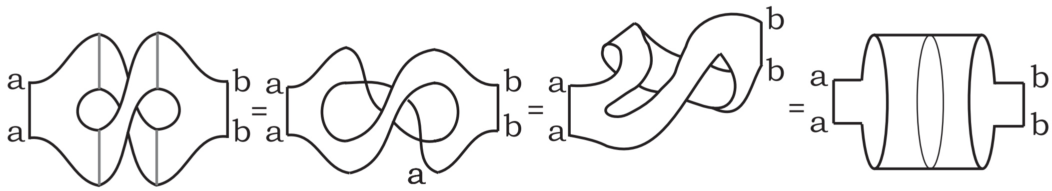


Figure 13: The Cardy-condition is expressing the factorization of the double-twist diagram in the closed string channel.

The next step is to verify the axioms.

1. $\left(Z(\mathbb{C}[G]), \theta_{Z(\mathbb{C}[G])}, 1_{Z(\mathbb{C}[G])}\right)$ is a Frobenius algebra.
   Let $I \subset \ker\left(\theta_{Z(\mathbb{C}[G])}\right)$ be an ideal of $Z(\mathbb{C}[G])$, and $\sum_g \alpha_g g \in I$. Then $\theta_{Z(\mathbb{C}[G])}\left(\sum_g \alpha_g g\right) = \frac{\alpha_1}{|G|} = 0$, hence $\alpha_1 = 0$. If $h \in G$ we have $\sum_g \alpha_g g h^{-1} \in I$,thus $\theta_{Z(\mathbb{C}[G])}\left(\sum_g \alpha_g g h^{-1}\right) = \frac{\alpha_h}{|G|} = 0$. For this reason $\alpha_h = 0$ for any $h \in G$. Then $I = \{0\}$.

2a. Notation $\mathcal{O}_{ij} = \mathrm{Hom}(E_i, E_j) = \begin{cases} \mathbb{C}\,\mathrm{Id}_{E_i} & \text{if } i = j, \\ 0 & \text{in other case.} \end{cases}$
   Then $\mathcal{O}_{ij} \otimes \mathcal{O}_{jk} \to \mathcal{O}_{ik}$ is zero except for $i = j = k$. In this case

$$\begin{aligned} \mathcal{O}_{ii} \otimes \mathcal{O}_{ii} &\to \mathcal{O}_{ii} \\ \lambda\,\mathrm{Id} \otimes \mu\,\mathrm{Id} &\mapsto \lambda\mu\,\mathrm{Id} \end{aligned}$$

2b. The trace $\theta_i : \mathcal{O}_{ii} \to \mathbb{C}$ is nondegenerate. Note that if $\psi \in \mathcal{O}_{ii}$ then there exists $\lambda \in \mathbb{C}$ such that $\psi = \lambda\,\mathrm{Id}_i$, hence $\ker(\theta_i) = \{0\}$.

2c. First, suppose that $i \neq j$ then

$$\mathcal{O}_{ij} \otimes \mathcal{O}_{ji} \to \mathcal{O}_{ii} \xrightarrow{\theta_i} \mathbb{C},$$
$$\mathcal{O}_{ji} \otimes \mathcal{O}_{ij} \to \mathcal{O}_{jj} \xrightarrow{\theta_j} \mathbb{C}$$

we have $\theta_i(\psi\phi) = 0 = \theta_j(\phi\psi)$.

If $i = j$ then $\mathcal{O}_{ii} \otimes \mathcal{O}_{ii} \to \mathcal{O}_{ii} \xrightarrow{\theta_i} \mathbb{C}$. In this case $\psi = \lambda\,\mathrm{Id}$ and $\phi = \mu\,\mathrm{Id}$, hence $\psi\phi = \phi\psi$, and as a consequence $\theta_i(\psi\phi) = \theta_i(\phi\psi)$.

3a. ι_E is an algebra homomorphism.

$$\iota_E\big((\sum_g \alpha_g g)(\sum_h \beta_h h)\big) = \iota_E\big(\sum \alpha_g \beta_h gh\big) = \sum \alpha_g \beta_h \rho_{gh}$$
$$\iota_E\big(\sum_g \alpha_g g\big)\iota_E\big(\sum_h \beta_h h\big) = \sum_g \alpha_g \rho_g \sum_h \alpha_h \rho_h = \sum \alpha_g \beta_h \rho_g \rho_h$$

This expressions are the same because ρ is a group homomorphism.

3b. The identity is preserved by definition ($\iota_E(e) = \mathrm{Id}_E$).

3c. The linear map ι_E is central i.e. $\iota_E\big(\sum_g \alpha_g g\big)\psi = \psi\iota_F\big(\sum_g \alpha_g g\big)$ with $\psi \in \mathcal{O}_{EF}$.
If $\psi \in \mathcal{O}_{ij}$, then $\psi = 0$ for $i \neq j$ or $\psi = \lambda\,\mathrm{Id}_i$ for $i = j$.
If $i \neq j$ the statement is true. Now we see the case $i = j$, but since we have $\psi = \lambda\,\mathrm{Id}$ then it follows.

3d. The linear maps ι_E and ι^E are adjoint, i.e. $\theta_{Z(\mathbb{C}[G])}\big(\iota^E(\psi)\phi\big) = \theta_E\big(\psi\iota_E(\phi)\big)$.

$$\begin{aligned}
\theta_E\big(\psi\iota_E(\phi)\big) &= \theta_E\big(\psi \sum_g \alpha_g \rho_g\big) = \theta_E\big(\sum_g \alpha_g \psi \rho_g\big) \\
&= \frac{1}{|G|}\operatorname{tr}\big(\sum_g \alpha_g \psi \rho_g\big) = \frac{1}{|G|}\sum_g \alpha_g \operatorname{tr}\big(\psi\rho_g\big) \\
\theta_{Z(\mathbb{C}[G])}\big(\iota^E(\psi)\phi\big) &= \theta_{Z(\mathbb{C}[G])}\big(\sum_g \alpha_g \iota^E(\psi) g\big) = \theta_{Z(\mathbb{C}[G])}\big(\sum_g \alpha_g \operatorname{tr}(\psi\rho_g)\big) \\
&= \frac{1}{|G|}\sum_g \alpha_g \operatorname{tr}\big(\psi\rho_g\big)
\end{aligned}$$

3e. First

$$\begin{tikzcd}
\mathcal{O}_{ii} \arrow[r] \arrow[rrr, bend left, "\pi^i_j"] & \mathcal{O}_{ij} \otimes \mathcal{O}_{ji} \arrow[r, "\tau"] & \mathcal{O}_{ji} \otimes \mathcal{O}_{ij} \arrow[r] & \mathcal{O}_{jj}
\end{tikzcd}$$

If $i \neq j$ then $\pi^i_j = 0$. If $i = j$ we have

$$\mathcal{O}_{ii} \to \mathcal{O}_{ii} \otimes \mathcal{O}_{ii} \to \mathcal{O}_{ii} \otimes \mathcal{O}_{ii} \to \mathcal{O}_{ii}$$

$$\lambda \operatorname{Id} \mapsto \lambda \operatorname{Id} \otimes \frac{|G|}{n_i} \operatorname{Id} \mapsto \frac{|G|}{n_i} \operatorname{Id} \otimes \lambda \operatorname{Id} \mapsto \frac{|G|}{n_i} \lambda \operatorname{Id}$$

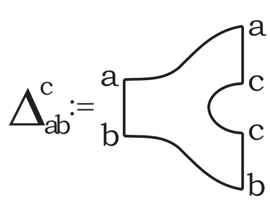
Then $\pi^i_i(\lambda \operatorname{Id}) = \frac{|G|}{n_i}\lambda$, where $n_i = \dim E_i$.
Now we need to study $\iota_i \iota^j$.
The map $\iota^i : \mathcal{O}_{ii} \to Z(\mathbb{C}[G])$ takes $\lambda \operatorname{Id}$ to $\sum_g \operatorname{tr}(\lambda \rho_g) g = \lambda \sum_g \chi_i(g) g$ and $\iota_j : Z(\mathbb{C}[G]) \to \mathcal{O}_{jj}$ takes $\sum_g \alpha_g g$ to $\sum_g \alpha_g \rho_g$. Consequently $\iota_i \iota^j(\lambda \operatorname{Id}) = \lambda \sum_g \chi_i(g) \rho_g : E_j \to E_j$.
For the map $\rho_g : E_j \to E_j$, with E_j an irreducible representation, there exists $\mu \in \mathbb{C}$ such that $\rho_g = \mu \operatorname{Id}_i$. Hence $\operatorname{tr}(\rho_g) = \mu \dim E_j$, so $\mu = \frac{1}{n_j}\chi_j(g)$. For this $\iota_i \iota^j(\lambda \operatorname{Id}) = \lambda \sum_g \chi_i(g) \frac{1}{n_j} \chi_j(g) \operatorname{Id}_j = \frac{\lambda}{n_j} \sum_g \chi_i(g) \chi_j(g) \operatorname{Id}_j = \frac{\lambda}{n_j} \delta_{i^* j} |G| \operatorname{Id}_j$.
Using that the representations are real, we have that $\chi_i(g) = \overline{\chi_i}(g)$, then $\delta_{i^* j} = \delta_{ij}$ and the maps coincide.

4.4 2D Open-Closed TFT with Positive Boundary

In a 2D open-closedTFT we have a family of maps $\Delta^c_{ab} : \mathcal{O}_{ab} \to \mathcal{O}_{ac} \otimes \mathcal{O}_{cb}$, which are called *coproducts*, with $a, b, c \in \mathcal{B}$. These are defined by the commutativity of the square

$$\begin{tikzcd}
\mathcal{O}_{ab} \arrow[rr, "\Delta^c_{ab}"] \arrow[d, "\Phi_{ab}"'] & & \mathcal{O}_{ac} \otimes \mathcal{O}_{cb} \\
\mathcal{O}^*_{ba} \arrow[r, "\eta^{c*}_{ba}"'] & \mathcal{O}^*_{bc} \otimes \mathcal{O}^*_{ca} \arrow[r, "\tau"'] & \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{bc} \arrow[u, "\Phi^{-1}_{ac} \otimes \Phi^{-1}_{cb}"']
\end{tikzcd}$$

where $\Phi_{ab} : \mathcal{O}_{ab} \to \mathcal{O}^*_{bc}$ is $\Phi_{ab}(x)(y) = \Theta_a(xy)$, for $x \in \mathcal{O}_{ab}$ and $y \in \mathcal{O}_{ba}$.



It is clear that Δ^c_{ab} is a linear map.

Remark 4.2. The spaces $\mathcal{O}_{\mathfrak{ab}}$ are of finite dimension with bilinear maps

$$\eta^{\mathfrak{c}}_{\mathfrak{ab}} : \mathcal{O}_{\mathfrak{ac}} \otimes \mathcal{O}_{\mathfrak{cb}} \to \mathcal{O}_{\mathfrak{ab}}.$$

In the case $\mathfrak{a} = \mathfrak{b} = \mathfrak{c}$, $\eta^{\mathfrak{a}}_{\mathfrak{aa}}$ is an associative product.
These maps satisfy the next commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathfrak{ab}} \otimes \mathcal{O}_{\mathfrak{bc}} \otimes \mathcal{O}_{\mathfrak{cd}} \arrow[r, "\eta^{\mathfrak{b}}_{\mathfrak{ac}} \otimes 1"] \arrow[d, "1 \otimes \eta^{\mathfrak{c}}_{\mathfrak{bd}}"'] & \mathcal{O}_{\mathfrak{ac}} \otimes \mathcal{O}_{\mathfrak{cd}} \arrow[d, "\eta^{\mathfrak{c}}_{\mathfrak{ad}}"] \\
\mathcal{O}_{\mathfrak{ad}} \otimes \mathcal{O}_{\mathfrak{bd}} \arrow[r, "\eta^{\mathfrak{b}}_{\mathfrak{ad}}"'] & \mathcal{O}_{\mathfrak{ad}}
\end{tikzcd}$$

Lemma 4.3. *The maps $\Delta^{\mathfrak{c}}_{\mathfrak{ab}}$ are* coassociative*, i.e. the next diagram commutes*

$$\begin{tikzcd}
\mathcal{O}_{\mathfrak{ab}} \arrow[r, "\Delta^{\mathfrak{d}}_{\mathfrak{ab}}"] \arrow[d, "\Delta^{\mathfrak{c}}_{\mathfrak{ab}}"'] & \mathcal{O}_{\mathfrak{ad}} \otimes \mathcal{O}_{\mathfrak{db}} \arrow[d, "\Delta^{\mathfrak{c}}_{\mathfrak{ad}} \otimes 1"] \\
\mathcal{O}_{\mathfrak{ac}} \otimes \mathcal{O}_{\mathfrak{cb}} \arrow[r, "1 \otimes \Delta^{\mathfrak{d}}_{\mathfrak{cb}}"'] & \mathcal{O}_{\mathfrak{ac}} \otimes \mathcal{O}_{\mathfrak{cd}} \otimes \mathcal{O}_{\mathfrak{db}}
\end{tikzcd}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{B}$.

Proof. Note that in the next diagram we need to prove that the external diagram commutes.

$$
\begin{tikzcd}
 & & & & \mathcal{O}^*_{bd}\otimes\mathcal{O}^*_{da} \arrow[dr, "\tau"] & & & & \\
 & & & \mathcal{O}^*_{ba} \arrow[ur, "\eta^{d*}_{ba}"] & & \mathcal{O}^*_{da}\otimes\mathcal{O}^*_{bd} \arrow[dr, "\Phi^{-1}_{ad}\otimes\Phi^{-1}_{db}"] \arrow[ddl, "\eta^{c*}_{da}\otimes 1^*"'] & & & \\
 & & \mathcal{O}_{ab} \arrow[ur, "\Phi_{ab}"] \arrow[dl, "\Phi_{ab}"'] & \textcircled{1} & & & \mathcal{O}_{ad}\otimes\mathcal{O}_{db} \arrow[dr, "\Phi_{ad}\otimes 1"] & & \\
 & \mathcal{O}^*_{ba} \arrow[uurr, bend right, "1"'] \arrow[dl, "\eta^{c*}_{ba}"'] & & \textcircled{2} & \mathcal{O}^*_{dc}\otimes\mathcal{O}^*_{ca}\otimes\mathcal{O}^*_{bd} \arrow[dd, "\tau\otimes 1"] & \textcircled{3} & & \mathcal{O}^*_{da}\otimes\mathcal{O}_{db} \arrow[dr, "\eta^{c*}_{da}\otimes 1"] & \\
\mathcal{O}^*_{bc}\otimes\mathcal{O}^*_{ca} \arrow[dr, "\tau"'] & & & \mathcal{O}^*_{ca}\otimes\mathcal{O}^*_{bd}\otimes\mathcal{O}^*_{dc} \arrow[dr, "1\otimes\tau"] & & & & & \mathcal{O}^*_{dc}\otimes\mathcal{O}^*_{ca}\otimes\mathcal{O}_{db} \arrow[dl, "\tau\otimes 1"] \\
 & \mathcal{O}^*_{ca}\otimes\mathcal{O}^*_{bc} \arrow[urr, "1\otimes\eta^{d*}_{bc}"] \arrow[dr, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{bc}"'] & & & \mathcal{O}^*_{ca}\otimes\mathcal{O}^*_{dc}\otimes\mathcal{O}^*_{bd} \arrow[rrr, "1\otimes 1\otimes\Phi^{-1}_{db}"] \arrow[d, "1\otimes\tau"] & & & \mathcal{O}^*_{ca}\otimes\mathcal{O}^*_{dc}\otimes\mathcal{O}_{db} \arrow[dl, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{cb}\otimes 1"] & \\
 & & \mathcal{O}_{ac}\otimes\mathcal{O}_{cb} \arrow[dr, "1\otimes\Phi_{cb}"'] & \textcircled{4} & \mathcal{O}^*_{ca}\otimes\mathcal{O}^*_{bd}\otimes\mathcal{O}^*_{dc} \arrow[dd, "\Phi^{-1}_{ac}\otimes 1\otimes 1"] & \textcircled{5} & \mathcal{O}_{ac}\otimes\mathcal{O}_{cd}\otimes\mathcal{O}_{db} & & \\
 & & & \mathcal{O}_{ac}\otimes\mathcal{O}^*_{bc} \arrow[dr, "1\otimes\eta^{d*}_{bc}"'] & & \mathcal{O}_{ac}\otimes\mathcal{O}^*_{dc}\otimes\mathcal{O}^*_{bd} \arrow[ur, "1\otimes\Phi^{-1}_{cd}\otimes\Phi^{-1}_{db}"'] & & & \\
 & & & & \mathcal{O}_{ac}\otimes\mathcal{O}^*_{bd}\otimes\mathcal{O}^*_{dc} \arrow[ur, "1\otimes\tau"'] & & & &
\end{tikzcd}
$$

Note that (1) commutes trivially. The diagram (2) can be divided into four commutative diagrams

$$\begin{tikzcd}
\mathcal{O}^*_{ba} \arrow[r, "\eta^{d*}_{ba}"] \arrow[d, "\eta^{c*}_{ba}"'] & \mathcal{O}^*_{bd} \otimes \mathcal{O}^*_{da} \arrow[r, "\tau"] \arrow[d, "1\otimes\eta^{c*}_{da}"'] & \mathcal{O}^*_{da} \otimes \mathcal{O}^*_{bd} \arrow[d, "\eta^{c*}_{da}\otimes 1"] \\
\mathcal{O}^*_{bc} \otimes \mathcal{O}^*_{ca} \arrow[r, "\eta^{d*}_{bc}\otimes 1"'] \arrow[d, "\tau"'] & \mathcal{O}^*_{bd} \otimes \mathcal{O}^*_{dc} \otimes \mathcal{O}^*_{ca} \arrow[r, "\tau"] \arrow[d, "\tau"'] & \mathcal{O}^*_{dc} \otimes \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{bd} \arrow[d, "\tau\otimes 1"] \\
\mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{bc} \arrow[r, "1\otimes\eta^{d*}_{bc}"'] & \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{bd} \otimes \mathcal{O}^*_{dc} \arrow[r, "1\otimes\tau"'] & \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{dc} \otimes \mathcal{O}^*_{bd}
\end{tikzcd}$$

The diagram (3) is the following

$$\begin{tikzcd}
\mathcal{O}^*_{da} \otimes \mathcal{O}^*_{bd} \arrow[r, "\Phi^{-1}_{ad}\otimes\Phi^{-1}_{db}"] \arrow[d, "\eta^{c*}_{da}\otimes 1"'] & \mathcal{O}_{ad} \otimes \mathcal{O}_{db} \arrow[r, "\Phi_{ad}\otimes 1"] & \mathcal{O}^*_{da} \otimes \mathcal{O}_{db} \arrow[d, "\eta^{c*}_{da}\otimes 1"] \\
\mathcal{O}^*_{dc} \otimes \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{bd} \arrow[d, "\tau\otimes 1"'] & & \mathcal{O}^*_{dc} \otimes \mathcal{O}^*_{ca} \otimes \mathcal{O}_{db} \arrow[d, "\tau\otimes 1"] \\
\mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{dc} \otimes \mathcal{O}^*_{bd} \arrow[rr, "1\otimes\Phi^{-1}_{db}"'] & & \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{dc} \otimes \mathcal{O}_{db}
\end{tikzcd}$$

and it commutes naturally. Now we check that the diagram (4) commutes

$$\begin{tikzcd}
\mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{bc} \arrow[r, "1\otimes\eta^{d*}_{bc}"] \arrow[d, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{cb}"'] & \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{bd} \otimes \mathcal{O}^*_{dc} \arrow[r, "1\otimes\tau"] \arrow[dr, "1"'] & \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{dc} \otimes \mathcal{O}^*_{bd} \arrow[d, "1\otimes\tau"] \\
\mathcal{O}_{ac} \otimes \mathcal{O}_{cb} \arrow[d, "1\otimes\Phi_{cb}"'] & & \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{bd} \otimes \mathcal{O}^*_{dc} \arrow[dl, "\Phi^{-1}_{ac}\otimes 1"] \\
\mathcal{O}_{ac} \otimes \mathcal{O}^*_{bc} \arrow[r, "1\otimes\eta^{d*}_{bc}"'] & \mathcal{O}_{ac} \otimes \mathcal{O}^*_{bd} \otimes \mathcal{O}^*_{dc} &
\end{tikzcd}$$

It commutes naturally. Finally, it remains to prove that the diagram (5) commutes. Then the external diagram commutes. The diagram (5) can be divided into the next diagrams.

$$\begin{tikzcd}
\mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{dc}\mathcal{O}^*_{bd} \arrow[rr, "1\otimes\Phi^{-1}_{db}"] \arrow[d, "1\otimes\tau"'] \arrow[ddr, "\Phi^{-1}_{ac}\otimes 1"] \arrow[ddrr, bend left, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{cd}\otimes\Phi^{-1}_{db}"] & & \mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{dc} \otimes \mathcal{O}_{db} \arrow[dd, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{cd}\otimes 1"] \\
\mathcal{O}^*_{ca} \otimes \mathcal{O}^*_{bd} \otimes \mathcal{O}^*_{dc} \arrow[d, "\Phi^{-1}_{ac}\otimes 1"'] & & \\
\mathcal{O}_{ac} \otimes \mathcal{O}^*_{bd} \otimes \mathcal{O}^*_{dc} \arrow[r, "1\otimes\tau"'] & \mathcal{O}_{ac} \otimes \mathcal{O}^*_{dc} \otimes \mathcal{O}^*_{bd} \arrow[r, "1\otimes\Phi^{-1}\otimes\Phi^{-1}"'] & \mathcal{O}_{ac} \otimes \mathcal{O}_{cd} \otimes \mathcal{O}_{db}
\end{tikzcd}$$

It is clear that they are commutative, and the coproducts are coassociative.

♣

Lemma 4.4. *Given the maps $\Theta_a : \mathcal{O}_{aa} \to \mathbb{k}$, we have that the triangles*

$$\begin{tikzcd}
\mathcal{O}_{ab} \arrow[r, "\Delta^b_{ab}"] \arrow[d, "\cong"'] & \mathcal{O}_{ab} \otimes \mathcal{O}_{bb} \arrow[dl, "1\otimes\Theta_b"] \\
\mathcal{O}_{ab} \otimes \mathbb{k} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_{ab} \arrow[r, "\Delta^a_{ab}"] \arrow[d, "\cong"'] & \mathcal{O}_{aa} \otimes \mathcal{O}_{ab} \arrow[dl, "\Theta_a\otimes 1"] \\
\mathbb{k} \otimes \mathcal{O}_{ab} &
\end{tikzcd}$$

commute.

Proof. Note the identity $\Theta_a = u^*_a \circ \Phi_a$. It is clear that the next diagram commutes,

$$\begin{tikzcd}
\mathcal{O}_{ab} \arrow[r, "\Phi_{ab}"] \arrow[dd, "\cong"'] & \mathcal{O}^*_{ba} \arrow[r, "\eta^{b*}_{ba}"] & \mathcal{O}^*_{bb} \otimes \mathcal{O}^*_{ba} \arrow[d, "\tau"] \\
 & & \mathcal{O}^*_{ba} \otimes \mathcal{O}^*_{bb} \arrow[dll, "\Phi^{-1}_{ab}\otimes u^*_b"'] \arrow[dl, "\Phi^{-1}_{ab}\otimes 1"] \arrow[d, "\Phi^{-1}_{ab}\otimes\Phi^{-1}_{bb}"] \\
\mathcal{O}_{ab} \otimes \mathbb{C} & \mathcal{O}_{ab} \otimes \mathcal{O}^*_{bb} \arrow[l, "1\otimes u^*_b"] & \mathcal{O}_{ab} \otimes \mathcal{O}_{bb} \arrow[l, "1\otimes\Phi_b"]
\end{tikzcd}$$

the reason is that the identity $\eta^b_{ba} \circ (u_b \otimes 1) = 1$ implies that $(u^*_b \otimes 1) \circ \eta^{b*}_{ba} = 1$ then

$$(\Phi^{-1}_{ab} \otimes u^*_b) \circ \tau \circ \eta^{b*}_{ba} = \tau \circ (1 \otimes \Phi^{-1}_{ab}) \circ (u^*_b \otimes 1) \circ \eta^{b*}_{ba} = \tau \circ (1 \otimes \Phi^{-1}_{ab})$$

This proves the lemma.

♣

Consider the maps

$$\overline{\eta^c_{ab}} : \mathcal{O}_{ab} \to \mathrm{Hom}(\mathcal{O}_{ca}, \mathcal{O}_{cb}) \cong \mathcal{O}_{cb} \otimes \mathcal{O}^*_{ca},$$

$$x \mapsto \cdot x : \mathcal{O}_{ca} \to \mathcal{O}_{cb}, \text{ product by the right of } x$$

$$\overline{\xi^c_{ab}} : \mathcal{O}_{ab} \to \mathrm{Hom}(\mathcal{O}_{bc}, \mathcal{O}_{ac}) \cong \mathcal{O}_{ac} \otimes \mathcal{O}^*_{bc},$$

$$x \mapsto x \cdot : \mathcal{O}_{bc} \to \mathcal{O}_{ac}, \text{ product by the left of } x$$

It is not difficult to prove that the next diagrams commute

$$\begin{tikzcd}
\mathcal{O}_{ab} \arrow[r, "\Phi_{ab}"] \arrow[d, "\Delta^c_{ab}"'] & \mathcal{O}^*_{ba} \arrow[r, "\Phi^{-1}_{ab}"] \arrow[d, "\eta^{c*}_{ba}"] & \mathcal{O}_{ab} \arrow[d, "\overline{\eta^c_{ab}}"] \\
\mathcal{O}_{ac} \otimes \mathcal{O}_{cb} \arrow[r, "(\Phi_{cb}\otimes\Phi_{ac})\circ\tau"'] & \mathcal{O}^*_{bc} \otimes \mathcal{O}^*_{ca} \arrow[r, "\Phi^{-1}_{cb}\otimes 1"'] & \mathcal{O}_{cb} \otimes \mathcal{O}^*_{ca}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_{ab} \arrow[r, "\Phi_{ab}"] \arrow[d, "\Delta^c_{ab}"'] & \mathcal{O}^*_{ba} \arrow[d, "\eta^{c*}_{ba}"] & \mathcal{O}_{ab} \arrow[l, "\Phi_{ab}"'] \arrow[d, "\overline{\xi^c_{ab}}"] \\
\mathcal{O}_{ac} \otimes \mathcal{O}_{cb} \arrow[r, "(\Phi_{cb}\otimes\Phi_{ac})\circ\tau"'] & \mathcal{O}^*_{bc} \otimes \mathcal{O}^*_{ca} & \mathcal{O}_{ac} \otimes \mathcal{O}^*_{bc} \arrow[l, "\tau\circ(\Phi_{ac}\otimes 1)"]
\end{tikzcd}$$

Proposition 4.5. *The coproduct Δ^c_{ab} is a morphism of $\mathcal{O}_{da} \times \mathcal{O}_{be}$-bimodules for all* d, e*, i.e. the squares*

$$\begin{tikzcd}
\mathcal{O}_{da} \otimes \mathcal{O}_{ab} \arrow[r, "\eta^a_{db}"] \arrow[d, "1\otimes\Delta^c_{ab}"'] & \mathcal{O}_{db} \arrow[d, "\Delta^c_{db}"] \\
\mathcal{O}_{da} \otimes \mathcal{O}_{ac} \otimes \mathcal{O}_{cb} \arrow[r, "\eta^a_{dc}\otimes 1"'] & \mathcal{O}_{dc} \otimes \mathcal{O}_{cb}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_{ab} \otimes \mathcal{O}_{be} \arrow[r, "\eta^b_{ae}"] \arrow[d, "\Delta^c_{ab}\otimes 1"'] & \mathcal{O}_{ab} \arrow[d, "\Delta^c_{ae}"] \\
\mathcal{O}_{ac} \otimes \mathcal{O}_{cb} \otimes \mathcal{O}_{be} \arrow[r, "1\otimes\eta^b_{ce}"'] & \mathcal{O}_{ac} \otimes \mathcal{O}_{ce}
\end{tikzcd}$$

commute.

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathcal{O}_{da}\otimes\mathcal{O}_{ab} \arrow[r, "\eta^a_{ab}"] \arrow[dl, "1\otimes\overline{\xi^c_{ab}}"'] \arrow[d, "1\otimes\Delta^c_{ab}"] \arrow[dr, phantom, "1"] & \mathcal{O}_{db} \arrow[d, "\Delta^c_{db}"'] \arrow[dr, "\overline{\xi^c_{db}}"] & \\
\mathcal{O}_{da}\otimes\mathcal{O}_{ac}\otimes\mathcal{O}^*_{bc} \arrow[dr, "\eta^a_{dc}\otimes 1^*"'] & \mathcal{O}_{da}\otimes\mathcal{O}_{ac}\otimes\mathcal{O}_{cb} \arrow[l, "1\otimes 1\otimes\Phi_{cb}"] \arrow[r, "\eta^a_{dc}\otimes 1^*"'] \arrow[dr, phantom, "4"] & \mathcal{O}_{dc}\otimes\mathcal{O}_{cb} \arrow[r, "1\otimes\Phi_{cb}"'] \arrow[d, "1\otimes\Phi_{cb}"'] & \mathcal{O}_{dc}\otimes\mathcal{O}^*_{bc} \arrow[dl, "1\otimes 1^*"] \\
 & \mathcal{O}_{dc}\otimes\mathcal{O}^*_{bc} \arrow[r, "1\otimes 1^*"'] & \mathcal{O}_{dc}\otimes\mathcal{O}^*_{bc} &
\end{tikzcd}$$

If we prove that the external diagram, and the diagrams 2, 3, 4, 5 commute then the diagram 1 commutes. Note that the diagramas 2 and 5 commute using the last statement. Clearly the diagrams 3 and 4 commute, and finally the external diagram commutes by definition of $\overline{\xi^c_{ab}}$.

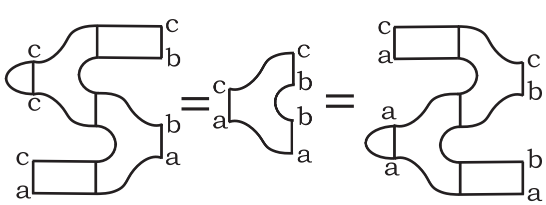
We use the next diagram to prove that the other diagram commutes.

$$\begin{tikzcd}
 & \mathcal{O}_{ab}\otimes\mathcal{O}_{be} \arrow[r, "\eta^b_{ab}"] \arrow[dl, "\tau\circ(\overline{\eta^c_{ab}}\otimes 1)"'] \arrow[d, "\Delta^c_{ab}\otimes 1"] & \mathcal{O}_{ae} \arrow[d, "\Delta^c_{ae}"] \arrow[dr, "\tau\circ\overline{\eta^c_{ae}}"] & \\
\mathcal{O}^*_{ca}\otimes\mathcal{O}_{cb}\otimes\mathcal{O}_{be} \arrow[dr, "1^*\otimes\eta^b_{ce}"'] & \mathcal{O}_{ac}\otimes\mathcal{O}_{cb}\otimes\mathcal{O}_{be} \arrow[l, "\Phi_{ac}\otimes 1\otimes 1"] \arrow[r, "1\otimes\eta^b_{ce}"'] & \mathcal{O}_{ac}\otimes\mathcal{O}_{ce} \arrow[r, "\Phi_{ac}\otimes 1"] \arrow[d, "\Phi_{ac}\otimes 1"'] & \mathcal{O}^*_{ca}\otimes\mathcal{O}_{ce} \arrow[dl, "1^*\otimes 1"] \\
 & \mathcal{O}^*_{ca}\otimes\mathcal{O}_{ce} \arrow[r, "1^*\otimes 1"'] & \mathcal{O}^*_{ca}\otimes\mathcal{O}_{ce} &
\end{tikzcd}$$

♣

Applying the Proposition 4.5 we have that the cobordisms of the figure 14 coincide.

Lemma 4.6.



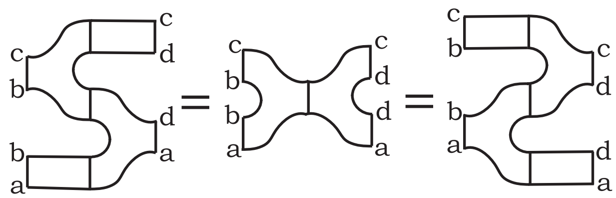
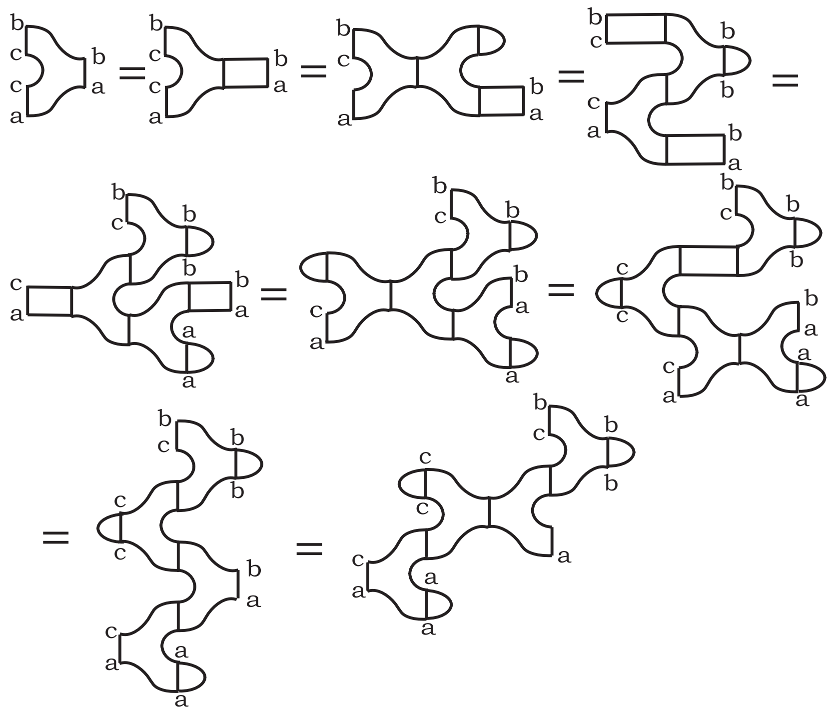
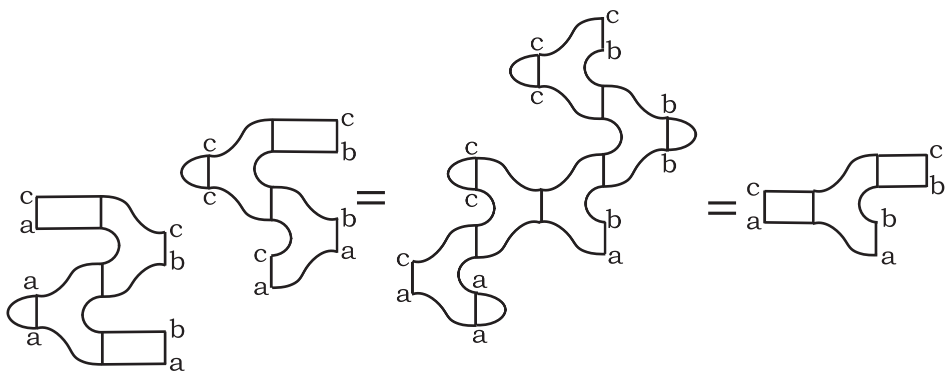
[htp] 

Figure 14: Abrams condition.

Proof.



hence



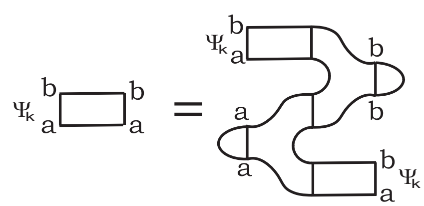
♣

Remark 4.7. Let $\Theta_{ab} : \mathbb{C} \to \mathcal{O}_{ab} \otimes \mathcal{O}_{ba}$ defined by

$$\Theta_{ab} = \Delta^{b}_{aa} \circ u_a,$$

where $u_a : \mathbb{C} \to \mathcal{O}_{aa}$ is the unit. Then $\Theta_{ab}(1) = \Sigma_i \Psi_i \otimes \Psi^i$, where $\{\Psi_i\}$ is a basis of $\mathcal{O}_{ab}$, and $\{\Psi^i\}$ is the dual basis of $\mathcal{O}_{ba}$, i.e. $\langle \Psi_i, \Psi^j \rangle = \delta_{ij}$.

Proof. Let be $\Theta_{ab}(1) = \Sigma_{i,j} \beta_{ij} \Psi_i \otimes \Psi^j$, where $\beta_{ij} \in \mathbb{C}$.



Then we have $(1 \otimes \Theta_b) \circ (1 \otimes \eta^{a}_{bb})(\Sigma_{ij} \beta_{ij} \Psi_i \otimes \Psi^j \otimes \Psi_k) = (1 \otimes \Theta_b)(\Sigma_{ij} \beta_{ij} \Psi_i \otimes \Psi^j \Psi_k) = \Sigma_{ij} \beta_{ij} \Theta_b(\Psi^j \Psi_k) \Psi_i = \Sigma_i \beta_{ik} \Psi_i = \Psi_k$ and hence $\beta_{ij} = \delta_{ij}$.

♣

Proposition 4.8. *We can modify the axiom 2 in the definition of Frobenius structure as follows (see Section 4.1): there exist a family of coassociative linear maps* $\Delta^{c}_{ab} : \mathcal{O}_{ab} \to \mathcal{O}_{ac} \otimes \mathcal{O}_{cb}$ *which are* $\mathcal{O}_{aa} \times \mathcal{O}_{bb}$*-bimodule morphisms and linear maps*

$\Theta_a : \mathcal{O}_{aa} \to \mathbb{C}$ *such that*

$$\begin{tikzcd}
\mathcal{O}_{ab} \arrow[r, "\Delta^b_{ab}"] \arrow[d, "\cong"'] & \mathcal{O}_{ab} \otimes \mathcal{O}_{bb} \arrow[dl, "1\otimes\Theta_b"] \\
\mathcal{O}_{ab} \otimes \mathbb{k} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_{ab} \arrow[r, "\Delta^a_{ab}"] \arrow[d, "\cong"'] & \mathcal{O}_{aa} \otimes \mathcal{O}_{ab} \arrow[dl, "\Theta_a\otimes 1"] \\
\mathbb{k} \otimes \mathcal{O}_{ab} &
\end{tikzcd}$$

commute.

Proof. We only need to prove that the trace $\Theta_a : \mathcal{O}_{aa} \to \mathbb{C}$ is non-degenerate. For this we consider the next commutative diagram

$$\begin{tikzcd}
 & & \mathcal{O}_{aa} \otimes \mathcal{O}_{aa} \otimes \mathcal{O}_{aa} \arrow[dr, "1\otimes\eta^a_{aa}"] & & \\
\mathbb{C} \otimes \mathcal{O}_{aa} \arrow[r, "u_a\otimes 1"] & \mathcal{O}_{aa} \otimes \mathcal{O}_{aa} \arrow[ur, "\Delta^a_{aa}\otimes 1"] \arrow[dr, "\eta^a_{aa}"'] & & \mathcal{O}_{aa} \otimes \mathcal{O}_{aa} \arrow[r, "1\otimes\Theta_a"] & \mathcal{O}_{aa} \otimes \mathbb{C} \\
 & & \mathcal{O}_{aa} \arrow[ur, "\Delta^a_{aa}"'] & &
\end{tikzcd}$$

This implies the next property

$$1 \otimes x \mapsto 1_a \otimes x \mapsto (\sum_i u_i \otimes e_i) \otimes x \mapsto \sum_i u_i \otimes e_i x \mapsto \sum_i \Theta_a(e_i x) u_i = x$$

where $\{e_i\}$ is a basis of $\mathcal{O}_{aa}$. Hence $\{u_i\}$ is also a basis of $\mathcal{O}_{aa}$.
If we take $x = u_j$, then $\Theta_a(e_i u_j) = \delta_{ij}$. We suppose $y = \sum_i \alpha_i e_i$ with the property that $\Theta_a(yx) = 0$ for all $x \in \mathcal{O}_{aa}$. Therefore, if we take $x = u_j$ hence $\sum_i \alpha_i \Theta_a(e_i u_j) = \alpha_j = 0$ for all j. This proves that $y = 0$ and consequently the trace is non-degenerate.

♣

Definition 4.9. We define a *positive (outgoing) boundary open-closed topological field theory* (2D OC-TFT$_+$) just as we defined a 2D OC-TFT with the difference that the morphisms have at least one outgoing boundary. In particular there is no linear form associated to the surfaces illustrated in the Figure 15. Namely, we no longer have traces. Now, we describe the algebraic axioms of this theory.

A positive boundary 2D open-closed TFT is given by the following algebraic data:

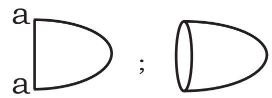


Figure 15: Traces in the open theory and closed theory.

1. $(\mathcal{A}, \Delta_{\mathcal{A}}, 1_{\mathcal{A}})$ is a commutative non compact Frobenius algebra.

2 $\mathcal{O}_{ab}$ is a collection of vector spaces for $a, b \in \mathcal{B}$.

2a. There is a family of associative linear maps

$$\eta^{b}_{ac} : \mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac} \tag{20}$$

2b. There is a family of co-associative linear maps

$$\Delta^{c}_{ab} : \mathcal{O}_{ab} \to \mathcal{O}_{ac} \otimes \mathcal{O}_{cb}.$$

2c. Moreover, Δ^{c}_{ab} is a morphism of $\mathcal{O}_{da} \times \mathcal{O}_{be}$-bimodule, i.e. the diagrams0

$$\begin{tikzcd}
\mathcal{O}_{da} \otimes \mathcal{O}_{ab} \arrow[r, "\eta^{a}_{db}"] \arrow[d, "1\otimes\Delta^{c}_{ab}"'] & \mathcal{O}_{db} \arrow[d, "\Delta^{c}_{db}"] \\
\mathcal{O}_{da} \otimes \mathcal{O}_{ac} \otimes \mathcal{O}_{cb} \arrow[r, "\eta^{a}_{dc}\otimes 1"'] & \mathcal{O}_{dc} \otimes \mathcal{O}_{cb}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_{ab} \otimes \mathcal{O}_{bb} \arrow[r, "\eta^{b}_{ae}"] \arrow[d, "\Delta^{c}_{ab}\otimes 1"'] & \mathcal{O}_{ae} \arrow[d, "\Delta^{c}_{ae}"] \\
\mathcal{O}_{ac} \otimes \mathcal{O}_{cb} \otimes \mathcal{O}_{be} \arrow[r, "1\otimes\eta^{b}_{ce}"'] & \mathcal{O}_{ac} \otimes \mathcal{O}_{ce}
\end{tikzcd}$$

commute.

3. There are linear maps:

$$\iota_a : \mathcal{A} \to \mathcal{O}_{aa}, \iota^{a} : \mathcal{O}_{aa} \to \mathcal{A} \tag{21}$$

such that

3a. ι_a is an algebra homomorphism

$$\iota_a(\phi_1 \phi_2) = \iota_a(\phi_1)\iota_a(\phi_2) \tag{22}$$

3b. The identity is preserved

$$\iota_a(1_{\mathcal{A}}) = 1_a \tag{23}$$

3c. Moreover, ι_a is central in the sense that

$$\iota_a(\phi)\psi = \psi\iota_b(\phi) \tag{24}$$

for all $\phi \in \mathcal{A}$ and $\psi \in \mathcal{O}_{ab}$.

3d. The "Cardy conditions". For this we define the map $\pi_b^a : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$ as follows. Since $\mathcal{O}_{ab}$ and $\mathcal{O}_{ba}$ are in duality (using θ_a or θ_b), if we let ψ_μ be a basis for $\mathcal{O}_{ba}$ then there is a dual basis ψ^μ for $\mathcal{O}_{ab}$. Then we set

$$\pi_b^a(\psi) = \sum_\mu \psi_\mu \psi \psi^\mu, \tag{25}$$

and the "Cardy condition" is

$$\pi_b^a = \iota_b \circ \iota^a. \tag{26}$$

Remark 4.10. This algebraic construction is equivalent to the categorical one in the 2D open-closed TFT case, with the restriction that it does not contain traces for the closed and the open part.

5 Virtual Fundamental Classes

5.1 Motivation

One of the most fruitful ways of constructing examples of TQFTs is by the method of the virtual fundamental classes on moduli spaces (of fields).

Moduli spaces often do not quite have a fundamental class (that we will require to do the integration). The problem is that roughly speaking $\mathcal{M}$ is given as the intersection of two submanifolds (equations) N_1 and N_2 of a larger manifold V (taking only two is possible by using the diagonal map trick, namely $N_1 \cap \ldots \cap N_r = (N_1 \times \cdots \times N_r) \cap \triangle(V^r)$). Often this intersection is not transversal. Therefore rather than a tangent we have a *virtual* tangent bundle (in K-theory)

$$[T\mathcal{M}]^{\mathrm{virt}} = [TN_1]|_{\mathcal{M}} + [TN_2]|_{\mathcal{M}} - [TV]|_{\mathcal{M}}$$

whose orientation (in cohomology, K-theory, complex cobordism) is called the *virtual fundamental class* $[\mathcal{M}]^{\mathrm{virt}}$. This is closely related to the theory of *derived manifolds* and could be reinterpreted in the language of [Spi10] but as it is not strictly necessary, we prefer to work in a more traditional topological language.

The basic example is afforded to us by Poincaré duality. This model written $(H^M, Z^M)_{1+1} \cong (A_M, \theta_M)$ depends only of a fixed oriented compact closed smooth manifold M and lives in dimension $1+1$. Let $\mathrm{Maps}^{\odot}(Y, M)$ be the space of constant maps from Y to M. Clearly if Y is connected (and non-empty), $\mathrm{Maps}^{\odot}(Y, M) \cong M$ and in fact this last homeomorphism is given by the map

$$\mathrm{ev}_y \colon \mathrm{Maps}^{\odot}(Y, M) \to M$$

that evaluates at $y \in Y$. For $Z \subset Y$ we will write $\mathrm{ev}_Z \colon \mathrm{Maps}^{\odot}(Y, M) \to \mathrm{Maps}^{\odot}(Z, M)$ to be the restriction map defined by $\mathrm{ev}_Z(f) = f|_Z$.

In this theory the fields are

$$\mathcal{F}(Y) = \mathrm{Maps}^{\odot}(Y, M),$$

namely the moduli space of constant maps from Y to M. We consider Y to be $(1+1)$-dimensional. Notice that

$$\mathrm{Maps}^{\odot}(Y, M) = M \times M \times \cdots \times M$$

where the product contains as many copies of M as connected components has Y. Consider now the situation in which $Y = P$ a 2-dimensional pair-of-pants (a 2-sphere with three small discs removed) with two incoming boundary components and one outgoing, and M is an oriented compact closed smooth manifold. Let a,b and c be three boundary components P each one diffeomorphic to S^1.

$$\begin{tikzcd} & \mathcal{F}(Y) \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ \mathcal{F}(\partial_0 Y) & & \mathcal{F}(\partial_1 Y) \end{tikzcd}$$

that is to say

$$\begin{tikzcd} & \mathrm{Maps}^{\odot}(P, M) \arrow[dl, "\mathrm{ev}_a \times \mathrm{ev}_b"'] \arrow[dr, "\mathrm{ev}_c"] & \\ \mathrm{Maps}^{\odot}(S^1, M) \times \mathrm{Maps}^{\odot}(S^1, M) & & \mathrm{Maps}^{\odot}(S^1, M) \end{tikzcd} \tag{27}$$

which becomes thus

$$\begin{tikzcd} & M \arrow[dl, "\triangle"'] \arrow[dr, "="] & \\ M \times M & & M \end{tikzcd}$$

and indeed, since that is a smooth correspondence of degree $-d$ we have that

$$\triangle_! = ev_c \circ (\mathrm{ev}_a \times \mathrm{ev}_b)_! \colon H_*(M) \otimes H_*(M) \to H_{*-d}(M)$$

is the induced homomorphism of degree $-d$ in homology. Namely, *the Feynman evolution for a pair of pants in this field theory is simply the intersection product in homology.*

We could have used the space $\mathbf{8}$ consisting of the wedge of two copies of S^1 instead of P (they are after all homotopy equivalent, we can define ev_c by choosing a quotient map $c \to \mathbf{8}$ identifying two points of c). Notice that by using pairs-of-pants we can recover any compact oriented 2-dimensional cobordism Y which is not boundaryless. In fact by using correspondences we can recover Ψ_Y^M for all Y that has at least one outgoing boundary component. In a sense correspondences encode a big portion of Poincaré duality this way, the so-called positive boundary sector of the TQFT.

For this model we have,

- $A_M = \mathcal{H}(\bullet) = H_*(M)$ (the homology of M which is graded).

- The mapping associated to the pair of pants

$$A_M \otimes A_M \to A_M \tag{28}$$

is the intersection product on the homology of the manifold (and is of degree $-d$).

- The trace is defined as $\theta_M \colon A_M = H_*(M) \to H_*(\bullet) \cong \mathbb{C}$ via the pushforward map associated to the canonical map $p : M \to \bullet$, i.e. $\theta_M(x) := p_*(x)$ The nondegeneracy of the trace is a consequence of Poincaré duality.

It may be instructive to see how the Pontrjagin-Thom construction and the Thom isomorphism can be used to induce the map (28). That basic idea is to use the *diagonal map*

$$\triangle \colon M \to M \times M.$$

$$m \mapsto (m, m)$$

The product on A_M is precisely the Gysin map $\triangle_!$ which can be defined using integration over the fiber, or as follows. It is not hard to verify that the normal bundle ν of $M = \triangle(M)$ in $M \times M$ is isomorphic to the tangent bundle TM of M. Let us write M_ϵ a small neighborhood of M in $M \times M$, and M^{TM} the Thom space on TM. Then we have a natural map

$$M \times M \longrightarrow M \times M/(M \times M - M_\epsilon) = M^{TM}$$

which by the use of the Thom isomorphism induces

$$\triangle_! \colon H_*(M) \otimes H_*(M) \longrightarrow H_{*-d}(M)$$

as desired.

Example 5.1. This is a famous example due to Chas and Sullivan [CS]. Following Cohen and Jones [CJ02] we do something rather drastic now and let the maps roam free, namely we write the correspondence (27) but with the whole mapping spaces rather than just the constant maps.

$$\begin{tikzcd}
& \mathrm{Maps}(\mathbf{8}, M) \arrow[dl, "\mathrm{ev}_a \times \mathrm{ev}_b"'] \arrow[dr, "\mathrm{ev}_c"] & \\
(\mathcal{L}M)^2 = \mathrm{Maps}(S^1, M) \times \mathrm{Maps}(S^1, M) & & \mathrm{Maps}(S^1, M) = \mathcal{L}M
\end{tikzcd} \tag{29}$$

which *is a degree* $-d$ *smooth correspondence.* We must replace the pair of pants P for the figure eight space $\mathbf{8}$ in order to ensure that $\mathrm{Maps}(\mathbf{8}, M) \to \mathcal{L}M \times \mathcal{L}M$ is a *finite* codimension embedding. This in turns implies the existence of the Gysin map

$$(\mathrm{ev}_a \times \mathrm{ev}_b)_! \colon H_*(\mathcal{L}M \times \mathcal{L}M) \to H_{*-d}(\mathrm{Maps}(\mathbf{8}, M)).$$

The induced map in homology

$$\bullet \colon H_*(\mathcal{L}M) \otimes H_*(\mathcal{L}M) \to H_{*-d}(\mathcal{L}M)$$

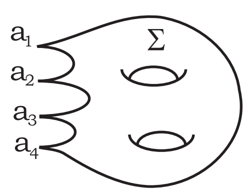
is called the *Chas-Sullivan* product on the homology of the free loop space of M. From the functoriality of correspondences it is not hard to verify that the product is associative.

Chas and Sullivan proved more, by defining a degree one map $\Delta \colon H_*(\mathcal{L}M) \to H_{*+1}(\mathcal{L}M)$ given by $\Delta(\sigma) = \rho_*(\theta \otimes \sigma)$ where $\rho \colon S^1 \times \mathcal{L}M \to \mathcal{L}M$ is the evaluation map and θ is the generator of $H^1(S^1, \mathbb{Z})$, they proved that $(H_*(M), \bullet, \Delta)$ is a Batalin-Vilkovisky algebra, namely

- $(H_{*-d}(M), \bullet)$ is a graded commutative algebra.
- $\Delta^2 = 0$
- The bracket $\{\alpha, \beta\} = (-1)^{|\alpha|}\Delta(\alpha \bullet \beta) - (-1)^{|\alpha|}\Delta(\alpha) \bullet \beta - \alpha \bullet \Delta(\beta)$ makes $H_{*-d}(M)$ into a graded Gerstenhaber algebra (namely it is a Lie bracket which is a derivation on each variable).

This statement amounts essentially to the construction of $\Psi_Y^{\mathcal{L}M}$ for all positive boundary genus zero $(1+1)$-dimensional cobordisms Y due to a theorem of Getzler (cf. [Get94]). The case of positive genus has been studied by Cohen and Godin [CG04].

Example 5.2. The Gromov-Witten invariants introduced by Ruan in [Rua96] can be understood in terms of a field theory [PSS96]. Now we consider a Riemmann surface $Y = \Sigma_g$ of genus g with k marked points. These marked points will take the place of $\partial_0 Y$ and for simplicity we will not consider outgoing boundary for now.



In this (1+1)-dimensional quantum field theory we start by considering a fixed symplectic manifold (M, ω). The space of fields is given (roughly speaking) by the space of J-holomorphic maps on the class $\beta \in H_2(M)$,

$$\mathcal{F}(Y) = \mathcal{M}_\Sigma = \mathrm{Hol}_\beta(\Sigma, M) = \{f \in \mathrm{Hol}(\Sigma, M) | f_*[\Sigma] = \beta\},$$

If we denote by $\mathrm{ev}_i \colon \mathcal{M}_\Sigma \to M$ the evaluation map at $a_i \in \Sigma$, then we have the correspondence diagram

$$\begin{tikzcd}
 & \mathcal{M}_\Sigma \arrow[dl, "\times_i \mathrm{ev}_i"'] \arrow[dr] & \\
M^k = \mathcal{F}(\amalg_i a_i) & & \mathcal{F}(\varnothing) = \bullet
\end{tikzcd}$$

Given k cohomology classes $u_1, \ldots, u_k \in H^*(M)$ we can let them evolve according to Feynman's pull-push formalism to obtain the corresponding *Gromov-Witten invariant*

$$\Phi_{g,\beta,k}(u_1, \ldots, u_k) = \int_{\mathcal{M}_\Sigma} \mathrm{ev}_1^* u_1 \wedge \ldots \wedge \mathrm{ev}_k^* u_k$$

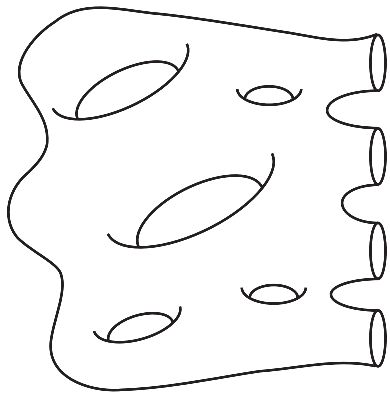
Here we should mention two important technical points regarding the moduli space $\mathcal{M}_\Sigma$. Firstly Kontsevich [Kon95] discovered that the most convenient space for defining this field theory is the moduli space of stable maps (where at most ordinary double points are allowed, and with finite automorphism groups). The moduli space turns out to be an orbifold, not a manifold. We will return to the definition of an orbifold later.

The corrected formula for the Gromov-Witten invariants is then

$$\Phi_{g,\beta,k}(u_1, \ldots, u_k) = \int_{[\mathcal{M}_\Sigma]^{\mathrm{virt}}} \mathrm{ev}_1^* u_1 \wedge \ldots \wedge \mathrm{ev}_k^* u_k.$$

Example 5.3. Floer theory is also a quantum field theory. Now we consider $Y = \Sigma_{g,k}$

to be a genus g Riemann surface with k small discs removed.



The fields are again holomorphic mappings $\mathcal{F}(Y) = \mathcal{M}_\Sigma$.

$$\begin{tikzcd} & \mathcal{M}_\Sigma \arrow[dl] \arrow[dr, "\times_i ev_i"] & \\ \mathcal{F}(\varnothing) = \bullet & & (\mathcal{L}M)^k = \mathcal{F}(\partial_1 \Sigma) \end{tikzcd}$$

In this case rather than simply considering the homology of $\mathcal{L}M$ we consider its semi-infinite (co)homology. This means that we consider the homology of cycles that are half-dimensional in $\mathcal{L}M$. The semi-infinite (co)homology $H^{si}_*(\mathcal{L}M)$ is also known as the *Floer* (co)homology $HF_*(M)$.

5.2 The Calculus of Obstruction Classes

We need the technical machinery of obstruction classes for our computations of virtual fundamental classes. The technical details of such theory are found in the last appendix of this book.

6 String Topology

String topology is the study of the topological properties of the free loop space $\mathcal{L}M$ of a smooth manifold M by the use of methods originating in quantum field string theories and in classical algebraic topology. Here $\mathcal{L}M$ is by definition the space $\mathrm{Maps}(S^1; M)$ of piecewise smooth maps from the unit circle S^1 to M. This study was initiated by Chas and Sullivan in their seminal paper [CS] where they defined a remarkable product $\circ$ on the homology $H_*(\mathcal{L}M)$ of the loop space of a smooth manifold. As we will see, string topology provides us with a family of TFTs, one for for each manifold M.

Let M be a smooth, orientable manifold of dimension n. The *space of free loop space* is

$$\mathcal{L}M = \{\alpha : S^1 \to M\}$$

where every loop is assumed piecewise smooth.
Chas and Sullivan in [CS] proved the next result.

Theorem 6.1 (Chas and Sullivan, 1999). *Let M be a compact, closed, smooth, orientable manifold of dimension d. There is a commutative and associative product*

$$H_p(\mathcal{L}M) \otimes H_q(\mathcal{L}M) \to H_{p+q-d}(\mathcal{L}M)$$

- *making $\mathbb{H}_*(\mathcal{L}M) := H_{*+d}(\mathcal{L}M)$ an associtive, commutative graded algebra and*
- *compatible with the intersection product on $H_*(M)$, i.e., the following diagram commutes.*

$$\begin{tikzcd} H_p(\mathcal{L}M) \otimes H_q(\mathcal{L}M) \arrow[r] \arrow[d, "ev_* \otimes ev_*"'] & H_{p+q-d}(\mathcal{L}M) \arrow[d, "ev_*"] \\ H_p M \otimes H_q M \arrow[r] & H_{p+q-d} M \end{tikzcd}$$

In this section we present a generalization of this result when M is not necessarily compact. Moreover, we will prove that $H_*(\mathcal{L}M)$ is a nearly Frobenius algebra. In particular, using the folk theorem we have an example of a 2D-TFT with positive boundary. In the next chapter, we will give an extension of the string theory that permits us to give a new example of 2D Open-Closed TFT with positive boundary.

6.1 Algebraic Structure

The Loop product: Following Cohen and Jones the Chas-Sullivan "loop product" in the homology (over a field $\Bbbk$ of zero characteristic) of the free loop space of a closed oriented d-manifold,

$$\mu : H_p(\mathcal{L}M) \otimes H_q(\mathcal{L}M) \to H_{p+q-d}(\mathcal{L}M)$$

is defined as follows.

Let $\mathrm{Map}(8, M)$ be the mapping space from the figure 8 (i.e the wedge of two circles) to the manifold M. Chose a basis point in the circle, notice that $\mathrm{Map}(8, M)$ can be viewed as the subspace of $\mathcal{L}M \times \mathcal{L}M$ consisting of those pair of loops that agree at the basepoint. In other words, there is a pullback square

$$\begin{tikzcd} \mathrm{Map}(8, M) \arrow[r, "e"] \arrow[d, "ev"'] & \mathcal{L}M \times \mathcal{L}M \arrow[d, "ev \times ev"] \\ M \arrow[r, "\Delta"'] & M \times M, \end{tikzcd} \tag{30}$$

where $\mathrm{ev} : \mathcal{L}M \to M$ is the fibration given by evaluating a loop at the basepoint. The map $\mathrm{ev} : \mathrm{Map}(8, M) \to M$ evaluates the map at the crossing point on the figure 8. Since $\mathrm{ev} \times \mathrm{ev}$ is a fibre bundle, $e : \mathrm{Map}(8, M) \hookrightarrow \mathcal{L}M \times \mathcal{L}M$ can be viewed as a codimension d embedding, with normal bundle $\mathrm{ev}^*(\nu_\Delta) \cong \mathrm{ev}^*(TM)$.

The existence of this pullback diagram of fiber bundles, means that there is a natural tubular neighborhood of the embedding $e : \mathrm{Map}(8, M) \to \mathcal{L}M \times \mathcal{L}M$. Namely, the inverse image of a tubular neighborhood of the diagonal embedding $\Delta : M \to M \times M$. That is, $\eta_e = (\mathrm{ev} \times \mathrm{ev})^{-1}(\eta_\Delta)$. Because ev is a locally trivial fibration, the tubular neighborhood η_e is homeomorphic to the total space of the normal bundle $\mathrm{ev}^*(TM)$. This induces a homeomorphism of the quotient space to the Thom space,

$$(\mathcal{L}M \times \mathcal{L}M)/((\mathcal{L}M \times \mathcal{L}M) - \eta_e) \cong (\mathrm{Map}(8, M))^{\mathrm{ev}^*(TM)}.$$

Combining this homeomorphism with the projection onto this quotient space, defines a Thom-collapse map

$$\tau_e : \mathcal{L}M \times \mathcal{L}M \to (\mathrm{Map}(8, M))^{\mathrm{ev}^*(TM)}.$$

For notation, we refer the Thom space of the pullback bundle $\mathrm{ev}^*(TM) \to \mathrm{Map}(8, M)$ as $\mathrm{Map}(8, M)^{TM}$.

There is a functorial construction in homology which goes in the wrong direction. This is called the *Gysin map* or *Umkher map*, see [CK09]. We define an umkehr map,

$$e_! : H_*(\mathcal{L}M \times \mathcal{L}M) \xrightarrow{\tau_e} H_*(\mathrm{Map}(8, M)^{TM}) \xrightarrow{\cap u} H_{*-d}(\mathrm{Map}(8, M))$$

where $u \in H^d(\mathrm{Map}(8, M)^{TM})$ is the Thom class.

Chas and Sullivan also observed that given a map from the figure 8 to M then one obtains a loop in M by starting at the intersection point, traversing the top loop of the 8, and then traversing the bottom loop, this defines a map

$$\rho : \mathrm{Map}(8, M) \to \mathcal{L}M.$$

Definition 6.2. We consider the next diagram

$$\begin{tikzcd} & \mathrm{Map}(8, M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & \\ \mathcal{L}M \times \mathcal{L}M & & \mathcal{L}M \end{tikzcd}$$

where e is defined in Diagram (30). The loop product in the homology of the loop space is the composition

$$\eta : H_*(\mathcal{L}M) \otimes H_*(\mathcal{L}M) \to H_*(\mathcal{L}M \times \mathcal{L}M) \xrightarrow{e_!} H_{*-d}(\mathrm{Map}(8, M)) \xrightarrow{\rho_*} H_{*-d}(\mathcal{L}M)$$

The Loop coproduct: Notice that $\mathrm{Map}(8, M)$ can be viewed as the subspace of $\mathcal{L}M$ consisting of loops that agree at 0 and at $\frac{1}{2}$. In other words, there is a pullback square

$$\begin{tikzcd} \mathrm{Map}(8, M) \arrow[r, "\rho"] \arrow[d, "ev_0"'] & \mathcal{L}M \arrow[d, "ev_0 \times ev_{\frac{1}{2}}"] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd}$$

where $ev_0 \times ev_{\frac{1}{2}} : \mathcal{L}M \to M \times M$ is the map given by evaluating a loop at 0 and $\frac{1}{2}$. Then we can define the umkehr map

$$\rho_! : H_*(\mathcal{L}M) \xrightarrow{\tau_\rho} H_*(\mathrm{Map}(8, M)^{TM}) \xrightarrow{\cap u} H_{*-d}(\mathrm{Map}(8, M)).$$

Definition 6.3. The loop coproduct for the homology of the loop space is the composition

$$\Delta : H_{*+d}(\mathcal{L}M) \xrightarrow{\rho_!} H_*(\mathrm{Map}(8, M)) \xrightarrow{e_*} H_*(\mathcal{L}M \times \mathcal{L}M) \cong H_*(\mathcal{L}M) \otimes H_*(\mathcal{L}M).$$

The unit and counit: Consider the disk D as a cobordism with zero incoming boundary component and one outgoing boundary component (see Figure 16). The restriction map to the zero incoming boundary is the map

$$\rho_{in} : \mathrm{Map}(D, M) \to \mathrm{Map}(\emptyset, M) = \text{point}.$$

Notice that the disc D is homotopy equivalent to a point, then the smooth mapping

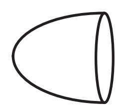


Figure 16: The disc D

space $\mathrm{Map}(D, M)$ is homotopy equivalent to the manifold M. The umkehr map in this setting is

$$(\rho_{in})_! : H_*(point) \to H_{*+d}(M),$$

which is defined by sending the generator to $[M] \in H_d(M)$. The restriction to the outgoing boundary component is the map

$$\rho_{out} : M \simeq \mathrm{Map}(D, M) \to \mathcal{L}M,$$

which is given by $\iota : M \hookrightarrow \mathcal{L}M$. Thus the unit is given by

$$u : (\rho_{out})_* \circ (\rho_{in})_! = \iota_* \circ (\rho_{in})_! : H_*(point) \to H_{*+d}(M) \to H_{*+d}(\mathcal{L}M),$$

which sends the generator to the image of the fundamental class.

The reason of the nonexistence of a counit in the Frobenius structure is formally the same to the existence of a unit. Namely, for this operation one must consider D as a cobordism with one incoming boundary, and zero outgoing boundary components. In this setting the role of the restriction maps ρ_{in} and ρ_{out} are reversed, and one obtains the diagram

$$\begin{tikzcd}
\mathrm{Map}(\emptyset, M) \arrow[d, "\|"'] & \mathrm{Map}(D, M) \arrow[l, "\rho_{out}"'] \arrow[r, "\rho_{in}"] \arrow[d, "\|"] & \mathcal{L}M \arrow[d, "\|"] \\
\text{point} & M \arrow[l, "\epsilon"] \arrow[r, "\iota"'] & \mathcal{L}M
\end{tikzcd}$$

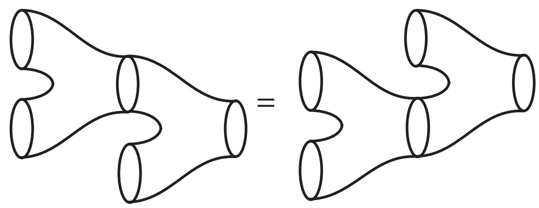
where $\epsilon : M \to$ point is the constant map. Now notice that in this case the embedding $\mathrm{Map}(D, M) \hookrightarrow \mathcal{L}M$ is of infinite codimension, and to our knowledge there is no way to define the umkher map. Ando and Morava [AM01] have given an argument that says that if one has a theory where this umkehr map exists, one would need that the Euler class of the normal bundle $e(\nu(\iota)) \in H^*(M)$ is invertible.

6.2 Verification of the nearly Frobenius Algebra Axioms

We will use the Proposition 22.6 to prove of the next theorem.

Theorem 6.4. $H_*(\mathcal{L}M)$ *is a nearly Frobenius algebra.*

Proof. 1. **Associativity of the loop product**



(1) (2)

The loop product is defined by the next diagram.

$$\begin{tikzcd}
& \mathrm{Map}(8, M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & \\
\mathcal{L}M \times \mathcal{L}M & & \mathcal{L}M
\end{tikzcd}$$

The associativity of the product is represented by the next two diagrams

(1)

$$\begin{tikzcd}
& & \mathrm{Map}(\clubsuit, M) \arrow[dll, bend right, "\varphi"'] \arrow[dl, "j"'] \arrow[dr, "i"] \arrow[drr, bend left, "\psi"] & & \\
& \mathrm{Map}(8, M) \times LM \arrow[dl, "e\times 1"'] \arrow[dr, "\rho\times 1"'] & & \mathrm{Map}(8, M) \arrow[dl, "e"] \arrow[dr, "\rho"] & \\
\mathcal{L}M \times \mathcal{L}M \times \mathcal{L}M & & \mathcal{L}M \times \mathcal{L}M & & \mathcal{L}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& & \mathrm{Map}(\clubsuit, M) \arrow[dll, bend right, "\varphi"'] \arrow[dl, "j'"'] \arrow[dr, "i"] \arrow[drr, bend left, "\psi"] & & \\
& \mathcal{L}M \times \mathrm{Map}(8, M) \arrow[dl, "1\times e"'] \arrow[dr, "1\times\rho"'] & & \mathrm{Map}(8, M) \arrow[dl, "e"] \arrow[dr, "\rho"] & \\
\mathcal{L}M \times \mathcal{L}M \times \mathcal{L}M & & \mathcal{L}M \times \mathcal{L}M & & \mathcal{L}M
\end{tikzcd}$$

We will use Quillen's result to prove this property.

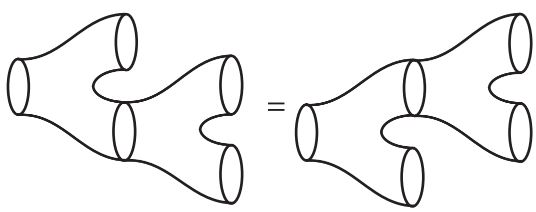
$$\begin{tikzcd}
\varphi^*(TM) = \kappa^* \operatorname{ev}^*(TM) \arrow[r, dashed] & \operatorname{Map}(\&, M) \arrow[d, "i"] & \\
\operatorname{ev}^*(TM) \arrow[r, dashed] & \operatorname{Map}(8, M) \arrow[r, "e"] \arrow[d, "\operatorname{ev}"'] & \mathcal{L}M \times \mathcal{L}M \arrow[d, "\operatorname{ev}_0 \times \operatorname{ev}_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

where $\varphi = \operatorname{ev} \circ \kappa$ and $\operatorname{ev}^*(TM)$ is the normal bundle of i.

$$\begin{tikzcd}
\operatorname{ev}^*(TM) \arrow[r, dashed] & \operatorname{Map}(\&, M) \arrow[r, "j"] \arrow[d, "\operatorname{ev}"] & \operatorname{Map}(8, M) \times \mathcal{L}M \arrow[d, "\operatorname{ev} \times \operatorname{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

(1) We have that $0 \to \operatorname{ev}^*(TM) \to \varphi^*(TM) \to F_1 \to 0$ is an exact sequence. Note that $\varphi = \operatorname{ev}$, then $F_1 = 0$. Similarly, for (2) we have $F_2 = 0$, then $e(F_1) = e(F_2)$.

2. **Coassociativity of the coproduct**



(1) (2)

(1)

$$\begin{tikzcd}[column sep=small]
 & & \operatorname{Map}(\&, \mathrm{M}) \arrow[dl, "i"'] \arrow[dr, "j'"] \arrow[dll, bend right, "\psi"'] \arrow[drr, bend left, "\varphi"] & & \\
 & \operatorname{Map}(8, M) \arrow[dl, "\rho"'] \arrow[dr, "e"] & & \mathcal{L}M \times \operatorname{Map}(8, M) \arrow[dl, "1 \times \rho"'] \arrow[dr, "1 \times e"] & \\
\mathcal{L}M & & \mathcal{L}M \times \mathcal{L}M & & \mathrm{LM} \times \mathrm{LM} \times \mathrm{LM}
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathrm{Map}(\clubsuit, M) \arrow[dl, "i"'] \arrow[dr, "j"] \arrow[dll, bend right, "\psi"'] \arrow[drr, bend left, "\varphi"] & & \\
 & \mathrm{Map}(8, M) \arrow[dl, "\rho"'] \arrow[dr, "e"] & & \mathrm{Map}(8, M) \times \mathrm{LM} \arrow[dl, "\rho\times 1"'] \arrow[dr, "e\times 1"] & \\
\mathcal{L}M & & \mathcal{L}M \times \mathcal{L}M & & \mathrm{LM} \times \mathrm{LM} \times \mathrm{LM}
\end{tikzcd}$$

(1) In the first case we have:

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[r, "i"] \arrow[d, "\mathrm{ev}"] & \mathrm{Map}(8, M) \arrow[d, "\mathrm{ev}_{\frac{1}{2}} \times \mathrm{ev}_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
j'^*(\mathrm{ev} \times \mathrm{ev})^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[d, "j'"] & \\
 & \mathcal{L}M \times \mathrm{Map}(8, M) \arrow[r, "1\times e"] \arrow[d, "\mathrm{ev} \times \mathrm{ev}"'] & \mathrm{LM} \times \mathrm{LM} \arrow[d, "\mathrm{ev} \times \mathrm{ev} \times \mathrm{ev}_{\frac{1}{2}}"] \\
TM \arrow[r, dashed] & M \times M \arrow[r, "1\times\Delta"] & M \times M \times M
\end{tikzcd}$$

Then, we have the next short exact sequence $0 \to \mathrm{ev}^*(TM) \to r_2^*(\mathrm{ev} \times \mathrm{ev})^*(TM) \to F_1 \to 0$. We conclude that $F_1 = 0$ since $\mathrm{ev}^*(TM) = r_2^*(\mathrm{ev} \times \mathrm{ev})^*(TM)$.

(2) In the other case there are the diagrams

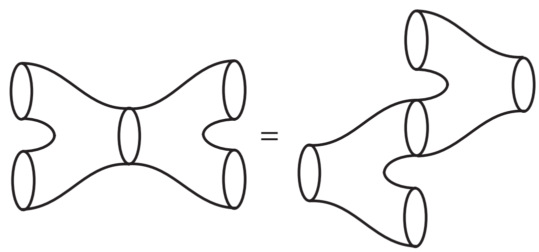
$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[r, "i"] \arrow[d, "\mathrm{ev}"] & \mathrm{Map}(8, M) \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_{\frac{1}{2}}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
j^*(\mathrm{ev}\times \mathrm{ev})^*(TM) \arrow[r, dashed] & \mathrm{Map}(\&, M) \arrow[d, "j_2"] & \\
& \mathrm{Map}(8, M)\times \mathcal{L}M \arrow[r, "e\times 1"] \arrow[d, "\mathrm{ev}\times\mathrm{ev}"'] & \mathrm{LM}\times\mathrm{LM} \arrow[d, "\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"] & M\times M\times M
\end{tikzcd}$$

Then we have the exact sequence $0 \to \mathrm{ev}^*(TM) \to j_2^*(\mathrm{ev}\times\mathrm{ev})^*(TM) \to F_2 \to 0$. Since $\mathrm{ev}^*(TM) = j_2^*(\mathrm{ev}\times\mathrm{ev})^*(TM)$ then $F_2 = 0$.

3. Abrams condition



(1) (2)

(1)

$$\begin{tikzcd}
& & \mathrm{Map}(\&, M) \arrow[dl, "i"'] \arrow[dr, "i"] \arrow[dll, bend right, "\psi"'] \arrow[drr, bend left, "\varphi"] & & \\
& \mathrm{Map}(8, M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & & \mathrm{Map}(8, M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & \\
\mathcal{L}M\times\mathcal{L}M & & \mathcal{L}M & & \mathcal{L}M\times\mathcal{L}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& & \mathrm{Map}(\&, M) \arrow[dl, "j'"'] \arrow[dr, "j"] \arrow[dll, bend right, "\psi"'] \arrow[drr, bend left, "\varphi"] & & \\
& \mathcal{L}M\times\mathrm{Map}(8, M) \arrow[dl, "1\times\rho"'] \arrow[dr, "1\times e"] & & \mathrm{Map}(8, M)\times\mathcal{L}M \arrow[dl, "e\times 1"'] \arrow[dr, "\rho\times 1"] & \\
\mathcal{L}M\times\mathcal{L}M & & \mathrm{LM}\times\mathrm{LM}\times\mathrm{LM} & & \mathcal{L}M\times\mathcal{L}M
\end{tikzcd}$$

In the first diagram we have

$$\begin{tikzcd}
i^*\,\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[d, "\kappa'"] & \\
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(8, M) \arrow[r] \arrow[d, "\mathrm{ev}"'] & \mathcal{L}M \arrow[d, "\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[r, "i"] \arrow[d, "\mathrm{ev}"] & \mathrm{Map}(8, M) \arrow[d, "\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

Then we have the exact sequence $0 \to \mathrm{ev}^*(TM) \to \kappa'^*\,\mathrm{ev}^*(TM) \to F_1 \to 0$. Since $\mathrm{ev}\circ\kappa' = \mathrm{ev}$ then $F_1 = 0$.

For the second diagram

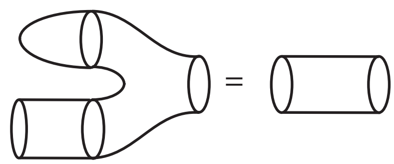
$$\begin{tikzcd}
j^*(\mathrm{ev}\times\mathrm{ev})^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[d, "j"] & \\
(\mathrm{ev}\times\mathrm{ev})^*(TM) \arrow[r, dashed] & \mathrm{Map}(8, M)\times\mathcal{L}M \arrow[r, "\rho\times 1"] \arrow[d, "\mathrm{ev}\times\mathrm{ev}"'] & \mathrm{LM}\times\mathrm{LM}\times\mathrm{LM} \arrow[d, "\mathrm{ev}\times\mathrm{ev}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"] & M\times M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[r, "j'"] \arrow[d, "\mathrm{ev}"] & \mathcal{L}M\times\mathrm{Map}(8, M) \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

Therefore we have the exact sequence $0 \to \mathrm{ev}^*(TM) \to j^*(\mathrm{ev}\times\mathrm{ev})^*(TM) \to F_2 \to 0$. Note that $\mathrm{ev}^*(TM) = j^*(\mathrm{ev}\times\mathrm{ev})^*(TM)$, then $F_2 = 0$.

4. **Unit axiom**



(1) (2)

$$\begin{tikzcd}
 & & & \mathcal{L}M \arrow[dll, "\epsilon_0\times\mathrm{id}"'] \arrow[drr, "k"] \arrow[ddlll, bend right, "\psi"'] \arrow[ddrrr, bend left, "\varphi"] & & & \\
 & M\times\mathcal{L}M \arrow[dl, "\rho_{in}\times 1"] \arrow[drr, "r\times 1"] & & & & \mathrm{Map}(8,M) \arrow[dll, "\rho_{in}"'] \arrow[dr, "\rho_{out}"'] & \\
\mathrm{pt}\times\mathcal{L}M & & & \mathcal{L}M\times\mathcal{L}M & & & \mathcal{L}M
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathcal{L}M \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[ddll, bend right, "\psi"'] \arrow[ddrr, bend left, "\varphi'"] & & \\
 & \mathcal{L}M \arrow[dl, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & & \mathcal{L}M \arrow[dl, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & \\
\mathcal{L}M & & \mathcal{L}M & & \mathcal{L}M
\end{tikzcd}$$

First, we note that φ and φ' are homotopic maps, then $\varphi_* = \varphi'_*$.
In (1) we have

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathcal{L}M \arrow[r, "\epsilon_0\times\mathrm{id}"] \arrow[d, "\mathrm{ev}"'] & M\times\mathcal{L}M \arrow[d, "\mathrm{id}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathcal{L}M \arrow[d, "k"] & \\
 & \mathrm{Map}(8,M) \arrow[r] \arrow[d, "\mathrm{ev}"'] & \mathcal{L}M\times\mathcal{L}M \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

Then $F_1 = 0$. In the second diagram is trivial to prove that $F_2 = 0$.

♣

From this we can conclude that string topology defines a TQFT+, this has been proved by different methods in [CG04]

6.3 String Topology as a non-Compact Calabi-Yau Category

Let $\mathcal{B}$ be the category of D-branes, the objects of this category are a collection of submanifolds of M,

$$\text{Obj}(\mathcal{B}) = \{D_i \subset M : \text{submanifold of } M\}.$$

Now we consider the path spaces, see Figure 17,

$$\mathcal{P}_M(D_i, D_j) = \{\gamma : [0,1] \to M \text{ picewise smooth } : \gamma(0) \in D_i, \gamma(1) \in D_j\}$$

Then, the morphisms of the category $\mathcal{B}$ are

$$\text{Hom}_{\mathcal{B}}(D_i, D_j) = H_*(\mathcal{P}_M(D_i, D_j)),$$

for $D_i, D_j \in \text{Obj}(\mathcal{B})$.

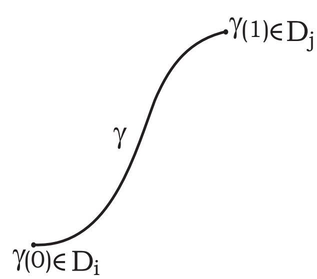
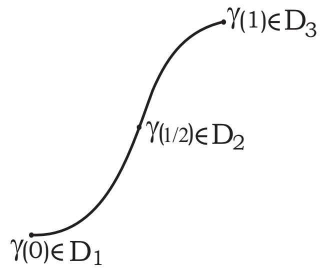


Figure 17: Space $\mathcal{P}_M(D_i, D_j)$.

We have already endowed the free loop space $(H_*(LM), \Delta, u)$ with the structure of a nearly Frobenius algebra. In waht follows we will describe the other structural maps.

Consider the path space

$$\mathcal{P}_M(D_1, D_2, D_3) = \left\{\alpha : [0,1] \to M : \alpha(0) \in D_1, \alpha\left(\frac{1}{2}\right) \in D_2, \alpha(1) \in D_3\right\}$$



Now we consider the next diagram

$$\begin{tikzcd}
 & \mathcal{P}_M(D_1,D_2,D_3) \arrow[dl, "j_{12}\times j_{23}"'] \arrow[dr, "i^2_{13}"] & \\
\mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_3) & & \mathcal{P}_M(D_1,D_3)
\end{tikzcd}$$

where $i^2_{13} : \mathcal{P}_M(D_1,D_2,D_3) \to \mathcal{P}_M(D_1,D_2)$ is the natural inclusion, $j_{12} : \mathcal{P}_M(D_1,D_2,D_3) \to \mathcal{P}_M(D_1,D_2)$ is defined by $j_{12}(\alpha)(t) := \alpha(\frac{t}{2})$, and $j_{23} : \mathcal{P}_M(D_1,D_2,D_3) \to \mathcal{P}_M(D_2,D_3)$ is defined by $j_{23}(\alpha)(t) := \alpha(\frac{1+t}{2})$.

The main idea to defining the product is to construct the umkehr map

$$(j_{12}\times j_{23})! : H_*(\mathcal{P}_M(D_1,D_2))\otimes H_*(\mathcal{P}_M(D_2,D_3)) \to H_*(\mathcal{P}_M(D_1,D_2,D_3))$$

and we define the product η^2_{13} as the composition

$$\eta^2_{13} = (i^2_{13})_* \circ (j_{12}\times j_{23})! : H_*(\mathcal{P}_M(D_1,D_2))\otimes H_*(\mathcal{P}_M(D_2,D_3)) \to H_*(\mathcal{P}_M(D_1,D_3)).$$

Now we observe that there is a pullback diagram of fibrations,

$$\begin{tikzcd}
\mathcal{P}_M(D_1,D_2,D_3) \arrow[r, "j_{12}\times j_{23}"] \arrow[d, "ev_{\frac{1}{2}}"'] & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_3)\ , \arrow[d, "ev_1\times ev_0"] \\
D_2 \arrow[r, "\Delta"'] & D_2\times D_2
\end{tikzcd}$$

this let us define the umkehr map $(j_{12}\times j_{23})!$.

As before we can consider the diagram

$$\begin{tikzcd}
 & \mathcal{P}_M(D_1,D_2,D_3) \arrow[dl, "i^2_{13}"'] \arrow[dr, "j_{12}\times j_{23}"] & \\
\mathcal{P}_M(D_1,D_3) & & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_3)
\end{tikzcd}$$

Then, we define a coproduct

$$\Delta_{13}^2 : H_*(\mathcal{P}_M(D_1, D_3)) \to H_*(\mathcal{P}_M(D_1, D_2)) \otimes H_*(\mathcal{P}_M(D_2, D_3))$$

as the composition $(j_{12} \times j_{23})_* \circ (i_{13}^2)! : H_*(\mathcal{P}_M(D_1, D_3)) \to H_*(\mathcal{P}_M(D_1, D_2, D_3)) \to H_*(\mathcal{P}_M(D_1, D_2)) \otimes H_*(\mathcal{P}_M(D_2, D_3))$.
We can define the umkehr map $(i_{13}^2)!$ because we have a pullback diagram of fibrations,

$$\begin{tikzcd}
\mathcal{P}_M(D_1, D_2, D_3) \arrow[r, "i_{13}^2"] \arrow[d, "ev_{\frac{1}{2}}"'] & \mathcal{P}_M(D_1, D_3) \arrow[d, "ev_{\frac{1}{2}} \times ev_{\frac{1}{2}}"] \\
D_2 \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

For the unit we consider the diagram

$$\begin{tikzcd}
& D \arrow[dl, "r"'] \arrow[dr, "i"] & \\
pt & & \mathcal{P}_M(D, D)
\end{tikzcd}$$

where $r : D \to pt$ is the constant map and $i : D \to \mathcal{P}_M(D, D)$ is the inclusion. This diagram defines the unit

$$u_D : H_*(pt) \to H_*(\mathcal{P}_M(D, D))$$

as $u_D := i_* \circ r!$, where $r! : H_*(pt) \to H_*(D)$ sends the generator to the fundamental class $[D]$.

To finish the construction we need to define the connection maps. Consider the open-closed cobordism i between an interval, whose boundary is labeled by a D-brane D, and a circle. This cobordism is pictured in the Figure 18. As in the

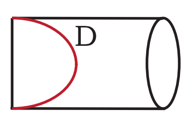


Figure 18: The cobordism i.

previous cases, we consider the space,

$$L_D(M) = \{\beta \in \mathcal{L}M : \beta(0) \in D\}$$

and the diagram

$$\begin{tikzcd} & \mathcal{L}M \arrow[dl, "i_D"'] \arrow[dr, "j_D"] & \\ \mathcal{L}M & & \mathcal{P}_M(D,D) \end{tikzcd}$$

We define the map ι^D by the composition,

$$\iota^D = (i_D)_* \circ (j_D)! : H_*(\mathcal{P}_M(D,D)) \to H_*(L_D(M)) \to H_*(\mathcal{L}M).$$

For defining the umkehr map we observe that there is a pullback square

$$\begin{tikzcd} L_D(M) \arrow[r, "j_D"] \arrow[d, "ev_0"'] & \mathcal{P}_M(D,D) \arrow[d, "ev_0 \times ev_1"] \\ D \arrow[r, "\Delta"'] & D \times D \end{tikzcd}$$

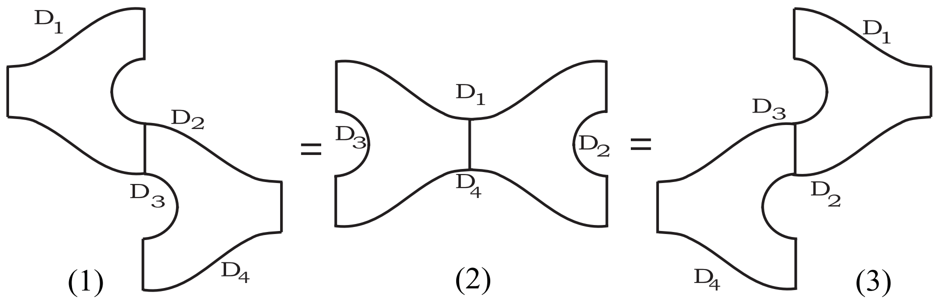
Finally we define the map $\iota_D = (j_D)_* \circ (i_D)_! : H_*(\mathcal{L}M) \to H_*(\mathcal{P}_M(D,D)) \to H_*(\mathcal{P}_M(D,D))$, where the umkehr map $(i_D)_!$ can be defined because the existence of a pullback square,

$$\begin{tikzcd} L_D(M) \arrow[r, "j_D"] \arrow[d, "ev_0"'] & \mathcal{L}M \arrow[d, "ev_0 \times ev_0"] \\ D \arrow[r, "\Delta"'] & M \times M \end{tikzcd}$$

Theorem 6.5. $(H_*(\mathcal{L}M), \mathcal{B})$ *is a 2D open-closed* TFT *with positive boundary.*

Proof. We only need to prove the open axioms since we have given already a proof for the closed axioms. We will use Proposition 22.6.

1. **Abrams condition.**
   This condition is represented in the next figure.



For this we just meed to prove that the maps for (1) and (2) are the same. The same applies for the the maps for (2) and (3). The next diagrams represent these composition maps.
(1)

$$\begin{tikzcd}
 & & & \mathcal{P}_M(D_1,D_2,D_3,D_4) \arrow[dll, "i^4_{123}\times i^{12}_{34}"'] \arrow[drr, "i^{34}_{12}\times i^1_{234}"] \arrow[ddlll, bend right, "\eta_1"'] \arrow[ddrrr, bend left, "\xi_1"] & & & \\
 & \mathcal{P}_M(D_1,D_2,D_3)\times\mathcal{P}_M(D_3,D_4) \arrow[dl, "i^2_{13}\times 1"'] \arrow[drr, "j_{12}\times j_{23}\times 1"] & & & & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_3,D_4) \arrow[dll, "1\times j_{23}\times j_{34}"'] \arrow[dr, "1\times i^3_{24}"] & \\
\mathcal{P}_M(D_1,D_3)\times\mathcal{P}_M(D_3,D_4) & & & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_3)\times\mathcal{P}_M(D_3,D_4) & & & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_4)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_M(D_1,D_2,D_3,D_4) \arrow[dl, "i^2_{134}"'] \arrow[dr, "i^3_{124}"] \arrow[ddll, bend right, "\eta_2"'] \arrow[ddrr, bend left, "\xi_2"] & & \\
 & \mathcal{P}_M(D_1,D_3,D_4) \arrow[dl, "j_{13}\times j_{34}"'] \arrow[dr, "i^3_{14}"] & & \mathcal{P}_M(D_1,D_2,D_4) \arrow[dl, "i^2_{14}"'] \arrow[dr, "j_{12}\times j_{24}"] & \\
\mathcal{P}_M(D_1,D_3)\times\mathcal{P}_M(D_3,D_4) & & \mathcal{P}_M(D_1,D_4) & & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_4)
\end{tikzcd}$$

First, we note that $\xi_1 = \xi_2$, $\eta_1 = \eta_2$ and that the squares are pullback squares. To prove that the composition maps coincide we only need to check that the Euler class of each square coincides.

(1) In the first diagram we have the next constructions

$$\begin{tikzcd}
(i^{34}_{12} \times i^{1}_{234})^*(\mathrm{ev} \times \mathrm{ev}_{\frac{1}{2}})^*(TD_3) \arrow[r, dashed] & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[d, "i^{34}_{12} \times i^{1}_{234}"] & \\
& \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_3, D_4) \arrow[r, "1 \times j_{23} \times j_{34}"] \arrow[d, "\mathrm{ev} \times \mathrm{ev}_{\frac{1}{2}}"'] & \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_3) \times \mathcal{P}_M(D_3, D_4) \arrow[d, "\mathrm{ev} \times \mathrm{ev} \times \mathrm{ev}"] \\
TD_3 \arrow[r, dashed] & D_2 \times D_3 \arrow[r, "1 \times \Delta"] & D_2 \times D_3 \times D_3
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{2}{3}})^*(TD_3) \arrow[r, dashed] & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[r, "i^{4}_{123} \times i^{12}_{34}"] \arrow[d, "\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{2}{3}}"] & \mathcal{P}_M(D_1, D_2, D_3) \times \mathcal{P}_M(D_3, D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{2}} \times \mathrm{ev} \times \mathrm{ev}"] \\
TD_3 \arrow[r, dashed] & D_2 \times D_3 \arrow[r, "1 \times \Delta"] & D_2 \times D_3 \times D_3
\end{tikzcd}$$

Note that $(\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{2}{3}})^*(TD_3) = (i^{34}_{12} \times i^{1}_{234})^*(\mathrm{ev} \times \mathrm{ev}_{\frac{1}{2}})^*(TD_3)$. Then

$$0 \to (\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{2}{3}})^*(TD_3) \to r_2^*(\mathrm{ev} \times \mathrm{ev}_{\frac{1}{2}})(TD_3) \to F_1 \to 0,$$

is exact where $F_1 = 0$.

(2) In the second case we have

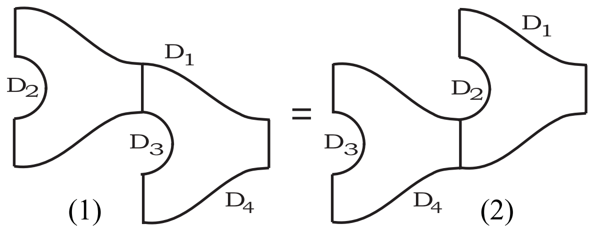
$$\begin{tikzcd}
(i^{3}_{124})^* \mathrm{ev}^*_{\frac{1}{2}}(\nu_2) \arrow[r, dashed] & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[d, "i^{3}_{124}"] & \\
& \mathcal{P}_M(D_1, D_2, D_4) \arrow[r, "i^{2}_{14}"] \arrow[d, "\mathrm{ev}_{\frac{1}{2}}"'] & \mathcal{P}_M(D_1, D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{2}} \times \mathrm{ev}_{\frac{1}{2}}"] \\
\nu_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{ev}^*_{\frac{1}{2}}(\nu_2) \arrow[r, dashed] & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[r, "i^{2}_{134}"] \arrow[d, "\mathrm{ev}_{\frac{1}{3}}"] & \mathcal{P}_M(D_1, D_3, D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{1}{3}}"] \\
\nu_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

As $(\mathrm{ev}_{\frac{1}{3}})^*(\nu_2) = (i^{3}_{124})^*(\mathrm{ev}_{\frac{1}{2}})^*(\nu_2)$, then $F_2 = 0$.

2. **Associativity of the product.**



(1)

$$\begin{tikzcd}
& & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[dl, "i^4_{123}\times i^{12}_{34}"'] \arrow[dr, "i^2_{134}"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
& \mathcal{P}_M(D_1, D_2, D_3) \times \mathcal{P}_M(D_3, D_4) \arrow[dl, "j_{12}\times j_{23}\times 1"'] \arrow[dr, "i^2_{13}\times 1"] & & \mathcal{P}_M(D_1, D_3, D_4) \arrow[dl, "j_{13}\times j_{34}"'] \arrow[dr, "i^3_{14}"] & \\
\mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_3) \times \mathcal{P}_M(D_3, D_4) & & \mathcal{P}_M(D_1, D_3) \times \mathcal{P}_M(D_3, D_4) & & \mathcal{P}_M(D_1, D_4)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[dl, "i^{34}_{12}\times i^{1}_{234}"'] \arrow[dr, "i^3_{124}"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
& \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_3, D_4) \arrow[dl, "1\times j_{23}\times j_{34}"'] \arrow[dr, "1\times i^3_{24}"] & & \mathcal{P}_M(D_1, D_2, D_4) \arrow[dl, "j_{12}\times j_{24}"'] \arrow[dr, "i^2_{14}"] & \\
\mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_3) \times \mathcal{P}_M(D_3, D_4) & & \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_4) & & \mathcal{P}_M(D_1, D_4)
\end{tikzcd}$$

First, we note that the external maps coincide.
In the diagram (1) we have

$$\begin{tikzcd}
\mathrm{ev}^*_{\frac{2}{3}}(TD_3) \arrow[r, dashed] & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[r, "i^4_{123}\times i^{12}_{34}"] \arrow[d, "\mathrm{ev}_{\frac{2}{3}}"'] & \mathcal{P}_M(D_1, D_2, D_3) \times \mathcal{P}_M(D_3, D_4) \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
TD_3 \arrow[r, dashed] & D_3 \arrow[r, "\Delta"] & D_3 \times D_3
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathfrak{i}^2_{134})^* \operatorname{ev}^*_{\frac{2}{3}}(\mathsf{T}\mathsf{D}_3) \arrow[r, dashed] & \mathcal{P}_{\mathsf{M}}(\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3, \mathsf{D}_4) \arrow[d, "\mathfrak{i}^2_{134}"'] & \\
& \mathcal{P}_{\mathsf{M}}(\mathsf{D}_1, \mathsf{D}_3, \mathsf{D}_4) \arrow[r, "\mathfrak{j}_{13} \times \mathfrak{j}_{34}"] \arrow[d, "\operatorname{ev}_{\frac{1}{2}}"'] & \mathcal{P}_{\mathsf{M}}(\mathsf{D}_1, \mathsf{D}_3) \times \mathcal{P}_{\mathsf{M}}(\mathsf{D}_3, \mathsf{D}_4) \arrow[d, "\operatorname{ev} \times \operatorname{ev}"] \\
\mathsf{T}\mathsf{D}_3 \arrow[r, dashed] & \mathsf{D}_3 \arrow[r, "\Delta"] & \mathsf{D}_3 \times \mathsf{D}_3
\end{tikzcd}$$

Note that $\operatorname{ev}_{\frac{1}{2}} \circ \mathfrak{i}^2_{134} = \operatorname{ev}_{\frac{2}{3}}$, then $\operatorname{ev}^*_{\frac{2}{3}}(\mathsf{T}\mathsf{D}_3) = (\operatorname{ev}_{\frac{1}{2}} \circ \mathfrak{i}^2_{134})^*(\mathsf{T}\mathsf{D}_3)$, and as a consequence $\mathsf{F}_1 = 0$.

In the second diagram we have

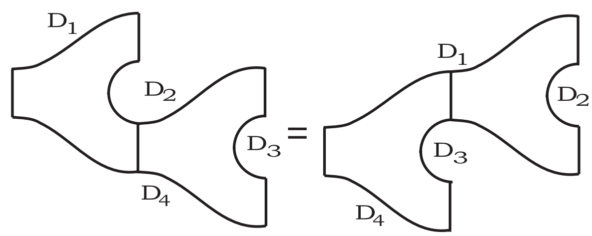
$$\begin{tikzcd}
\operatorname{ev}^*_{\frac{1}{3}}(\mathsf{T}\mathsf{D}_2) \arrow[r, dashed] & \mathcal{P}_{\mathsf{M}}(\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3, \mathsf{D}_4) \arrow[r, "\mathfrak{i}^{34}_{12} \times \mathfrak{i}^{1}_{234}"] \arrow[d, "\operatorname{ev}_{\frac{1}{3}}"'] & \mathcal{P}_{\mathsf{M}}(\mathsf{D}_1, \mathsf{D}_2) \times \mathcal{P}_{\mathsf{M}}(\mathsf{D}_2, \mathsf{D}_3, \mathsf{D}_4) \arrow[d, "\operatorname{ev} \times \operatorname{ev}"] \\
\mathsf{T}\mathsf{D}_2 \arrow[r, dashed] & \mathsf{D}_2 \arrow[r, "\Delta"] & \mathsf{D}_2 \times \mathsf{D}_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathfrak{i}^3_{124})^* \operatorname{ev}^*_{\frac{1}{2}}(\mathsf{T}\mathsf{D}_2) \arrow[r, dashed] & \mathcal{P}_{\mathsf{M}}(\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3, \mathsf{D}_4) \arrow[d, "\mathfrak{i}^3_{124}"'] & \\
& \mathcal{P}_{\mathsf{M}}(\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_4) \arrow[r] \arrow[d, "\operatorname{ev}_{\frac{1}{2}}"'] & \mathcal{P}_{\mathsf{M}}(\mathsf{D}_1, \mathsf{D}_2) \times \mathcal{P}_{\mathsf{M}}(\mathsf{D}_2, \mathsf{D}_4) \arrow[d, "\operatorname{ev} \times \operatorname{ev}"] \\
\mathsf{T}\mathsf{D}_2 \arrow[r, dashed] & \mathsf{D}_2 \arrow[r, "\Delta"] & \mathsf{D}_2 \times \mathsf{D}_2
\end{tikzcd}$$

We note that $\operatorname{ev}_{\frac{1}{3}} = \operatorname{ev}_{\frac{1}{2}} \circ \mathfrak{i}^3_{124}$. Then $\operatorname{ev}^*_{\frac{1}{3}}(\mathsf{T}\mathsf{D}_2) = (\operatorname{ev}_{\frac{1}{2}} \circ \mathfrak{i}^3_{124})^*(\mathsf{T}\mathsf{D}_2)$ and $\mathsf{F}_2 = 0$.

3. **Coassociativity of the coproduct.**



(1) (2)

(1)

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[dl, "i^1_{234}"'] \arrow[dr, "i^3_{124}"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
 & \mathcal{P}_M(D_1, D_2, D_4) \arrow[dl, "i^2_{14}"'] \arrow[dr, "j_{12} \times j_{24}"] & & \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_3, D_4) \arrow[dl, "1 \times i^3_{24}"'] \arrow[dr, "1 \times j_{23} \times j_{34}"] & \\
\mathcal{P}_M(D_1, D_4) & & \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_4) & & \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_3) \times \mathcal{P}_M(D_3, D_4)
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[dl, "j_1"'] \arrow[dr, "j_2"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
 & \mathcal{P}_M(D_1, D_3, D_4) \arrow[dl, "i^3_{14}"'] \arrow[dr, "j_{13} \times j_{34}"] & & \mathcal{P}_M(D_1, D_2, D_3) \times \mathcal{P}_M(D_3, D_4) \arrow[dl, "i^2_{13} \times 1"'] \arrow[dr, "j_{12} \times j_{23} \times 1"] & \\
\mathcal{P}_M(D_1, D_4) & & \mathcal{P}_M(D_1, D_3) \times \mathcal{P}_M(D_3, D_4) & & \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_3) \times \mathcal{P}_M(D_3, D_4)
\end{tikzcd}
$$

In the first case we have

$$
\begin{tikzcd}
\mathrm{ev}^*_{\frac{2}{3}}(\mu) \arrow[r, dashed] & \mathcal{P}_M(D_1, D_2, D_3, D_4) \arrow[r, "i^3_{124}"] \arrow[d, "\mathrm{ev}_{\frac{2}{3}}"'] & \mathcal{P}_M(D_1, D_2, D_3) \arrow[d, "\mathrm{ev}_{\frac{2}{3}} \times \mathrm{ev}_{\frac{2}{3}}"] \\
\mu \arrow[r, dashed] & D_3 \arrow[r, "\Delta"] & M \times M
\end{tikzcd}
$$

and

$$\begin{tikzcd}
(i^1_{234})^*(\mathrm{ev}\times \mathrm{ev}_{\frac{1}{2}})^*(\mu) \arrow[r, dashed] & \mathcal{P}_M(D_1,D_2,D_3,D_4) \arrow[d, "i^1_{234}"'] & \\
 & \mathcal{P}_M(D_1,D_2)\times \mathcal{P}_M(D_2,D_3,D_4) \arrow[r, "1\times i^3_{24}"] \arrow[d, "\mathrm{ev}\times \mathrm{ev}_{\frac{1}{2}}"'] & \mathcal{P}_M(D_1,D_2)\times \mathcal{P}_M(D_2,D_4) \arrow[d, "\mathrm{ev}\times \mathrm{ev}_{\frac{1}{2}}\times \mathrm{ev}_{\frac{1}{2}}"] \\
\mu \arrow[r, dashed] & D_2\times D_3 \arrow[r, "1\times\Delta"] & D_2\times M\times M
\end{tikzcd}$$

Then the sequence $0 \to \mathrm{ev}^*_{\frac{2}{3}}(\mu) \to i^*_2(\mathrm{ev}\times \mathrm{ev}_{\frac{1}{2}})^*(\mu) \to F_1 \to 0$ is exact, with $(i^1_{234})^*(\mathrm{ev}\times \mathrm{ev}_{\frac{1}{2}})^*(\mu) = \mathrm{ev}^*_{\frac{2}{3}}(\mu)$. And for that reason we conclude $F_1 = 0$.

In the second case, there is the diagram

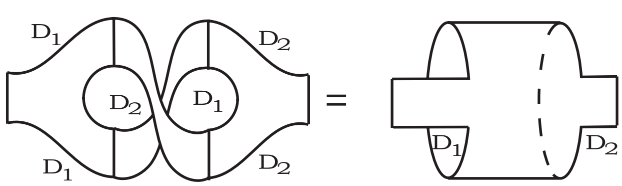
$$\begin{tikzcd}
\mathrm{ev}^*_{\frac{1}{3}}(\nu) \arrow[r, dashed] & \mathcal{P}_M(D_1,D_2,D_3,D_4) \arrow[r, "i^2_{134}"] \arrow[d, "\mathrm{ev}_{\frac{1}{3}}"'] & \mathcal{P}_M(D_1,D_3,D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{3}}\times \mathrm{ev}_{\frac{1}{3}}"] \\
\nu \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(i^4_{123})^*(\mathrm{ev}_{\frac{1}{2}}\times \mathrm{ev})^*(\nu) \arrow[r, dashed] & \mathcal{P}_M(D_1,D_2,D_3,D_4) \arrow[d, "i^4_{123}"'] & \\
 & \mathcal{P}_M(D_1,D_2,D_3)\times \mathcal{P}_M(D_3,D_4) \arrow[r, "i^2_{13}\times 1"] \arrow[d, "\mathrm{ev}_{\frac{1}{2}}\times \mathrm{ev}"'] & \mathcal{P}_M(D_1,D_3)\times \mathcal{P}_M(D_3,D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{2}}\times \mathrm{ev}_{\frac{1}{2}}\times \mathrm{ev}"] \\
\nu \arrow[r, dashed] & D_2\times D_3 \arrow[r, "1\times\Delta"] & M\times M\times D_3,
\end{tikzcd}$$

therefore $\mathrm{ev}^*_{\frac{1}{3}}(\nu) = (i^4_{123})^*(\mathrm{ev}_{\frac{1}{2}}\times \mathrm{ev})^*(\nu)$. Consequently $F_2 = 0$.

4. Cardy condition



(1) (2)

Consider

$$L_M(D_1, D_2) = \left\{\alpha : S^1 \to M : \ \alpha(0) \in D_1 \text{ and } \alpha\left(\frac{1}{2}\right) \in D_2\right\}$$

(1)

$$\begin{tikzcd}
 & & L_M(D_1,D_2) \arrow[dl, "i_2"'] \arrow[dr, "i_1"] \arrow[dll, bend right, "\eta"'] \arrow[drr, bend left, "\xi"] & & \\
 & \mathcal{P}_M(D_1,D_2,D_1) \arrow[dl, "i^2_{11}"] \arrow[dr, "\tau\circ(j_{12}\times j_{21})"] & & \mathcal{P}_M(D_2,D_1,D_2) \arrow[dl, "j_{21}\times j_{12}"'] \arrow[dr, "i^1_{22}"'] & \\
\mathcal{P}_M(D_1,D_1) & & \mathcal{P}_M(D_2,D_1)\times\mathcal{P}_M(D_1,D_2) & & \mathcal{P}_M(D_2,D_2)
\end{tikzcd}$$

where τ is the transposition map.

(2)

$$\begin{tikzcd}
 & & L_M(D_1,D_2) \arrow[dl, "j_1"'] \arrow[dr, "j_2"] \arrow[dll, bend right, "\eta"'] \arrow[drr, bend left, "\xi"] & & \\
 & L_{D_1}(M) \arrow[dl, "j_{D_1}"] \arrow[dr, "i_{D_1}"] & & L_{D_2}(M) \arrow[dl, "i_{D_2}"'] \arrow[dr, "j_{D_2}"'] & \\
\mathcal{P}_M(D_1,D_1) & & \mathcal{L}M & & \mathcal{P}_M(D_2,D_2)
\end{tikzcd}$$

Note that the next diagram is a pullback square

$$\begin{tikzcd}
L_M(D_1,D_2) \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathcal{P}_M(D_2,D_1,D_2) \arrow[d, "j_{21}\times j_{12}"] \\
\mathcal{P}_M(D_1,D_2,D_1) \arrow[r, "\tau\circ(j_{12}\times j_{21})"'] & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_2)
\end{tikzcd}$$

Then, for the first case

$$\begin{tikzcd}
ev^*(TD_1) \arrow[r, dashed] & L_M(D_1, D_2) \arrow[r, "i_1"] \arrow[d, "ev"'] & \mathcal{P}_M(D_1, D_2, D_1) \arrow[d, "ev_0 \times ev_1"] \\
TD_1 \arrow[r, dashed] & D_1 \arrow[r, "\Delta"'] & D_1 \times D_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
(i_2)^* ev^*(TD_1) \arrow[r, dashed] & L_M(D_1, D_2) \arrow[d, "i_2"'] & \\
 & \mathcal{P}_M(D_2, D_1, D_2) \arrow[r, "\tau\circ(j_{21}\times j_{12})"] \arrow[d, "ev"'] & \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_1) \arrow[d, "ev \times ev"] \\
TD_1 \arrow[r, dashed] & D_1 \arrow[r, "\Delta"'] & D_1 \times D_1
\end{tikzcd}$$

The next equality holds $ev^*(TD_1) = (ev \circ i_2)^*(TD_1)$. And we conclude $F_1 = 0$. In the second case

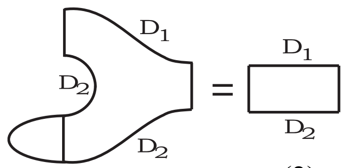
$$\begin{tikzcd}
ev^*(\zeta) \arrow[r, dashed] & L_M(D_1, D_2) \arrow[r, "j_1"] \arrow[d, "ev"'] & L_{D_1}(M) \arrow[d, "ev_{\frac{1}{2}} \times ev_{\frac{1}{2}}"] \\
\zeta \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
j_2^* ev^*(\zeta) \arrow[r, dashed] & L_M(D_1, D_2) \arrow[d, "j_2"'] & \\
 & L_{D_2}(M) \arrow[r, "i_{D_2}"] \arrow[d, "ev"'] & \mathcal{L}M \arrow[d, "ev \times ev"] \\
\zeta \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

In the same way $ev^*(\zeta) = (ev \circ j_2)^*(\zeta)$, then $F_2 = 0$.

5. **Unit axiom**



(1) (2)

(1)

$$\begin{tikzcd}
 & & \mathcal{P}_M(D_1,D_2) \arrow[dl, "1\times\epsilon_1"'] \arrow[dr, "i"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
 & \mathcal{P}_M(D_1,D_2)\times D_2 \arrow[dl, "1\times r"'] \arrow[dr, "1\times\iota"] & & \mathcal{P}_M(D_1,D_2,D_2) \arrow[dl, "j_{12}\times j_{22}"'] \arrow[dr, "i^2_{12}"] & \\
\mathcal{P}_M(D_1,D_2)\times \mathrm{pt} & & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_2) & & \mathcal{P}_M(D_1,D_2)
\end{tikzcd}$$

First, we note that the next diagram is a pullback square.

$$\begin{tikzcd}
\mathcal{P}_M(D_1,D_2) \arrow[r, "i"] \arrow[d, "1\times\epsilon_1"'] & \mathcal{P}_M(D_1,D_2,D_2) \arrow[d, "j_{12}\times j_{22}"] \\
\mathcal{P}_M(D_1,D_2)\times D_2 \arrow[r, "1\times\iota"'] & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_2)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_M(D_1,D_2) \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[ddll, bend right, "\mathrm{id}"'] \arrow[ddrr, bend left, "\mathrm{id}"] & & \\
 & \mathcal{P}_M(D_1,D_2) \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & & \mathcal{P}_M(D_1,D_2) \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
\mathcal{P}_M(D_1,D_2) & & \mathcal{P}_M(D_1,D_2) & & \mathcal{P}_M(D_1,D_2)
\end{tikzcd}$$

It is clear that for the second diagram we have $F_2 = 0$. Basically we have $\eta = \mathtt{id}$ and $\xi \simeq \mathtt{id}$, then $\xi_* = \mathtt{id}_*$. In the first diagram the umkher map $(1 \times \epsilon_1)!$ due to the next square

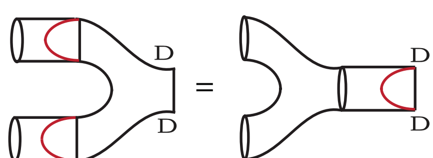
$$\begin{tikzcd}
ev^*(TD_2) \arrow[r, dashed] & \mathcal{P}_M(D_1, D_2) \arrow[r, "1\times\epsilon_1"] \arrow[d, "ev"'] & \mathcal{P}_M(D_1, D_2) \times D_2 \arrow[d, "ev \times \mathtt{id}"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & D_2 \times D_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
i^* \, ev^*_{\frac{1}{2}}(TD_2) \arrow[r, dashed] & \mathcal{P}_M(D_1, D_2) \arrow[d, "i"] & \\
 & \mathcal{P}_M(D_1, D_2, D_2) \arrow[r, "j_{12}\times j_{22}"] \arrow[d, "ev_{\frac{1}{2}}"'] & \mathcal{P}_M(D_1, D_2) \times \mathcal{P}_M(D_2, D_2) \arrow[d, "ev \times ev"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & D_2 \times D_2
\end{tikzcd}$$

Since $(ev_{\frac{1}{2}} \circ i)^*(TD_2) = ev^*(TD_2)$, then $F_1 = 0$.

6. **ι_D is morphism of algebras**



(1) (2)

Let be

$$\mathrm{Map}_D(8, M) = \{\alpha : 8 \to M : \ \alpha(0) \in D\}$$

(1)

$$
\begin{tikzcd}
 & & \mathrm{Map}_{D}(8, M) \arrow[dl, "i"'] \arrow[dr, "j"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
 & \mathrm{Map}(8, M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & & L_D(M) \arrow[dl, "i_D"'] \arrow[dr, "j_D"] & \\
\mathcal{L}M \times \mathcal{L}M & & \mathcal{L}M & & \mathcal{P}_M(D, D)
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
 & & \mathrm{Map}_{D}(8, M) \arrow[dl, "i_1 \times i_1"'] \arrow[dr, "j_1"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
 & L_D(M) \times L_D(M) \arrow[dl, "i_D \times i_D"'] \arrow[dr, "j_D \times j_D"] & & \mathcal{P}_M(D, D, D) \arrow[dl, "j_{11} \times j_{11}"'] \arrow[dr, "i^1_{11}"] & \\
\mathcal{L}M \times \mathcal{L}M & & \mathcal{P}_M(D, D) \times \mathcal{P}_M(D, D) & & \mathcal{P}_M(D, D)
\end{tikzcd}
$$

In the first diagram there is the square

$$
\begin{tikzcd}
\mathrm{ev}^*(\rho) \arrow[r, dashed] & \mathrm{Map}_{D}(8, M) \arrow[r, "i"] \arrow[d, "\mathrm{ev}"'] & \mathrm{Map}(8, M) \arrow[d, "\mathrm{ev} \times \mathrm{ev}"] \\
\rho \arrow[r, dashed] & D \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
j^* \,\mathrm{ev}^*(\rho) \arrow[r, dashed] & \mathrm{Map}_{D}(8, M) \arrow[d, "j"'] & \\
 & L_D(M) \arrow[r, "i_D"] \arrow[d, "\mathrm{ev}"'] & \mathcal{L}M \arrow[d, "\mathrm{ev} \times \mathrm{ev}"] \\
\rho \arrow[r, dashed] & D \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}
$$

Clearly $\mathrm{ev}^*(\rho) = j^* \,\mathrm{ev}^*(\rho)$. Then $F_1 = 0$.

On the other hand in (2) we have

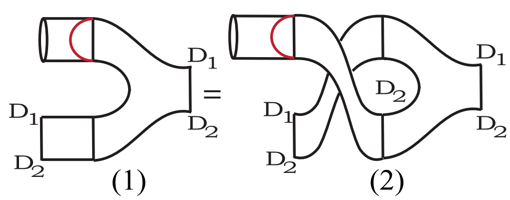
$$\begin{tikzcd}
\mathrm{ev}^*(\mathsf{TD}) \arrow[r, dashed] & \mathrm{Map}_{\mathsf{D}}(8, M) \arrow[r] \arrow[d, "\mathrm{ev}"'] & \mathsf{L}_{\mathsf{D}}(M) \times \mathsf{L}_{\mathsf{D}}(M) \arrow[d, "\mathrm{ev} \times \mathrm{ev}"] \\
\mathsf{TD} \arrow[r, dashed] & \mathsf{D} \arrow[r, "\Delta"] & \mathsf{D} \times \mathsf{D}
\end{tikzcd}$$

and

$$\begin{tikzcd}
j'^* \, \mathrm{ev}^*_{\frac{1}{2}}(\mathsf{TD}) \arrow[r, dashed] & \mathrm{Map}_{\mathsf{D}}(8, M) \arrow[d, "j_1"'] & \\
& \mathcal{P}_M(\mathsf{D}, \mathsf{D}, \mathsf{D}) \arrow[r, "j_{11} \times j_{11}"] \arrow[d, "\mathrm{ev}_{\frac{1}{2}}"'] & \mathcal{P}_M(\mathsf{D}, \mathsf{D}) \times \mathcal{P}_M(\mathsf{D}, \mathsf{D}) \arrow[d, "\mathrm{ev}_1 \times \mathrm{ev}_0"] \\
\mathsf{TD} \arrow[r, dashed] & \mathsf{D} \arrow[r, "\Delta"] & \mathsf{D} \times \mathsf{D}
\end{tikzcd}$$

As before, $j_1^* \, \mathrm{ev}^*(\mathsf{TD}) = \mathrm{ev}^*(\mathsf{TD})$. Consequently $F_2 = 0$.

7. ι is a central morphism



(1)

$$\begin{tikzcd}[column sep=small]
& & & \mathsf{L}_{\mathsf{D}_1}(\mathsf{T})_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_M(\mathsf{D}_1, \mathsf{D}_2) \arrow[ddlll, bend right, "\eta"'] \arrow[dll, "i"'] \arrow[drr, "j"] \arrow[ddrrr, bend left, "\xi"] & & & \\
& \mathsf{L}_{\mathsf{D}_1}(M) \times \mathcal{P}_M(\mathsf{D}_1, \mathsf{D}_2) \arrow[dl, "i_{\mathsf{D}_1} \times 1"] \arrow[drr, "j_{\mathsf{D}_1} \times 1"] & & & & \mathcal{P}_M(\mathsf{D}_1, \mathsf{D}_1, \mathsf{D}_2) \arrow[dll, "j_{11} \times j_{12}"'] \arrow[dr, "i^1_{12}"] & \\
\mathcal{L}M \times \mathcal{P}_M(\mathsf{D}_1, \mathsf{D}_2) & & & \mathcal{P}_M(\mathsf{D}_1, \mathsf{D}_1) \times \mathcal{P}_M(\mathsf{D}_1, \mathsf{D}_2) & & & \mathcal{P}_M(\mathsf{D}_1, \mathsf{D}_2)
\end{tikzcd}$$

(2)

$$
\begin{tikzcd}
 & & \mathcal{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0} L_{D_2}(M) \arrow[dl, "\tau\circ i'"'] \arrow[dr, "j'"] \arrow[ddll, bend right, "\eta'"'] \arrow[ddrr, bend left, "\xi'"] & & \\
 & L_{D_2}(M)\times \mathcal{P}_M(D_1,D_2) \arrow[dl, "i_{D_2}\times 1"'] \arrow[dr, "\tau\circ(j_{D_2}\times 1)"] & & \mathcal{P}_M(D_1,D_2,D_2) \arrow[dl, "j_{12}\times j_{22}"'] \arrow[dr, "i^2_{12}"] & \\
\mathcal{L}M\times \mathcal{P}_M(D_1,D_2) & & \mathcal{P}_M(D_1,D_2)\times \mathcal{P}_M(D_2,D_2) & & \mathcal{P}_M(D_1,D_2)
\end{tikzcd}
$$

Note that in the last case we have that the pullback spaces are different. For this particular case we use the corollary 22.7, for this, we first need to prove that $L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_M(D_1,D_2)$ and $L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_M(D_1,D_2)$ are homotopically equivalent spaces. For this we construct the maps.

We define the map

$$
\begin{array}{rccc}
\varphi: & \varphi: L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_M(D_1,D_2) & \longrightarrow & \mathcal{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0} L_{D_2}(T) \\
 & (\alpha,\beta) & \longmapsto & (\beta,\overline{\beta}*\alpha*\beta),
\end{array}
$$

and in the same way define

$$
\begin{array}{rccc}
\psi: & \mathcal{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0} L_{D_2}(T) & \longrightarrow & L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_M(D_1,D_2) \\
 & (\gamma,\delta) & \longmapsto & (\gamma*\delta\overline{\gamma},\gamma).
\end{array}
$$

See these maps in Figure 19.

Now we check that these maps determine a homotopy equivalence.

$$
\begin{aligned}
\psi\circ\varphi(\alpha,\beta) &= \psi(\beta,\overline{\beta}*\alpha*\beta) = (\alpha,\overline{\alpha}*\alpha*\beta*\overline{\alpha}*\alpha)\simeq(\alpha,\beta)\\
\varphi\circ\psi(\gamma,\delta) &= \varphi(\gamma*\delta*\overline{\gamma},\gamma) \ = (\gamma,\overline{\gamma}*\gamma*\delta*\overline{\gamma}*\gamma)\simeq(\gamma,\delta).
\end{aligned}
$$

Finally we need to check that the external maps are homotopic.

$$
\begin{aligned}
\eta'\circ\varphi(\alpha,\beta) &= \eta'(\beta,\overline{\beta}*\alpha*\beta) \quad (\overline{\beta}*\alpha*\beta,\beta)\simeq(\alpha,\beta)\\
\eta(\alpha,\beta) &= (\alpha,\beta)\\
\xi'\circ\varphi(\alpha,\beta) &= \xi'(\beta,\overline{\beta}*\alpha*\beta) = \beta*\overline{\beta}*\alpha*\beta\simeq\alpha*\beta\\
\xi(\alpha,\beta) &= (\alpha*\beta)\\
\eta\circ\psi(\gamma,\delta) &= \eta(\gamma*\delta*\overline{\gamma},\gamma) \ = (\gamma*\delta*\overline{\gamma},\gamma)\simeq(\delta,\gamma)\\
\eta'(\gamma,\delta) &= (\delta,\gamma)\\
\xi\circ\psi(\gamma,\delta) &= \xi(\gamma*\delta*\overline{\gamma},\gamma) \ = \gamma*\delta*\overline{\gamma}*\gamma\simeq\gamma*\delta\\
\xi'(\gamma,\delta) &= \gamma*\delta
\end{aligned}
$$

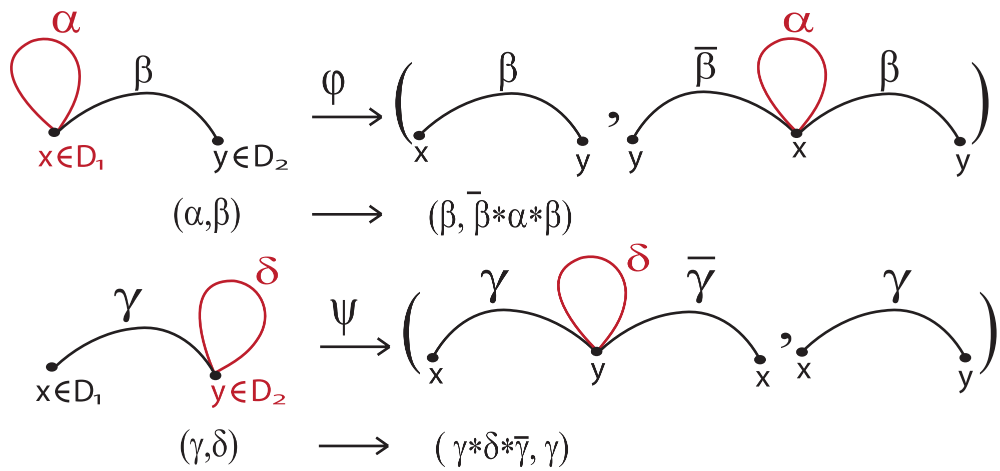


Figure 19: `The map` $\varphi : L_{D_1}(T)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_M(D_1, D_2) \to L_{D_1}(T)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_M(D_1, D_2)$

Then, we can use Corollary 22.7. It remains to calculate the Euler classes.

In the first diagram we have

$$\begin{tikzcd}
ev_\infty^*(TD_1) \arrow[r, dashed] & L_{D_1}(T)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_M(D_1, D_2) \arrow[r, "i"] \arrow[d, "ev_\infty"'] & L_{D_1} \times \mathcal{P}_M(D_1, D_2) \arrow[d, "\epsilon_1 \times \epsilon_0"] \\
TD_1 \arrow[r, dashed] & D_1 \arrow[r, "\Delta"'] & D_1 \times D_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
j^* ev_{\frac{1}{2}}^*(TD_1) \arrow[r, dashed] & L_{D_1}(T)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_M(D_1, D_2) \arrow[d, "j"'] & \\
 & \mathcal{P}_M(D_1, D_1, D_2) \arrow[r, "j_{11} \times j_{12}"] \arrow[d, "ev_{\frac{1}{2}}"'] & \mathcal{P}_M(D_1, D_1) \times \mathcal{P}_M(D_1, D_2) \arrow[d, "\epsilon_1 \times \epsilon_0"] \\
TD_1 \arrow[r, dashed] & D_1 \arrow[r, "\Delta"'] & D_1 \times D_1
\end{tikzcd}$$

Note that $j^* ev_{\frac{1}{2}}^*(TD_1) = ev_\infty^*(TD_1)$. Then $F_1 = 0$.

In the second diagram there is the square

$$\begin{tikzcd}
ev_\infty^*(TD_2) \arrow[r, dashed] \arrow[d, dashed] & \mathcal{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0} L_{D_2}(M) \arrow[r, "\tau\circ i'"] \arrow[d, "ev_\infty"] & L_{D_2}(M)\times\mathcal{P}_M(D_1,D_2) \arrow[d, "\epsilon_0\times\epsilon_1"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & D_2\times D_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
j'^*\,ev_{\frac{1}{2}}^*(TD_2) \arrow[r, dashed] & \mathcal{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0} L_{D_2}(M) \arrow[d, "j'"] & \\
 & \mathcal{P}_M(D_1,D_2,D_2) \arrow[r, "j_{12}\times j_{22}"] \arrow[d, "ev_{\frac{1}{2}}"] & \mathcal{P}_M(D_1,D_2)\times\mathcal{P}_M(D_2,D_2) \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & D_2\times D_2
\end{tikzcd}$$

Clearly $j'^*\,ev_{\frac{1}{2}}^*(TD_2)$ and $ev_\infty^*(TD_2)$ coincide, then $F_2 = 0$.
Finally, we need to determine that $\nu_\varphi = 0$. For this we will construct the next homotopy.

$$\begin{array}{rccc}
H: & I\times(L_{D_1}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_M(D_1,D_2) & \longrightarrow & \mathcal{L}M_{\epsilon_1}\times_{\epsilon}\mathcal{P}_M(D_1,D_2)\times I \\
 & (s,(\alpha,\beta)) & \longmapsto & (\overline{\beta_s}*\alpha*\beta_s,\beta,s)
\end{array}$$

where the map $\epsilon : I\times\mathcal{P}_M(D_1,D_2)\to M$ is given by $\epsilon(s,\beta) := \beta(s)$, and the curve $\beta_s : I\to M$ is $\beta_s(t) = \beta(st)$ for all $t,s\in I$.

Note that $H(0,(\alpha,\beta)) = (\alpha,\beta)$ and $H(1,(\alpha,\beta)) = (\overline{\beta}*\alpha*\beta,\beta) = \tau\circ\varphi(\alpha,\beta)$. Now we need to prove that these spaces of infinite dimension have a smooth structure i.e. a infinite dimensional manifold; see [KM91]. The space $W := \mathcal{L}M_{\epsilon_1}\times_{\epsilon}\mathcal{P}_M(D_1,D_2)\times I$ is determined by the next pullback square.

$$\begin{tikzcd}
W = \mathcal{L}M_{\epsilon_1}\times_{\epsilon}\mathcal{P}_M(D_1,D_2)\times I \arrow[r] \arrow[d, "\epsilon\times 1"] & \mathcal{L}M\times\mathcal{P}_M(D_1,D_2)\times I \arrow[d, "\epsilon_0\times\epsilon\times 1"] \\
M\times I \arrow[r, "\Delta\times 1"'] & M\times M\times I
\end{tikzcd}$$

Then W is a infinite dimensional manifold. In the other hand, the next pullback square give us that the spaces $Z_s := L_{D_1}M_{\epsilon_1}\times_{\epsilon_s}\mathcal{P}_M(D_1,D_2)$ are submanifolds of W of codimension one.

$$\begin{tikzcd} Z_s = L_{D_1}M_{\epsilon_1} \times_{\epsilon_s} \mathcal{P}_M(D_1, D_2) \times \{s\} \arrow[r] \arrow[d, "\epsilon_\infty \times s"'] & \mathcal{L}M_{\epsilon_1} \times_{\epsilon} \mathcal{P}_M(D_1, D_2) \times I \arrow[d, "\epsilon_\infty \times 1"] \\ M \times \{s\} \arrow[r, hook] & M \times I \end{tikzcd}$$

In particular we have the next situation

$$\begin{tikzcd} Z_0 = L_{D_1}M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_M(D_1, D_2) \arrow[d, "\mathrm{Id}"'] & & Z_0 = L_{D_1}M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_M(D_1, D_2) \arrow[d, "\varphi"] \\ {} \arrow[r, phantom, "\overset{H}{\underset{\simeq}{\Longrightarrow}}"] & {} & {} \\ Z_0 = L_{D_1}M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_M(D_1, D_2) & & Z_1 = \mathcal{P}_M(D_1, D_2)_{\epsilon_1} \times_{\epsilon_0} L_{D_2}M \end{tikzcd}$$

Then $\nu_\varphi = 0$ and $e(\nu_\varphi) = 1$.

♣

This result holds as well at the chain level (cf. [BCT09]).

7 Orbifolds and their Mapping Spaces

7.1 Orbifolds

The notion of orbifold was first introduced by Satake in his seminal paper [Sat56]. In this 1956 paper Satake defines for the very first time the concept of an orbifold by means of orbifold atlases whose charts Satake calls local uniformizing systems. The name that orbifolds take in this early work are $\mathsf{V}-$*manifolds.* Quite remarkably he already works with a version of Čech groups. He goes on to prove the De Rham theorem and Poincaré duality with rational coefficients. For about two decades the japanese school carried out brilliantly the study of orbifolds. It deserves special mention the work of Tetsuro Kawasaki. In his papers of the late 70's Kawasaki generalizes index theory to the orbifold setting [Kaw78, Kaw79, Kaw81]. Another important work along these veins in the work of Thurston specially his concept of orbifold fundamental group [Thu97]

Somewhat independently the algebraic geometers developed the concept of *stack* in order to deal with moduli problems. As it happens orbifolds arise quite naturally from the very same moduli problems and it did not take long to realize that the theory of stacks provided another way of understanding the category of orbifolds, and viceversa. For example, the Deligne-Mumford moduli stack $\mathcal{M}_{\mathsf{g}}$ for genus g curves [DM69] is in fact an orbifold. This is one of the reasons for the importance of orbifolds, many moduli spaces are better understood as orbifolds. The paper of Artin [Art74] is the place where a very explicit conection with groupoid atlases takes place for the first time. Implicitly these ideas are already present in Grothendieck's toposes [Gro72]. The groupoiod approach to orbifolds is finally carried out by Haefliger [Hae84] and by Moerdijk and his collaborators [Moe91, MP99, CM00, MP97]. In this work they put forward the important concept of Morita equivalence.

The interest of orbifolds in physics can be traced back to the work of Dixon, Harvey, Vafa and Witten [DHVW85, DHVW86] who where motivated by superstring compactification to introduce an orbifold theory using a K3 with 27 singular points. It is there that the orbifold Euler characteristic is defined motivated by the physics. It is a remarkably insightful notion of their work to realize that their results depend only on the orbifold and not on group actions, for all their examples are global orbifolds. This work produced an explosion of activity related to orbifolds in the physics community. The introduction to the mathematics side of the geometrization of many of these ideas and results is due to Chen and Ruan. Their highly influential papers [CR04a, Rua02b] introduced many concepts from the physics literature rigorously into symplectic and algebraic geometry. In this book orbifolds are often completely general, not necessarily global quotients.

7.1.1 Group Actions

Given a space M we often want to study all its self-transformations that preserve some of its properties. Often such transformations are called *symmetries* and often they are also called *automorphisms*.

Example 7.1. Consider a triangle T as a subset of $\mathbb{R}^2$. We may ask how many mappings $g \colon T \to T$ there are with the property

$$d(x, y) = d(g(x), g(y))$$

for every pair of points in the triangle, where d denotes the usual distance. Such a map is called an *isometry* of the triangle.

The answer of course depends very much on the triangle.

- If the triangle is scalene only the identity is an isometry of T.
- If the triangle is isosceles then there are two such isometries.
- If the triangle is equilateral there are six isometries of T.

This can be verified by noticing that an isometry is completely determined by its restriction to the vertices.

Here, as we all know, we can take a remarkable conceptual leap: *we decide to remember how the different symmetries interact* rather than the symmetries themselves. For this we observe that

- If g and h are symmetries of T so is $g \circ h = gh$.
- $(gh)k = g(hk)$
- There is always the identity symmetry 1_T.
- Given a symmetry g there is another symmetry k such that $gk = kg = 1_T$.

This motivates the definition of (abstract) **group** [Lan02]. A group is a set of things, that together with a composition law that satisfy all the previous axioms. We say for example that the isometries of T form a group.

Once we have this definition we end up with groups that are (at first) not naturally the symmetries of anything. For example, the fundamental group of a space X is at first an abstract group formed with homotopy classes of paths. In this case it may come as a surprise to learn that $\pi_1(X)$ in fact acts as some sort of symmetry, namely as *deck* transformations of the universal cover $M = \widetilde{X}$. It is often important to realize that an (abstract) group is indeed a group of transformations of some space M.

Definition 7.1. We say that the group G *acts on* the object M if we are given a homomorphism

$$\psi\colon G \to \mathrm{Aut}(M),$$

Namely, for every $g \in G$ and every $m \in M$ we have

- $mg = \psi(g)(m) \in M$ such that
- $m1_M = m$
- $(mg)h = m(gh)$

Definition 7.2. We say that the group G acts *effectively* on the object M if $\psi\colon G \to \mathrm{Aut}(M)$ is injective, namely for all $g \in G, g \neq 1$ there is an $m \in M$ so that $mg \neq m$.

Definition 7.3. The equivalence relation *induced* by the action of G on M is the relation generated by

$$x \sim xg.$$

The quotient $M/\sim$ is also written

$$M/G.$$

The equivalence classes of this relation are called the *orbits* of the action. They are written

$$[m] = m \cdot G = \{mg | g \in G\}.$$

If there is only one equivalence class (orbit) for the action we say that G acts *transitively* on M.

Definition 7.4. The stabilizer subgroup of $m \in M$ is

$$G_m = \{g \in G | mg = m\}.$$

Notice that even effective actions often have nontrivial stabilizers.

Proposition 7.5. *If* G *acts on* M *then*

$$G/G_m \simeq m \cdot G$$

as sets.

Example 7.2. Let $M = P$ be the set of all lines in $\mathbb{R}^3$ containing the origin. Then the group of all linear automorphisms of $\mathbb{R}^3$, $G = \mathrm{GL}_3(\mathbb{R})$ acts on M. Let $m \in M$ be the x axis. Then it is not hard to see that $G_m = \mathrm{GL}_2(\mathbb{R})$ and therefore

$$P = \mathrm{GL}_3(\mathbb{R})/\mathrm{GL}_2(\mathbb{R})$$

We write

$$p \colon M \to M/G$$

for the mapping

$$m \mapsto [m]$$

If M is a topological space and G acts on M then we can put a natural topology on M/G, namely a subset U of M/G is declared to be open if and only if $p^{-1}(U)$ is open in M.

Example 7.3. $\widetilde{X}/\pi_1(X) \simeq X$.

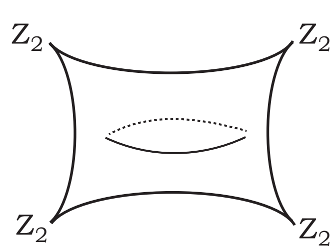
There are quotients in the category of sets, and also in the category of topological spaces.

But the category of smooth manifolds is quite unlike the category of sets or of topological spaces (for manifolds have structure sheafs).

7.1.2 Examples

Let $M = T^2 = S^1 \times S^1$ be a two-dimensional torus, and let $G = \mathbb{Z}_2$ be the *finite* subgroup of diffeomorphisms of M given by the action

$$(z, w) \mapsto (\bar{z}, \bar{w})$$



Example 7.4. Show that while the quotient space $\mathsf{X} = \mathsf{M}/\mathsf{G}$ is topologically a sphere it is impossible to put a smooth structure on X so that the quotient map $\mathsf{M} \longrightarrow \mathsf{X}$ will become smooth. It is in this sense that we say that X is not a smooth manifold.

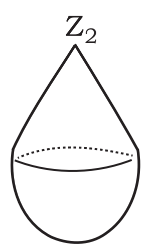
What will enlarge the category of smooth manifolds to a bigger category is called the category of *orbifolds*. Once we do this, when we consider the orbifolds M and $\mathcal{X}$ then the natural orbifold morphism $\mathsf{M} \longrightarrow \mathcal{X}$ becomes smooth.

While the orbifold M contains exactly the same amount of information as M the orbifold $\mathcal{X} = [\mathsf{M}/\mathsf{G}]$ (known as a pillowcase) contains more information that the quotient space $\mathsf{X} = \mathsf{M}/\mathsf{G}$. For instance $\mathcal{X}$ remembers that the action had 4 fixed points each with stabilizer G. It remembers in fact the stabilizer of every point, and how these stabilizers fit together. On the other hand $\mathcal{X}$ does not remember neither the manifold M nor the group G. In fact if we define N to be two disjoint copies of M and $\mathsf{H} = \mathsf{G} \times \mathsf{G}$ to act on M by letting $\mathsf{G} \times 1$ act by complex conjugation on both copies as before, and $1 \times \mathsf{G}$ act by swapping the copies then

$$\mathcal{X} = [\mathsf{M}/\mathsf{G}] \cong [\mathsf{N}/\mathsf{H}].$$

Not every orbifold can be obtained from a finite group acting on a manifold. An orbifold is always *locally* the quotient of a manifold by a finite group but this may fail globally.

For example consider the teardrop $\mathsf{W}(1, 2)$:



This orbifold may be obtained by gluing two global quotients. Consider the orbifold $\mathcal{X}_1 = [\mathbb{C}/\mathbb{Z}_2]$ where $\mathbb{Z}_2$ acts by the holomorphic automorphism $z \mapsto -z$. Let $\mathcal{X}_2 = \mathbb{C}$ simply be the complex line. Then we have in the category of orbifolds a diagram of inclusions

$$\mathcal{X}_1 \longleftarrow \mathbb{C}^* \longrightarrow \mathcal{X}_2$$

and therefore we can glue $\mathcal{X}_1$ and $\mathcal{X}_2$ along $\mathbb{C}^*$ to obtaining the teardrop $\mathcal{X}$.

Example 7.5. An important remark: there are orbifolds X that *cannot* be represented by a groupoid of the form $[M/G]$. In other words, in spite of the fact that there is indeed a groupoid representing X, nevertheless there is no manifold M with a *finite* group action G so that $X \cong [M/G]$. We say in this situation that the orbifold in question is *not* a *global quotient*. Examples are given by the toric orbifolds $W(a_0, \dots, a_n)$ whose quotient spaces are the weighted projective spaces $\mathbf{P}(a_0, \dots, a_n)$ (here a_i are coprime positive integers). For simplicity, let us discuss the case of the orbifold $W(1, 2)$ whose quotient space is the weighted projective line $\mathbf{P}(1, 2) \cong \mathbf{P}^1$. One way to describe $W(1, 2)$ is through the system of local charts:

$$\begin{tikzcd} & {[\mathbb{C}^\times/\{1\}]} \arrow[dl, "z\mapsto 1/z^2"'] \arrow[dr, "z\mapsto z"] & \\ {[\mathbb{C}/\mathbb{Z}_2]} & & {[\mathbb{C}/\{1\}].} \end{tikzcd}$$

If $W(1, 2)$ were Morita equivalent to a groupoid $[M/G]$, then this would induce a homomorphism $\rho\colon G \to \mathbb{Z}_2$ (this follows by looking at the unique point in $W(1, 2)$ with isotropy $\mathbb{Z}_2$). Therefore the orbifold $[M'/\mathbb{Z}_2]$ with $M' := M/\ker(\rho)$ would be equivalent to $W(1, 2)$. But this is a contradiction because any action of $\mathbb{Z}_2$ in a compact surface cannot have only one fixed point.

This example might be a source of misunderstanding because weighted projective spaces are indeed quotient varieties of manifolds by actions of finite groups. For instance, in our example, $\mathbf{P}(1, 2)$ is isomorphic to the quotient of $\mathbf{P}^1$ by $\mathbb{Z}/2\mathbb{Z}$ under the action $[x, y] \mapsto [x, -y]$ in homogeneous coordinates. On the other hand, although the orbifold $W(1, 2)$ can be presented as a quotient of a manifold by an action of a Lie group, namely $[\mathbb{C}^2 - \{0\}/\mathbb{C}^\times]$ with $\lambda \cdot (x, y) \mapsto (\lambda^2 x, \lambda y)$, it is not equivalent to global quotient by a finite group. It is worth pointing out that it is still an open question whether every compact orbifold can be presented (up to Morita equivalence) as the quotient of a manifold by a Lie group [HM04].

There are several definitions of the concept of an orbifold. The first one due to Satake [Sat57] was written using the so-called orbifold atlases, unfortunately quite a few concepts are a bit cumbersome using this definition. We opt to think of an orbifold as a certain kind of category following Grothendieck, Haefliger and Moerdijk [Moe02].

7.1.3 Groupoids

In this section we construct the category of orbifolds. It contains the category of finite groups and also the category of manifolds. The category of orbifolds extends both categories at the same time.

Example 7.6. The most familiar situation in physics is that of an orbifold of the type $\mathcal{X} = [\mathsf{M}/\mathsf{G}]$, where M is a smooth manifold and G is a finite group acting smoothly on M; namely, we give ourselves a homomorphism $\mathsf{G} \to \mathrm{Diff}(\mathsf{M})$. We will consider mostly right actions. Thus, instead of writing gx for the action of g in x we will write xg, the action being $(x, g) \mapsto xg$. We make a point of distinguishing the orbifold $\mathcal{X} = [\mathsf{M}/\mathsf{G}]$ from its quotient space (also called orbit space) $\mathsf{X} = \mathsf{M}/\mathsf{G}$. As a set, as we know, a point in X is an orbit of the action: that is, a typical element of M/G is $\mathrm{Orb}(x) = \{xg \mid g \in \mathsf{G}\}$.

For us an orbifold $\mathcal{X} = [\mathsf{M}/\mathsf{G}]$ is a smooth category (actually a topological groupoid) whose objects are the points of M, $\mathcal{X}_0 = \mathrm{Obj}(\mathcal{X}) = \mathsf{M}$, and we insist on remembering that $\mathsf{X}_0 = \mathrm{Obj}(\mathcal{X})$ is a *smooth manifold*. The arrows of this category are $\mathcal{X}_1 = \mathrm{Mor}(\mathcal{X}) = \mathsf{M} \times \mathsf{G}$ again thinking of it as a smooth manifold. A typical arrow in this category is

$$x \xrightarrow{(x,g)} xg,$$

and the composition of two arrows looks like

$$\begin{tikzcd} x \arrow[r, "{(x,g)}"] \arrow[rr, bend right, "{(x,gh)}"'] & xg \arrow[r, "{(xg,h)}"] & xgh. \end{tikzcd}$$

As we have already pointed out, an important property of this category is that it is actually a groupoid: indeed, every arrow (x, g) has an inverse (depending smoothly on (x, g)), to wit $(x, g)^{-1} = (xg, g^{-1})$.

To be fair, the definition of an orbifold is somewhat more complicated. First, we must impose some technical conditions on the groupoids that we will be working with. Second, we must consider an equivalence relation (usually called *Morita equivalence*, related to equivalence of categories) on the family of all smooth groupoids. Then one can roughly say that an orbifold is an equivalence class of groupoids [Moe02, LU04a]. For a nice motivation to the definition of a groupoid see [Wei96] and [Wei01]. Choosing a particular groupoid to represent an orbifold is akin to choosing coordinates for a physical system, and clearly the theories we are interested in should be invariant under such freedom of choice.

For example, consider the manifold $\mathsf{N} = \mathsf{M} \times \mathbb{Z}_2$ consisting of two disjoint copies of M, and the group $\mathsf{H} = \mathsf{G} \times \mathbb{Z}_2$, and let H act on N by the formula

$$(m, \epsilon_0) \cdot (g, \epsilon_1) = (mg, \epsilon_0\epsilon_1).$$

Then not only are $\mathsf{N}/\mathsf{H} \cong \mathsf{M}/\mathsf{G}$ homeomorphic, but moreover $\mathsf{X} \cong [\mathsf{N}/\mathsf{H}] \cong [\mathsf{M}/\mathsf{G}]$ are equivalent groupoids, while clearly $\mathsf{N} \neq \mathsf{M}$ and $\mathsf{H} \neq \mathsf{G}$.

Definition 7.6. A Lie groupoid G is a category in which every morphism is invertible such that G_0 and G_1, the sets of objects and morphism respectively, are *smooth manifolds.* We will denote the structure maps by:

$$G_1 \, {}_t\!\times_s G_1 \xrightarrow{m} G_1 \xrightarrow{i} G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0 \xrightarrow{e} G_1$$

where s and t are the source and the target maps, m is the composition (we can compose two arrows whenever the target of the first equals the source of the second), i gives us the inverse arrow, and e assigns the identity arrow to every object. We will assume that all the structure maps are smooth maps. We also require the maps s and t to be submersions, so that $G_1 \, {}_t\!\times_s G_1$ is also a manifold.

Definition 7.7. The *stabilizer* G_x of a groupoid G on $x \in G_0$ is the set of arrows whose source and target are both x. Notice that G_x is a group.

Definition 7.8. A topological (Lie) groupoid is called *étale* if the source and target maps s and t are local homeomorphisms (local diffeomorphisms).

For an étale groupoid we will mean a topological étale groupoid.

We will always denote groupoids by letters of the type G, H, S.

We will also assume that the anchor map $(s,t) : G_1 \to G_0 \times G_0$ is proper, groupoids with this property are called *proper groupoids.* This will force all stabilizers to be finite.

Definition 7.9. A morphism of groupoids $\Psi : H \to G$ is a pair of maps $\Psi_i : H_i \to G_i$ $i = 0, 1$ such that they commute with the structure maps. The maps Ψ_i will be required to be smooth.

The morphism Ψ is called *Morita* if the following square is a cartesian square .

$$\begin{tikzcd} H_1 \arrow[r, "\Psi_1"] \arrow[d, "(s,t)"'] & G_1 \arrow[d, "(s,t)"] \\ H_0 \times H_0 \arrow[r, "\Psi_0 \times \Psi_0"] & G_0 \times G_0 \end{tikzcd} \tag{31}$$

and if $s \circ \pi_2 : H_0 \, {}_{\Psi_0}\!\times_t G_1 \to G_0$ is an open surjection.

Two groupoids G and H are Morita equivalent if there exist another groupoid K with Morita morphisms $G \xleftarrow{\simeq} K \xrightarrow{\simeq} H$.

A theorem of Moerdijk [Moe02] states that the category of orbifolds is equivalent to a quotient category of the category of proper étale groupoids after formally inverting the Morita morphisms.

Whenever we write orbifold, we will choose a proper étale smooth groupoid representing it (up to Morita equivalence).

Example 7.7. Consider again the pillowcase (as in Example 7.4). Define the following groupoids.

- The groupoid G whose space of objects are elements $m \in M$ with the topology of M, and whose space of arrows is the set of pairs (m, g) with the topology of $M \times G$. We have the diagrams

$$m \overset{(m,g)}{\to} mg$$

  and the composition law

$$(m, g) \circ (mg, h) = (m, gh).$$

- Similarly we define the groupoid H using the action of H in N with objects $n \in N$ and arrows $(n, h) \in N \times H$.

The orbifold $\mathcal{X}$ is the equivalence class of the groupoid G. Since there is a Morita morphism $H \to G$, we can say also that $\mathcal{X}$ is the equivalence class of H. By abuse of notation we will often say that G is an orbifold when we really mean that its equivalence class is the orbifold.

Example 7.8. Smooth manifolds provide a natural source of groupoids. Let M be a smooth manifold. It is well known that a smooth manifold is a pair $(M, \mathcal{U})$ of a (Hausdorff, paracompact) topological space M together with an atlas $\mathcal{U} = (U_i)_{i \in I}$, and is only by abuse of notation that we speak of a manifold M. In fact a smooth manifold is actually an equivalence class of a pair $[M, \mathcal{U}]$ where we say that $(M, \mathcal{U}_1) \sim (M, \mathcal{U}_2)$ if and only if there is a common refinement $(M, \mathcal{U}_3)$ of the atlas. We can say this in a slightly different way that will be easier to generalize to the case of orbifolds. To have a pair $(M, \mathcal{U})$ is the same thing as to have a small topological category $M_{\mathcal{U}}$ defined as follows.

- Objects: Pairs (m, i) so that $m \in U_i$. We endow the space of objects with the topology

$$\coprod_i U_i.$$

- Arrows: Triples (m, i, j) so that $m \in U_i \cap U_j = U_{ij}$. An arrow acts according to the following diagram.

$$(x, i) \overset{(x,i,j)}{\to} (x, j).$$

- The composition of arrows is given by

$$(x, i, j) \circ (x, j, k) = (x, i, k)$$

The topology of the space of arrows in this case is

$$\coprod_{(i,j)} U_{ij}.$$

The category M is actually a **groupoid**, in fact

$$(x, i, j) \circ (x, j, i) = (x, i, i) = \mathrm{Id}_{(x,i)}.$$

We will therefore define a manifold to be the equivalence class of the groupoid $M_{\mathcal{U}}$ by an equivalence relation called Morita equivalence (that will amount exactly to the equivalence of atlases in this case).

Example 7.9. More generally, let M be a smooth manifold and $G \subset \mathrm{Diff}(M)$ be a finite group acting on it.

- We say that the orbifold $[M/G]$ is the equivalence class of the groupoid $\mathcal{X}$ with objects $m \in M$ and arrows $(m, g) \in M \times G$.
- We can define another groupoid representing the same orbifold as follows. Take a contractible open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M such that all the finite intersections of the cover are either contractible or empty, and with the property that for any $g \in G$ and any $i \in I$ there exists $j \in I$ so that $U_i g = U_j$. Define G_0 as the disjoint union of the U_i's with $G_0 \xrightarrow{\rho} M = \mathcal{X}_0$ the natural map. Take G_1 as the pullback square

$$\begin{tikzcd} G_1 \arrow[r] \arrow[d] & M \times G \arrow[d, "s \times t"] \\ G_0 \times G_0 \arrow[r, "\rho \times \rho"] & M \times M \end{tikzcd}$$

where $s(m, g) = m$ and $t(m, g) = mg$. From the construction of G we see that we can think of G_1 as the disjoint union of all the intersections of two sets on the base times the group G, i.e.

$$G_1 = \left(\coprod_{(i,j) \in I \times I} U_i \cap U_j \right) \times G$$

where the arrows in $U_i \cap U_j \times \{g\}$ start in $U_i|_{U_j}$ and end in $(U_j|_{U_i})g$. This defines a proper étale Leray groupoid G and by definition it is Morita equivalent to $\mathcal{X}$.

7.1.4 Moduli Spaces

Moduli spaces are often given by orbifolds. Moduli spaces are "spaces" that contain the universal family of objects of certain kind. If $\mathcal{X}$ is the moduli space of objects of certain kind we want

$$\mathrm{Maps}(S, \mathcal{X})$$

to classify families of objects of this kind over S. This is akin to the situation in topology in which we represent, for example n-dimensional vector bundles over M *up to isomorphism* by *homotopy classes* of maps to a certain universal space $BU(n)$. Remember that $BU(n) = \mathrm{Gr}_n(\mathbb{C}^\infty)$. Moduli spaces are often not spaces at all but rather *orbifolds*.

Example 7.10. Let us consider the **moduli space of triangles** $\mathcal{T}$. We identify an Euclidean triangle T with a triple

$$T = (a, b, c)$$

satisfying the triangle inequalities

$$a + b > c,$$

$$b + c > a,$$

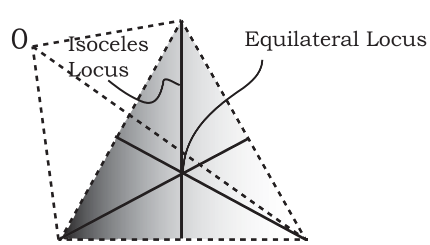
and

$$c + a > b.$$

The *set* M of all such T is diffeormorphic to

$$M \approx \Delta \times \mathbb{R}^+.$$

It is a positive cone over an equilateral triangle (of triangles of fixed perimeter $a + b + c$) that we denote by Δ.



The Moduli space of Triangles

The is a natural action of $\mathfrak{S}_3$ on M by multiplication of the corresponding permutation matrix. The moduli orbifold of triangles is

$$\mathcal{T} = [M/\mathfrak{S}_3]$$

Now the class of *smooth* families of triangles over the circle $S = S^1$ is now endowed naturally with the structure of an orbifold:

$$\mathcal{T}^S = [P/\mathfrak{S}_3]$$

where P is the family of paths $I = [0, 1] \to M$ so that

$$\gamma(1) = \gamma(0) \cdot g$$

for some $g \in \mathfrak{S}_3$. This is what we have called the *loop orbifold* [LU02b, LUX08]. We will come back to this later.

7.1.5 Almost Free Lie Group Actions

We will suppose now that K is a Lie group. Let M be a smooth manifold in which K is acting. We say that M is a K-manifold.

A map $\phi\colon M \to N$ between K-manifolds is said to be *equivariant* if

$$\phi(xg) = \phi(x)g.$$

We say that a vector bundle $E \longrightarrow M$ is a K-vector bundle if K acts linearly on the fibers and the projection map is equivariant.

Stabilizers K_m of Lie group actions are closed subgroups and hence Lie groups. Stabilizers of points in the same orbit are conjugate to each other:

$$K_{mg} = g^{-1} K_m g$$

The conjugacy class of a subgroup H will be written (H). Hence (K_m) only depends on the orbit of m and not on m. Given $m \in M$ he map

$$f_m\colon K/K_m \longrightarrow M$$

given by

$$f_m(\overline{g}) = mg,$$

is an injective immersion. It does not follow that $m \cdot K \subseteq M$ is a submanifold. Just think of a torus with an irrational flow. Nonetheless, of course, if K is *compact* then $m \cdot K \subseteq M$ is always a submanifold. If K is compact M/K is Hausdorff and $p\colon M \to M/K$ is proper and closed. So, from now on we shall suppose that K is compact. Fix $m \in M$ and let

$$V_m = T_m M / T_m(mK).$$

Notice that for $g \in K_m$ we have

$$d_m g\colon T_m M \longrightarrow T_{mg} M = T_m M$$

Therefore

$$K_m \longrightarrow \mathrm{Aut}(V_m).$$

Also K_m acts freely on $K \times V_m$, by $h(g, v) = (gh^{-1}, hv)$. This defines a vector bundle $K \times_{K_m} V_m \longrightarrow K/K_m$.

Theorem 7.10 (The Slice Theorem (Koszul 1953) [Kos53])**.** *There exists an equivariant diffeomorphism from an equivariant open neighborhood of the zero section of* $K \times_{K_m} V_m \longrightarrow K/K_m$ *to an open neighborhood of* $mK \subseteq M$*, sending the zero section to* mK *by* f_m*.*

The union of all the orbits of a given type is a submanifold of M. If M is compact there are only finitely many orbit types.

From now on we will suppose that all K_m are finite, and that M/K is connected. Then there exists a finite group G so that the set of points in M with stabilizers conjugate to G (denoted by $M_{(G)}$) is open and dense in M. (Prove it by induction over the dimension of the manifold M, and consider the sphere bundle of the neighborhoods provided by the Slice Theorem.)

If K is a compact Lie group acting on M, and each stabilizer K_x is finite, then $K \ltimes M$ is an orbifold groupoid. Observe that the slice theorem for compact group actions gives for each point x a 'slice" $V_x \subseteq M$ for which the action defines a diffeomorphism $K \times_{K_x} V_x \hookrightarrow M$ onto a saturated open neighborhood U_x of x. Then $K_x \ltimes V_x$ is an étale groupoid which is Morita equivalent to $K \ltimes U_x$. Patching these étale groupoids together for sufficiently many slices V_x yields an étale groupoid Morita equivalent to $K \ltimes M$ [AR03].

Definition 7.11. A orbivector bundle over $\mathcal{X}$ is a pair (E, θ) where E is an ordinary vector bundle over $\mathcal{X}_0$ and θ is an isomorphism $s^*E \cong t^*E$. (Here we are choosing a representative of the Morita class.)

Example 7.11. This recovers the usual definition for a manifold acted on by the identity group.

Example 7.12. For the groupoid $M \rtimes G$ this gives the usual definition of an equivariant vector bundle. The *tangent bundle* $T\mathcal{X}$ of an orbifold $\mathcal{X}$ is a orbibundle over $\mathcal{X}$.

Example 7.13. If $U = [V/G]$ is a local chart (namely the restriction of the groupoid to a very small neighborhood), then a corresponding local uniformizing system for $T\mathcal{X}$ will be $[TV/G]$ with the action $g \cdot (x, \nu) = (gx, dg_x(\nu))$.

Definition 7.12. Given an orbifold $\mathcal{X}$ we say that the space $X = \mathcal{X}_1/\sim$ is its coarse topological space, or quotient space. Here $x \sim y$ whenever there is an arrow from x to y. We will often write $\pi\colon \mathcal{X}_0 \to X$ to denote the canonical projection.

Definition 7.13. Given a point $x \in X$ and an open neighborhood $x \in U \subseteq X$ we define $\mathcal{X}_U$ to be the restricted groupoid, namely its objects are $V = \pi^{-1}U$ and its arrows are all arrows α such that both $\pi(s(\alpha))$ and $\pi(t(\alpha))$ are in U. It is easy to show that for a sufficiently small U we have that $\mathcal{X}_U$ is isomorphic to $[V/G]$ for some finite group G acting on the manifold V. Such orbifold $[V/G]$ is called a local orbifold chart, or sometimes, a uniformizing system. An orbifold is called effective if at every point of X we can find a local orbifold chart where the action of G in V is effective.

Similarly the *frame bundle* $P(\mathcal{X})$ is a principal orbibundle over $\mathcal{X}$. The local uniformizing system is $U \times O(n)/G$ with local action $g \cdot (x, A) = (gx, dg \circ A)$. Notice that if the orbifold is effective then $P(\mathcal{X})$ is always a *smooth manifold* for the local action is free and $(s,t)\colon \mathcal{X}_1 \to \mathcal{X}_0 \times \mathcal{X}_0$ is one-to-one. From this we deduce that $\mathcal{X} = [P(\mathcal{X})/O(n)]$. This proves the following proposition.

Proposition 7.14. *Every effective orbifold arises from the almost free action of a Lie group on a manifold.*

7.1.6 The Homotopy Type of Orbifolds

Define

$$\mathcal{X}^{(n)} := \underbrace{\mathcal{X}_1 \,{}_t\!\times_s \cdots \,{}_t\!\times_s \mathcal{X}_1}_{n}.$$

In the case in which $\mathcal{X}_1$ is a set then $\mathcal{X}^{(n)}$ is the set of sequences $(\gamma_1, \gamma_2, \dots, \gamma_n)$ so that we can form the composition $\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_n$.

With this data we can form a simplicial set [Seg68a].

Definition 7.15. A *(semi-)simplicial set (resp. group, space, scheme)* $X_\bullet$ is a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ (resp. groups, spaces, schemes) together with maps

$$X_0 \leftrightarrows X_1 \leftrightarrows X_2 \leftrightarrows \cdots \leftrightarrows X_m \leftrightarrows \cdots$$

$$\partial_i\colon X_m \to X_{m-1}, \quad s_j\colon X_m \to X_{m+1}, \quad 0 \le i, j \le m.$$

called *boundary* and *degeneracy* maps, satisfying

$$\begin{aligned}
\partial_i\partial_j &= \partial_{j-1}\partial_i \quad \text{if } i < j \\
s_i s_j &= s_{j+1} s_i \quad \text{if } i < j \\
\partial_i s_j &= \begin{cases} s_{j-1}\partial_i & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ s_j \partial_{i-1} & \text{if } i > j+1 \end{cases}
\end{aligned}$$

The nerve of a category (following Segal [Seg68a]) is a semi-simplicial set $N\mathcal{C}$ where the objects of $\mathcal{C}$ are the vertices, the morphisms the 1-simplices, the triangular commutative diagrams the 2-simplices, and so on.

For a category coming from a groupoid then the corresponding simplicial object will satisfy $N\mathcal{C}_n = X_n = \mathcal{X}^{(n)}$.

We can define the boundary maps $\partial_i : \mathcal{X}^{(n)} \to \mathcal{X}^{(n-1)}$ by:

$$\partial_i(\gamma_1, \ldots, \gamma_n) = \begin{cases} (\gamma_2, \ldots, \gamma_n) & \text{if } i = 0 \\ (\gamma_1, \ldots, m(\gamma_i, \gamma_{i+1}), \ldots, \gamma_n) & \text{if } 1 \le i \le n-1 \\ (\gamma_1, \ldots, \gamma_{n-1}) & \text{if } i = n \end{cases}$$

and the degeneracy maps by

$$s_j(\gamma_1, \ldots, \gamma_n) = \begin{cases} (e(s(\gamma_1)), \gamma_1, \ldots, \gamma_n) & \text{for } j = 0 \\ (\gamma_1, \ldots, \gamma_j, e(t(\gamma_j)), \gamma_{j+1}, \ldots, \gamma_n) & \text{for } j \ge 1 \end{cases}$$

We will write Δ^n to denote the standard n-simplex in $\mathbb{R}^n$. Let $\delta_i\colon \Delta^{n-1} \to \Delta^n$ be the linear embedding of Δ^{n-1} into Δ^n as the i-th face, and let $\sigma_j\colon \Delta^{n+1} \to \Delta^n$ be the linear projection of Δ^{n+1} onto its j-th face.

Definition 7.16. The *geometric realization* $|X_\bullet|$ of the simplicial object $X_\bullet$ is the space

$$|X_\bullet| = \left(\coprod_{n\in\mathbb{N}} \Delta^n \times X_n \right) \Big/ \begin{array}{l} (z, \partial_i(x)) \sim (\delta_i(z), x) \\ (z, s_j(x)) \sim (\sigma_j(z), x) \end{array}$$

Notice that the topologies of X_n are relevant to this definition.

The simplicial object $\mathsf{N}\mathcal{C}$ determines $\mathcal{C}$ and its topological realization is called $B\mathcal{C}$, the *classifying space of the category.* Again in our case $\mathcal{C}$ is a *topological category* in Segal's sense.

Definition 7.17. For a groupoid $\mathcal{X}$ we will call $B\mathcal{X} = |\mathsf{N}\mathcal{X}|$ the *classifying space of the orbifold.*

The following proposition establishes that B is a functor from the category of groupoids to that of topological spaces. Recall that we say that two morphisms of groupoids are Morita related if the corresponding functors for the associated categories are connected by a morphism of functors.

Proposition 7.18. *A morphism of groupoids $\mathcal{X}_1 \to \mathcal{X}_2$ induces a continuous map $B\mathcal{X}_1 \to B\mathcal{X}_2$. Two morphism that are Morita related will produce homotopic maps. In particular a Morita equivalence $\mathcal{X}_1 \sim \mathcal{X}_2$ will induce a homotopy equivalence $B\mathcal{X}_1 \simeq B\mathcal{X}_2$. This assignment is functorial.*

Example 7.14. For the groupoid $\bar{\mathsf{G}} = (\star \times \mathsf{G} \rightrightarrows \star)$ the space $B\bar{\mathsf{G}}$ coincides with the classifying space $B\mathsf{G}$ of G.

Consider now the groupoid $\mathcal{X} = (\mathsf{G}\times\mathsf{G} \rightrightarrows \mathsf{G})$ where $s(g_1, g_2) = g_1$, $t(g_1, g_2) = g_2$ and $m((g_1, g_2); (g_2, g_3)) = (g_1, g_3)$ then it is easy to see that $B\mathcal{X}$ is contractible and has a G action. Usually $B\mathcal{X}$ is written EG; here one has to be careful with the local triviality for the map $\mathsf{EG} \to \mathsf{BG}$ and this is studied and resolved by Segal in [Seg68a].

A morphism of groupoids $\mathcal{X} \to \bar{\mathsf{G}}$ is the same thing as a principal G bundle over $\mathcal{X}$ and therefore can be written by means of a map $\mathsf{G} \times \mathsf{G} \to \mathsf{G}$. If we choose $(g_2, g_2) \mapsto g_1^{-1} g_2$ the induced map of classifying spaces

$$\mathsf{EG} \longrightarrow \mathsf{BG}$$

is the universal principal G-bundle fibration over BG.

Example 7.15. Consider a smooth manifold X and a good open cover $\mathcal{U} = \{U_\alpha\}_\alpha$. Consider the groupoid $\mathcal{G} = (\mathcal{G} \rightrightarrows \mathcal{G}_0)$ where $\mathcal{G}_1$ consists on the disjoint union of the double intersections $U_{\alpha\beta}$. Segal calls X_U the corresponding topological category. Then Segal proves [Seg68a] that $B\mathcal{G} = B\mathsf{X}_U \simeq \mathsf{X}$.

If we are given a principal G bundle over $\mathcal{G}$ then we have a morphism $\mathcal{G} \to \bar{\mathsf{G}}$ of groupoids, that in turn induces a map $\mathsf{X} \to \mathsf{BG}$. Suppose that in the previous example we take $\mathsf{G} = GL_n(\mathbb{C})$. Then we get a map $\mathsf{X} \to BGL_n(\mathbb{C}) = BU$.

Example 7.16. Consider a groupoid $\mathcal{X}$ of the form $\mathsf{M} \times \mathsf{G} \rightrightarrows \mathsf{M}$ where G is acting on M continuously. Then $B\mathcal{X} \simeq \mathsf{EG} \times_{\mathsf{G}} \mathsf{M}$ is the Borel construction for the action $\mathsf{M} \times \mathsf{G} \to \mathsf{M}$.

Definition 7.19. The fundamental group of $\mathcal{X}$ is defined to be $\pi_1(\mathcal{X}) = \pi_1(B\mathcal{X})$. Similarly for the cohomology $H^*(\mathcal{X}) = H^*(B\mathcal{X})$.

This last definition of cohomology is a bit too naive whenever we have obtained our orbifold by some geometric procedures. For example, as the space of solutions of algebraic equations. We will return to this issue later once we have the perspective given to us by topological quantum field theories.

7.2 Loop Orbifolds

7.2.1 The Definition of the Loop Groupoid

The loop space is slightly more complicated in the case of an orbifold.

To generalize this situation to an orbifold $\mathcal{X}$ (replacing the rôle of M above), we must be able to say what is the candidate to replace $\mathcal{L}M$. This was done for a general orbifold in [LU02b]. The basic idea is that to a groupoid $\mathcal{X}$ we must assign a new (infinite-dimensional) groupoid $L\mathcal{X}$ that takes the place of the free loopspace of M in a functorial manner

$$L\colon \textbf{Orbifolds} \to \textbf{Orbifolds}.$$

In the case in which $\mathcal{X} = [M/G]$, we proved that $L\mathcal{X}$ admits a much smaller and very concrete model defined as follows. The objects of the loop groupoid are given by

$$(L\mathcal{X})_0 := \bigsqcup_{g \in G} \mathcal{P}_g,$$

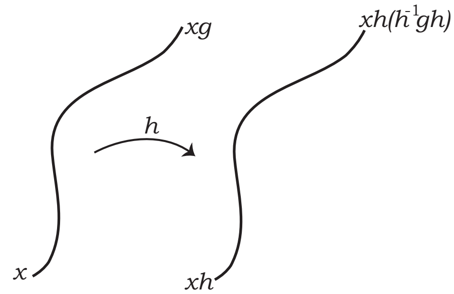
where $\mathcal{P}_g$ is the set of all pairs (γ, g) with $\gamma : \mathbb{R} \to X$ and $g \in G$ with $\gamma(t)g = \gamma(2\pi + t)$. The space of arrows of the loop groupoid is

$$(L\mathcal{X})_1 := \bigsqcup_{g \in G} \mathcal{P}_g \times G,$$

and the action of G in $\mathcal{P}_g$ is by translation in the first coordinate; and by conjugation in the second; that is, a typical arrow in the loop groupoid looks like

$$(\gamma, g) \xrightarrow{((\gamma,g);h)} (\gamma \cdot h, h^{-1}gh),$$

or pictorially:



7.2.2 The Loop Space as a Classifying Space for the Loop Orbifold

The following result describes the relation between $\mathcal{L}\mathrm{B}\mathcal{X}$ and $\mathsf{L}\mathcal{X}$

Theorem 7.20. *There is a canonical map*

$$\tau\colon \mathcal{L}\mathrm{B}\mathcal{X} \longrightarrow \mathsf{BL}\mathcal{X}$$

that induces a weak homotopy equivalence.

Proof. We will construct two Serre fibrations over $\mathrm{B}\mathcal{X}$.

- Consider $\mathsf{BL}\mathcal{X}$.

  Define a morphism of groupoids

  $$\widetilde{ev}_0\colon \mathsf{L}\mathcal{X} \to \mathcal{X}$$

  induced by the equivariant map of G-spaces

  $$ev_0\colon \mathcal{P}_{\mathsf{G}}(\mathsf{M}) \longrightarrow \mathsf{M}$$

  given by evaluation at 0,

  $$ev_0(\gamma, g) := \gamma(0).$$

  This morphism induces a map at the level of classifying spaces

  $$|ev_0|\colon \mathsf{BL}\mathcal{X} \to \mathrm{B}\mathcal{X}.$$

  If we interpret the classifying spaces in terms of the Borel construction we have $\mathsf{BL}\mathcal{X} = \mathcal{P}_{\mathsf{G}}(\mathsf{M}) \times_{\mathsf{G}} \mathsf{EG}$ and $\mathrm{B}\mathcal{X} = \mathsf{M} \times_{\mathsf{G}} \mathsf{EG}$. For a point $z \in \mathrm{B}\mathcal{X}$ with $z = [m, \xi]$, the following holds

  $$|ev|^{-1}(z) = [\mathcal{P}^{m}_{\mathsf{G}}(\mathsf{M}) \times \{\xi\}]$$

where

$$\mathcal{P}^m_G(M) := \bigsqcup_{g \in G} \mathcal{P}^m_g(M) \times \{g\}$$

with

$$\mathcal{P}^m_g(M) = \{\gamma \in \mathcal{P}_g(M) | \gamma(0) = m\}.$$

- On $\mathcal{L}B\mathcal{X}$.

  Take the map

  $$\epsilon_0 \colon \mathcal{L}B\mathcal{X} \longrightarrow B\mathcal{X}$$

  which evaluates a free loop at 0, i.e. for $\sigma \colon S^1 \to B\mathcal{X}$ then $\epsilon_0(\sigma) := \sigma(0)$. Then

  $$\epsilon_0^{-1}(z) = \Omega_z(B\mathcal{X}) := \mathcal{P}^z_z(B\mathcal{X})$$

  is the space of loops based at z.

Now let us define the map τ. Consider the fixed $z = [m, \xi]$ as above and for $\sigma \in \mathcal{L}B\mathcal{X}$, lift it to $\tilde{\sigma}$ making the following diagram commutative

$$\begin{tikzcd}
[0,1] \arrow[r, "\tilde{\sigma}"] \arrow[d, "\exp(2\pi_\ )"'] & M \times EG \arrow[d, "p"] \\
S^1 \arrow[r, "\sigma"] & M \times_G EG
\end{tikzcd}$$

such that $\tilde{\sigma}(0) = (m, \xi)$ (the construction follows from the fact that the map p is a G-principal bundle and G is finite). Since G acts freely on EG there exists a unique element k in G such that $\tilde{\sigma}(0)k = \tilde{\sigma}(1)$. Define τ in the following way

$$\tau(\sigma) \colon = [(\pi_1 \circ \tilde{\sigma}, k), \xi] \in BL\mathcal{X},$$

where $\pi_1 : M \times EG \to M$ is the projection on the first coordinate. From the definition of τ it follows that it is well defined and that $\pi_1 \circ \tilde{\sigma} \in \mathcal{P}_k(M)$. Moreover the following diagram is commutative

$$\begin{tikzcd}
\mathcal{L}B\mathcal{X} \arrow[rr, "\tau"] \arrow[dr, "\epsilon_0"'] & & BL\mathcal{X} \arrow[dl, "|ev_0|"] \\
& B\mathcal{X}. &
\end{tikzcd}$$

Let us denote by $\tau_z := \tau|_{\epsilon_o^{-1}(z)}$, then

Lemma 7.21. *The map*

$$\tau_z \colon \epsilon_0^{-1}(z) \longrightarrow |ev_0|^{-1}(z)$$

is a homotopy equivalence.

Proof. From the definition of τ it is clear that τ_z is surjective. Let us now check the homotopy type of the inverse image of a point. Recall from above that the map τ_z goes from $\Omega_z(M \times_G EG)$ to $[\mathcal{P}^m_G(M) \times \{\xi\}]$. Take $(\gamma, g) \in \mathcal{P}^m_G(M)$. From the definition of τ above it follows that

$$\tau_z^{-1}([((\gamma, g), \xi)]) \cong \mathcal{P}^{\xi}_{g^{-1}\xi}(EG)$$

where $\mathcal{P}^{\xi}_{g^{-1}\xi}(EG)$ stands for the paths in EG that go from ξ to $(g^{-1}\xi)$.

The space $\mathcal{P}^{\xi}_{g^{-1}\xi}(EG)$ is independent of the choice of representative in $[((\gamma, g), \xi)]$.

As the space $\mathcal{P}^{\xi}_{g^{-1}\xi}(EG)$ is contractible then it follows that τ_z induces a homotopy equivalence.

♣

As τ induces a homotopy equivalence on the fibers of the Serre fibrations given by ϵ_0 and $|ev_0|$, then the Theorem 7.20 follows from a theorem of Dold [Dol63]. Hence τ induces a weak homotopy equivalence between $\mathcal{L}B\mathcal{X}$ and $BL\mathcal{X}$.

♣

7.2.3 The Circle Action

We have seen that the map $\tau \colon \mathcal{L}B\mathcal{X} \longrightarrow B(L\mathcal{X})$ is a weak homotopy equivalence, and it is natural to wonder whether the equivalence is S^1-equivariant. The answer turns out to be negative as we will see shortly.

There is a natural action of S^1 onto $\mathcal{L}B\mathcal{X}$ by rotating the loop, but the action does not get carried into $BL\mathcal{X}$ via τ. The reason is the following, the loop orbifold $L\mathcal{X}$ comes provided with a natural action of the orbifold $[\mathbb{R}/\mathbb{Z}]$ which is a *stack* model for the circle. The action of $\mathbb{R}$ into the orbifold loops of $\mathcal{P}_G(M)$ is the obvious one, the map gets shifted by the parameter in $\mathbb{R}$. The subtlety arises here, once we act on the orbifold loop by $1 \in \mathbb{R}$, we do not end up with the orbifold loop from the beginning, but instead we get one that is related to the initial one via an arrow of the loop orbifold category. This arrow in the loop orbifold is where $1 \in \mathbb{Z}$ gets mapped. By the way, precisely this fact was the one that allowed us to define the loop orbifold in a non trivial way, namely a loop on the orbifold was not a map from the circle to the orbifold, but a functor from $[\mathbb{R}/\mathbb{Z}]$ to the orbifold.

More accurately, to define the action of $[\mathbb{R}/\mathbb{Z}]$ on $\mathsf{L}\mathcal{X} = [\mathcal{P}_G(M)/G]$ we first define an action of $\mathbb{R}$ on $\mathcal{P}_G(M)$ in the natural way, namely, take $\gamma \in \mathcal{P}_k(M)$ and $s \in \mathbb{R}$ and define

$$(s \cdot \gamma)(t) := \gamma_s(t) = \gamma(t + s - \lfloor t+s \rfloor) k^{\lfloor t+s \rfloor}$$

where $\lfloor \cdot \rfloor$ is the least integer function. Then for each $(\gamma, k) \in \mathcal{P}_G(M)$ and $1 \in \mathbb{Z}$ we choose the arrow of $\mathsf{L}\mathcal{X}$ that relates the orbifold loops (γ, k) and (γ_1, k), this is the arrow $((\gamma, k), k) \in \mathcal{P}_G(M) \times G$. The source of $((\gamma, k), k)$ is (γ, k) and the target is $(\gamma \cdot k, k) = (\gamma_1, k)$ the loop shifted by 1.

Using the construction of section 7.2.2 we have that

$$\tau(\sigma) : = [((\pi_1 \circ \tilde{\sigma}, k), \xi)] \,,$$

and denote $\pi_1 \circ \tilde{\sigma}$ by γ. For $s \in \mathbb{R}$,

$$\tau(s \cdot \sigma) = [(\gamma_s, k), \xi)]$$

and $1 \cdot \sigma = \sigma$, but $\tau(1 \cdot \sigma) \neq \tau(\sigma)$. Instead $\tau(1 \cdot \sigma)$ and $\tau(\sigma)$ are related by an arrow.

Nevertheless, if we take the coarse moduli space of $\mathsf{L}\mathcal{X}$ (that we will write $\mathsf{L}\mathcal{X}/\sim = \mathcal{P}_G(M)/G$), the map induced by τ is S^1-equivariant. For in $\mathsf{L}\mathcal{X}/\sim = \mathcal{P}_G(M)/G$ the elements $\tau(1 \cdot \sigma)$ and $\tau(\sigma)$ become by definition the same. Then we can conclude

Lemma 7.22. *The space* $\mathsf{L}\mathcal{X}/\sim = \mathcal{P}_G(M)/G$ *has a natural* S^1 *action and the map*

$$\tilde{\tau} : \mathcal{L}B\mathcal{X} \longrightarrow \mathsf{L}\mathcal{X}/\sim = \mathcal{P}_G(M)/G$$

which is the composition of τ *with the projection* $B\mathsf{L}\mathcal{X} \to \mathsf{L}\mathcal{X}/\sim$*, is* S^1*-equivariant.*

Corollary 7.23. *The map* $\tilde{\tau}$ *induces an isomorphism in homology*

$$\tilde{\tau}_* : H_*(\mathcal{L}B\mathcal{X}; \mathbb{Q}) \xrightarrow{\cong} H_*(\mathsf{L}\mathcal{X}/\sim; \mathbb{Q}),$$

and in equivariant homology

$$\tilde{\tau}_* : H_*^{S^1}(\mathcal{L}B\mathcal{X}; \mathbb{Q}) \xrightarrow{\cong} H_*^{S^1}(\mathsf{L}\mathcal{X}/\sim; \mathbb{Q})$$

Proof. As τ is a weak homotopy equivalence, then

$$\tau_* : H_*(\mathcal{L}B\mathcal{X}; \mathbb{Z}) \xrightarrow{\cong} H_*(B\mathsf{L}\mathcal{X}; \mathbb{Z}),$$

and as the group G is finite then

$$\tilde{\tau}_* : H_*(BL\mathcal{X};\mathbb{Q}) \xrightarrow{\cong} H_*(L\mathcal{X}/\sim;\mathbb{Q}).$$

The second isomorphism follows from the isomorphism of spectral sequences with real coefficients associated to the each of the following fibrations

$$\begin{tikzcd} \mathcal{L}B\mathcal{X} \times_{S^1} ES^1 \arrow[rr] \arrow[dr] & & L\mathcal{X}/\sim \times_{S^1} ES^1 \arrow[dl] \\ & BS^1. & \end{tikzcd}$$

♣

7.2.4 Cyclic Equivariant Loops

There is an alternative description of $\mathcal{P}_g(M)$ that although essentially obvious nevertheless relates it to some models that have been studied before.

Given an element $g \in G$ it generates a cyclic group $< g > \subseteq G$. Let m be the order of g in G. Then there is a natural injective morphism of groups

$$\zeta \colon < g > \to S^1$$

given by $\zeta(g) = \exp(2\pi i/m)$.

We define the space $\mathcal{L}_g M$ of g-*equivariant loops* in M to be the subspace of $\mathcal{L}M := \text{Maps}(S^1; M)$ of loops ϕ satisfying the following equation for every $z \in S^1$:

$$\phi(z \cdot \zeta(g)) = \phi(z) \cdot g.$$

The space of *cyclic equivariant loops* of M is defined to be simply

$$\mathcal{L}_G M := \bigsqcup_{g \in G} \mathcal{L}_g M \times \{g\}.$$

It is, again, naturally endowed with a G-action $((\phi, h); g) \mapsto (\phi g, g^{-1} h g)$.

The natural restriction map

$$\Psi \colon \mathcal{L}_g(M) \longrightarrow \mathcal{P}_g(M)$$

given by

$$\gamma(t) = \phi(\exp(2\pi i t/m)) = \phi(\zeta(g)^t)$$

is a diffeomorphism, and moreover it induces a G-equivariant diffeomorphism

$$\Psi\colon \mathcal{L}_G(M) \longrightarrow \mathcal{P}_G(M).$$

We conclude this subsection by pointing out that as a consequence of these remarks we have the following equality

$$\mathcal{L}_G(M) \times_G EG \simeq \mathcal{L}(M_G) = \mathcal{L}B\mathcal{X}.$$

7.2.5 Principal bundles

Let us consider G-principal bundles on S^1 and their relation to the various models of the loop orbifold. We are interested in the category of G-principal bundles $\pi\colon Q \to S^1$ over S^1 endowed with a marked point $q_0 \in Q$ so that $\pi(q_0) = 0 \in S^1$, and such that π is a local isometry.

Whenever we have such a pair (Q, q_0) we have a well-defined lift $\tilde{e}\colon [0,1] \to Q$, $\tilde{e}(0) = q_0$, of the exponential map $e\colon [0,1] \to S^1$ given by $t \mapsto \exp(2\pi i t)$, making the following diagram commutative:

$$\begin{tikzcd}
{[0,1]} \arrow[rr, "\tilde{e}"] \arrow[dr, "e"'] & & Q \arrow[dl, "\pi"] \\
& S^1. &
\end{tikzcd}$$

Since $\tilde{e}(0)$ and $\tilde{e}(1)$ belong to $\pi^{-1}(0)$ there is a $g \in G$ so that

$$\tilde{e}(1) = \tilde{e}(0) \cdot g.$$

We well call this $g \in G$ the *holonomy* of Q.

The isomorphism classes of G-principal bundles with a marked point are classified by their holonomy, for the set $\mathrm{Bun}_G(S^1)$ of such classes is given by

$$\mathrm{Bun}_G(S^1) = \pi_1 BG = G.$$

The following proposition is very easy.

Proposition 7.24. *The natural action of* G *on* $\mathrm{Bun}_G(S^1)$ *under the holonomy isomorphism* $\mathrm{hol}\colon \mathrm{Bun}_G(S^1) \longrightarrow G$ *becomes the action of* G *on* G *by conjugation.*

This proposition can be slightly generalized as follows. Consider now the space $\mathrm{Bun}_G(S^1, M)$ of isomorphism classes of G-equivariant maps from a principal G-bundle Q over the circle to M. This space has a natural G-action defined as follows. If Q_g denotes the principal bundle with holonomy g then the pair

$$[(\beta\colon Q_g \to M); k] \in \mathrm{Bun}_G(S^1, M) \times G$$

gets mapped by conjugation to

$$(\beta_k\colon Q_{k^{-1}gk} \to M) \in \mathrm{Bun}_G(S^1, M).$$

Proposition 7.25. *The loop orbifold* $\mathsf{L}\mathcal{X} = [\mathcal{P}_G(M)/G]$ *is isomorphic to the orbifold* $[\mathrm{Bun}_G(S^1, M)/G]$*, and therefore*

$$\mathrm{Bun}_G(S^1, M) \times_G EG \simeq \mathcal{L}(M_G).$$

Proof. It is enough to give a G-equivariant diffeomorphism

$$\mathrm{Bun}_G(S^1, M) \longrightarrow \mathcal{P}_G(M),$$

this can be achieved by the following formula

$$(\beta\colon Q_g \to M) \mapsto \gamma = \beta \circ \tilde{e}.$$

Since $\tilde{e}(1) = \tilde{e}(0) \cdot g$, then $\gamma(1) = \gamma(0) \cdot g$.

♣

To finish this section let us define $\mathrm{Bun}_g(S^1, M)$ to be the space of isomorphism classes of G-equivariant maps from a principal G-bundle Q_g with holonomy g to M. Then we have that

$$\mathrm{Bun}_G(S^1, M) = \bigsqcup_{g \in G} \mathrm{Bun}_g(S^1, M),$$

and in fact

$$\mathrm{Bun}_g(S^1, M) \cong \mathcal{P}_g(M).$$

7.3 Stacks

The yoga of stacks starts with the Yoneda Lemma. So say you have a (locally small)[2] category $\mathcal{C}$ and you fix an object $X \in \mathcal{C}$. We define its functor of points $P_X \colon \mathcal{C} \to \mathbf{Sets}$ by $Y \mapsto P_X(Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$. The *Yoneda lemma* states the somewhat surprising fact that one can recover X from P_X. To state the lemma write $F(\mathcal{C}, \mathbf{Sets})$ to denote the category of functors from $\mathcal{C}$ to **Sets** where objects are said functors and morphisms are natural transformations.

Theorem 7.26. *The functor $\mathcal{C} \to F(\mathcal{C}, \boldsymbol{Sets})$ sending X to P_X embeds $\mathcal{C}$ into $F(\mathcal{C}, \boldsymbol{Sets})$ fully faithfully.*

The proof is tautological and it is just a fun exercise.

Let us consider the example of manifolds. Say you have two manifolds X and Y. What this is saying is that it is exactly the same to have a smooth map $f : X \to Y$ that is it to have a natural transformation $\xi_f \colon P_X \to P_Y$, which sound slightly odd but is nevertheless tautological.

Functors of the form P_X behave like a sheaf living on the category $\mathcal{C}$. And they behave even more so when $\mathcal{C}$ is the category of smooth manifolds **Man**.

Proposition 7.27. *For a given manifold X the functor $P := P_X$ satisfies:*

- *For every object $Y \in \boldsymbol{Man}$ we have that $P(Y)$ is non-empty.*
- *For every object $Y \in \boldsymbol{Man}$ we have that $P(Y)$ is a set contained in the set $\mathrm{Hom}_{\boldsymbol{Sets}}(Y, X)$.*
- *The functor P is a sheaf: whenever we glue two manifolds Z and Y along an open submanifold W of both, and say that we have $f \in P(Y)$ and $g \in P(Z)$ so that $f|_W = g|_W$, then there exists $F \in P(Z \cup_W Y)$ gluing both f and g.*
- *If $f \in P(Y)$ and $g \in C^\infty(Z, Y)$ then $g \circ f \in P(Z)$.*

Definition 7.28. A functor $P : \mathbf{Man} \to \mathbf{Sets}$ satisfying all of the properties of the previous proposition is called a *diffeology*

Notice that it is enough to have the values $P(U)$ of P at all open sets U of euclidean spaces for manifolds can be obtained gluing those. A map $p \in P(U)$ is called a *plot* of the diffeology P.

[2] Meaning that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set for every two objects X and Y.

It is an interesting fact that not every diffeology P is of the form $P = P_X$ for some manifold X, and effective orbifolds can be modelled with diffeologies.

We often write $X := P(\bullet)$ for the value of P at a point. By abuse of notation one writes X instead of P_X and speaks of the diffeological space X instead of of the diffeology P. It is the same to say that $p \in P(U)$ than to say that $p : U \to X$ is a plot on X. Observe that the full functor P can be recovered by having the set X and the prescription to decide when a map of sets $U \to X$ is a plot.

Iglesias, Karshon and Zadka have proposed a definition for an effective orbifold in terms of diffeologies [IKZ10]. The following definitions are theirs:

Definition 7.29. Let X be a diffeological space, let $\sim$ be an equivalence relation on X, and let $\pi\colon X \to Y := X/\sim$ be the quotient map. The *quotient diffeology* on Y is the diffeology in which $p\colon U \to Y$ is a plot if and only if each point in U has a neighborhood $V \subset U$ and a plot $\tilde{p} : V\colon X$ such that $p|_V = \pi \circ \tilde{p}$.

Definition 7.30. A diffeological space X is *locally diffeomorphic* to a diffeological space Y at a point $x \in X$ if and only if there exists a subset A of X, containing x, and there exists a one-to-one function $f\colon A \to Y$ such that

1. for any plot $p\colon U \to X$, the composition $f \circ p$ is a plot of Y;
2. for any plot $q\colon V \to Y$, the composition $f^{-1} \circ q$ is a plot of X.

An n dimensional manifold can be interpreted as a diffeological space which is locally diffeomorphic to $\mathbb{R}^n$ at each point.

Definition 7.31. A *diffeological orbifold* is a diffeological space which is locally diffeomorphic at each point to a quotient $\mathbb{R}^n/\Gamma$, for some n, where Γ is a finite group acting linearly on $\mathbb{R}^n$.

As expected diffeological orbifolds, with differentiable maps, form a subcategory of the category of diffeological spaces.

Unfortunately this does not work so well for orbifolds that are non-effective as for example $\mathcal{X} := [\bullet/G]$. The category of diffeological orbifolds contains the category of manifolds, but the category of finite groups does not fit nicely on this approach. To deal with non-effective orbifolds we must think of functors

$$P\colon \mathbf{Man} \to \mathbf{Groupoids}.$$

rather than of functors $P\colon \mathbf{Man} \to \mathbf{Sets}$.

A stack can be thought of as a generalization of a diffeology where the values of the functor P are *discrete* groupoids rather than sets. The main difficulty in making this work is the sheaf condition. The gluing now occurs up to isomorphisms rather than on the nose [Hei05].

Definition 7.32. A stack $\mathcal{P}$ is a 2-functor

$$\mathcal{P}\colon \textbf{Man} \to \textbf{Groupoids},$$

so that:

- We can glue objects: For an open cover $\{U_i\}$ of Y and local maps $P_i \in \mathcal{P}(U_i)$ that are *isomorphic* along the intersections $\phi_{ij} : P_i \cong P_j$ satisfying the cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ along triple intersections, then there is a global object $P \in \mathcal{P}(Y)$ together with local isomorphisms $\phi_i\colon P|_{U_i} \cong P_i$ and $\phi_{ij} = \phi_j \circ \phi_i^{-1}$.
- We can glue morphisms: Given two objects $P, P' \in \mathcal{P}(Y)$, an open cover $\{U_i\}$ of Y and local isomorphisms $\phi_i\colon P|_{U_i} \cong P|_{U_i'}$ such that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ then there is a global isomorphism $\phi\colon P \cong P'$ such that $\phi_i = \phi|_{U_i}$.

Example 7.17. Consider the orbifold $\mathcal{B}G := [\bullet/G]$. Its associated stack is

$$\mathcal{P}(Y) = \{P \to Y\colon P \text{ is a } G-\text{principal bundle}\},$$

namely $C^\infty(Y, \mathcal{B}G)$ is the discrete groupoid of principal G-bundles over Y together with isomorphisms of G-principal bundles. Here we stress that $C^\infty(Y, \mathcal{B}G)$ is a groupoid and not only a set.

Let $\mathcal{C}_Y$ be the category of open sets on Y (together with inclusions). Let $\mathcal{S}_{(Y,\mathcal{B}G)}$ be the category of G-principal bundles over open sets of Y. To have the forgetful functor $p : \mathcal{S}_{(Y,\mathcal{B}G)} \to \mathcal{C}_Y$ which remembers only the base of the bundle is the same as to have $\mathcal{P}$. This can be seen by setting $\mathcal{P}(U) := \mathcal{S}_U = p^{-1}(U)$.

Example 7.18. Consider the global quotient orbifold $\mathcal{X} := [M/G]$. Its associated stack is the groupoid

$$\mathcal{P}(Y) = \{(P \to Y, f\colon P \to M)\colon E \text{ is a } G-\text{principal bundle and } f(pg) = f(p)g\},$$

namely $C^\infty(Y, \mathcal{X})$ is the discrete groupoid of principal G-bundles over Y equipped with equivariant maps to M, together with isomorphisms of G-principal bundles. Here we stress again that $C^\infty(Y, \mathcal{X})$ is a groupoid and not only a set. Let $\mathcal{S}_{(Y,\mathcal{X})}$ be the category of G-principal bundles over open sets of Y together with equivariant maps to M. To have the forgetful functor $p : \mathcal{S}_{(Y,\mathcal{X})} \to \mathcal{C}_Y$ which remembers only the base of the bundle is the same as to have $\mathcal{P}$. This can be seen by setting $\mathcal{P}(U) := \mathcal{S}_U = p^{-1}(U)$.

There is yet one more way to understand stacks.

Let $\mathcal{C}, \mathcal{S}$ be a pair of categories and $p : \mathcal{S} \to \mathcal{C}$ a functor. For each $U \in \mathrm{Ob}(\mathcal{C})$ we denote $\mathcal{S}_U = p^{-1}(U)$.

Definition 7.33. The category $\mathcal{S}$ is *fibered by groupoids* over $\mathcal{C}$ if

- For all $\phi : U \to V$ in $\mathcal{C}$ and $y \in \mathrm{Ob}(\mathcal{S}_V)$ there is a morphism $f : x \to y$ in $\mathcal{S}$ with $p(f) = \phi$.
- For all $\psi : V \to W$, $\phi : U \to W$, $\chi : U \to V$, $f : x \to y$ and $g : y \to z$ with $\phi = \psi \circ \chi$, $p(f) = \phi$ and $p(g) = \psi$ there is a unique $h : x \to z$ such that $f = g \circ h$ and $p(h) = \chi$.

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[dr, "h"] \arrow[dd] & & y \arrow[dd] \\
& z \arrow[ur, "g"] \arrow[dd] & \\
U \arrow[rr, "\phi"] \arrow[dr, "\chi"] & & W \\
& V \arrow[ur, "\psi"] &
\end{tikzcd}$$

The conditions imply that the existence of the morphism $f : x \to y$ is unique up to canonical isomorphism. Then for $\phi : U \to V$ and $y \in \mathrm{Ob}(\mathcal{S}_V)$, $f : x \to y$ has been chosen; x will be written as $\phi^* y$ and ϕ^* is a functor from $\mathcal{S}_V$ to $\mathcal{S}_U$.

Definition 7.34. A *Grothendieck Topology* (G.T.) over a category $\mathcal{C}$ is a prescription of coverings $\{U_\alpha \to U\}_\alpha$ such that:

- $\{U_\alpha \to U\}_\alpha$ & $\{U_{\alpha\beta} \to U_\alpha\}_\beta$ implies $\{U_{\alpha\beta} \to U\}_{\alpha\beta}$
- $\{U_\alpha \to U\}_\alpha$ & $V \to U$ implies $\{U_\alpha \times_U V \to V\}_\alpha$
- $V \xrightarrow{\cong} U$ isomorphism, implies $\{V \longrightarrow U\}$

A category with a Grothendieck Topology is called a *Site*.

Example 7.19. $\mathcal{C} = \mathrm{Top}$, $\{U_\alpha \to U\}_\alpha$ if U_α is homeomorphic to its image and $U = \bigcup_\alpha \mathrm{im}(U_\alpha)$.

Definition 7.35. A *Sheaf* $\mathcal{F}$ over a site $\mathcal{C}$ is a functor p:$\mathcal{F} \to \mathcal{C}$ such that

- For all $S \in \mathrm{Ob}(\mathcal{C})$, $x \in \mathrm{Ob}(\mathcal{F}_S)$ and $f : T \to S \in \mathrm{Mor}(\mathcal{C})$ there exists a unique $\phi : y \to x \in \mathrm{Mor}(\mathcal{F})$ such that $p(\phi) = f$.
- For every cover $\{S_\alpha \to S\}_\alpha$, the following sequence is exact

$$\mathcal{F}_S \to \prod \mathcal{F}_{S_\alpha} \rightrightarrows \prod \mathcal{F}_{S_\alpha \times_S S_\beta}$$

Definition 7.36. A *Stack in groupoids* over $\mathcal{C}$ is a functor $p : \mathcal{S} \to \mathcal{C}$ such that

- $\mathcal{S}$ is fibered in groupoids over $\mathcal{C}$.
- For any $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}(\mathcal{S}_U)$, the functor

$$\mathbf{U} \to Sets$$

$$\phi : V \to U \mapsto \mathrm{Hom}(\phi^* x, \phi^* y)$$

  is a sheaf. $(\mathrm{Ob}(\mathbf{U}) = \{(S, \chi) | S \in \mathrm{Ob}(\mathcal{C}), \chi \in \mathrm{Hom}(S, U)\})$.

- If $\phi_i : V_i \to U$ is a covering family in $\mathcal{C}$, any descent datum relative to the ϕ_i's, for objects in $\mathcal{S}$, is effective.

Example 7.20. For X a G-set (provided with a G action over it) let $\mathcal{C} = \mathsf{Top}$, the category of topological spaces, and $S = [X/G]$ the category defined as follows:

$$\mathrm{Ob}([X/G])_S = \{f : E_S \to X\}$$

the set of all G-equivariant maps from principal G-bundles E_S over $S \in \mathrm{Ob}(\mathsf{Top})$, and

$$\mathrm{Mor}([X/G]) \subseteq \mathrm{Hom}_{BG}(E_S, E'_S)$$

given by

$$\begin{tikzcd} E_S \arrow[r, leftrightarrow] \arrow[d, "(\mathrm{proj}, f)"'] & S \times_{S'} E_{S'} \arrow[d, "1 \times f'"] \\ S \times X \arrow[r, leftrightarrow] & S \times X \end{tikzcd}$$

With the functor

$$\begin{array}{rcl} p : [X/G] & \to & \mathsf{Top} \\ (f : E_S \to X) & \mapsto & S \end{array}$$

By definition $[X/G]$ is a category fibered by groupoids, and if the group G is finite $[X/G]$ is a stack.

We can define the stack associated to an orbifold. Let X be an orbifold with $\{(V_p, G_p, \pi_p)\}_{p\in X}$ a set of orbifold charts. Let $\mathcal{C}$ be the category of all open subsets of X with the inclusions as morphisms and for $U \subset X$, let $\mathcal{S}_U$ be the category of all uniformizing systems of U such that they are equivalent for every $q \in U$ to the orbifold structure, in other words

$$\mathcal{S}_U = \{(W, H, \tau) | \forall q \in U, (V_q, G_q, \pi_q)\&(W, H, \tau) \text{ are equivalent at } q\}$$

It is clear that the category $\mathcal{S}$ is fibered by groupoids. It is known, and this requires more work, that this system $\mathcal{S} \to \mathcal{C}$ is also an stack, often called a C^∞-Deligne-Mumford stack.

The most complete reference for stacks is *The stacks project* an online wiki site at Columbia:

`http://stacks.math.columbia.edu/browse`

Some other excellent references are [Mum65], [Hei05],[Fan01], [Vis89], and [LMB99]

7.4 The Localization Principle

Theorem 7.37 (The Localization Principle [dFLNU]). *Let $\mathcal{X}$ be an orbifold and $\mathsf{L}\mathcal{X}$ its loop orbifold. Then the fixed orbifold under the natural circle action by rotation of loops is*

$$(\mathsf{L}\mathcal{X})^{S^1} = \mathrm{I}(\mathcal{X}) \tag{32}$$

where the groupoid $\mathrm{I}(\mathcal{X})$ has as its space of objects

$$\mathrm{I}(\mathcal{X})_0 = \{\alpha \in \mathcal{X}_1 \colon s(\alpha) = t(\alpha)\} = \coprod_{m\in\mathcal{X}_0} \mathrm{Aut}_{\mathcal{X}}(m)$$

and its space of arrows is

$$\mathrm{I}(\mathcal{X})_1 = Z(\mathrm{I}(\mathcal{X}_0)) = \{g \in \mathcal{X}_1 \colon \alpha \in \mathrm{I}(\mathcal{X})_0 \Rightarrow g^{-1}\alpha g \in \mathrm{I}(\mathcal{X})_0\},$$

a typical arrow in $\mathrm{I}(\mathcal{X})$ from α_0 to α_1 looks like

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha_0"] \arrow[r, bend left, "g"] & \circ \arrow[l, bend left, "g^{-1}"] \arrow[loop right, "\alpha_1^{-1}"] \end{tikzcd}$$

While for a smooth manifold the space of constant maps is

$$\mathsf{M} = (\mathcal{L}\mathsf{M})^{\mathsf{S}^1}$$

we have in contrast

$$\mathcal{X} \subset \mathsf{I}(\mathcal{X}) = (\mathsf{L}\mathcal{X})^{\mathsf{S}^1}.$$

In [LU04b] we define the *ghost loop space* $\mathcal{L}_s\mathrm{B}\mathcal{X}$ as the subspace of elements $\gamma \in \mathcal{L}\mathrm{B}\mathcal{X}$ so that the composition with the canonical projection $\pi_{\mathcal{X}} \colon \mathrm{B}\mathcal{X} \to \mathsf{X}$, $\pi_{\mathcal{X}} \circ \gamma$ is constant. In that paper it is proved the following homotopy equivalence

Theorem 7.38. *There is a homotopy equivalence between the classifying space of the inertia orbifold and the ghost loop space:*

$$\mathsf{BI}(\mathcal{X}) \simeq \mathcal{L}_s\mathrm{B}\mathcal{X}.$$

Example 7.21. Let us consider now a Riemannian metric on M. There is then a family of canonically defined operators: the Laplacians on k-forms Δ^{k}. These are related to a quantum field theory whose fields are maps from intervals the circle to M. Roughly speaking, the Lagrangian of the theory is given by

$$\mathsf{L}(\phi) = \frac{1}{2}\int |d\phi|^2.$$

All the information of such quantum theory is contained in the spectrum of the Laplacian. Recovering the classical theory from the quantum one is “hearing the shape of the drum.” In any case, the Feynman functional integration approach for the theory allows us to compute an integral over the free loop space of the manifold $\mathcal{L}(\mathsf{M}) = \mathrm{Maps}(\mathsf{S}^1; \mathsf{M})$ by stationary phase approximation as an integral over M.

This quantum field formalism is related to the heat equation

$$\partial_t \omega + \Delta^k \omega = 0,$$

whose solution is given by the heat flow $e^{-t\Delta^k}$. In particular the fundamental solution for the trace of the heat kernels is given by

$$\sum (-1)^k \mathrm{Tr}(e^{-t\Delta^k}) = \int_{\mathcal{L}\mathsf{M}} e^{t^{-1}\mathsf{L}(\phi)} \mathcal{D}\phi,$$

where $\mathcal{D}\phi$ is the formal part of the Wiener measure on $\mathcal{L}\mathsf{M}$.

It turns out that the the sum of the traces of the heat kernels is independent of t. The long time limit of this sum equals the Euler characteristic (by recalling

Hodge's theorem, which identifies the k-th Betti number of M as the dimension of the kernel of Δ^k), and the short time behaviour is given by an integral of a complicated curvature expression.

If the dimension of the manifold is 2, this equality of long and short time behaviour of the heat flow leads to the Gauss-Bonet theorem

$$\int_M K dA = \chi(M), \tag{33}$$

where K is the Gaussian curvature and dA is the volume element.

In fact we have oversimplified: we can do better than to simply recover the Euler characteristic. Suppose that M is a spin manifold; then we can recover through this procedure the *index of the Dirac operator* and this is oulined in the Appendix 20 on Orbifold Index Theory. But before we do that let us see how we stand in the orbifold case.

To try to apply these methods to an orbifold $\mathcal{X}$ (replacing the rôle of M above), we must replace $\mathcal{L}M$ for the loop orbifold.

Recall that while for a smooth manifold we have

$$M = (\mathcal{L}M)^{S^1},$$

we have, by contrast

$$\mathcal{X} \subset \mathrm{I}(\mathcal{X}) = (L\mathcal{X})^{S^1},$$

so we expect the Euler characteristic, the K-theory, and so, on to localize in $\mathrm{I}(\mathcal{X})$ rather than in X. While the orbifold $\mathrm{I}(\mathcal{X})$ is called in the mathematical literature the *inertia orbifold* of $\mathcal{X}$, and it is, as Chen and Ruan [CR04b] have pointed out (and as is reflected in their terminology), the classical geometrical manifestation of the *twisted sectors* of orbifold string theory [DHVW86].

Indeed, we have that for a general orbifold

$$\chi_{\mathbf{Orb}}(\mathcal{X}) = \chi(\mathrm{I}(\mathcal{X}))$$

and as explained in Appendix 18:

$$K^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C} \cong K^*(\mathrm{I}(\mathcal{X})) \otimes \mathbb{C}.$$

For example, in the case of a global quotient $\mathcal{X} = [M/G]$, one can readily verify that

$$\mathrm{I}(\mathcal{X}) = \coprod_{(g)} [M^g/C(g)], \tag{34}$$

recovering thus Segal's localization formula and the orbifold Euler characteristic (see Section 18.2).

8 Orbifolding Calabi-Yau Categories

8.1 Equivariant Closed Theories

Let us begin with some general remarks. In n-dimensional topological field theory one begins with a category nCob whose objects are oriented $(n-1)$-manifolds and whose morphisms are oriented cobordisms. Physicists say that a theory admits a group G as a global symmetry if G acts on the vector space associated to each $(n-1)$-manifold and the linear operator associated to each cobordism is a G-equivariant map. When we have such a global symmetry group G we can ask whether the symmetry can be *gauged*, i.e. whether elements of G can be applied independently in some sense at each point of space-time. Mathematically the process of gauging has a very elegant description: it amounts to extending the field theory functor from the category nCob to the category nCob$_G$ whose objects are $(n-1)$-manifolds equipped with a principal G-bundle, and whose morphisms are cobordisms with a G-bundle.

We have another interpretation of this category, this view is due to Turaev [Tur99] and it consists on working in the language of pointed homotopy theory (smooth version). For this, we consider a path-connected topological space X with a base point $x \in X$. We define an X-*manifold* to be a pair consisting of a pointed closed oriented manifold M and a characteristic map $g_M : M \to X$. We say that M is the base of the X-manifold g_M. For M and M' as before we can talk of a X-diffeomorphisms between them. A cobordism W from M_0 to M_1 is endowing with a map $W \to M$ sending the basis point of the boundary components into x. Both basis M_0 and M_1 are considered as X-manifolds with characteristic maps obtained by restricting the given map $W \to M$. An X-diffeomorphism of a X-cobordisms $f : (W, M_0, M_1) \to (W', M'_0, M'_1)$ is an orientation preserving diffeomorphism inducing a X-diffeomorphisms $M_0 \to M'_0$, $M_1 \to M'_1$ and such that $g_W = g_{W'} f$ where g_W, $g_{W'}$ are the characteristic maps of W, W' respectively.

We can glue X-cobordisms along the base. If (W_0, M_0, N), (W_1, N', M_1) are X-cobordisms and $f : N \to N'$ is an X-diffeomorphism then the gluing of W_0 with W_1 along f yields a new X-cobordism with base boundaries M_0 and M_1.

If we make a quotient by identifying diffeomorphic objects, hence any diffeomorphism becomes an identity. When we take $X = BG$ we get an alternative viewpoint for $n\mathsf{Cob}_G$.

Yet another equivalent interpretation of $n\mathsf{Cob}_G$ comes from considering it as a category of cobordisms of $\mathcal{B}G$-manifolds where $\mathcal{B}G$ is defined as the orbifold $\mathcal{B}G :=$ $[\bullet/G]$. For a manifold with a map to $\mathcal{B}G$ is the same as a manifold equipped with a G-principal bundle.

Definition 8.1. A G-*equivariant* TFT is a symmetrical monoidal functor from $\mathrm{nCob}_{\mathsf{G}}$ to $\mathrm{Vect}_{\mathbb{C}}$.

8.2 G-Frobenius Algebras

We start with the definition of the algebraic data, with a proposition that relates the Frobenius structure of the G-invariant part and with the equivariant version for the Abrams theorem. This definition was done in the paper of Moore and Segal [MS].

Definition 8.2. A G-*Frobenius algebra* is an algebra $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$, where $\mathcal{C}_g$ is a vector space of finite dimension for all $g \in G$ such that

1. There is a homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{C})$, see Figure 20, where $\mathrm{Aut}(\mathcal{C})$ is the algebra of homomorphisms of $\mathcal{C}$ such that

$$\alpha_h : \mathcal{C}_g \to \mathcal{C}_{hgh^{-1}},$$

and for every $g \in G$ we have

$$\alpha_g|_{\mathcal{C}_g} = 1_{\mathcal{C}_g}.$$

Note that $\alpha_e : \mathcal{C}_g \to \mathcal{C}_g$ is the identity map.

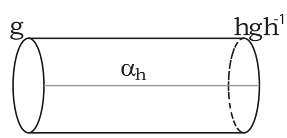


Figure 20: The action $\alpha_h : \mathcal{C}_g \to \mathcal{C}_{hgh^{-1}}$.

2. There is a G-invariant trace or counit $\varepsilon : \mathcal{C}_e \to \mathbb{C}$ which induce nondegenerate pairings, see Figure 21,

$$\theta_g : \mathcal{C}_g \otimes \mathcal{C}_{g^{-1}} \to \mathbb{C}.$$

3. For all $x \in \mathcal{C}_g$ and $y \in \mathcal{C}_h$ we have that the product is twisted commutative (see Figure 22), i.e.

$$xy = \alpha_g(y)x.$$

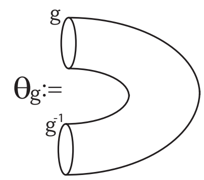


Figure 21: The pairing $\theta_g : \mathcal{C}_g \otimes \mathcal{C}_{g^{-1}} \to \mathbb{C}$.

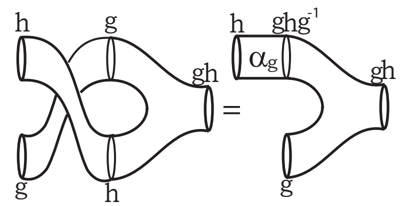


Figure 22: The twisted commutativity of the product.

4. Let $\Delta_g = \sum_i \xi_i^g \otimes \xi_i^{g^{-1}} \in \mathcal{C}_g \otimes \mathcal{C}_{g^{-1}}$ be the *Euler element*, where $\left\{\xi_i^g\right\}$ is a basis of $\mathcal{C}_g$ and $\left\{\xi_i^{g^{-1}}\right\}$ is the dual basis of $\mathcal{C}_{g^{-1}}$. For all $g, h \in G$ (see Figure 23) the identity
$$\sum_i \alpha_h(\xi_i^g)\xi_i^{g^{-1}} = \sum_i \xi_i^h \alpha_g(\xi_i^{h^{-1}})$$
holds.

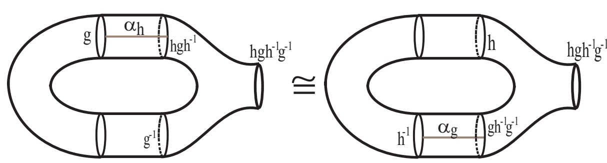


Figure 23: Torus axiom.

The next proposition provides us a natural consequence of this definition. It says that the G-invariant part of the G-Frobenius algebra $\mathcal{C}^G$ is a Frobenius algebra.

Proposition 8.3. *For $\mathcal{C}$ a G-Frobenius algebra, the G-invariant part of this algebra, denoted by $\mathcal{C}_{orb}$, is a Frobenius algebra.*

Proof. Let be $\mathcal{C}_{orb} := \mathcal{C}^G = (\oplus_{g\in G}\mathcal{C}_g)^G$. Note that $\mathcal{C}_{orb} \cong \oplus_{g\in T}\mathcal{C}_g^{C(g)}$ where T is a set of representatives for the conjugacy classes in G and $C(g)$ is the centralizer of

$g \in G$. The maps that define this isomorphism are

$$\begin{array}{rccc}\Psi: & \bigoplus_{g\in T}\mathcal{C}_g^{C(g)} & \longrightarrow & \left(\bigoplus_{b\in G}\mathcal{C}_g\right)^G \\ & \sum_{g\in G} y_g & \longmapsto & \sum_{g\in T}\sum_{h\in[g],h=kgk^{-1}} \alpha_k(y_g)\end{array}$$

and

$$\begin{array}{rccc}\Upsilon: & \left(\bigoplus_{b\in G}\mathcal{C}_g\right)^G & \longrightarrow & \bigoplus_{g\in T}\mathcal{C}_g^{C(g)} \\ & \sum_{g\in G} x_g & \longmapsto & \sum_{g\in T} x_g.\end{array}$$

First, we prove that $\mathcal{C}_{orb}$ is an algebra. The product is simply the restriction of the product in $\mathcal{C}$, this is because for $x, y \in \mathcal{C}_{orb}$ we have that $g \cdot x = \alpha_g(x) = x$ and $g \cdot y = \alpha_g(y) = y$ for all $g \in G$, then $g \cdot xy = \alpha_g(xy) = \alpha_g(x)\alpha_g(y) = xy$. An additional property is the commutative of the product, to check this we take $x = \sum_{g\in G} x_g$ and $y = \sum_{h\in G} y_h \in \mathcal{C}_{orb}$. The calculations are as follows:

$$xy = \sum_{g\in G}\sum_{h\in G} x_g y_h = \sum_{g,h\in G} \alpha_g(y_h)x_g = \sum_{g\in G}\alpha_g\left(\sum_{h\in G} y_h\right)x_g = \sum_{g\in G} yx_g = yx.$$

For the Frobenius structure we define the trace $\varepsilon : \mathcal{C}_{orb} \to \mathbb{C}$ as the restriction of $\varepsilon : \mathcal{C} \to \mathbb{C}$ with the value zero on $\mathcal{C}_g$ with $g \neq e$. To complete the proof we need to prove that the induced pairing is non-degenerate.
Let $x = \sum_{g\in G} x_g \in \mathcal{C}_{orb}$ and suppose $\varepsilon(xy) = 0$, for all $y \in \mathcal{C}_{orb}$. We need to prove that $x = 0$. If we show that $x_g = 0$ for all $g \in T$, we would be finished, and this holds because $x = \sum_{g\in T}\sum_{h\in[g],h=kgk^{-1}} \alpha_k(x_g)$. We can consider $y_h \in \mathcal{C}_h$, where h is the representative of $[h] \in T$, then $y := \sum_{k\in[h],k=lhl^{-1}} \alpha_l(y_h) \in \mathcal{C}_{orb}$. Now

$$\varepsilon(xy) = |[h]|\varepsilon(x_{h^{-1}}(y_h))$$

Hence $\varepsilon(x_{h^{-1}}y_h) = 0$ for all $y_h \in \mathcal{C}_h$, and then $x_{h^{-1}} = 0$ for every $h \in T$. Finally $x = 0$.

♣

Corollary 8.4. *The coproduct in* $\mathbb{C}_{orb}$ *is*

$$\Delta = (m \otimes 1) \cdot (1 \otimes \Theta)$$

where $\Theta : \mathbb{C} \to \mathbb{C}_{orb} \otimes \mathbb{C}_{orb}$ *is the copairing.*

Proof. We only need to construct a basis of $\mathbb{C}_{orb}$.
Let be $\{e_i^g\}$ a basis of $\mathbb{C}_g$ such that $\alpha_k(e_i^g) = e_i^{kgk^{-1}}$ is a basis of $\mathbb{C}_{kgk^{-1}}$.
For $x \in \mathbb{C}_{orb}$ there is the identity

$$x = \sum_{g\in T} \sum_{h\in[g],h=kgk^{-1}} \alpha_k(x_g),$$

where $x_g = \sum_i \lambda_i^g e_i^g \in \mathbb{C}_g$. Therefore

$$x = \sum_{g\in T} \sum_{h\in[g],h=kgk^{-1}} \sum_i \lambda_i^g \alpha_k(e_i^g) = \sum_{g\in T} \sum_i \lambda_i^g \sum_{h\in[g],h=kgk^{-1}} e_i^{kgk^{-1}} = \sum_{g\in T} \sum_i \lambda_i^g E_{i,g}$$

where $E_{i,g} = \sum_{h\in[g]} e_i^h$. This proves that $\{E_{i,g}\}$ is a generator of $\mathbb{C}_{orb}$. Now we prove that this set is linearly independent. Suppose that $\sum_{g\in T, i\in I_g} \beta_{i,g} E_{i,g} = 0$, then $\sum_{g\in T, i\in I_g} \sum_{h\in[g]} \beta_{i,g} e_i^h = \sum_{g\in G} \left(\sum_{i\in I_g} \beta_{i,g} e_i^h \right) = 0$, where $\beta_{i,g} = \beta_{i,h}$ if h and g are in the same conjugation class. As $\sum_{i\in I_g} \beta_{i,g} E_{i,g} \in \mathbb{C}_g$ hence $\sum_{i\in I_g} \beta_{i,g} E_{i,g} = 0$ for all $g \in G$. We use that e_i^g is a basis of $\mathbb{C}_g$, to prove that $\beta_{i,g} = 0$ for all $g \in T$, $i \in I_g$.

Note that for $E_{i,g} \in \mathbb{C}_{orb}$ and $k \in G$ we have $k \cdot E_{i,g} = \sum_{h\in[g]} \alpha_k(e_i^h) = \sum_{h\in[g]} e_i^{khk^{-1}} = \sum_{l\in[g]} e_i^l = E_{i,g}$, where $l = khk^{-1} \in [g]$.

We can construct $\left\{ E_{i,g}^{\#} \right\} = \frac{1}{|[g]|} \sum_{h\in[g]} e_i^{h^{-1}}$ as the dual basis of $\mathbb{C}_{orb}$. Then

$$\Theta(1) = \sum_{g\in T, i\in I_g} E_{i,g}^{\#} \otimes E_{i,g}$$

and

$$\Delta(x) = \sum_{g\in T, i\in I_g} x E_{i,g}^{\#} \otimes E_{i,g} = \sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} \frac{1}{|[g]|} x e_i^{h^{-1}} \otimes e_i^k.$$

♣

Theorem 8.5. *(Abrams equivariant case) Let $\mathbb{C} = \oplus_{g\in G} \mathbb{C}_g$ be an algebra with an associative product $m_{g,h} : \mathbb{C}_g \otimes \mathbb{C}_h \to \mathbb{C}_{gh}$ and a unit $u : \mathbb{C} \to \mathbb{C}_e$, where every $\mathbb{C}_g$ is a finite dimension space. We have that a trace $\varepsilon : \mathbb{C}_e \to \mathbb{C}$ is non-degenerate if and only if it has a coassociative coproduct $\Delta_{g,h} : \mathbb{C}_{gh} \to \mathbb{C}_g \otimes \mathbb{C}_h$, with ε as its counit,* *such that for every* $g, h, k \in G$ *the following diagrams commute:*

$$\begin{tikzcd} \mathbb{C}_g \otimes \mathbb{C}_{hk} \arrow[r, "m_{g,hk}"] \arrow[d, "1\otimes\Delta_{h,k}"'] & \mathbb{C}_{ghk} \arrow[d, "\Delta_{gh,k}"] \\ \mathbb{C}_g \otimes \mathbb{C}_h \otimes \mathbb{C}_k \arrow[r, "m_{g,h}\otimes 1"'] & \mathbb{C}_{gh} \otimes \mathbb{C}_k \end{tikzcd} \quad \begin{tikzcd} \mathbb{C}_{gh} \otimes \mathbb{C}_{k} \arrow[r, "m_{gh,k}"] \arrow[d, "\Delta_{g,h}\otimes 1"'] & \mathbb{C}_{ghk} \arrow[d, "\Delta_{g,hk}"] \\ \mathbb{C}_g \otimes \mathbb{C}_h \otimes \mathbb{C}_k \arrow[r, "1\otimes m_{h,k}"'] & \mathbb{C}_{g} \otimes \mathbb{C}_{hk} \end{tikzcd} \tag{35}$$

Proof. The necessity is the nontrivial part and for this we define the coproduct

$$\begin{tikzcd} \mathbb{C}_{gh} \arrow[rrr, "\Delta_{g,h}"] \arrow[d, "\Phi_f"'] & & & \mathbb{C}_g \otimes \mathbb{C}_h \\ \mathbb{C}^*_{h^{-1}g^{-1}} \arrow[r, "m_{h^{-1},g^{-1}}{}^*"'] & \mathbb{C}^*_{h^{-1}} \otimes \mathbb{C}^*_{g^{-1}} \arrow[rr, "\tau"'] & & \mathbb{C}^*_{g^{-1}} \otimes \mathbb{C}^*_{h^{-1}} \arrow[u, "\Phi_g^{-1}\otimes\Phi_h^{-1}"'] \end{tikzcd}$$

where $\Phi_f(x)(y) = \varepsilon(m_{f,f^{-1}}(x \otimes y))$. This coproduct is coassociative and satisfies the two Diagrams of (35).

♣

Theorem 8.6. *Every 2D G-equivariant topological field theory defines a G-Frobenius algebra from which it can be recovered, i.e. the categories 2d G-TQFT and G-Frobenius algebras are equivalent*

The proof of this theorem is very similar to the proofs we have already presented in the non-equivariant case and we refer the reader to [MS] for full details.

8.3 Nearly G-Frobenius Algebras

Definition 8.7. A *nearly* G*-Frobenius algebra* is an algebra $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$, where $\mathcal{C}_g$ is a vector space for all $g \in G$ such that

1. There is a homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{C})$, where $\mathrm{Aut}(\mathcal{C})$ is the algebra of homomorphisms of $\mathcal{C}$, such that

   $$\alpha_h : \mathcal{C}_g \to \mathcal{C}_{hgh^{-1}},$$

   for every $g \in G$ we have

   $$\alpha_g|_{\mathcal{C}_g} = \mathrm{Id}_{\mathcal{C}_g}.$$

   Note that $\alpha_e : \mathcal{C}_g \to \mathcal{C}_g$ is the identity map.

2. For all $x \in \mathcal{C}_g$ and $y \in \mathcal{C}_h$ we have that the product is twisted commutative, i.e.
$$xy = \alpha_g(y)x.$$

3. There are coproducts $\Delta_{g,h} : \mathbb{C}_{gh} \to \mathbb{C}_g \otimes \mathbb{C}_h$ such that the following diagrams commute.

$$\begin{tikzcd}
\mathbb{C}_g \otimes \mathbb{C}_{hf} \arrow[r, "m_{g,hf}"] \arrow[d, "1\otimes\Delta_{hf}"'] & \mathbb{C}_{ghf} \arrow[d, "\Delta_{gh,f}"] \\
\mathbb{C}_{gh} \otimes \mathbb{C}_{h^{-1}} \otimes \mathbb{C}_{hf} \arrow[r, "m_{g,h}\otimes 1"'] & \mathbb{C}_{gh} \otimes \mathbb{C}_f
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}_g \otimes \mathbb{C}_{hf} \arrow[r, "m_{g,hf}"] \arrow[d, "\Delta_{gh,h^{-1}}\otimes 1"'] & \mathbb{C}_{ghf} \arrow[d, "\Delta_{gh,f}"] \\
\mathbb{C}_{gh} \otimes \mathbb{C}_{h^{-1}} \otimes \mathbb{C}_{hf} \arrow[r, "1\otimes m_{h^{-1},hf}"'] & \mathbb{C}_{gh} \otimes \mathbb{C}_f
\end{tikzcd}$$

See Figure 24.

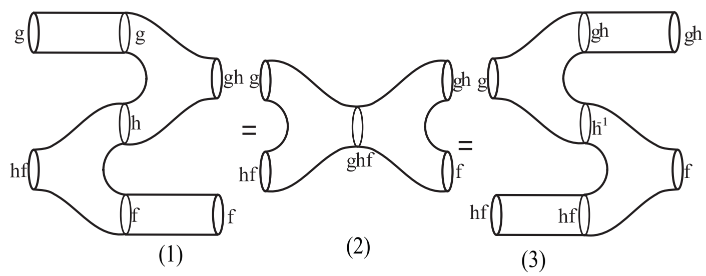


Figure 24: Abrams condition.

4. These coproducts have the next properties
For every $g, h \in G$ the next diagram commutes

$$\begin{tikzcd}
\mathbb{C}_e \arrow[r, "\Delta_h"] \arrow[d, "\Delta_g"'] & \mathbb{C}_h \otimes \mathbb{C}_{h^{-1}} \arrow[r, "1\otimes\alpha_g"] & \mathbb{C}_h \otimes \mathbb{C}_{gh^{-1}g^{-1}} \arrow[d, "m_{h,gh^{-1}g^{-1}}"] \\
\mathbb{C}_g \otimes \mathbb{C}_{g^{-1}} \arrow[r, "\alpha_h\otimes 1"'] & \mathbb{C}_{hgh^{-1}} \otimes \mathbb{C}_{g^{-1}} \arrow[r, "m_{hgh^{-1},g^{-1}}"'] & \mathbb{C}_{hgh^{-1}g^{-1}}
\end{tikzcd}$$

Remark 8.8. Note that the condition 3 implies the next particular case. We take the particular commutative diagrams

$$
\begin{tikzcd}
\mathbb{C}_g \otimes \mathbb{C}_e \arrow[r, "m_{g,e}"] \arrow[d, "1\otimes\Delta_{h^{-1},h}"'] & \mathbb{C}_g \arrow[d, "\Delta_{gh^{-1},h}"] \\
\mathbb{C}_g \otimes \mathbb{C}_{h^{-1}} \otimes \mathbb{C}_h \arrow[r, "m_{g,h^{-1}}\otimes 1"'] & \mathbb{C}_{gh^{-1}} \otimes \mathbb{C}_h
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}_e \otimes \mathbb{C}_g \arrow[r, "m_{e,g}"] \arrow[d, "\Delta_{gh^{-1},hg^{-1}}\otimes 1"'] & \mathbb{C}_g \arrow[d, "\Delta_{gh^{-1},h}"] \\
\mathbb{C}_{gh^{-1}} \otimes \mathbb{C}_{hg^{-1}} \otimes \mathbb{C}_g \arrow[r, "1\otimes m_{hg^{-1},g}"'] & \mathbb{C}_{gh^{-1}} \otimes \mathbb{C}_h
\end{tikzcd}
$$

and $x_g \in \mathbb{C}_g$, then the next equality is satisfied

$$\sum_i x_g e_i^{h^{-1}} \otimes e_i^h = \sum_i e_i^{gh^{-1}} \otimes e_i^{hg^{-1}} x_g,$$

where $\{e_i^h\}$ is a basis of $\mathbb{C}_h$, which is a generalized condition of Lauda (see Figure 25).

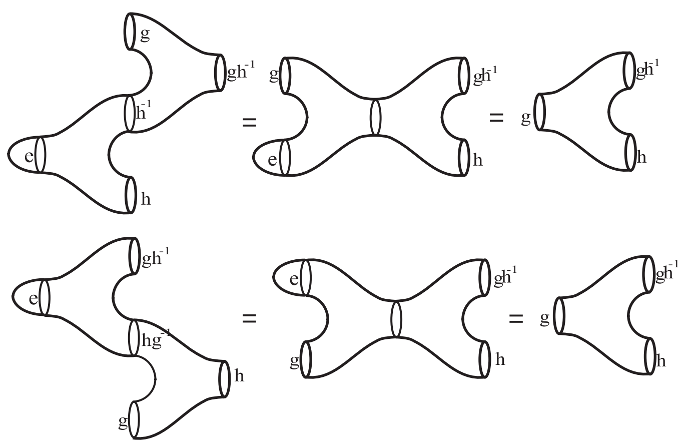


Figure 25: Generalized Lauda condition.

Theorem 8.9. *If $\mathcal{C}$ is a nearly G-Frobenius algebra then its G-invariant part, denoted by $\mathcal{C}_{orb}$, is a nearly Frobenius algebra.*

Proof. We define the coproduct

$$\Delta : \mathbb{C}_{orb} \to \mathbb{C}_{orb} \otimes \mathbb{C}_{orb}$$

similarly as in Corollary 8.4. This is $\Delta(x) = \sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} x e_i^{h^{-1}} \otimes e_i^k$.
To prove that $(\mathbb{C}_{orb}, \Delta)$ is a nearly Frobenius algebra we only need to prove the

Lauda condition, i.e.

$$\sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} x e_i^{h^{-1}} \otimes e_i^k = \sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} e_i^{h^{-1}} \otimes e_i^k x.$$

If $x = \sum_{l\in G} x_l$, then

$$\sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} x e_i^{h^{-1}} \otimes e_i^k = \sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} \sum_{l\in G} x_l e_i^{h^{-1}} \otimes e_i^k.$$

By Remark 8.8 we have $\sum_i x_g e_i^{h^{-1}} \otimes e_i^h = \sum_i e_i^{gh^{-1}} \otimes e_i^{hg^{-1}} x_g$. If we act on the second component by $\alpha_r : \mathbb{C}_h \to \mathbb{C}_{rhr^{-1}} = \mathbb{C}_k$ then the next identity is satisfied

$$\sum_i x_g e_i^{h^{-1}} \otimes \alpha_r(e_i^h) = \sum_i e_i^{gh^{-1}} \otimes \alpha_r(e_i^{hg^{-1}} x_g),$$

hence

$$\sum_i x_g e_i^{h^{-1}} \otimes e_i^k = \sum_i e_i^{gh^{-1}} \otimes e_i^{rhg^{-1}r^{-1}} \alpha_r(x_g).$$

Therefore

$$\sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} \sum_{l\in G} x_l e_i^{h^{-1}} \otimes e_i^k = \sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} \sum_{l\in G} e_i^{lh^{-1}} \otimes e_i^{rhl^{-1}r^{-1}} \alpha_r(x_l).$$

We use that lh^{-1} and $rhl^{-1}r^{-1} = krl^{-1}r^{-1}$ are in the same conjugacy class and lh^{-1} and $rhl^{-1}r^{-1}$ vary over all G, so we can change the variables h, k for u, v. Then

$$\begin{aligned}
\Delta(x) &= \sum_{g\in T, i\in I_g} \sum_{u,v\in[g]} \sum_{l\in G} e_i^{u^{-1}} \otimes e_i^v \alpha_r(x_l) \\
&= \sum_{g\in T, i\in I_g} \sum_{u,v\in[g]} e_i^{u^{-1}} \otimes e_i^v \alpha_r\left(\sum_{l\in G} x_l\right) \\
&= \sum_{g\in T, i\in I_g} \sum_{u,v\in[g]} e_i^{u^{-1}} \otimes e_i^v \alpha_r(x) \\
&= \sum_{g\in T, i\in I_g} \sum_{u,v\in[g]} e_i^{u^{-1}} \otimes e_i^v x.
\end{aligned}$$

♣

8.4 Examples of (nearly) G-Frobenius Algebras.

8.4.1 Chen-Ruan Cohomology

We will give now the definition of the Chen-Ruan cohomology following [CR04b]. First we need to define the degree shifting and the obstruction bundle for the Chen-Ruan theory.

The definition of the degree shifting is local so it is enough to define it in the case of a global quotient (cf. [FG03]).

Consider Y an almost complex G-manifold with G a finite group. Given $g \in G$ and $y \in Y^g$ we define $a(g, y)$ the *age* of g at y as follows. Diagonalize the action of g in T_yY to obtain

$$g = \mathrm{diag}(\exp(2\pi i r_1), \dots, \exp(2\pi i r_n)),$$

with $0 \le r_i < 1$ and set

$$a(g, y) := \sum_i r_i.$$

The age $a(g, y)$ only depends on the connected component Y^g_o of Y^g in which y lies. For this reason we can simply write $a(g, Y^g_o)$ or even $a(g)$ when there is no confusion.

Note that the age has the following interesting property

$$a(g, Y^g_o) + a(g^{-1}, Y^g_o) = \mathrm{codim}(Y^g_o, Y).$$

The *Chen-Ruan degree shifting number* is defined then as

$$s_g := 2a(g).$$

As a rational vector space the Chen-Ruan orbifold cohomology is

$$H^*_{CR}(Y, G) := H^*(Y, G)[s] = \bigoplus_{g \in G} H^*(Y^g, \mathbb{C})[s_g]$$

or more generally

$$H^*_{CR}(\mathcal{G}) := H^*(\Lambda\mathcal{G})[s].$$

The definition of the obstruction bundle is modeled on the definition of the virtual fundamental class on the moduli of curves for quantum cohomology.

Let $\bar{\mathcal{M}}_3(\mathcal{G})$ be the moduli space of ghost representable orbifold morphisms f_y from $\mathbb{P}^1_3$ to $\mathcal{G}$, where $\mathrm{im}(f) = y \in \mathcal{G}_0$ and the marked orbifold Riemann surface

$\mathbb{P}^1_3$ has three marked points, z_1, z_2, and z_3, with multiplicities m_1, m_2, and m_3, respectively. In [ALR07] it is proved that

$$\bar{\mathcal{M}}_3(\mathcal{G}) = \mathcal{G}^2.$$

Let us fix a connected component $\mathcal{G}^2_o$ of $\mathcal{G}^2$.

To define the Chen-Ruan obstruction bundle $\mathcal{E}_o \to \mathcal{G}^2_o$ we consider the elliptic complex

$$\bar{\partial}_y : \Omega^0(f_y^* T\mathcal{G}) \longrightarrow \Omega^{0,1}(f_y^* T\mathcal{G}).$$

Chen and Ruan proved that $\operatorname{coker}(\bar{\partial}_y)$ has constant dimension along components and forms an orbivector bundle $\mathcal{E}_o \to \mathcal{G}^2_o$.

The formula for the Chen-Ruan product is then (see Section 11.3)

$$H^*_{\mathrm{CR}}(\mathcal{G}) \otimes H^*_{\mathrm{CR}}(\mathcal{G}) \longrightarrow H^*_{\mathrm{CR}}(\mathcal{G})$$

given by

$$\alpha \star \beta := (e_{12})_*(e_1^* \alpha \cdot e_2^* \beta \cdot e(\mathcal{E})).$$

The following is a theorem of Chen and Ruan [CR04b] (cf. [Kau03].)

Theorem 8.10. $(H^*_{\mathrm{CR}}(\mathcal{G}), \star)$ *is a graded associative algebra, moreover it has a natural Frobenius algebra structure compatible with this product.*

We will study this theory in more detail in Chapter 11.

8.4.2 Stringy K-theory

Here we should mention that both the Chen-Ruan and the virtual orbifold theories can be written in K-theory without much modification in the formulæ [JKK07]. One just needs to change the Euler classes $e(\mathcal{V})$ and $e(\mathcal{E})$ for the corresponding Euler classes in K-theory $\lambda_{-1}(\mathcal{V})$ and $\lambda_{-1}(\mathcal{E})$ respectively. As $\mathbb{Z}$-modules we have $K^*_{\mathrm{virt}}(\Lambda\mathcal{G}) := K^*(\Lambda\mathcal{G})$ and $K^*_{\mathrm{JKK}}(\mathcal{G}) := K^*(\Lambda\mathcal{G})$. The corresponding expressions for the products in K-theory are:

$$V \times W := (e_{12})_*(e_1^* V \otimes e_2^* W \otimes \lambda_{-1}(\mathcal{V})),$$

and

$$V \star W := (e_{12})_*(e_1^* V \otimes e_2^* W \otimes \lambda_{-1}(\mathcal{E})),$$

respectively.

Theorem 8.11 (Jarvis-Kaufmann-JKK [JKK07]). *There exists a stringy Chern character*

$$\mathbf{Ch}_{\mathrm{JKK}} : \mathsf{K}^*_{\mathrm{JKK}}(\mathcal{G}) \otimes \mathbb{C} \longrightarrow \mathsf{H}^*_{\mathrm{CR}}(\mathcal{G}, \mathbb{C})$$

that is a Frobenius algebra isomorphism

Thsi theory will be thoroughly studied in Chapter 11.

8.4.3 Virtual Orbifold Cohomology

Let S be a complex manifold and let S_1 and S_2 be closed submanifolds that intersect *cleanly*; that is, $U := S_1 \cap S_2$ is a submanifold of S and at each point x of U the tangent space of U is the intersection of the tangent spaces of S_1 and S_2. Let $E(S, S_1, S_2)$ be the *excess* bundle of the intersection, i.e., the vector bundle over U which is the quotient of the tangent bundle of S by the sum of the tangent bundles of S_1 and S_2 restricted to U. Thus $E(S, S_1, S_2) = 0$ if and only if S_1 and S_2 intersect transversally. In the Grothendieck group of vector bundles over U the excess bundle becomes

$$E(S, S_1, S_2) = T_S|_U + T_U - T_{S_1}|_U - T_{S_2}|_U.$$

Denote by $e(S, S_1, S_2)$ the Euler class of $E(S, S_1, S_2)$ and by

$$\begin{tikzcd} U \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, "h"] & S_1 \arrow[d, "j_1"] \\ S_2 \arrow[r, "j_2"'] & S \end{tikzcd} \tag{36}$$

the relevant inclusion maps. Then for any cohomology class $\alpha \in H^*(S_1)$ the following *excess intersection formula* [Qui71, Prop. 3.3] holds in the cohomology ring of S_2:

$$j_2^* j_{1*} \alpha = i_{2*} \left(e(S, S_1, S_2) i_1^*(\alpha) \right). \tag{37}$$

Consider the orbifold $[Y/G]$ where Y is an almost complex manifold and G acts preserving the almost complex structure. Define the groups

$$H^*(Y, G) := \bigoplus_{g \in G} H^*(Y^g) \times \{g\}$$

where Y^g is the fixed point set of the element g. The group G acts in the natural way. Denote by $Y^{g,h} = Y^g \cap Y^h$ and suppose that for every $g, h \in G$ we have cohomology classes $\nu(g,h) \in H^*(Y^{g,h})$, which are G-equivariant in the sense that $w^*\nu(k^{-1}gk, k^{-1}hk) = \nu(g,h)$ where $w : Y^{k^{-1}gk,k^{-1}hk} \to Y^{g,h}$ takes x to $w(x) := xk$. Define the map

$$\begin{aligned} \times : H^*(Y^g) \times H^*(Y^h) &\to H^*(Y^{gh}) \\ (\alpha, \beta) &\mapsto i_* \left(\alpha|_{Y^{g,h}} \cdot \beta|_{Y^{g,h}} \cdot \nu(g,h) \right) \end{aligned}$$

where $i : Y^{g,h} \to Y^{gh}$ is the natural inclusion.

Let us define now a degree shift σ on $H^*(Y, G)$. We will declare that the degree of a class $\alpha_g \in H^*(Y^g) \subset H^*(Y, G)[\sigma]$ is

$$i + \sigma_g$$

where

$$\sigma_g := 2(\dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} Y^g),$$

and i is the ordinary degree of α_g. In this paper all dimensions and codimensions are complex. Virtual orbifold cohomology was introduced in [LUX07]. There it was shown that:

Theorem 8.12. *For the cohomology classes* $\nu(g,h) = e(Y, Y^g, Y^h)$ *the map* $\times$ *defines an associative graded product on* $H^*_{\mathrm{virt}}(Y, G) := H^*(Y, G)[\sigma]$.

We will prove and generalize this result in Chapter 10.

Definition 8.13. In the case when $\nu(g,h) = e(Y, Y^g, Y^h)$, we will call the product $\times$ in $H^*(Y, G)$ the *virtual intersection product* and we will write $H^*_{\mathrm{virt}}(Y, G) := (H^*(Y, G)[\sigma], \times)$. Given that $H^*(Y, G; \mathbb{R})^G \cong H^*(I[Y/G]; \mathbb{R})$, the product $\times$ induces a ring structure on the orbifold cohomology of $[Y/G]$. We will call this ring the virtual intersection ring of a global orbifold and we will denote it by $H^*_{\mathrm{virt}}(\Lambda[Y/G])$.

The definition of the virtual ring generalizes to a non-global orbifold. To do this we use the language of groupoids, and follow the notation of Adem-Ruan-Zhang [ARZ07]. The Lemma 7.2 of [ARZ07] is the generalization of the clean intersection formula of Quillen to the category of orbifolds. In the notation of [ARZ07] we must replace Y^g and Y^h by two copies of $\Lambda\mathcal{G}$, and $Y^{g,h}$ by a copy of $\mathcal{G}^2$. We define in general the virtual obstruction orbibundle $\mathcal{V} \to \mathcal{G}^2$ as the excess bundle of the diagram of embeddings:

$$\begin{tikzcd} \mathcal{G}^2 \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, "h"] & \Lambda\mathcal{G} \arrow[d, "j_1"] \\ \Lambda\mathcal{G} \arrow[r, "j_2"'] & \mathcal{G} \end{tikzcd} \tag{38}$$

The definition of the degree shifting is local so we can use the same definition. We set

$$H^*_{\mathrm{virt}}(\Lambda\mathcal{G}) := H^*(\Lambda\mathcal{G})[\sigma].$$

The formula for the product in general becomes

$$H^*_{\mathrm{virt}}(\Lambda\mathcal{G}) \otimes H^*_{\mathrm{virt}}(\Lambda\mathcal{G}) \longrightarrow H^*_{\mathrm{virt}}(\Lambda\mathcal{G})$$

given by

$$\alpha \times \beta := (e_{12})_*(e_1^*\alpha \cdot e_2^*\beta \cdot e(\mathcal{V})),$$

where $e_{12} : \mathcal{G}^2 \to \Lambda\mathcal{G}$ is the natural map that locally can be seen as the map $Y^{g,h} \to Y^{gh}$.

We will study this theory in more detail in chapter 10.

8.5 G-OC-TFT with Positive Boundary

As before we define the notion of a G-open-closed theory with positive boundary as a G-open-closed theory but with the restriction that the morphisms have at least one outgoing boundary.
The algebraic characterization is the following.

1. A nearly G-Frobenius algebra associated to the circle.
2. For each pair a, b of labels a vector space $\mathcal{O}_{ab}$ with a G-action

   $$\rho : G \to \mathrm{Aut}(\mathcal{O}_{ab})$$

   such that

   $$\rho_g(\eta^c_{ab}(\varphi_1 \otimes \varphi_2)) = \eta^c_{ab}(\rho_g(\varphi_1) \otimes \rho_g(\varphi_2)),$$
   $$\Delta^c_{ab}(\rho_g(\varphi)) = (\rho_g \otimes \rho_g)\Delta^c_{ab}(\varphi),$$

   for $\varphi_1 \in \mathcal{O}_{ac}$, $\varphi_2 \in \mathcal{O}_{cb}$, $\varphi \in \mathcal{O}_{ab}$ and $g \in G$. This conditions are represented in the figures 26 and 27.
3. For every label a the vector space $\mathcal{O}_{aa}$ is non necessarily a commutative nearly Frobenius algebra.
4. There are also G-twisted open-closed transition maps

   $$\iota_{g,a} : \mathbb{C}_g \to \mathcal{O}_{aa},$$

   $$\iota^{g,a} : \mathcal{O}_{aa} \to \mathbb{C}_g,$$

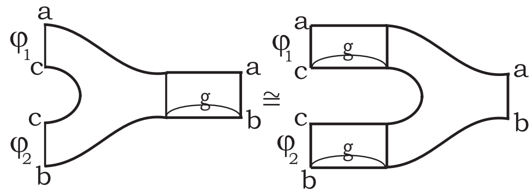


Figure 26: The product is a G-morphism with the diagonal action.

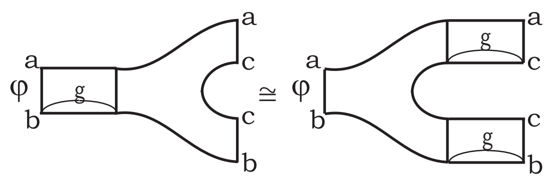


Figure 27: The coproduct is a G-morphism with the diagonal action.

which are equivariant.
The map $\iota : \mathbb{C} \to \mathcal{O}$ is obtained by putting the ι_g together, i.e. $\iota = \oplus_{g \in G} \iota_g$ is a ring homomorphisms, then

$$\iota_{g_1}(\Phi_1)\iota_{g_2}(\Phi_2) = \iota_{g_2 g_1}(\Phi_2 \Phi_1),$$

with $\Phi_1 \in \mathbb{C}_{g_1}$ and $\Phi_2 \in \mathbb{C}_{g_2}$. Moreover $\iota_e(1_{\mathbb{C}}) = 1_{\mathcal{O}_{aa}}$. The G-twisted centrality condition is

$$\iota_g(\Phi)(\rho_g \Psi) = \Psi \iota_g(\Phi),$$

where $\Phi \in \mathbb{C}_g$ y $\Psi \in \mathcal{O}_{aa}$.

5. The G-twisted Cardy conditions. For each $g \in G$ we must have

$$\pi^a_{g,b} = \iota_{g,b} \iota^{g,a}.$$

Hence $\pi^a_{g,b}$ is defined by

$$\pi^a_{g,b} := \eta^a_{bb} \circ \tau \circ (1 \otimes \rho_g) \circ \Delta^b_{aa} : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$$

where $\tau : \mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \to \mathcal{O}_{ba} \otimes \mathcal{O}_{ab}$ is the transposition map, see Figure 28.

Theorem 8.14. *The* G*-invariant part of a* G*-OC TFT with positive boundary is an OC-TFT with positive boundary.*

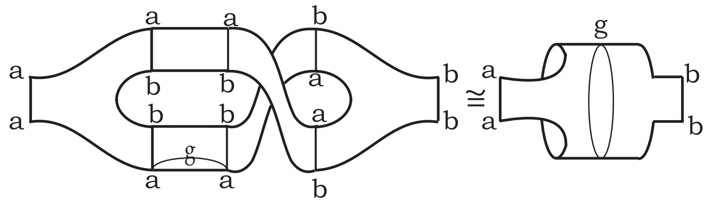


Figure 28: G-twisted Cardy condition.

9 Orbifold String Topology

Let M be a smooth, compact, connected, oriented manifold and let G be a finite group acting on M. We will consider the global quotient orbifold $\mathsf{X} = [\mathsf{M}/\mathsf{G}]$. We define now the *loop orbifold* LX for X as follows:

Consider the space

$$\mathcal{P}_\mathsf{G}(\mathsf{M}) := \bigsqcup_{g \in \mathsf{G}} \mathcal{P}_g(\mathsf{M}) \times \{g\}$$

where

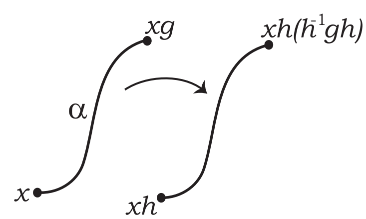
$$\mathcal{P}_g(\mathsf{M}) = \{\gamma : [0,1] \to \mathsf{Y} : \gamma(0)g = \gamma(1)\},$$

together with the G-action given by

$$\mathsf{G} \times \bigsqcup_{g \in \mathsf{G}} \mathcal{P}_g(\mathsf{M}) \times \{g\} \to \bigsqcup_{g \in \mathsf{G}} \mathcal{P}_g(\mathsf{M}) \times \{g\}$$

$$(h, (\gamma, g)) \mapsto (\gamma_h, h^{-1}gh)$$

where $\gamma_h(t) := \gamma(t)h$.



Then we define the loop orbifold as

$$\mathsf{LX} := [\mathcal{P}_\mathsf{G}(\mathsf{M})/\mathsf{G}].$$

In this section we associate a nearly G-Frobenius algebra to the loop orbifold LX. This is $H_*(\mathcal{P}_G(M)) = \bigoplus_{g\in G} H_*(\mathcal{P}_g(M))$, with the G-action

$$\alpha_h : H_*(\mathcal{P}_g(M)) \to H_*(\mathcal{P}_{hgh^{-1}}(M))$$

$$\alpha_h([\gamma]) = [\gamma_h]$$

It is important to mention that ordinary string topology is included since $\mathcal{P}_e(M) = \mathcal{L}M$ with $e \in G$ the identity element.

We will describe the structure maps in the next section.

9.1 Algebraic Structure

Orbifold string product: We will suppose that M is oriented and G acts by orientation preserving diffeomorphisms. Now we define the product for the homology of $\mathcal{P}_G(M)$. We start by defining a composition of path maps

$$\circledast : \mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) \to \mathcal{P}_{gh}(M)$$

where $\epsilon_t : \mathcal{P}_k(M) \to M$ is the evaluation map at t, given by $\gamma \mapsto \gamma(t)$ and

$$\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) = \{(\gamma_0, \gamma_1) : \gamma_0(1) = \gamma_1(0)\}.$$

The map $\circledast$ is given by

$$(\gamma_0 \circledast \gamma_1)(t) := \begin{cases} \gamma_0(2t), & 0 \le t \le \frac{1}{2} \\ \gamma_1(2t-1), & \frac{1}{2} < t \le 1 \end{cases}$$

Notice that the following diagram is a pullback square

$$\begin{tikzcd} \mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g(M) \times \mathcal{P}_h(M) \arrow[d, "\epsilon_1\times\epsilon_0"] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd} \tag{39}$$

where j is the inclusion, Δ is the diagonal map and $\epsilon_\infty(\gamma_0, \gamma_1) = \gamma_0(1) = \gamma_1(0)$. We observe that due to the pullback square (39) we can construct a Thom-Pontryagin map

$$\tau : \mathcal{P}_g(M) \times \mathcal{P}_h(M) \to (\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM},$$

where $(\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM}$ denotes the Thom space of the pullback bundle $\epsilon_\infty^*(TM)$. This is the normal bundle of the embedding j.

Let $(\mathcal{P}_{gh}(M))^{TM}$ be the Thom space of the bundle $\epsilon^*_{\frac{1}{2}}(TM)$ with $\epsilon_{\frac{1}{2}} : \mathcal{P}_{gh}(M) \to M$. The map $\circledast$ induces a map of Thom spaces

$$\tilde{\circledast} : (\mathcal{P}_g(M)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h(M))^{TM} \to (\mathcal{P}_{gh}(M))^{TM},$$

and therefore the next diagram is commutative

$$\begin{tikzcd}
\mathcal{P}_g(M) \times \mathcal{P}_h(M) \arrow[r, "\tau"] \arrow[d, "\epsilon_1 \times \epsilon_0"'] & (\mathcal{P}_g(M)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h(M))^{TM} \arrow[r, "\tilde{\circledast}"] \arrow[d, "\epsilon_\infty"'] & (\mathcal{P}_{gh}(M))^{TM} \arrow[d, "\epsilon_{\frac{1}{2}}"] \\
M \times M \arrow[r, "\tau"] & M^{TM} \arrow[r, "="] & M^{TM}
\end{tikzcd}$$

Then, we can consider the composition

$$\eta_{g,h} : H_p(\mathcal{P}_g(M)) \otimes H_q(\mathcal{P}_h(M)) \xrightarrow{\times} H_{p+q}(\mathcal{P}_g(M) \times \mathcal{P}_h(M)) \xrightarrow{(\tilde{\circledast}\circ\tau)_*}$$
$$H_{p+q}((\mathcal{P}_{gh}(M))^{TM}) \xrightarrow{\tilde{u}_*} H_{p+q-d}(\mathcal{P}_{gh}(M)),$$

where $\tilde{u}_*$ is the Thom isomorphism. Adding over all elements $g \in G$ we obtain the map

$$\eta : H_p(\mathcal{P}_G(M)) \otimes H_q(\mathcal{P}_G(M)) \to H_{p+q-d}(\mathcal{P}_G(M))$$

which we call the *G-string product*.

Orbifold string coproduct: First, we note that the next diagram is a pullback square

$$\begin{tikzcd}
\mathcal{P}_g(M)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h(M) \arrow[r, "\circledast"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{gh}(M) \arrow[d, "\epsilon_{\frac{1}{2}}, \epsilon_0 . g"] \\
M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

Then, we can consider the map

$$\overline{\circledast} : \mathcal{P}_{gh}(M) \to (\mathcal{P}_g(M)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h(M))^{TM}$$

where $(\mathcal{P}_g(M)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h(M))^{TM}$ denotes the Thom space of the pullback bundle $\epsilon^*_\infty(TM)$, which is the normal bundle of $\circledast$.
Then, we can consider the composition

$$\Delta_{g,h} : H_{p+q+d}(\mathcal{P}_{gh}(M)) \xrightarrow{\overline{\circledast}} H_{p+q+d}\left((\mathcal{P}_g(M)_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h(M))^{TM}\right) \xrightarrow{\tilde{u}_*} H_{p+q}(\mathcal{P}_g(M)_{\epsilon_i} \times_{\epsilon_0} \mathcal{P}_h(M)) \xrightarrow{j_*}$$
$$H_{p+q}(\mathcal{P}_g(M) \times \mathcal{P}_h(M)) \longrightarrow \bigoplus_{i+j=p+q} H_i(\mathcal{P}_g(M)) \otimes H_j(\mathcal{P}_h(M)).$$

Adding over all elements $g \in G$ we obtain the map

$$\Delta : H_*(\mathcal{P}_G(M)) \to H_*(\mathcal{P}_G(M)) \otimes H_*(\mathcal{P}_G(M))$$

We will call Δ the *G-string coproduct*.

The unit: We consider the next diagram

$$\begin{tikzcd} & M \arrow[dl, "r"'] \arrow[dr, "i_c"] & \\ \{pt\} & & \mathcal{P}_e(M) \end{tikzcd}$$

where $r : M \to \{pt\}$, the constant map and $i_c : M \to \mathcal{P}_e(M)$ is defined by $i_c(y) = \alpha : I \to M$ such that $\alpha(t) = y$ is the constant loop.

Then $u : H_*(\{pt\}) = k \xrightarrow{r!} H_*(M) \xrightarrow{i_{c*}} H_*(\mathcal{P}_e(M)) \to H_*(\mathcal{P}_G(M))$.

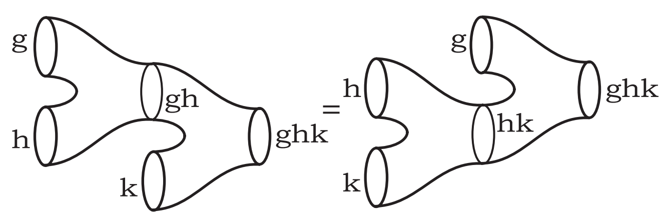
$$u : \mathbb{k} \to H_*(\mathcal{P}_G(M)).$$

Note that since $M \to \mathcal{P}_e(M)$ has infinite codimension we cannot define a trace map. This same feature is in the String Topology algebra.

Theorem 9.1. $H_*(\mathcal{P}_G(M))$ *is a nearly* G*-Frobenius algebra.*

Proof. We will check all the axioms.

1. Associativity of the product



(1) (2)

Remember that the product is defined from the next diagram

$$\begin{tikzcd} & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \arrow[dl, "j"'] \arrow[dr, "*"] & \\ \mathcal{P}_g M \times \mathcal{P}_h M & & \mathcal{P}_{gh} M \end{tikzcd}$$

The associativity is encoded in the next two diagrams.

(1)

$$
\begin{tikzcd}
 & & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "1\times j"'] \arrow[dr, "*\times 1"] & & \\
 & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M\times\mathcal{P}_k M \arrow[dl, "j\times 1"'] \arrow[dr, "*\times 1"] & & \mathcal{P}_{gh} M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_g M\times\mathcal{P}_h M\times\mathcal{P}_k M & & \mathcal{P}_{gh} M\times\mathcal{P}_k M & & \mathcal{P}_{ghk} M
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
 & & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "j\times 1"'] \arrow[dr, "1\times *"] & & \\
 & \mathcal{P}_g M\times\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "1\times j"'] \arrow[dr, "1\times *"] & & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{hk} M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_g M\times\mathcal{P}_h M\times\mathcal{P}_k M & & \mathcal{P}_g M\times\mathcal{P}_{hk} M & & \mathcal{P}_{ghk} M
\end{tikzcd}
$$

The first case involved the next constructions

$$
\begin{tikzcd}
(*\times 1)^*\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[d, "*\times 1"'] & \\
\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_{gh} M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{gh} M\times\mathcal{P}_k M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}
$$

and

$$\begin{tikzcd}
(\epsilon_\infty \times \epsilon_\infty)^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[r, "1\times j"] \arrow[d, "\epsilon_\infty \times \epsilon_\infty"'] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \times \mathcal{P}_k M \arrow[d, "\epsilon_\infty \times \epsilon_1 \times \epsilon_0"] \\
TM \arrow[r, dashed] & M \times M \arrow[r, "1\times\Delta"'] & M \times M \times M
\end{tikzcd}$$

We note that $(* \times 1)^* \epsilon_\infty^*(TM) = (\epsilon_\infty \times \epsilon_\infty)^*(TM)$. Then $F_1 = 0$.

In the second diagram we have the next constructions

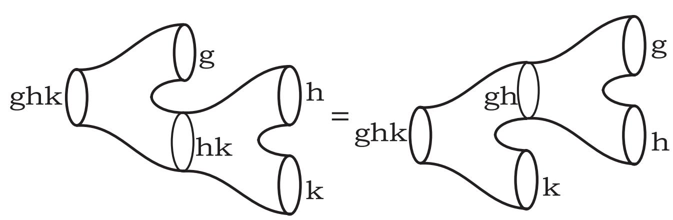
$$\begin{tikzcd}
(1 \times *)^* \epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[d, "1\times *"'] & \\
\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_{hk} M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g M \times \mathcal{P}_{hk} M \arrow[d, "\epsilon_1 \times \epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\epsilon_\infty \times \epsilon_\infty)^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[r, "j\times 1"] \arrow[d, "\epsilon_\infty \times \epsilon_\infty"'] & \mathcal{P}_g M \times \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[d, "\epsilon_1 \times \epsilon_0 \times \epsilon_\infty"] \\
TM \arrow[r, dashed] & M \times M \arrow[r, "\Delta\times 1"'] & M \times M \times M
\end{tikzcd}$$

Similarly as before, we note that $(1 \times *)^* \epsilon_\infty^*(TM) = (\epsilon_\infty \times \epsilon_\infty)^*(TM)$. Then $F_2 = 0$. Therefore the product is associative.

2. Coassociativity of the coproduct



(1) (2)

In the same way as the product, the coproduct is defined from the diagram

$$\begin{tikzcd}
& \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \arrow[dl, "*"'] \arrow[dr, "j"] & \\
\mathcal{P}_{gh} M & & \mathcal{P}_g M \times \mathcal{P}_h M
\end{tikzcd}$$

The diagrams that represent this property are
(1)

$$\begin{tikzcd}
& & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[dl, "* \times 1"'] \arrow[dr, "1 \times j"] & & \\
& \mathcal{P}_{gh} M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[dl, "*"'] \arrow[dr, "j"] & & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \times \mathcal{P}_k M \arrow[dl, "* \times 1"'] \arrow[dr, "j \times 1"] & \\
\mathcal{P}_{ghk} M & & \mathcal{P}_{gh} M \times \mathcal{P}_k M & & \mathcal{P}_g M \times \mathcal{P}_h M \times \mathcal{P}_k M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[dl, "1 \times *"'] \arrow[dr, "j \times 1"] & & \\
& \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_{hk} M \arrow[dl, "*"'] \arrow[dr, "j"] & & \mathcal{P}_g M \times \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[dl, "1 \times *"'] \arrow[dr, "1 \times j"] & \\
\mathcal{P}_{ghk} M & & \mathcal{P}_g M \times \mathcal{P}_{hk} M & & \mathcal{P}_g M \times \mathcal{P}_h M \times \mathcal{P}_k M
\end{tikzcd}$$

In the first case we have the next constructions

$$\begin{tikzcd}
(1 \times j)^*(\epsilon_\infty \times \epsilon_0)^* \eta \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[d, "1 \times j"'] & \\
(\epsilon_\infty \times \epsilon_0)^* \eta \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \times \mathcal{P}_k M \arrow[r, "* \times 1"] \arrow[d, "\epsilon_\infty \times \epsilon_0"'] & \mathcal{P}_{gh} M \times \mathcal{P}_k M \arrow[d, "(\epsilon_{\frac{1}{2}}, \epsilon_0 g) \times \epsilon_0"] \\
\eta \arrow[r, dashed] & M \times M \arrow[r, "\Delta \times 1"'] & M \times M \times M
\end{tikzcd}$$

and

$$
\begin{tikzcd}
(\epsilon_\infty \times \epsilon_0)^*\eta \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[r, "*\times 1"] \arrow[d, "\epsilon_\infty\times\epsilon_0"'] & \mathcal{P}_{gh} M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[d, "(\epsilon_{\frac{1}{2}},\epsilon_0 g)\times\epsilon_0"] \\
\eta \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"'] & M\times M\times M
\end{tikzcd}
$$

We note that $(1\times j)^*(\epsilon_\infty\times\epsilon_0)^*\eta = (\epsilon_\infty\times\epsilon_0)^*\eta$. Then $F_1 = 0$.

The second diagram has the next constructions

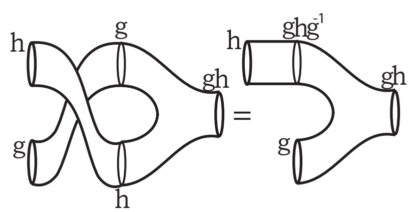
$$
\begin{tikzcd}
(j\times 1)^*(\epsilon_1\times\epsilon_\infty)^*\eta \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[d, "j\times 1"'] & \\
(\epsilon_1\times\epsilon_\infty)^*\eta \arrow[r, dashed] & \mathcal{P}_g M \times \mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[r, "1\times *"] \arrow[d, "\epsilon_1\times\epsilon_\infty"'] & \mathcal{P}_g M\times\mathcal{P}_{hk} M \arrow[d, "\epsilon_1\times(\epsilon_{\frac{1}{2}},\epsilon_0 h)"] \\
\eta \arrow[r, dashed] & M\times M \arrow[r, "1\times\Delta"'] & M\times M\times M
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(\epsilon_1\times\epsilon_\infty)^*\eta \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[r, "1\times *"] \arrow[d, "\epsilon_1\times\epsilon_\infty"'] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{hk} M \arrow[d, "\epsilon_1\times(\epsilon_{\frac{1}{2}},\epsilon_0 h)"] \\
\eta \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"'] & M\times M\times M
\end{tikzcd}
$$

In the same way as before, we note that $(j\times 1)^*(\epsilon_1\times\epsilon_\infty)^*\eta = (\epsilon_1\times\epsilon_\infty)^*\eta$. Then $F_2 = 0$.

3. Graded commutativity of the product



This property is represented by the next diagrams

(1)

$$\begin{tikzcd}
 & & \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_1} \mathcal{P}_g M \arrow[dl, "j\circ(1\times\alpha_{g^{-1}})\circ\tau"'] \arrow[dr, "\tau\circ(\alpha_{h^{-1}}\times 1)"] & & \\
 & \mathcal{P}_h M \times \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "\tau\circ(\alpha_{h^{-1}}\times\alpha_g)"] & & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_h M \times \mathcal{P}_g M & & \mathcal{P}_g M \times \mathcal{P}_h M & & \mathcal{P}_{gh} M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_1} \mathcal{P}_g M \arrow[dl, "i"'] \arrow[dr, "\alpha_g\times 1"] & & \\
 & \mathcal{P}_h M \times \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "(\alpha_g\times 1)\circ\tau"] & & \mathcal{P}_{ghg^{-1}} M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_g M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_h M \times \mathcal{P}_g M & & \mathcal{P}_{ghg^{-1}} M \times \mathcal{P}_g M & & \mathcal{P}_{gh} M
\end{tikzcd}$$

First, we need to check that the maps $* \circ (\alpha_g \times 1)$ and $* \circ \tau \circ (\alpha_{h^{-1}} \times 1)$ are homotopic maps and the same for $j \circ (1 \times \alpha_{g^{-1}})$ and i. In each case, we will construct the homotopy. In the first case we define

$$H : I \times (\mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_1} \mathcal{P}_g M) \to \mathcal{P}_{gh} M$$

by

$$H(s, (\gamma, \beta))(t) := \alpha_g(\gamma) * \beta * \alpha_{h^{-1}}(\gamma) \left(\frac{s + 2t}{3} \right)$$

Note that $H(0, (\gamma, \beta))(t) = \alpha_g(\gamma) * \beta * \alpha_{h^{-1}}(\gamma) \left(\frac{2t}{3}\right) = \alpha_g(\gamma) * \beta(t) = (* \circ (\alpha_g \times 1))(\gamma, \beta)(t)$, and $H(1, (\gamma, \beta))(t) = \alpha_g(\gamma) * \beta * \alpha_{h^{-1}}(\gamma) \left(\frac{1+2t}{3}\right) = \beta * \alpha_{h^{-1}}(\gamma)(t) = (* \circ \tau(\alpha_{h^{-1}} \times 1))(\gamma, \beta)(t)$.
In the second case the next map

$$F : I \times (\mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_1} \mathcal{P}_g M) \to \mathcal{P}_h M \times \mathcal{P}_g M$$

is defined by

$$F(s,(\gamma,\beta))(r,t)=\left(\gamma(r),\beta*\alpha_{g^{-1}}(\beta)\left(\frac{s+t}{2}\right)\right)$$

Note that $F(0,(\gamma,\beta))(r,t)=\left(\gamma(r),\beta*\alpha_{g^{-1}}(\beta)\left(\frac{t}{2}\right)\right)=(\gamma(r),\beta(t))=i(\gamma,\beta)(r,t)$, and $F(1,(\gamma,\beta))(r,t)=\left(\gamma(r),\beta*\alpha_{g^{-1}}(\beta)\left(\frac{1+t}{2}\right)\right)=(\gamma(r),\alpha_{g^{-1}}(\beta)(t))=j\circ(1\times\alpha_{g^{-1}})(\gamma,\beta)(r,t)$.

Now, we can determine the Euler classes. In the first case we have

$$\begin{tikzcd}
(\epsilon_\infty\circ\tau\circ(\alpha_{h^{-1}}\times 1))^*(TM) \arrow[r, dashed] & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_1}\mathcal{P}_gM \arrow[d, "\tau\circ(\alpha_{h^{-1}}\times 1)"'] & \\
& \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_gM\times\mathcal{P}_hM \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_1^*(TM) \arrow[r, dashed] & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_1}\mathcal{P}_gM \arrow[r, "i"] \arrow[d, "\epsilon_1"'] & \mathcal{P}_hM\times\mathcal{P}_gM \arrow[d, "\epsilon_1\times\epsilon_1"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

We note that $\epsilon_1=\epsilon_\infty\circ\tau\circ(\alpha_{h^{-1}}\times 1)$, then $F_1=0$.

For the second case

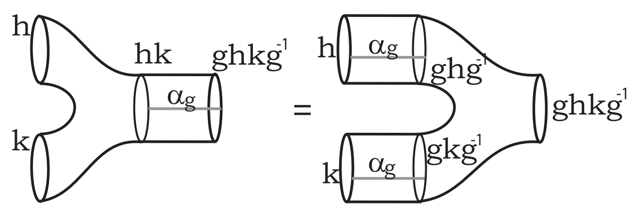
$$\begin{tikzcd}
(\epsilon_\infty\circ(\alpha_g\times 1))^*(TM) \arrow[r, dashed] & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_1}\mathcal{P}_gM \arrow[d, "\tau\circ(\alpha_g\times 1)"'] & \\
& \mathcal{P}_{ghg^{-1}}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_gM \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{ghg^{-1}}M\times\mathcal{P}_gM \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_1^*(TM) \arrow[r, dashed] & \mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_1}\mathcal{P}_g M \arrow[r, "i"] \arrow[d, "\epsilon_1"'] & \mathcal{P}_h M \times \mathcal{P}_g M \arrow[d, "\epsilon_1\times\epsilon_1"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

Similarly we note that $\epsilon_1 = \epsilon_\infty \circ (\alpha_g \times 1)$, then $F_2 = 0$.

4. The action is an algebra homomorphism



(1) (2)

This property is described by the next diagrams.

(1)

$$\begin{tikzcd}
 & & \mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "1"'] \arrow[dr, "*"] & & \\
 & \mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "j"'] \arrow[dr, "*"] & & \mathcal{P}_{hk} M \arrow[dl, "1"'] \arrow[dr, "\alpha_g"] & \\
\mathcal{P}_h M \times \mathcal{P}_k M & & \mathcal{P}_{hk} M & & \mathcal{P}_{ghkg^{-1}} M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}[column sep=tiny]
 & & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "j"'] \arrow[dr, "\alpha_g\times\alpha_g"] & & \\
 & \mathcal{P}_hM\times\mathcal{P}_kM \arrow[dl, "1"'] \arrow[dr, "\alpha_g\times\alpha_g"] & & \mathcal{P}_{ghg^{-1}}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{gkg^{-1}}M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_hM\times\mathcal{P}_kM & & \mathcal{P}_{ghg^{-1}}M\times\mathcal{P}_{gkg^{-1}}M & & \mathcal{P}_{ghkg^{-1}}M
\end{tikzcd}$$

In the first case is clearly that $F_1 = 0$ because the normal bundle is zero. Now we study the second case. This is

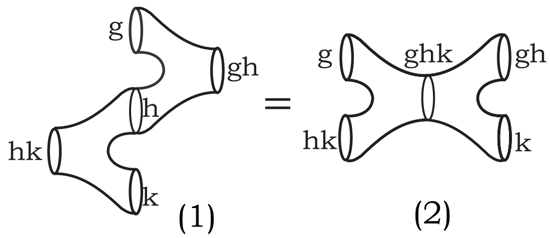
$$\begin{tikzcd}
(\alpha_g\times\alpha_g)^*\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[d, "\alpha_g\times\alpha_g"'] & \\
 & \mathcal{P}_{ghg^{-1}}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{gkg^{-1}}M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{ghg^{-1}}M\times\mathcal{P}_{gkg^{-1}}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_hM\times\mathcal{P}_kM \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

Note that $\epsilon_\infty^*(TM) = (\alpha_g\times\alpha_g)^*\epsilon_\infty^*(TM)$, then $F_2 = 0$.

5. Abrams condition



This property is modeled by the next diagrams
(1)

$$
\begin{tikzcd}
 & & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "j\times 1"'] \arrow[dr, "1\times j"] & & \\
 & \mathcal{P}_gM\times\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "1\times *"'] \arrow[dr, "1\times j"] & & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM\times\mathcal{P}_kM \arrow[dl, "j\times 1"'] \arrow[dr, "*\times 1"] & \\
\mathcal{P}_gM\times\mathcal{P}_{hk}M & & \mathcal{P}_gM\times\mathcal{P}_hM\times\mathcal{P}_kM & & \mathcal{P}_{gh}M\times\mathcal{P}_kM
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
 & & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "1\times *"'] \arrow[dr, "*\times 1"] & & \\
 & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{hk}M \arrow[dl, "j"'] \arrow[dr, "*"] & & \mathcal{P}_{gh}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "*"'] \arrow[dr, "j"] & \\
\mathcal{P}_gM\times\mathcal{P}_{hk}M & & \mathcal{P}_{ghk}M & & \mathcal{P}_{gh}M\times\mathcal{P}_kM
\end{tikzcd}
$$

The first case involves the following

$$\begin{tikzcd}
((\epsilon_\infty \times \epsilon_0) \circ (1 \times j))^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[d, "1\times j"'] & \\
& \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \times \mathcal{P}_k M \arrow[r, "j\times 1"] \arrow[d, "\epsilon_\infty \times \epsilon_0"'] & \mathcal{P}_g M \times \mathcal{P}_h M \times \mathcal{P}_k M \arrow[d, "\epsilon_1 \times \epsilon_0 \times \epsilon_0"] \\
TM \arrow[r, dashed] & M \times M \arrow[r, "\Delta\times 1"'] & M \times M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\epsilon_\infty \times \epsilon_0)^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[d, "\epsilon_\infty \times \epsilon_0"'] & \mathcal{P}_g M \times \mathcal{P}_h M \times \mathcal{P}_k M \arrow[d, "\epsilon_1 \times \epsilon_0 \times \epsilon_0"] \\
TM \arrow[r, dashed] & M \times M \arrow[r, "\Delta\times 1"'] & M \times M \times M
\end{tikzcd}$$

It is clear that $(\epsilon_\infty \times \epsilon_0)^*(TM) = ((\epsilon_\infty \times \epsilon_0) \circ (1 \times j))^*(TM)$, then $F_1 = 0$.

In the second case we have

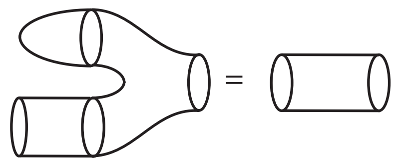
$$\begin{tikzcd}
(\epsilon_\infty \circ (* \times 1))^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[d, "*\times 1"'] & \\
& \mathcal{P}_{gh} M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[r, "*"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{ghk} M \arrow[d, "\epsilon_{\frac{1}{2}} \times \epsilon_0 gh"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\epsilon_1 \times \epsilon_\infty)^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_k M \arrow[r, "1\times *"] \arrow[d, "\epsilon_1 \times \epsilon_\infty"'] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_{hk} M \arrow[d, "\epsilon_1 \times \epsilon_{\frac{1}{2}} \times \epsilon_0 h"] \\
TM \arrow[r, dashed] & M \times M \arrow[r, "1\times\Delta"'] & M \times M \times M
\end{tikzcd}$$

Finally $(\epsilon_1 \times \epsilon_\infty)^*(TM) = (\epsilon_\infty \circ (* \times 1)^*(TM)$, and then $F_2 = 0$.

6. Unit axiom



(1) (2)

Remember that the unit map is defined from the next diagram

$$\begin{tikzcd}
 & M \arrow[dl, "r"'] \arrow[dr, "i_c"] & \\
pt & & \mathcal{P}_e M
\end{tikzcd}$$

where $r : M \to \text{pt}$ is the constant map, $\mathcal{P}_e M = \{\alpha : I \to M : \ \alpha(1) = \alpha(0)\} = \mathcal{L}M$, and $i_c : M \hookrightarrow \mathcal{L}M$ in the natural inclusion. Then $u : H_*(\text{pt}) \to H_*(\mathcal{L}M) = H_*(\mathcal{P}_e M)$ is the next composition map

$$H_*(\text{pt}) \xrightarrow{r!} H_*(M) \xrightarrow{i_{c*}} \mathcal{L}M.$$

The diagrams that represent the unit axiom are

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "1"] & & \\
 & \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "1"] & & \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "1"] & \\
\mathcal{P}_g M & & \mathcal{P}_g M & & \mathcal{P}_g M
\end{tikzcd}$$

It is clear that $F_2 = 0$.
(1)

$$\begin{tikzcd}
 & & \mathcal{P}_g M \arrow[dl, "1\times\epsilon_1"'] \arrow[dr, "1\times i_c\circ\epsilon_1"] \arrow[dll, bend right, "\varphi"'] \arrow[drr, bend left, "\psi"] & & \\
 & \mathcal{P}_g M \times M \arrow[dl, "1\times r"'] \arrow[dr, "1\times i_c"] & & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_e M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_g M \times \{\mathrm{pt}\} & & \mathcal{P}_g M \times \mathcal{P}_e M & & \mathcal{P}_g M
\end{tikzcd}$$

First, we note that the map ψ is homotopic to the identity $\mathrm{Id} : \mathcal{P}_g M \to \mathcal{P}_g M$, this is because

$$\psi : \alpha \mapsto (\alpha, i_c(\alpha(1))) \mapsto \alpha * i_c(\alpha(1)) \simeq \alpha.$$

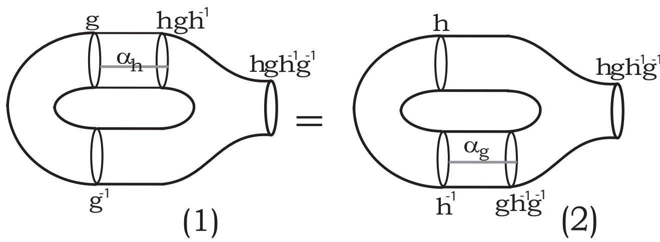
Clearly the map φ is the identity map.
Now, we determine the class of the square.

$$\begin{tikzcd}
\epsilon_1^*(TM) \arrow[r, dashed] & \mathcal{P}_g M \arrow[r, "1\times\epsilon_1"] \arrow[d, "\epsilon_1"'] & \mathcal{P}_g M \times M \arrow[d, "\epsilon_1\times 1"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

$$\begin{tikzcd}
(1\times\epsilon_1)^*\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_g M \arrow[d, "1\times\epsilon_1"'] & \\
 & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_e M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g M \times \mathcal{P}_e M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

In this case we note that $\epsilon_\infty \circ (1 \times \epsilon_1) = \epsilon_1$, this implies $\epsilon_1^*(TM) = (1 \times \epsilon_1)^*\epsilon_\infty^*(TM)$, and then $F_1 = 0$.

7. Torus axiom



The co-pairing map $\Theta_g : \mathbb{k} \to H_*(\mathcal{P}_g M) \otimes H_*(\mathcal{P}_{g^{-1}} M)$ is defined as the composition of the unit and the coproduct as follows,

$$\mathbb{k} \xrightarrow{u} H_*(\mathcal{P}_e M) \overset{\Delta_{g,g^{-1}}}{\longrightarrow} H_*(\mathcal{P}_g M) \otimes H_*(\mathcal{P}_{g^{-1}} M).$$

Now, we describe this map.

$$\begin{tikzcd}
 & & M^g \arrow[dl, "i_g"'] \arrow[dr, "f_g"] & & \\
 & M \arrow[dl, "r"'] \arrow[dr, "i_c"] & & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_{g^{-1}} M \arrow[dl, "*"'] \arrow[dr, "j"] & \\
pt & & \mathcal{P}_e M & & \mathcal{P}_g M \times \mathcal{P}_{g^{-1}} M
\end{tikzcd}$$

where the map $i_g : M^g \to M$ is the inclusion, and $f_g : M^g \to \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_{g^{-1}} M$ is given by $x \mapsto (\alpha_x, \alpha_x)$ with α_x the constant loop. The Quillen's class of this square is described as follows:

$$\nu_{i_g} \longrightarrow M^g \overset{i_g}{\hookrightarrow} M$$

and

$$\begin{tikzcd}
f_g^* \epsilon_0^* (\nu_{(1 \times \alpha_g)}) \arrow[r] & M^g \arrow[d, "f_g"] & \\
 & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_{g^{-1}} M \arrow[r, "*"] \arrow[d, "\epsilon_0"] & \mathcal{P}_e M \arrow[d, "\epsilon_0 \times \epsilon_{\frac{1}{2}}"] \\
\nu_{(1 \times \alpha_g)} \arrow[r] & M \arrow[r, "1 \times \alpha_g"'] & M \times M
\end{tikzcd}$$

Note that $\epsilon_0 \circ f_g(x) = x$, this implies that $\epsilon_0 \circ f_g = i_g$ and $f_g^*\epsilon_0^*(\nu_{(1\times\alpha_g)}) = i_g^*(\nu_{(1\times\alpha_g)})$. Therefore F_g is given by the next exact sequence

$$0 \longrightarrow \nu_{i_g} \longrightarrow i_g^*(\nu_{(1\times\alpha_g)}) \longrightarrow F_g \longrightarrow 0.$$

In the next step we determine the diagram associated to the first figure.
(1)

$$\begin{tikzcd}
 & & & M^{g,h} \arrow[dl, "i^g_{g,h}"'] \arrow[ddrr, "j_{g,h}"] \arrow[ddddrrr, bend left, "i"] \arrow[ddddlll, bend right, "r"'] & & & \\
 & & M^g \arrow[dl, "i_g"'] \arrow[dr, "f_g"] & & & & \\
 & M \arrow[ddl, "r"'] \arrow[ddr, "i_c"] & & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{g^{-1}}M \arrow[ddl, "*"'] \arrow[dr, "j"] & & \mathcal{P}_{hgh^{-1}}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{g^{-1}}M \arrow[dd, "j"] \arrow[ddr, "*"] & \\
 & & & & \mathcal{P}_g M\times\mathcal{P}_{g^{-1}}M \arrow[dr, "\alpha_h\times 1"] & & \\
pt & & \mathcal{P}_e M & & & \mathcal{P}_{hgh^{-1}}M\times\mathcal{P}_{g^{-1}}M & \mathcal{P}_{hgh^{-1}g^{-1}}M
\end{tikzcd}$$

The class F_1 is given by

$$\nu_{i^g_{g,h}} \longrightarrow M^{g,h} \xhookrightarrow{i^g_{g,h}} M^g$$

and

$$\begin{tikzcd}
j^*_{g,h}\epsilon^*_\infty(TM) \arrow[r] & M^{g,h} \arrow[d, "j_{g,h}"] & \\
 & \mathcal{P}_{hgh^{-1}}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{g^{-1}}M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{hgh^{-1}}M\times\mathcal{P}_{g^{-1}}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

Note that $\epsilon_\infty \circ j_{g,h}(x) = \epsilon_\infty(\alpha_h(\alpha_x), \alpha_x) = x$, then $\epsilon_\infty \circ j_{g,h} = i_{g,h}$ and we have the next exact sequence

$$0 \longrightarrow \nu_{i^g_{g,h}} \longrightarrow i^*_{g,h}(TM) \longrightarrow F_1 \longrightarrow 0.$$

The second diagram is the following
(2)

$$\begin{tikzcd}
 & & & M^{g,h} \arrow[dl, "i^h_{g,h}"'] \arrow[ddrr, "j_{g,h}"] \arrow[ddddlll, bend right, "r"'] \arrow[ddddrrr, bend left, "i"] & & & \\
 & & M^h \arrow[dl, "i_h"'] \arrow[dr, "f_h"] & & & & \\
 & M \arrow[ddl, "r"] \arrow[ddr, "i_c"] & & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{h^{-1}}M \arrow[ddl, "*"'] \arrow[dr, "j"] & & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{gh^{-1}g^{-1}}M \arrow[dd, "j"'] \arrow[ddr, "*"] & \\
 & & & & \mathcal{P}_hM\times\mathcal{P}_{h^{-1}}M \arrow[dr, "1\times\alpha_g"] & & \\
pt & & \mathcal{P}_eM & & & \mathcal{P}_hM\times\mathcal{P}_{gh^{-1}g^{-1}}M & \mathcal{P}_{hgh^{-1}g^{-1}}M
\end{tikzcd}$$

The class F_2 is associate to the next map

$$\nu_{i^h_{g,h}} \longrightarrow M^{g,h} \xhookrightarrow{i^h_{g,h}} M^h$$

in this case we have

$$\begin{tikzcd}
j^*_{g,h}\epsilon^*_\infty(TM) \arrow[r] & M^{g,h} \arrow[d, "j_{g,h}"] & \\
 & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{gh^{-1}g^{-1}}M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_hM\times\mathcal{P}_{gh^{-1}g^{-1}}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

As before there is the identity $j^*_{g,h}\epsilon^*_\infty(TM) = i^*_{g,h}(TM)$. Then

$$0 \longrightarrow \nu_{i^h_{g,h}} \longrightarrow i^*_{g,h}(TM) \longrightarrow F_2 \longrightarrow 0.$$

Applying the Quillen's formulae we conclude

$$\circledast_* j!((\alpha_h\times 1)j)_* \circledast\, !i_{c*}r!(1) = i_*(r!(1)\cap(e(i^{g*}_{g,h}(F_g))\cup e(F_1)))$$

and

$$\circledast_* j!((1\times\alpha_g)j)_* \circledast !i_{c*}r!(1) = i_*(r!(1)\cap(e(i^{h*}_{g,h}(F_h))\cup e(F_2)))$$

To prove the axiom we need to check that

$$e(i^{g*}_{g,h}(F_g))\cup e(F_1) = e(i^{h*}_{g,h}(F_h))\cup e(F_2),$$

or equivalently

$$i^{g*}_{g,h}(F_g)\oplus F_1 \cong i^{h*}_{g,h}(F_h)\oplus F_2.$$

The bundles are the following:

$$\begin{array}{lllllll} E_1 & = & i^{g*}_{g,h}(F_g)\oplus F_1 & = & \frac{i^*_{g,h}(TM)}{i^{g*}_{g,h}(\nu_{i_g})} & \oplus & \frac{i^*_{g,h}(TM)}{\nu_{i^g_{g,h}}} \\ E_2 & = & i^{h*}_{g,h}(F_h)\oplus F_2 & = & \frac{i^*_{g,h}(TM)}{i^{h*}_{g,h}(\nu_{i_h})} & \oplus & \frac{i^*_{g,h}(TM)}{\nu_{i^h_{g,h}}} \end{array}$$

The information is represented in the next diagrams

$$\begin{tikzcd}
i^*_{g,h}(TM) \arrow[dr, dotted] & i^{g*}_{g,h}(\nu_{i_g}) \arrow[d, dotted] & \nu_{i_g} \arrow[d, dotted] & TM \arrow[d, dotted] \\
\nu_{i^g_{g,h}} \arrow[r, dotted] & M^{g,h} \arrow[r, hook, "i^g_{g,h}"'] & M^g \arrow[r, hook, "i_g"'] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
i^*_{g,h}(TM) \arrow[dr, dotted] & i^{h*}_{g,h}(\nu_{i_h}) \arrow[d, dotted] & \nu_{i_h} \arrow[d, dotted] & TM \arrow[d, dotted] \\
\nu_{i^h_{g,h}} \arrow[r, dotted] & M^{g,h} \arrow[r, hook, "i^h_{g,h}"'] & M^h \arrow[r, hook, "i_h"'] & M
\end{tikzcd}$$

Using that all the maps are inclusions we have that $i^*_{g,h}(TM) = TM|_{M^{g,h}}$ and $i^{g*}_{g,h}(\nu_{i_g}) = \nu_{i_g}|_{M^{g,h}}$. On other hand, we observe that

$$TM|_{M^{g,h}} = TM^{g,h}\oplus\nu_{i^g_{g,h}}\oplus\nu_{i_g}|_{M^{g,h}},$$

and

$$TM|_{M^{g,h}} = TM^{g,h}\oplus\nu_{i^h_{g,h}}\oplus\nu_{i_h}|_{M^{g,h}}.$$

Then

$$\nu_{i^g_{g,h}}\oplus\nu_{i_g}|_{M^{g,h}} \cong \nu_{i^h_{g,h}}\oplus\nu_{i_h}|_{M^{g,h}}$$

and in particular $E_1\cong E_2$. This proves that $e(E_1) = e(E_2)$ and the torus axiom is satisfied.

♣

9.2 Open-closed Orbifold String Topology

In the previous section we saw that the homology of the Loop Orbifold has the structure of a G-topological field theory with positive boundary. Now we describe the open part of this theory.
The category of branes is the following:

$$\mathcal{B} = \{X \subset M \text{ G-invariant submanifold with } X \pitchfork Y \text{ transverse for } X \neq Y\}$$

Now we consider the sets $\mathcal{P}_{X,Y}M = \{\alpha : I \to M : \ \alpha(0) \in X, \alpha(1) \in Y\}$, for $X, Y \in \mathcal{B}$. We define $\mathrm{Hom}_{\mathcal{B}}(X, Y) = H_*(\mathcal{P}_{X,Y}M)$. Note that G acts in $H_*(\mathcal{P}_{X,Y}M)$ as follows

$$\begin{aligned} \rho : G \to & \mathrm{Aut}(H_*(\mathcal{P}_{X,Y}M)) \\ g \mapsto & \ \rho_g : H_*(\mathcal{P}_{X,Y}M) \to H_*(\mathcal{P}_{X,Y}M) \\ & \qquad\qquad\qquad \alpha \mapsto \alpha.g \end{aligned}$$

where $\alpha.g(t) = \alpha(t)g$ for $t \in I$.
The product and coproduct are the same as the product and coproduct defined in the open-closed string topology.
Now we describe the connection maps. For this we consider the next diagram

$$\begin{tikzcd} & \mathcal{P}_g^X M \arrow[dl, "j"'] \arrow[dr, "i"] & \\ \mathcal{P}_g M & & \mathcal{P}_{X,X}M \end{tikzcd}$$

where $\mathcal{P}_g^X M = \{\alpha : I \to M : \ \alpha(1) = \alpha(0)g, \ \alpha(0) \in X\}$.
First, we will prove that the map $j_! : H_*(\mathcal{P}_g M) \to H_*(\mathcal{P}_g^X M)$ exists. This is because the next diagram is a pullback square.

$$\begin{tikzcd} \mathcal{P}_g^X M \arrow[r, "j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g M \arrow[d, "\epsilon_0 \times \epsilon_1"] \\ X \arrow[r, hook, "(\mathrm{id}, g)"'] & M \times M \end{tikzcd}$$

Clearly the map $(\mathrm{id}, g) : X \to M \times M$ is an embedding. Then, we can define the map $\iota_{g,X}$ as the composition

$$H_*(\mathcal{P}_g M) \xrightarrow{j_!} H_*(\mathcal{P}_g^X M) \xrightarrow{i_*} H_*(\mathcal{P}_{X,X}M).$$

For the other map we consider the same diagram

$$\begin{tikzcd} & \mathcal{P}_g^X M \arrow[dl, "j"'] \arrow[dr, "i"] & \\ \mathcal{P}_g M & & \mathcal{P}_{X,X} M \end{tikzcd}$$

and we use the next pullback square

$$\begin{tikzcd} \mathcal{P}_g^X M \arrow[r, "i"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{X,X} M \arrow[d, "\epsilon_0 \times \epsilon_1"] \\ X \arrow[r, "(\mathrm{id}, g)"'] & X \times X \end{tikzcd}$$

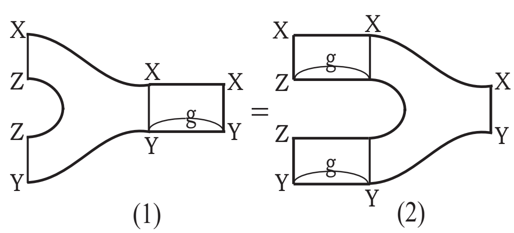
to define the map $\iota^{g,X}$ as the composition

$$\iota^{g,X} : \mathrm{H}_*(\mathcal{P}_{X,X} M) \xrightarrow{i_!} \mathrm{H}_*(\mathcal{P}_g^X M) \xrightarrow{j_*} \mathrm{H}_*(\mathcal{P}_g M).$$

Theorem 9.2. *The homology* $\mathrm{H}_*(\mathcal{P}_G(M))$*, together with the graded vector spaces* $\mathrm{Hom}_{\mathcal{B}}(X, Y)$ *for all* $X, Y \in \mathcal{B}$*, becomes a* G-OC-TFT *with positive boundary.*

Proof. We will check the open axioms.

1. The action respects the product



The property is the following

(1)

$$\begin{tikzcd}
 & & \mathcal{P}_{XZY}M \arrow[dl, "1"'] \arrow[dr, "i_{XY}"] & & \\
 & \mathcal{P}_{XZY}M \arrow[dl, "j_{XZ}\times j_{ZY}"'] \arrow[dr, "i_{XY}"] & & \mathcal{P}_{XY}M \arrow[dl, "1"'] \arrow[dr, "\rho_g"] & \\
\mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M & & \mathcal{P}_{XY}M & & \mathcal{P}_{XY}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_{XZY}M \arrow[dl, "j_{XZ}\times j_{ZY}"'] \arrow[dr, "\rho_g"] & & \\
 & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M \arrow[dl, "1"'] \arrow[dr, "\rho_g\times\rho_g"'] & & \mathcal{P}_{XZY}M \arrow[dl, "j_{XZ}\times j_{ZY}"'] \arrow[dr, "i_{XZ}"] & \\
\mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M & & \mathcal{P}_{XY}M
\end{tikzcd}$$

In the first diagram is clear that $F_1 = 0$, this because the normal bundles are zero. In the second diagram we have

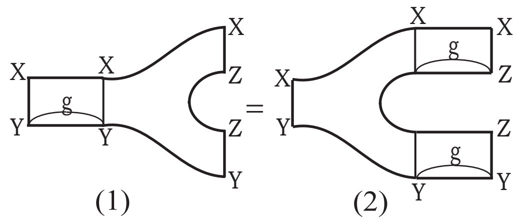
$$\begin{tikzcd}
(\rho_g \circ \epsilon_{\frac{1}{2}})^*(\eta) \arrow[r, dashed] & \mathcal{P}_{XZY}M \arrow[d, "\rho_g"] & \\
 & \mathcal{P}_{XZY}M \arrow[r, "j_{XZ}\times j_{ZY}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
\eta \arrow[r, dashed] & Z \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon^*_{\frac{1}{2}}(\eta) \arrow[r, dashed] & \mathcal{P}_{XZY}M \arrow[r, "j_{XZ}\times j_{ZY}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XZ} \times \mathcal{P}_{ZY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
\eta \arrow[r, dashed] & Z \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

We note, as before, that $(\rho_g \circ \epsilon_{\frac{1}{2}})^*(\eta) = \epsilon^*_{\frac{1}{2}}(\eta)$. Then $F_2 = 0$.

2. The action respects the coproduct



(1) (2)

(1)

$$
\begin{tikzcd}
 & & & \mathcal{P}_{XZY}M \arrow[dll, "i_{XY}"'] \arrow[drr, "\rho_g"] & & & \\
 & \mathcal{P}_{XY}M \arrow[dl, "1"'] \arrow[drr, "\rho_g"] & & & & \mathcal{P}_{XZY}M \arrow[dll, "i_{XY}"'] \arrow[dr, "j_{XZ}\times j_{ZY}"] & \\
\mathcal{P}_{XY}M & & & \mathcal{P}_{XY}M & & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
 & & & \mathcal{P}_{XZY}M \arrow[dll, "1"'] \arrow[drr, "j_{XZ}\times j_{ZY}"] & & & \\
 & \mathcal{P}_{XZY}M \arrow[dl, "i_{XY}"'] \arrow[drr, "j_{XZ}\times j_{ZY}"] & & & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M \arrow[dll, "1"'] \arrow[dr, "\rho_g\times\rho_g"] & \\
\mathcal{P}_{XY}M & & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M & & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M
\end{tikzcd}
$$

In the second diagram it is clear that $F_2 = 0$. In the first diagram the calculus as the following

$$
\begin{tikzcd}
(\epsilon_{\frac{1}{2}} \circ \rho_g)(\vartheta) \arrow[r, dashed] & \mathcal{P}_{XZY}M \arrow[d, "\rho_g"'] & \\
 & \mathcal{P}_{XZY}M \arrow[r, "i_{XY}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XY}M \arrow[d, "\epsilon_{\frac{1}{2}}\times\epsilon_{\frac{1}{2}}"] \\
\vartheta \arrow[r, dashed] & Z \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}
$$

and

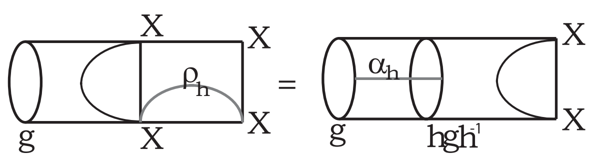
$$\begin{tikzcd}
\epsilon_{\frac{1}{2}}^{*}(\vartheta) \arrow[r, dashed] & \mathcal{P}_{XZY}M \arrow[r, "i_{XY}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XY}M \arrow[d, "\epsilon_{\frac{1}{2}} \times \epsilon_{\frac{1}{2}}"] \\
\vartheta \arrow[r, dashed] & Z \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

Since ρ_g is a isomorphism, then the next bundles are isomorphic,

$$\epsilon_{\frac{1}{2}}^{*}(\vartheta) \simeq (\epsilon_{\frac{1}{2}} \circ \rho_g)(\vartheta)$$

hence $F_1 = 0$.

3. The map ι_g is an equivariant map



(1) (2)

Remember that the connection maps are defined using the next diagram

$$\begin{tikzcd}
& \mathcal{P}_g^X M \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathcal{P}_g M & & \mathcal{P}_{XX} M
\end{tikzcd}$$

where $\mathcal{P}_g^X M = \{\alpha : I \to M : \ \alpha(1) = \alpha(0)g, \ \alpha(0) \in X\}$.
We defined $\iota_{g,X}$ by the composition

$$H_*(\mathcal{P}_g M) \xrightarrow{j_!} H_*(\mathcal{P}_g^X M) \xrightarrow{i_*} H_*(\mathcal{P}_{XX} M)$$

The diagrams that model this properties are:

(1)

$$
\begin{tikzcd}
 & & \mathcal{P}_g^X M \arrow[dl, "1"'] \arrow[dr, "i"] & & \\
 & \mathcal{P}_g^X M \arrow[dl, "j"'] \arrow[dr, "i"] & & \mathcal{P}_{XX} M \arrow[dl, "1"'] \arrow[dr, "\rho_h"] & \\
\mathcal{P}_g M & & \mathcal{P}_{XX} M & & \mathcal{P}_{XX} M
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}
 & & \mathcal{P}_g^X M \arrow[dl, "i"'] \arrow[dr, "\alpha_h"] & & \\
 & \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "\alpha_h"] & & \mathcal{P}^X_{hgh^{-1}} M \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathcal{P}_g M & & \mathcal{P}_{hgh^{-1}} M & & \mathcal{P}_{XX} M
\end{tikzcd}
$$

In the first case it is clear that $F_1 = 0$. This because the normal bundles are zero. For the second case we have

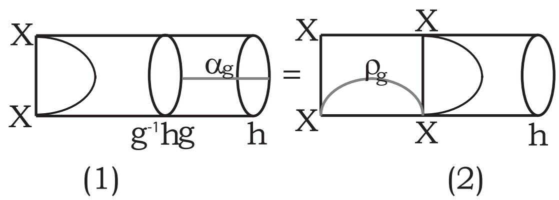
$$
\begin{tikzcd}
(\epsilon_0 \circ \alpha_h)^*(\vartheta) \arrow[r, dashed] & \mathcal{P}_g^X M \arrow[d, "\alpha_h"'] & \\
 & \mathcal{P}^X_{hgh^{-1}} M \arrow[r, "j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{hgh^{-1}} M \arrow[d, "\epsilon_0"] \\
\vartheta \arrow[r, dashed] & X \arrow[r, "\iota"'] & M
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\epsilon_0^*(\vartheta) \arrow[r, dashed] & \mathcal{P}_g^X M \arrow[r, "i"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g M \arrow[d, "\epsilon_0"] \\
\vartheta \arrow[r, dashed] & X \arrow[r, "\iota"'] & M
\end{tikzcd}
$$

The bundles $\epsilon_0^*(\vartheta)$ and $(\epsilon_0 \circ \alpha_h)^*(\vartheta)$ are isomorphic because the action α_h is a diffemorphism. Then, in particular is $F_2 = 0$.

4. The map ι^h is an equivariant map



The diagrams are the following

(1)

$$\begin{tikzcd}
 & & \mathcal{P}^X_{g^{-1}hg}M \arrow[dl, "1"'] \arrow[dr, "j"] & & \\
 & \mathcal{P}^X_{g^{-1}hg}M \arrow[dl, "i"'] \arrow[dr, "j"] & & \mathcal{P}_{g^{-1}hg}M \arrow[dl, "1"'] \arrow[dr, "\alpha_g"] & \\
\mathcal{P}_{XX}M & & \mathcal{P}_{g^{-1}hg}M & & \mathcal{P}_hM
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}^X_{g^{-1}hg}M \arrow[dl, "i"'] \arrow[dr, "\alpha_g"] & & \\
 & \mathcal{P}_{XX}M \arrow[dl, "1"'] \arrow[dr, "\rho_g"] & & \mathcal{P}^X_hM \arrow[dl, "i"'] \arrow[dr, "j"] & \\
\mathcal{P}_{XX}M & & \mathcal{P}_{XX}M & & \mathcal{P}_hM
\end{tikzcd}$$

For the first case, it is an easy consequence that $F_1 = 0$. This because the normal bundles are zero.

The second case involves the following diagrams

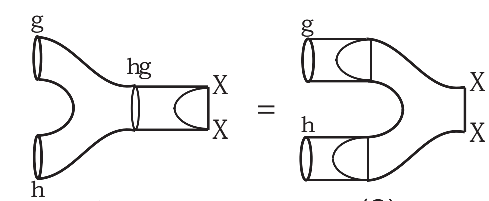
$$\begin{tikzcd}
(\epsilon_0 \circ \alpha_g)^*(\vartheta) \arrow[r, dashed] & \mathcal{P}^X_{g^{-1}hg}M \arrow[d, "\alpha_g"'] & \\
& \mathcal{P}^X_h M \arrow[r, "i"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{XX}M \arrow[d, "\epsilon_0 \times \epsilon_1"] \\
\vartheta \arrow[r, dashed] & X \arrow[r, "1 \times \alpha_h"'] & X \times X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_0^*(\vartheta) \arrow[r, dashed] & \mathcal{P}^X_{g^{-1}hg}M \arrow[r, "i"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{XX}M \arrow[d, "\epsilon_0 \times \epsilon_1"] \\
\vartheta \arrow[r, dashed] & X \arrow[r, "1 \times \alpha_{g^{-1}hg}"'] & X \times X
\end{tikzcd}$$

Note that the bundles $\epsilon_0^*(\vartheta) \simeq (\epsilon_0 \circ \alpha_g)^*(\vartheta)$ since α_g is a diffeomorphism. Then $F_2 = 0$.

5. The map ι_g is a ring homomorphism



(1) (2)

In this case the diagrams that model this property are the following
(1)

$$\begin{tikzcd}[column sep=small]
& & \mathcal{P}^X_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}^X_h M \arrow[dl, "j \times j"'] \arrow[dr, "*"] & & \\
& \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \arrow[dl, "j"'] \arrow[dr, "*"] & & \mathcal{P}^X_{gh}M \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathcal{P}_g M \times \mathcal{P}_h M & & \mathcal{P}_{gh}M & & \mathcal{P}_{XX}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}^X_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}^X_h M \arrow[dl, "j\times j"'] \arrow[dr, "*"] & & \\
 & \mathcal{P}^X_g M \times \mathcal{P}^X_h M \arrow[dl, "j\times j"'] \arrow[dr, "i\times i"] & & \mathcal{P}_{XXX} M \arrow[dl, "j_{12}\times j_{23}"'] \arrow[dr, "j_{13}"] & \\
\mathcal{P}_g M \times \mathcal{P}_h M & & \mathcal{P}_{XX} M \times \mathcal{P}_{XX} M & & \mathcal{P}_{XX} M
\end{tikzcd}$$

For the first case we have

$$\begin{tikzcd}
(\epsilon_0 \circ *)^*(\eta) \arrow[r, dashed] & \mathcal{P}^X_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}^X_h M \arrow[d, "*"'] & \\
 & \mathcal{P}^X_{gh} M \arrow[r, "j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{gh} M \arrow[d, "\epsilon_0\times\epsilon_1"] \\
\eta \arrow[r, dashed] & X \arrow[r, "1\times g"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_0^*(\eta) \arrow[r, dashed] & \mathcal{P}^X_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}^X_h M \arrow[r, "j\times j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g M_{\epsilon_0} \times_{\epsilon_1} \mathcal{P}_h M \arrow[d, "\epsilon_0\times\epsilon_1"] \\
\eta \arrow[r, dashed] & X \arrow[r, "1\times g"'] & M \times M
\end{tikzcd}$$

We note that $\epsilon_0 \circ * = \epsilon_0$, then $\epsilon_0^*(\eta) = (\epsilon_0 \circ *)^*(\eta)$ and $F_1 = 0$.
The second case has the following diagrams

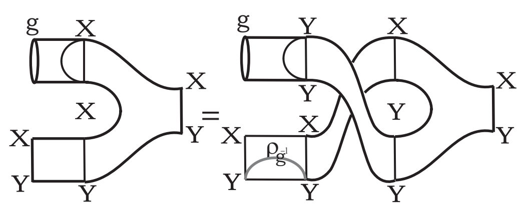
$$\begin{tikzcd}
(\epsilon_{\frac{1}{2}} \circ *)^*(TX) \arrow[r, dashed] & \mathcal{P}^X_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}^X_h M \arrow[d, "*"'] & \\
 & \mathcal{P}_{XXX} M \arrow[r, "j_{12}\times j_{23}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XX} M \times \mathcal{P}_{XX} M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TX \arrow[r, dashed] & X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_\infty^*(TX) \arrow[r, dashed] & \mathcal{P}_g^X M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h^X M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g^X M \times \mathcal{P}_h M \arrow[d, "\epsilon_1 \times \epsilon_0"] \\
TX \arrow[r, dashed] & X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

We note that $\epsilon_{\frac{1}{2}} \circ * = \epsilon_\infty$, hence $\epsilon_\infty^*(\eta) = (\epsilon_{\frac{1}{2}} \circ *)^*(\eta)$ and $F_2 = 0$.

6. G-twisted centrality condition



(1) (2)

This condition is modeled by the next diagrams.
(1)

$$\begin{tikzcd}
 & & \mathcal{P}_g^X M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_{XY} M \arrow[dl, "j'"'] \arrow[dr, "* \circ \iota"] \arrow[ddll, bend right, "\psi_1"'] \arrow[ddrr, bend left, "\varphi_1"] & & \\
 & \mathcal{P}_g^X M \times \mathcal{P}_{XY} M \arrow[dl, "i \times 1"'] \arrow[dr, "j \times 1"] & & \mathcal{P}_{XXY} M \arrow[dl, "j_{12} \times j_{23}"'] \arrow[dr, "j_{13}"] & \\
\mathcal{P}_g M \times \mathcal{P}_{XY} M & & \mathcal{P}_{XX} M \times \mathcal{P}_{XY} M & & \mathcal{P}_{XY} M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_{XY} M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_g^Y M \arrow[dl, "\tau \circ (1 \times \rho_g)"'] \arrow[dr, "* \circ \iota'"] \arrow[ddll, bend right, "\psi_2"'] \arrow[ddrr, bend left, "\varphi_2"] & & \\
 & \mathcal{P}_g^Y M \times \mathcal{P}_{XY} M \arrow[dl, "i \times 1"'] \arrow[dr, "\tau \circ (j \times \rho_{g^{-1}})"] & & \mathcal{P}_{XYY} M \arrow[dl, "j_{12} \times j_{23}"'] \arrow[dr, "j_{13}"] & \\
\mathcal{P}_g M \times \mathcal{P}_{XY} M & & \mathcal{P}_{XY} M \times \mathcal{P}_{XYY} M & & \mathcal{P}_{XY} M
\end{tikzcd}$$

We first check that the spaces $\mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M$ and $\mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M$ are homotopic. We define the maps as follow:

$$\begin{array}{rccc} \varphi: & \mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M & \longrightarrow & \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M \\ & (\alpha,\beta) & \longmapsto & (\beta, \overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta)) \end{array}$$

$$\begin{array}{rccc} \psi: & \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M & \longrightarrow & \mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M \\ & (\gamma,\delta) & \longmapsto & (\rho_g(\gamma)*\rho_g(\delta)*\overline{\gamma},\gamma) \end{array}$$

$$\psi\circ\varphi(\alpha,\beta) = \psi(\beta,\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta)) = (\rho_g(\beta)*\rho_g(\overline{\beta})*\alpha*\beta*\overline{\beta},\beta) \simeq (\alpha,\beta),$$

$$\varphi\circ\psi(\gamma,\delta) = \varphi(\rho_g(\gamma)*\rho_g(\delta)*\overline{\gamma},\gamma) = (\gamma,\overline{\gamma}*\gamma*\delta*\rho_{g^{-1}}(\overline{\gamma})*\rho_{g^{-1}}(\gamma)) \simeq (\gamma,\delta).$$

Then

$$\psi\circ\varphi \simeq \mathrm{Id} \quad \text{and} \quad \varphi\circ\psi \simeq \mathrm{Id}\,.$$

Now we check the external maps for the diagrams (1) and (2).

- $\varphi_2\circ\varphi(\alpha,\beta) = \varphi_2(\beta,\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta)) = \beta*\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta) \simeq \alpha*\beta$,
- $\varphi_1(\alpha,\beta) = \alpha*\beta$.
- $\psi_2\circ\varphi(\alpha,\beta) = \psi_2(\beta,\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta)) = (\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta),\rho_g(\beta)) \simeq (\alpha,\beta)$,
- $\psi_1(\alpha,\beta) = (\alpha,\beta)$.
- $\varphi_1\circ\psi(\gamma,\delta) = \varphi_1(\rho_g(\gamma)*\rho_g(\delta)*\overline{\gamma},\gamma) = \rho_g(\gamma)*\rho_g(\delta)*\overline{\gamma}*\gamma \simeq \gamma*\delta$,
- $\varphi_2(\gamma,\delta) = \gamma*\delta$.
- $\psi_1\circ\psi(\gamma,\delta) = \psi_1(\rho_g(\gamma)*\rho_g(\delta)*\overline{\gamma},\gamma) = (\rho_g(\gamma)*\rho_g(\delta)*\overline{\gamma},\gamma) \simeq (\delta,\rho_g(\gamma))$,
- $\psi_2(\gamma,\delta) = (\delta,\rho_g(\gamma))$.

Finally we need to calculate the Euler class in each diagram. For the first case we have

$$\begin{tikzcd} (\epsilon_{\frac{1}{2}}\circ*\circ\iota)^*(TX) \arrow[r, dashed] & \mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M \arrow[d, "*\circ\iota"'] & \\ & \mathcal{P}_{XXY}M \arrow[r, "j_{12}\times j_{23}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XX}M\times\mathcal{P}_{XY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\ TX \arrow[r, dashed] & X \arrow[r, "\Delta"'] & X\times X \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small] \epsilon_\infty^*(TX) \arrow[r, dashed] & \mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M \arrow[r, "j'"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g^X M \times \mathcal{P}_{XY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\ TX \arrow[r, dashed] & X \arrow[r, "\Delta"'] & X\times X \end{tikzcd}$$

We note that $\epsilon_\infty^*(TX) = (\epsilon_{\frac{1}{2}} \circ * \circ \iota)^*(TX)$. Then $F_1 = 0$.
The second case has associated the next diagrams

$$\begin{tikzcd}[column sep=small] (\epsilon_{\frac{1}{2}} \circ * \circ \iota')^*(TY) \arrow[r, dashed] & \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M \arrow[d, "*\circ\iota'"'] & \\ & \mathcal{P}_{XYY}M \arrow[r, "j_{12}\times j_{23}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XY}M \times \mathcal{P}_{YY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\ TY \arrow[r, dashed] & Y \arrow[r, "\Delta"'] & Y\times Y \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small] \epsilon_\infty^*(TY) \arrow[r, dashed] & \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M \arrow[r, "\tau\circ(1\times\rho_g)"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g^Y M \times \mathcal{P}_{XY}M \arrow[d, "\epsilon_0\times\epsilon_1"] \\ TY \arrow[r, dashed] & Y \arrow[r, "\Delta"'] & Y\times Y \end{tikzcd}$$

As before the identity holds $\epsilon_\infty^*(TY) = (\epsilon_{\frac{1}{2}} \circ * \circ \iota')^*(TY)$. Then $F_2 = 0$.
To finish the proof we only need to check that $\nu_\varphi = 0$. For this, we construct the next homotopy:

$$\begin{array}{rccc} H: & I \times (\mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M) & \longrightarrow & \mathcal{P}_{XY}M_{\epsilon}\times_{\epsilon_0}\mathcal{P}_g M \times I \\ & (s,(\alpha,\beta)) & \longmapsto & (\beta, \overline{\beta_s} * \rho_{g^{-1}}(\alpha) * \rho_{g^{-1}}(\beta), s) \end{array}$$

where $\epsilon : I\times\mathcal{P}_{XY}M \to M$ is given by $\epsilon(s,\beta) := \beta(s)$. The next pullback square proves that $W := \mathcal{P}_{XY}M_{\epsilon}\times_{\epsilon_0}\mathcal{P}_g M \times I$ is an infinite manifold.

$$\begin{tikzcd} W = \mathcal{P}_{XY}M_{\epsilon}\times_{\epsilon_0}\mathcal{P}_g M \times I \arrow[r] \arrow[d, "\epsilon_\infty\times 1"'] & \mathcal{P}_{XY}M \times \mathcal{P}_g M \times I \arrow[d, "\epsilon\times\epsilon_0\times 1"] \\ M\times I \arrow[r, "\Delta\times 1"'] & M\times M\times I \end{tikzcd}$$

Similarly, the next pullback square

$$\begin{tikzcd}
Z_s := \mathcal{P}_{XY}M_{\epsilon_s} \times_{\epsilon_0} \mathcal{P}_g M \times \{s\} \arrow[r] \arrow[d, "\epsilon_s \times \{s\}"'] & \mathcal{P}_{XY}M_{\epsilon} \times_{\epsilon_0} \mathcal{P}_g M \times I \arrow[d, "\epsilon \times 1"] \\
M \times \{s\} \arrow[r, hook] & M \times I
\end{tikzcd}$$

proves that Z_s is an inclusion of codimension one on W for all s.
Note that the homotopy H satisfies that

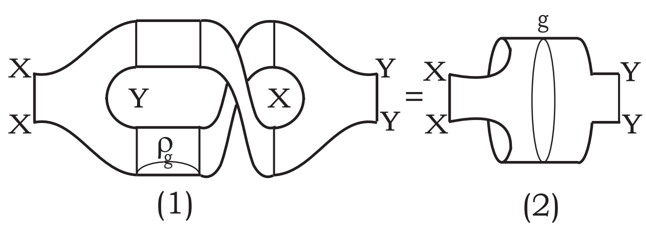
$$\begin{array}{rclcl}
H(0,(\alpha,\beta)) & = & (\beta, \rho_{g^{-1}}(\alpha)) & = & (1 \times \rho_{g^{-1}}) \circ \tau(\alpha,\beta) \\
H(1,(\alpha,\beta)) & = & (\beta, \overline{\beta} * \rho_{g^{-1}}(\alpha) * \rho_{g^{-1}}(\beta)) & = & \varphi(\alpha,\beta)
\end{array}$$

Then, in particular we have the next situation

$$\begin{tikzcd}
\mathcal{P}^X_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_{XY}M \arrow[d, "(1\times\rho_{g-1})\circ\tau"', "\text{diffemorphism}"] \arrow[r, phantom, "\overset{H}{\underset{\simeq}{\Longrightarrow}}"] & \mathcal{P}^X_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_{XY}M \arrow[d, "\varphi"] \\
Z_0 = \mathcal{P}_{XY}M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}^Y_g M & Z_1 = \mathcal{P}_{XY}M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}^Y_g M
\end{tikzcd}$$

Since $\nu_{(1\times\rho_{g-1})\circ\tau} = 0$ then $\nu_\varphi = 0$ and $e(\nu_\varphi) = 1$.

7. Cardy condition



(1)

$$\begin{tikzcd}
 & & \mathcal{P}_g^{X\cap Y}M \arrow[dl, "i_X"'] \arrow[dr, "i_Y"] \arrow[ddll, bend right, "g_1"'] \arrow[ddrr, bend left, "f_1"] & & \\
 & \mathcal{P}_g^{X}M \arrow[dl, "i"'] \arrow[dr, "j"] & & \mathcal{P}_g^{Y}M \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathcal{P}_{XX}M & & \mathcal{P}_g M & & \mathcal{P}_{YY}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}^Y_{g,X}M \arrow[dl, "i"'] \arrow[dr, "*\circ(\rho_g\times 1)\circ\tau\circ(j\times j)"] \arrow[ddll, bend right, "g_2"'] \arrow[ddrr, bend left, "f_2"] & & \\
 & \mathcal{P}_{XYX}M \arrow[dl, "i^Y_{XX}"'] \arrow[dr, "\tau\circ(1\times\rho_g)\circ(j\times j)"] & & \mathcal{P}_{YXY}M \arrow[dl, "j\times j"'] \arrow[dr, "i^X_{YY}"] & \\
\mathcal{P}_{XX}M & & \mathcal{P}_{YX}M\times\mathcal{P}_{XY}M & & \mathcal{P}_{XX}M
\end{tikzcd}$$

In this particular case, the maps are illustrated in Figure 29, and they are homotopic to the cobordism illustrated in Figure 30. We will suppose that

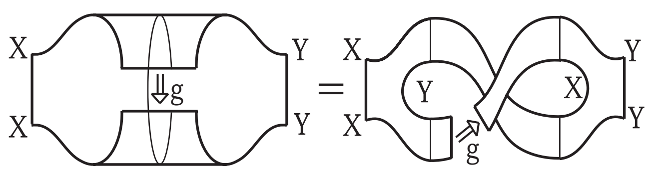


Figure 29: The composition maps in the Cardy condition.

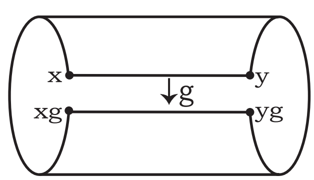


Figure 30: The cobordisms associated to the compositions.

the intersection $X\cap Y$ is non-empty, this because if it is empty then the two composition maps are zero. In the second cobordism the composition is zero by definition, and in the first this is because for an empty intersection the composition of the umhker maps is zero since the tubular neighborhoods are disjoint.

We prove that $\mathcal{P}_g^{X\cap Y}M$ and $\mathcal{P}_{g,X}^{Y}M$ are homotopically equivalent spaces. First we describe the maps between the spaces. Suppose that $z \in X \cap Y$, and if we take M path-connected then for $x \in M$ there exists $\eta : I \to M$ such that $\eta(0) = z$ and $\eta(1) = x$.

$$\begin{array}{rccc} \varphi: & \mathcal{P}_g^{X\cap Y}M & \longrightarrow & \mathcal{P}_{g,Y}^{X}M \\ & \alpha & \longmapsto & \alpha * \overline{\alpha} * \alpha \\ \psi: & \mathcal{P}_{g,Y}^{X}M & \longrightarrow & \mathcal{P}_g^{X\cap Y}M \\ & \delta & \longmapsto & \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \end{array}$$

The composition maps are

$$\psi \circ \varphi(\alpha) = \psi(\alpha * \overline{\alpha} * \alpha) = \eta * \alpha * \overline{\alpha} * \alpha * \rho_{g^{-1}}(\overline{\eta}) \simeq \eta * \rho_{g^{-1}}(\overline{\eta}) \simeq \alpha.$$

$$\begin{aligned} \varphi \circ \psi(\delta) = \varphi(\eta * \delta * \rho_{g^{-1}}(\overline{\eta})) &= \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) * \rho_{g^{-1}}(\eta) * \overline{\delta} * \overline{\eta} * \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \\ &\simeq \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \simeq \delta. \end{aligned}$$

The composition with the external maps is the following. First we note that the maps $f_1 : \mathcal{P}_g^{X\cap Y}M \hookrightarrow \mathcal{P}_{XX}M$, $g_1 : \mathcal{P}_g^{X\cap Y}M \hookrightarrow \mathcal{P}_{YY}M$ and $g_2 : \mathcal{P}_{g,Y}^{X}M \hookrightarrow \mathcal{P}_{XX}M$ are natural inclusion maps. Finally, the map $f_2 : \mathcal{P}_{g,Y}^{X}M \to \mathcal{P}_{YY}M$ is given by $f_2(\alpha * \beta) = \rho_g(\beta) * \alpha$. Then

$$\delta \xmapsto{\psi} \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \xmapsto{g_1} \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \simeq \delta$$

$$\delta \xmapsto{g_2} \delta$$

$$\delta \xmapsto{\psi} \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \xmapsto{f_1} \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) = \eta * \alpha * \beta * \rho_{g^{-1}}(\overline{\eta}) \simeq \rho_g(\beta) * \alpha$$

$$\delta = \alpha * \beta \xmapsto{f_2} \rho_g(\beta) * \alpha$$

$$\alpha \xmapsto{g_1} \alpha$$

$$\alpha \xmapsto{\varphi} \alpha * \overline{\alpha} * \alpha \xmapsto{g_2} \alpha * \overline{\alpha} * \alpha \simeq \alpha$$

$$\alpha \xmapsto{f_1} \alpha$$

$$\alpha \xmapsto{\varphi} \alpha * \overline{\alpha} * \alpha \xmapsto{f_2} \rho_g(\overline{\alpha} * \alpha) * \alpha \simeq \alpha$$

Now we need to determine the Euler class for this case. First, we calcule that $e(\nu_\psi) = 0$. Let be the homotopy

$$\begin{array}{rccc} H: & I \times \mathcal{P}^X_{g,Y}M & \longrightarrow & \mathcal{P}^I_{g,Y,X}M \times I \\ & (s,\delta) & \longmapsto & (\eta_s * \delta * \rho_{g^{-1}}(\overline{\eta_s}), s) \end{array}$$

where $\eta_s : I \to M$ is given by $\eta_s(t) = \eta((1-s)t + s)$, then $\eta_s(0) = \eta(s)$ and $\eta_s(1) = \eta(1)$. See Figure 31.

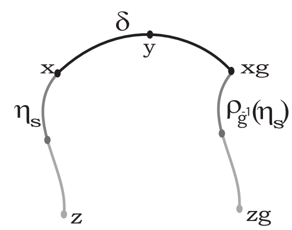


Figure 31: The homotopy H.

Note that $H(0,\delta) = (\eta_0 * \delta * \rho_{g^{-1}}(\overline{\eta_0}), 0) = (\eta * \delta * \rho_{g^{-1}}(\overline{\eta}) = \psi(\delta)$, and $H(1,\delta) = (\eta_1 * \delta * \rho_{g^{-1}}(\overline{\eta_1}), 1) = \delta = \mathrm{Id}(\delta)$. Then, we have the next situation

$$\begin{tikzcd} \mathcal{P}^X_{g,Y}M \arrow[d, "\mathrm{Id}"'] \arrow[r, phantom, "\overset{H}{\underset{\simeq}{\Longrightarrow}}"] & \mathcal{P}^X_{g,Y}M \arrow[d, "\psi"] \\ Z_0 = \mathcal{P}^X_{g,Y}M & Z_1 = \mathcal{P}^{X\cap Y}_g M \end{tikzcd}$$

For the space $Z_s := \mathcal{P}^s_{g,Y,X}M \times \{s\} = \{\eta_s * \delta * \rho_{g^{-1}}(\overline{\eta_s}) : \delta \in \mathcal{P}^X_{g,Y}M\} \times \{s\} \subset W := \mathcal{P}^I_{g,Y,X}M \times I = \{\eta_s * \delta * \rho_{g^{-1}}(\overline{\eta_s}) : \ s \in I\} \times I$ we have that Z_s gives an inclusion on W of codimension one. This is because the next diagram is a pullback square:

$$\begin{tikzcd} Z_s := \mathcal{P}^s_{g,Y,X}M \times \{s\} \arrow[r] \arrow[d, "\epsilon_\infty \times \{s\}"'] & \mathcal{P}^I_{g,Y,X}M \times I \arrow[d, "\epsilon_\infty \times 1"] \\ X \times \{s\} \arrow[r, hook] & X \times I \end{tikzcd}$$

Then $\nu_\psi = \nu_{\mathrm{Id}} = 0$ and $e(\nu_\psi) = 1$.
Finally, we need to determine the Euler class of the following two diagrams. The first diagram is

$$\begin{tikzcd}
\epsilon_0^*(\nu_{i_2}) \arrow[r] & \mathcal{P}_g^{X\cap Y}M \arrow[d, "i_Y"] & \\
& \mathcal{P}_g^{Y}M \arrow[r, "j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g M \arrow[d, "\epsilon_0"] \\
\nu_{i_2} \arrow[r] & Y \arrow[r, hook, "i_2"'] & M
\end{tikzcd}$$

and the second

$$\begin{tikzcd}
\epsilon_0^*(\nu_{i_1}) \arrow[r] & \mathcal{P}_g^{X\cap Y}M \arrow[r, "i_X"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g^{X}M \arrow[d, "\epsilon_0"] \\
\nu_{i_1} \arrow[r] & X \cap Y \arrow[r, hook, "i_1"'] & X
\end{tikzcd}$$

If we suppose that $X \pitchfork Y$ then $e(\epsilon_0^*(\nu_{i_1})) = e(\epsilon_0^*(\nu_{i_2}))$, and $F_1 = 0$.
In the second case we have

$$\begin{tikzcd}
\epsilon_0^*(\nu_{(1\times\alpha_g)}) \arrow[r] & \mathcal{P}_{g,Y}^{X}M \arrow[r] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{XYX}M \arrow[d, "\epsilon_0\times\epsilon_1"] \\
\nu_{(1\times\alpha_g)} \arrow[r] & X \arrow[r, "1\times\alpha_g"'] & X \times X
\end{tikzcd}$$

and

$$\begin{tikzcd}
f^*\epsilon_{\frac{1}{2}}^*(TX) \arrow[r] & \mathcal{P}_{g,Y}^{X}M \arrow[d, "f=*\circ(\rho_g\times 1)\circ\tau\circ(j\times j)"] & \\
& \mathcal{P}_{YXY}M \arrow[r] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{YX}M \times \mathcal{P}_{XY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TX \arrow[r] & X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

Note that $f^*\epsilon_{\frac{1}{2}}^*(TX) \simeq \epsilon_0^*(\nu_{1\times\alpha_g})$, this is because $\nu_{(1\times\alpha_g)} \cong TX$. Then $F_2 = 0$.

♣

9.3 BV-Structure

The Chas-Sullivan string product on $H_*(\mathcal{L}M)$ was only part of a very interesting structure unveiled in their work; for example Chas and Sullivan defined a degree one map

$$\Delta\colon H_*(\mathcal{L}M) \to H_{*+1}(\mathcal{L}M)$$

given by

$$\Delta(\sigma) = \rho_*(d\theta \otimes \sigma)$$

where $\rho\colon S^1 \times \mathcal{L}M \to \mathcal{L}M$ is the evaluation map and $d\theta$ is the fundamental class of S^1. One of the main theorems of [CS] is the following one

Theorem 9.3 (Chas-Sullivan [CS]). *The triple*

$$(H_*(\mathcal{L}M), \circ, \Delta)$$

is a Batalin-Vilkovisky algebra, namely

- $(H_{*-d}(\mathcal{L}M), \circ)$ *is a graded commutative algebra.*
- $\Delta^2 = 0$.
- *The bracket*

  $$\{\alpha, \beta\} = (-1)^{|\alpha|}\Delta(\alpha \circ \beta) - (-1)^{|\alpha|}\Delta(\alpha) \circ \beta - \alpha \circ \Delta(\beta)$$

  makes $H_{*-d}(M)$ *into a graded Gerstenhaber algebra (namely it is a Lie bracket which is a derivation on each variable).*

This establishes a striking relation between algebraic topology and recent findings in quantum field theory and string theory [BV85, Get94].

Cohen and Jones [CJ02] discovered that a very rich part of this structure was available at a more homotopy-theoretic level and reinterpreted the BV-algebra structure in terms of an action of the cactus operad on a certain prospectrum associated to M. They showed moreover that the Chas-Sullivan string product was the natural product in the Hochschild cohomology interpretation of the homology of the loop space of M [Jon87]. Cohen and Godin [CG04] studied interactions with the study of the homology of moduli spaces of Riemann surfaces, establishing a direct connection to topological quantum field theories. Cohen and Godin used the concept of Sullivan chord diagram in their work. Cohen, Jones and Yan [CJY04] provided more explicit calculations of the product by the careful use of the spectral sequence associated to the fibration

$$\Omega M \longrightarrow \mathcal{L}M \longrightarrow M$$

induced by the evaluation map. In particular they computed the Chas-Sullivan product on the homology of the free loop space of spheres and complex projective spaces.

In [LUX08] we generalize several of the fundamental results of string topology by showing that they remain true if we replace the manifold M by an orientable orbifold $\mathcal{X} = [M/G]$, where G is a finite group acting by orientation preserving diffeomorphisms on M. More precisely the following theorem is the main result in [LUX08] and can be seen as a generalization of Theorem 9.3 to the orbifold context.

Theorem 9.4. *Let $\mathcal{X} = [M/G]$ be an orientable orbifold, then*

$$A_{L\mathcal{X}} := H_*(\mathcal{L}(M \times_G EG); \mathbb{Q})$$

has the structure of a Batalin-Vilkovisky algebra.

This BV-algebra can be identified in two extreme cases:

- When $G = \{1\}$ and for arbitrary M then $A_{L\mathcal{X}}$ coincides with the Chas-Sullivan BV-algebra.
- When $M = \{m_0\}$ is a single point and for arbitrary finite G then $A_{L\mathcal{X}}$ is isomorphic to the center of the group algebra of G.

9.4 Examples

In this paragraph we illustrate how one computes the pair of pants product in orbifold string topology.

Example 9.1. Let M be a smooth manifold and consider $X = [M/\{1\}]$ (in other words we consider the case when $G = \{1\}$). Then it is clear that $\mathcal{P}_g(M) = \mathcal{P}_G(M) = \mathcal{L}M$ is simple the free loop space and $H_*(LX) = H_*(\mathcal{L}M)$. By the work of Cohen and Jones we recover the Chas-Sullivan BV-algebra in this case.

Example 9.2. Let G be a finite group and consider $X = [\bullet/G]$ be the orbifold consisting of a point $M = \bullet$ being acted by G. Sometimes this orbifold is denoted by $\mathcal{B}G$ (not to be confused with BG the classifying space of G). Clearly every loop and every path in this case is constant, namely the space $\mathcal{P}_g(M) = \star_g$ is a point, and so

$\mathcal{P}_G(M)$ is in one-to-one correspondence with G. Therefore the category $[\mathcal{P}_G(M)/G]$ is equivalent to the category $[G/G]$ of G acting on G by conjugation, for we have

$$h(\star_g) = \star_{hgh^{-1}}.$$

For each $g \in G$ the stabilizer of this action is the centralizer

$$C(g) = \{h \in G | hgh^{-1} = g\}.$$

Now, in the category $[G/G]$ an object $g \in G$ is isomorphic to $g' \in G$ if and only if g and g' are conjugate. Therefore we have the equivalence of categories

$$LX \simeq [\mathcal{P}_G(M)/G] \simeq [G/G] \simeq \coprod_{(g)} [\star_g/C(g)].$$

Here (g) runs through the conjugacy classes of elements in $g \in G$. From this we can conclude that the equivalence

$$\mathcal{L}BX = BLX$$

becomes in this particular case (cf. [LU04b])

$$\mathcal{L}BG \simeq \coprod_{(g)} BC(g)$$

This equation becomes at the level of homology with complex coefficients the center of the group algebra

$$H_*(\mathcal{L}BG) \cong Z(\mathbb{C}[G])$$

and in fact $H_*(\mathcal{L}BG)$ is simply the Frobenius algebra of Dijkgraaf and Witten [DW90].

The reader may be interested in comparing this result with that of [ACG+08].

Let X be a topological space endowed with the action of a connected Lie group Γ. Take $G \subset \Gamma$ finite and consider the quotient X/G and the map $\pi : X \to X/G$ the projection.

Lemma 9.5. *The projection map induces an isomorphism*

$$\pi_* : H_*(X; \mathbb{Q}) \xrightarrow{\cong} H_*(X/G; \mathbb{Q}).$$

Proof. Take $g \in G$ and its induced action $g : X \to X$. We claim that $g_* : H_*(X) \xrightarrow{=} H_*(X)$ is the identity. Join the identity of Γ with g with a path $\alpha_t \in \Gamma$ (i.e. $\alpha_0 = \mathrm{id}_\Gamma$ and $\alpha_1 = g$), hence α_t is a homotopy between the identity and g, therefore $g_* = \mathrm{id}$.

Taking the averaging operator

$$\begin{array}{rcl} H_*(X;\mathbb{Q}) & \stackrel{\alpha}{\to} & H_*(X;\mathbb{Q})^G \\ x & \mapsto & \left(\frac{1}{|G|}\sum_{g\in G} g_* x\right) (= x) \end{array}$$

and using that $H_*(X;\mathbb{Q})^G \stackrel{\iota}{\cong} H_*(X/G;\mathbb{Q})$ the isomorphism follows, for it is not hard to check that $\pi_* = \iota\alpha$.

♣

With the same hypothesis as before consider now the orbifold loops, namely $\mathcal{P}_g X = \{f : [0,1] \to X | f(0)g = f(1)\}$.

Lemma 9.6. *There is a* $C(g)$*-equivariant homotopy equivalence between* $\mathcal{L}X$ *and* $\mathcal{P}_g X$.

Proof. Let $\alpha_t : [0,1] \to G$ be the map defined in Lemma 9.5. Consider the maps

$$\rho : \mathcal{P}_g X \to \mathcal{L}X \qquad \text{and} \qquad \tau : \mathcal{L}X \to \mathcal{P}_g X \tag{40}$$

where

$$\rho(f)(s) := \begin{cases} f(2s) & \text{if} \quad 0 \le s \le \frac{1}{2} \\ f(1)\alpha_{2s-1}^{-1} & \text{if} \quad \frac{1}{2} \le s \le 1 \end{cases}$$

and

$$\tau(\sigma)(s) := \begin{cases} \sigma(2s) & \text{if} \quad 0 \le s \le \frac{1}{2} \\ \sigma(1)\alpha_{2s-1} & \text{if} \quad \frac{1}{2} \le s \le 1. \end{cases}$$

The composition $\rho \circ \tau : \mathcal{L}X \to \mathcal{L}X$ is clearly homotopic to the identity. The same holds for $\tau \circ \rho$. The maps ρ and τ are trivially $C(g)$-equivariant.

♣

Corollary 9.7. *The group structure of the loop homology of* $[X/G]$ *can be seen as*

$$H_*(L[X/G];\mathbb{Q}) \cong \bigoplus_{(g)} H_*(\mathcal{L}X;\mathbb{Q}).$$

Proof. It follows from the lemmas 9.5 and 9.6 and the fact that

$$H_*(L[X/G]; \mathbb{Q}) \cong \bigoplus_{(g)} H_*(\mathcal{P}_g X/C(g); \mathbb{Q}).$$

♣

Notation: Let X be an orbifold of dimension d. Let us denote the loop homology of X by

$$\mathbb{H}_*(X) := H_{*+d}(LX; \mathbb{Q}).$$

In this way the orbifold string product $\mathbb{H}_*(X)$ is graded associative.

Example 9.3. The loop homology of the lens spaces $L_{(n,p)} = S^n/\mathbb{Z}_p$ (n odd, $p > 0$) is

$$\mathbb{H}_*(L_{(n,p)}) = H_*(\mathcal{L}L_{(n,p)}) = \Lambda[a] \otimes \mathbb{Q}[u, v]/(v^p = 1)$$

with $a \in \mathbb{H}_{-n}(L_{(n,p)})$, $v \in \mathbb{H}_0(L_{(n,p)})$ and $u \in \mathbb{H}_{n-1}(L_{(n,p)})$.

Proof. As the action of $\mathbb{Z}_p$ on S^n comes from the action of S^1 on S^n via the Hopf fibration, we can use Corollary 9.7 . Let g be a generator of $\mathbb{Z}_p$, then

$$\mathbb{H}_*(L_{(n,p)}) \cong \mathbb{H}_*([S^n/\mathbb{Z}_p]) \cong \bigoplus_{j=0}^{p-1} H_*(\mathcal{P}_{g^j} S^n)^{\mathbb{Z}_p},$$

as graded vector spaces.

As $H_*(\mathcal{P}_{g^j} S^n)^{\mathbb{Z}_p} \cong H_*(\mathcal{P}_{g^j} S^n)$ the string product $\circ$ could be calculated from the following commutative diagram

$$\begin{tikzcd}
H_*(\mathcal{P}_{g^j} S^n)^{\mathbb{Z}_p} \times H_*(\mathcal{P}_{g^k} S^n)^{\mathbb{Z}_p} \arrow[r, "\circ"] \arrow[d, "\cong"'] & H_*(\mathcal{P}_{g^{j+k}} S^n)^{\mathbb{Z}_p} \arrow[d, "\cong"] \\
H_*(\mathcal{P}_{g^j} S^n) \times H_*(\mathcal{P}_{g^k} S^n) \arrow[r, "\circ"] & H_*(\mathcal{P}_{g^{j+k}} S^n).
\end{tikzcd}$$

The map $\tau^j : \mathcal{L}S^n \to \mathcal{P}_{g^j} S^n$ defined in (40) gives an isomorphism in homology, so we can define the generators of the homology of $\mathcal{P}_{g^j} S^n$ via the map τ^j and the loop homology of the sphere, namely $\mathbb{H}_*(S^n) = H_*(\mathcal{L}S^n) \cong \Lambda[a] \otimes \mathbb{Q}[u]$ (see [CJY04]). Denote then by σ^j_k the generator of the group $H_{k+n}(\mathcal{P}_{g^j} S^n)$ and using that $(\tau^j)_*$ is an isomorphism one gets that $\sigma^j_{(n-1)l-n} = \tau^j_*(au^l)$, $\sigma^j_{(n-1)l} = \tau^j_*(u^l)$ and $\sigma^j_m = 0$ for all other values of m.

We claim now that
$$\sigma^j_l \circ \sigma^k_m = \sigma^{j+k}_{l+m}.$$
The identity follows from the fact that

$$\sigma^j_l \circ \sigma^k_m = \tau^j_*(\sigma^0_l) \circ \tau^k_*(\sigma^0_m) = \tau^{j+k}_*(\sigma^0_l \circ \sigma^0_m) = \tau^{j+k}_*(\sigma^0_{l+m}) = \sigma^{j+k}_{l+m}$$

where the second identity follows from the definition of the maps τ and the third identity follows from the algebraic structure of $\Lambda[a] \otimes \mathbb{Q}[u]$.

From this we can deduce that the map $\tau_{j*} : H_*(\mathcal{L}S^n) \to H_*(\mathcal{P}_{g^j} S^n)$ maps $\sigma^0_k \mapsto \sigma^0_k \circ \sigma^j_0$ where σ^j_0 is the n-simplex of paths that to every x in S^n assigns the path that goes from x to xg^j through the S^1 action.

We are only left to prove that when $j + k = p$ the formula $\sigma^j_l \circ \sigma^k_m = \sigma^0_{l+m}$ holds. So, let $\beta : S^n \to \mathcal{L}S^n$ be the map that to a point x in the sphere associates the free loop defined that starts and ends in x and travels in the direction of the S^1 action. Now define the map $\phi : \mathcal{L}S^n \to \mathcal{L}S^n$ that takes a loop γ to $\gamma \circ \beta$. The map ϕ is homotopic to the identity because the cycle β is homotopic to the cycle of constant loops over the sphere (one way to prove this uses the fact that the odd dimensional spheres have two orthogonal never vanishing vector fields). Therefore we have that $\tau^p : \mathcal{L}S^n \to \mathcal{P}_{g^p} S^n = \mathcal{L}S^n$ is homotopic to the identity.

We can conclude then that the elements $a = \sigma^0_{-n}$, $v = \sigma^1_0$ and $u = \sigma^0_{n-1}$ generate the loop homology of $L_{(n,p)}$, and the only extra condition is that $v^p = 1$. Therefore

$$\mathbb{H}_*(L_{(n,p)}) = \Lambda[a] \otimes \mathbb{Q}[u, v]/(v^p = 1)$$

♣

Example 9.4. Take the orbifold defined by the action of $\mathbb{Z}_p$ onto S^2 given by rotation of $2\pi/p$ radians with respect to the z-axis. Then the loop homology of $[S^2/\mathbb{Z}_p]$ is

$$\mathbb{H}_*([S^2/\mathbb{Z}_p]) = \Lambda[b] \otimes \mathbb{Q}[a, v, y]/(a^2, ab, av, y^p - 1)$$

Proof. The action of $\mathbb{Z}_p$ comes from the S^1 action on S^2 given by rotation about the z-axis. therefore the calculation of the loop homology product follows the same argument as in the Example 9.3. To make the notation simpler we will work with $p = 2$ ($\mathbb{Z}_2 = \{1, g\}$); the other cases are similar.

From [CJY04] we know that the loop homology of S^2 is given by

$$\mathbb{H}_*(S^2) = \Lambda[b] \otimes \mathbb{Z}[a, v]/(a^2, ab, 2av) \tag{41}$$

with $|b| = 1$, $|a| = -2$, $|v| = 2$. Since $\tau : \mathcal{L}S^2 \to \mathcal{P}_g S^2$ is a homotopy equivalence, we will follow the argument of Example 9.3. The only different argument is on the behavior of the map $\phi := \tau^2 : \mathcal{L}S^2 \to \mathcal{L}S^2$. In homology, ϕ_* maps $\alpha \in H_k(\mathcal{L}S^2)$ to $\alpha \circ \beta \in H_k(\mathcal{L}S^2)$ where $\beta \in H_2(\mathcal{L}S^2) = \mathbb{H}_0(S^2)$ is the class of the map $S^2 \to \mathcal{L}S^2$ that assigns to every point x the loop that starts at x and rotates around the z axis, and $\circ$ is the homology string product.

We claim that $\beta = 1 + av$ in the notation of (41), (the proof of this fact will be postponed to Lemma 9.8). As av is a torsion class, i.e. $2av = 0$, then in rational homology ϕ_* is the identity map. As in Example 9.3, we can add a new variable y that behaves like a root of unity, and we conclude that

$$\mathbb{H}_*([S^2/\mathbb{Z}_2]) = \Lambda[b] \otimes \mathbb{Q}[a, v, y]/(a^2, ab, av, y^2 - 1).$$

♣

Lemma 9.8. *The homology class* $\beta \in H_2(\mathcal{L}S^2) = \mathbb{H}_0(S^2)$ *of the map* $S^2 \to \mathcal{L}S^2$ *that to a point* x *assigns the loop that starts at* x *and winds around the sphere once by the* S^1 *action, and the homology class* $1 + av \in H_2(\mathcal{L}S^2) = \mathbb{H}_0(S^2)$ *as in (41), are equal.*

Proof. When we contract all the loops of β through the north pole we end up with the homology class $[S^2] + \xi$, where $[S^2]$ is the fundamental class of the sphere (constant loops) and therefore the unit in $1 = [S^2] \in \mathbb{H}_0(S^2)$, and ξ is defined in what follows. For $\theta \in S^1$ and P_S the south pole, consider the map $f : S^1 \times S^1 \to \mathcal{L}S^2$ such that the function $f_\theta = f(\cdot, \theta) : S^1 \to \mathcal{L}S^2$ is the loop of based loops that starts at the constant loop in P_S and goes around the sphere (as a rubber band) at the angle θ. The class $f_{\theta *}([S^1])$ is the generator of $H_1(\mathcal{L}S^2)$, and the class $f_*([S^1 \times S^1])$ is ξ. We claim that $\xi = av$.

We know that the homology spectral sequence of the Serre fibration $\Omega S^2 \to \mathcal{L}S^2 \to S^2$ has for E_2-term

$$E_2^{p,q} = H_p(S^2) \otimes H_q(\Omega S^2)$$

with non trivial differential $d^2(u \otimes x^{2k+1}) = 2\iota \otimes x^{2k+2}$ where $x \in H_1(\Omega S^2)$, $\iota \in H_0(S^2)$, $1_\Omega \in H_0(\Omega S^2)$ and $u \in H_2(S^2)$ are generators respectively. Also we know from [CJY04] that $av = \iota \otimes x^2$.

Denote by $\dot{T}S^2 \xrightarrow{\pi} S^2$ the sphere bundle of the tangent bundle $TS^2 \to S^2$. The map π is an S^1-fibration and a point in $\dot{T}S^2$ consists of a pair (z, v) where $z \in S^2$ and v is a unit vector tangent to S^2 at z. For each point (z, v) we can define a map

$h_{(z,v)} : S^1 \to \mathcal{L}S^2$ in the same way that the function f_θ was defined two paragraphs above; namely, $h_{(z,v)}$ is the loop of loops that starts with the constant loop at z and sweeps the sphere as a rubber band, following the direction of the oriented maximum circle tangent to the vector v. We can assemble all the functions $h_{(z,v)}$ by letting (z, v) vary and we can obtain a function

$$\psi : S^1 \times \dot{T}S^2 \to \mathcal{L}S^2$$

such that $\psi(\phi, (z, v)) = h_{(z,v)}(\phi)$.

The map ψ defines a map of Serre fibrations

$$\begin{tikzcd} S^1 \times S^1 \arrow[r] \arrow[d] & \Omega S^2 \arrow[d] \\ S^1 \times \dot{T}S^2 \arrow[r, "\psi"] \arrow[d] & \mathcal{L}S^2 \arrow[d] \\ S^2 \arrow[r, "="] & S^2 \end{tikzcd} \tag{42}$$

that induces a map in spectral sequences. If $\epsilon \in H_0(S^1) \otimes H_0(S^1)$, $a \in H_1(S^1) \otimes H_0(S^1)$, $b \in H_0(S^1) \otimes H_1(S^1)$, $c \in H_1(S^1) \otimes H_1(S^1)$, are the generators in homology, at the second term of the map of spectral sequences

$$\psi_* : H_p(S^2) \otimes H_q(S^1 \times S^1) \to H_p(S^2) \otimes H_q(\Omega S^2)$$

induces the following identities:

- $\psi_*(\epsilon) = 1_\Omega$,
- $\psi_*(b) = 0$ and
- $\psi_*(a) = x$ because the functions f_θ determine the generator x of $H_1(\Omega S^2)$.

We also know that $d^2(u \otimes a) = 2(\iota \otimes c)$ because $\dot{T}S^2 = SO(3)$ and its fundamental group is $\mathbb{Z}_2$.

Therefore we have the following set of identities:

$$\begin{aligned} 2(\iota \otimes x^2) &= d^2(u \otimes x) \\ &= d^2(\psi_*(u \otimes a)) \\ &= \psi_*(d^2(u \otimes a)) \\ &= \psi_* 2(\iota \otimes c) \end{aligned}$$

and this implies that $\psi_*(\iota \otimes c) = \iota \otimes x^2$. Since $\iota \otimes c$ represents the class $[S^1 \times S^1]$ we can conclude that $f_*([S^1 \times S^1]) = \psi_*(\iota \otimes c) = \iota \otimes x^2 = a\nu$.

♣

10 Virtual Orbifold Cohomology

10.1 Virtual Cohomology as a TQFT+

Now we introduce a new structure which was defined in [LUX07] and further developed in [RU08, GLS$^+$07]. The virtual orbifold cohomology could be understood as the algebraic information which can be obtained from the Orbifold String Topology [LUX08] if we restrict our attention only to constant loops. The virtual orbifold cohomology will provide us with an important example of a nearly G-Frobenius algebra.

This nearly Frobenius algebra generalizes two diferent families of nearly Frobenius algebras. The first one is the *Poincaré algebra* of an oriented smooth manifold M and the second one is the Frobenius algebra of the *Dijkgraaf-Witten model* associated to a finite group G. We can relate these two structures through the diagram

$$\begin{tikzcd} & \mathsf{G} \circlearrowleft \mathsf{M} & \\ \mathsf{M} \arrow[ur] & & \mathsf{G}. \arrow[ul] \end{tikzcd}$$

We will work as before with the global quotient orbifold $[\mathsf{M}/\mathsf{G}]$, where M is a smooth manifold and G is a finite group acting by diffeomorphisms on M.

Denote $\mathsf{M}^{g} := \{x \in \mathsf{M} : xg = x\}$ the set of *fixed* points of $g \in \mathsf{G}$.

Definition 10.1. As graded groups we can define the G-*virtual cohomology*

$$\mathrm{H}^*_{\mathrm{virt}}(\mathsf{M}, \mathsf{G}) := \bigoplus_{g \in \mathsf{G}} \mathrm{H}^*(\mathsf{M}^{g}; \mathbb{C}).$$

The next diagram defines the *virtual product* in $\mathrm{H}^*_{\mathrm{virt}}(\mathsf{M}, \mathsf{G})$ in the following way: take $g, h \in \mathsf{G}$ and $\mathsf{M}^{g,h} := \mathsf{M}^{g} \cap \mathsf{M}^{h}$ with inclusion maps

$$\begin{tikzcd} \mathsf{M}^{g} & \mathsf{M}^{g,h} \arrow[l, "e_g"'] \arrow[dl, "e_h"'] \arrow[dr, "e_{gh}"] & \\ \mathsf{M}^{h} & & \mathsf{M}^{gh} \end{tikzcd}$$

for $\alpha \in \mathrm{H}^*(\mathsf{M}^{g})$ and $\beta \in \mathrm{H}^*(\mathsf{M}^{h})$ define the virtual product by

$$\alpha \star \beta := e_{gh*}\left(e_g^*\alpha \cdot e_h^*\beta \cdot \mathsf{Eu}(\nu(g,h))\right),$$

where $\mathsf{Eu}(\nu(g,h))$ is the Euler class of the excess bundle $\nu(g,h) = \frac{TM|_{M^{g,h}}}{TM^g|_{M^{g,h}} + TM^h|_{M^{g,h}}}$, which is called *the excess intersection class* of the diagram

$$\begin{tikzcd} TM|_{M^{g,h}} & TM^g|_{M^{g,h}} \arrow[l] \\ TM^h|_{M^{g,h}} \arrow[u] & TM^{g,h}. \arrow[l] \arrow[u] \end{tikzcd}$$

In the Grothendieck group of vector bundles over $M^{g,h}$ the class of the Excess intersection bundle becomes

$$\nu(g,h) = TM|_{M^{g,h}} \oplus TM^{g,h} \ominus TM^g|_{M^{g,h}} \ominus TM^h|_{M^{g,h}}.$$

This product becomes graded when we endow it with the degree shift

$$\dim_{virt}(\alpha) = |\alpha| + \mathrm{cod}_{\mathbb{R}}(M^g \subseteq M).$$

We have a natural action of the group G on $\mathrm{H}^*_{\mathrm{virt}}(M;G)$

$$\alpha_g : \mathrm{H}^*(M^h) \to \mathrm{H}^*(M^{ghg^{-1}})$$

where this map is induced by the natural action $M^{ghg^{-1}} \to M^h$, $x \mapsto xg$. Note that $\alpha_g|_{\mathrm{H}^*(M^g)} = \mathrm{id}_{\mathrm{H}^*(M^g)}$.

Now we define the *virtual coproduct* associated to the diagram

$$\begin{tikzcd} M^{gh} & M^{g,h} \arrow[l, "e_{gh}"'] \arrow[r, "e_g"] \arrow[dr, "e_h"] & M^g \\ & & M^h \end{tikzcd}$$

as follows: for $\alpha \in \mathrm{H}^*(M^{gh})$ define the coproduct of α in $H^*(M^g) \otimes H^*(M^h)$ by

$$\Delta^{g,h}_{gh}(\alpha) := \big(e_g \boxtimes e_h\big)_*\big(e^*_{gh}(\alpha) \cdot \mathsf{Eu}\big(\mu(g,h)\big)\big)$$

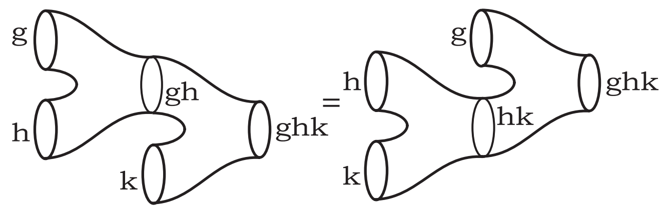
where $e_g \boxtimes e_h$ denotes the map $e_g \boxtimes e_h : M^{g,h} \to M^g \times M^h, x \mapsto (x,x)$, and

$\mu(g,h) = e\left(\frac{TM|_{M^{g,h}}}{TM^{gh}|_{M^{g,h}}} + TM^{g,h}\right)$ is the sum of the normal bundle of the embedding $M^{gh} \to M$ restricted to $M^{g,h}$ together with the tangent bundle of $M^{g,h}$.

Theorem 10.2. *The graded groups* $\mathrm{H}^*_{\mathrm{virt}}(\mathsf{M};\mathsf{G})$ *endowed with virtual product and the virtual coproducs is a nearly* G*-Frobenius algebra. We will call* $\mathrm{H}^*_{\mathrm{virt}}(\mathsf{M};\mathsf{G})$ *the virtual cohomology of* $[\mathsf{M}/\mathsf{G}]$.

Proof. We will make use of Proposition 22.3 to prove the properties.

1. **Associativity of the virtual product** We have to prove that the virtual product satisfies the property determined by the diagram:



(1) (2)

The information associated to the diagram (1) is:

$$\begin{tikzcd}[column sep=small]
 & & \mathsf{M}^{g,h,k} \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & & \\
 & \mathsf{M}^{g,h}\times\mathsf{M}^{k} \arrow[dl, "e_g\times e_h\times 1"'] \arrow[dr, "e_{gh}\times 1"] & & \mathsf{M}^{gh,k} \arrow[dl, "e_{gh}\times e_k"'] \arrow[dr, "e_{ghk}"] & \\
\mathsf{M}^{g}\times\mathsf{M}^{h}\times\mathsf{M}^{k} & & \mathsf{M}^{gh}\times\mathsf{M}^{k} & & \mathsf{M}^{ghk}
\end{tikzcd}$$

while the information associated to the diagram (2) is:

$$\begin{tikzcd}[column sep=small]
 & & \mathsf{M}^{g,h,k} \arrow[dl, "e_1'"'] \arrow[dr, "e_2'"] & & \\
 & \mathsf{M}^{g}\times\mathsf{M}^{h,k} \arrow[dl, "1\times e_h\times e_k"'] \arrow[dr, "1\times e_{hk}"] & & \mathsf{M}^{g,hk} \arrow[dl, "e_g\times e_{hk}"'] \arrow[dr, "e_{ghk}"] & \\
\mathsf{M}^{g}\times\mathsf{M}^{h}\times\mathsf{M}^{k} & & \mathsf{M}^{g}\times\mathsf{M}^{hk} & & \mathsf{M}^{ghk}
\end{tikzcd}$$

In order to prove $(\alpha\star\beta)\star\gamma = \alpha\star(\beta\star\gamma)$ it is enough to show that the Euler classes of the different intersection bundles behave well when restricted to $\mathsf{M}^{g,h,k}$.

From diagram (1) we get the cohomology class

$$e_2^*\big(\mathrm{Eu}(\nu(gh,k))\big)\mathrm{Eu}(F_1)e_1^*(\mathrm{Eu}(\nu(g,h))\times 1)$$

where $\mathrm{Eu}(F_1)=\mathrm{Eu}\left(\frac{TM^{gh}\times M^k|_{M^{g,h,k}}}{TM^{g,h}\times M^k|_{M^{g,h,k}}+TM^{gh,k}|_{M^{g,h,k}}}\right)$, $\mathrm{Eu}\big(\nu(gh,k)\big)=\mathrm{Eu}\left(\frac{TM|_{M^{gh,k}}}{TM^{gh}|_{M^{gh,k}}+TM^k|_{M^{gh,k}}}\right)$, and $\mathrm{Eu}\big(\nu(g,h)\big)=\mathrm{Eu}\left(\frac{TM|_{M^{g,h}}}{TM^g|_{M^{g,h}}+TM^h|_{M^{g,h}}}\right)$.
Noting that $e_1^*\left(\mathrm{Eu}(\nu(g,h))\times 1\right)=\mathrm{Eu}\big(\nu(g,h)|_{M^{g,h,k}}\big)$, then we see that the cohomology class defined above is the Euler class of an element in K-theory of $M^{g,h,k}$ which is

$$\langle 1\rangle+\langle gh,k\rangle-\langle gh\rangle-\langle k\rangle+\langle gh\rangle+\langle k\rangle+\langle g,h,k\rangle-\langle g,h\rangle-\langle k\rangle-\langle gh,k\rangle+\langle 1\rangle+\langle g,h\rangle-\langle g\rangle-\langle h\rangle$$

$$=\langle 2\rangle+\langle g,h,k\rangle-\langle g\rangle-\langle h\rangle-\langle k\rangle.$$

once we have denoted $=\langle k_1,k_2,...\rangle:=TM^{k_1,k_2,\dots}|_{M^{g,h,k}}$.
From diagram (2) we get the cohomology class

$$e_2'^*\big(\mathrm{Eu}(\nu(g,hk))\big)\mathrm{Eu}(F_2)e_1'^*(1\times\mathrm{Eu}(\nu(g,h)))$$

where $\mathrm{Eu}(F_2)=\mathrm{Eu}\left(\frac{TM^g\times M^{hk}|_{M^{g,h,k}}}{TM^g\times M^{h,k}|_{M^{g,h,k}}+TM^{g,hk}|_{M^{g,h,k}}}\right)$, $\mathrm{Eu}(\nu(h,k))=\mathrm{Eu}\left(\frac{TM|_{M^{h,k}}}{TM^h|_{M^{h,k}}+TM^k|_{M^{h,k}}}\right)$, and $\mathrm{Eu}(\nu(g,hk))=\mathrm{Eu}\left(\frac{TM|_{M^{g,hk}}}{TM^g|_{M^{g,hk}}+TM^{hk}|_{M^{g,hk}}}\right)$.
So, we get that this cohomology class is the Euler class of the bundle that in K-theory becomes

$$\langle 1\rangle+\langle g,hk\rangle-\langle g\rangle-\langle hk\rangle+\langle g\rangle+\langle hk\rangle+\langle g,h,k\rangle-\langle g\rangle-\langle h,k\rangle-\langle g,hk\rangle+\langle 1\rangle+\langle h,k\rangle-\langle h\rangle-\langle k\rangle$$

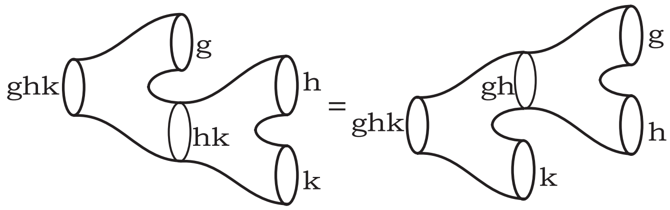
$$=\langle 2\rangle+\langle g,h,k\rangle-\langle g\rangle-\langle h\rangle-\langle k\rangle.$$

Since the elements in K-theory associated to both diagrams (1) and (2) agree, we get the the desired equality:

$$e_2^*(\mathrm{Eu}(\nu(gh,k)))\mathrm{Eu}(F_1)e_1^*(\mathrm{Eu}(\nu(g,h))\times 1)=e_2'^*(\mathrm{Eu}(\nu(g,hk)))\mathrm{Eu}(F_2)e_1'^*(1\times\mathrm{Eu}(\nu(g,h))).$$

2. **Coassociativity of the virtual coproduct**

The outline of the proof will follow the same steps as the one before. An equivalence between two surfaces will determine the property to show, and this property boils down to show that two cohomology classes match. The cohomology classes to compare are highlated by $\bullet$, and the equality of these classes is shown by comparing the elements in K-theory that define these classes.



(1) (2)

(1)

$$\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & & \\
 & M^{g,hk} \arrow[dl, "e_{ghk}"'] \arrow[dr, "e_g \times e_{hk}"] & & M^g \times M^{h,k} \arrow[dl, "1\times e_{hk}"'] \arrow[dr, "1\times e_h \times e_k"] & \\
M^{ghk} & & M^g \times M^{hk} & & M^g \times M^h \times M^k
\end{tikzcd}$$

• $e_1^*(\mathrm{Eu}(\mu(g,hk)))\mathrm{Eu}(F_1)e_2^*(1\times \mathrm{Eu}(\mu(h,k)))$
where $\mathrm{Eu}(F_1) = \mathrm{Eu}\left(\frac{TM^g \times M^{hk}|_{M^{g,h,k}}}{TM^{g,hk}|_{M^{g,h,k}} + TM^g \times M^{h,k}|_{M^{g,h,k}}}\right)$, $\mathrm{Eu}(\mu(g,hk)) = \mathrm{Eu}\left(\frac{TM|_{M^{g,hk}}}{TM^{ghk}|_{M^{g,hk}}} + TM^{g,hk}\right)$,
and $\mathrm{Eu}(\mu(h,k)) = \mathrm{Eu}\left(\frac{TM|_{M^{h,k}}}{TM^{hk}|_{M^{h,k}}} + TM^{h,k}\right)$.
If we realize the calculations in K-theory, then

$$\begin{aligned}\langle 1\rangle + \langle h,k\rangle - \langle hk\rangle + \langle 1\rangle - \langle ghk\rangle + \langle g,hk\rangle + \langle g\rangle + \langle hk\rangle + \langle g,h,k\rangle - \langle g,hk\rangle - \langle g\rangle - \langle h,k\rangle \\ = \langle 2\rangle + \langle g,h,k\rangle - \langle ghk\rangle.\end{aligned}$$

(2)

$$\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl, "e_1'"'] \arrow[dr, "e_2'"] & & \\
 & M^{gh,k} \arrow[dl, "e_{ghk}"'] \arrow[dr, "e_{gh} \times e_k"] & & M^{g,h} \times M^{k} \arrow[dl, "e_{gh}\times 1"'] \arrow[dr, "e_g\times e_h \times 1"] & \\
M^{ghk} & & M^{gh} \times M^{k} & & M^g \times M^h \times M^k
\end{tikzcd}$$

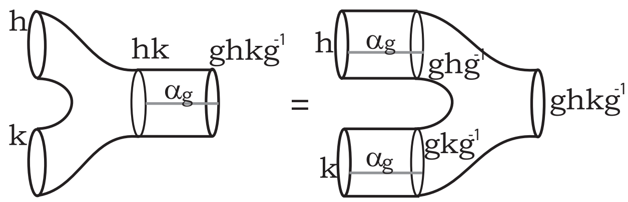
• $e_1'^*(\mathrm{Eu}(\mu(gh,k)))\mathrm{Eu}(F_2)e_2'^*(\mathrm{Eu}(\mu(g,h))\times 1)$
where $\mathrm{Eu}(F_2) = \mathrm{Eu}\left(\frac{TM^{gh} \times M^{k}|_{M^{g,h,k}}}{TM^{g,h}\times M^k|_{M^{g,h,k}} + TM^{gh,k}|_{M^{g,h,k}}}\right)$, $\mathrm{Eu}(\mu(g,h)) = \mathrm{Eu}\left(\frac{TM|_{M^{g,h}}}{TM^{gh}|_{M^{g,h}}} + TM^{g,hk}\right)$,

and $\mathsf{Eu}(\nu(gh,k)) = \mathsf{Eu}\left(\frac{TM|_{M^{gh,k}}}{TM^{ghk}|_{M^{gh,k}}} + TM^{gh,k}\right)$.
In K-theory

$$\langle 1\rangle+\langle g,h\rangle-\langle gh\rangle+\langle 1\rangle+\langle gh,k\rangle-\langle ghk\rangle+\langle gh\rangle+\langle k\rangle+\langle g,h,k\rangle-\langle gh,k\rangle-\langle g,h\rangle-\langle k\rangle$$

$$= \langle 2\rangle + \langle g,h,k\rangle - \langle ghk\rangle.$$

3. **The action is an algebra homomorphism**



(1) (2)

(1)

$$\begin{tikzcd}[column sep=small]
 & & M^{ghg^{-1},gkg^{-1}} \arrow[dl, "\lambda_g"'] \arrow[dr, "i_{ghg^{-1},gkg^{-1}}"] & & \\
 & M^{h,k} \arrow[dl, "e_h \times e_k"'] \arrow[dr, "e_{hk}"] & & M^{ghkg^{-1}} \arrow[dl, "\alpha_g"'] \arrow[dr, "1"] & \\
M^h \times M^k & & M^{hk} & & M^{ghkg^{-1}}
\end{tikzcd}$$

- $\mathsf{Eu}(F_1) = \mathsf{Eu}\left(\frac{TM^{hk}|_{M^{ghg^{-1},gkg^{-1}}}}{TM^{h,k}|_{M^{ghg^{-1},gkg^{-1}}} + TM^{ghkg^{-1}}|_{MM^{ghg^{-1},gkg^{-1}}}}\right)$

and $\mathsf{Eu}(\nu(h,k)) = \mathsf{Eu}\left(\frac{TM|_{M^{h,k}}}{TM^h|_{M^{h,k}} + TM^k|_{M^{h,k}}}\right)$.
Then in K-theory the calculations are

$$\langle 1\rangle + \langle h,k\rangle - \langle h\rangle - \langle k\rangle + \langle hk\rangle + \langle ghg^{-1}, gkg^{-1}\rangle - \langle h,k\rangle - \langle ghkg^{-1}\rangle$$

$$= \langle 1\rangle - \langle h\rangle - \langle k\rangle - \langle h,k\rangle.$$

(2)

$$\begin{tikzcd}
 & & M^{ghg^{-1},gkg^{-1}} \arrow[dl, "e_{ghg^{-1}}\times e_{gkg^{-1}}"'] \arrow[dr, "1"] & & \\
 & M^{ghg^{-1}}\times M^{gkg^{-1}} \arrow[dl, "\alpha_g\times\alpha_g"'] \arrow[dr, "1"] & & M^{ghg^{-1},gkg^{-1}} \arrow[dl, "e_{ghg^{-1}}\times e_{gkg^{-1}}"'] \arrow[dr, "e_{ghkg^{-1}}"] & \\
M^h\times M^k & & M^{ghg^{-1}}\times M^{gkg^{-1}} & & M^{ghkg^{-1}}
\end{tikzcd}$$

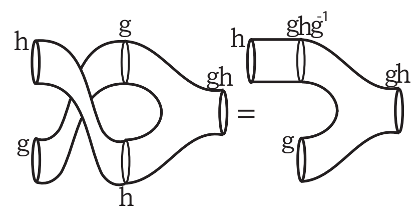
• $\mathrm{Eu}(F_2)=\mathrm{Eu}\left(\dfrac{TM^{ghg^{-1}}\times TM^{gkg^{-1}}|_{M^{ghg^{-1},gkg^{-1}}}}{TM^{ghg^{-1}}\times TM^{gkg^{-1}}|_{M^{ghg^{-1},gkg^{-1}}}+TM^{ghg^{-1},gkg^{-1}}|_{M^{M^{ghg^{-1},gkg^{-1}}}}}\right)$

and $\mathrm{Eu}(\nu(ghg^{-1},gkg^{-1}))=\mathrm{Eu}\left(\dfrac{TM|_{M^{ghg^{-1},gkg^{-1}}}}{TM^{ghg^{-1}}|_{M^{h,k}}+TM^{gkg^{-1}}|_{M^{h,k}}}\right)$

Then in K-theory

$$\begin{aligned}\langle 1\rangle+\langle ghg^{-1},gkg^{-1}\rangle-\langle ghg^{-1}\rangle-\langle gkg^{-1}\rangle+\langle ghg^{-1},gkg^{-1}\rangle+\langle ghg^{-1}\rangle\\
+\langle gkg^{-1}\rangle-\langle ghg^{-1}\rangle-\langle gkg^{-1}\rangle-\langle ghg^{-1},gkg^{-1}\rangle\\
=\langle 1\rangle-\langle h\rangle-\langle k\rangle-\langle h,k\rangle.\end{aligned}$$

4. **Graded commutativity of the product**



(2)

$$\begin{tikzcd}
 & & M^{ghg^{-1},g}=M^{g,h} \arrow[ddll, bend right, "\delta_{h,g}"'] \arrow[ddrr, bend left, "i_{g,h}"] \arrow[dl, "e_{ghg^{-1}}\times e_g"'] \arrow[dr, "1"] & & \\
 & M^{ghg^{-1}}\times M^g \arrow[dl, "\alpha_g\times 1"'] \arrow[dr, "1"] & & M^{ghg^{-1},g} \arrow[dl, "e_{ghg^{-1}}\times e_g"'] \arrow[dr, "e_{gh}"] & \\
M^h\times M^g & & M^{ghg^{-1}}\times M^g & & M^{gh}
\end{tikzcd}$$

• $\mathsf{Eu}(\nu(ghg^{-1},g))\mathsf{Eu}(F_2)1$
where $\mathsf{Eu}(F_2) = \mathsf{Eu}\left(\frac{TM^{ghg^{-1}} \times M^g|_{M^{g,h}}}{TM^{ghg^{-1},g}|_{M^{g,h}} + TM^{ghg^{-1}} \times M^g|_{M^{g,h}}}\right) = \mathsf{Eu}(0) = 1.$
In K-theory

$$\langle 1\rangle + \langle g,h\rangle - \langle ghg^{-1}\rangle - \langle g\rangle.$$

(1)

$$\begin{tikzcd}
 & & M^{ghg^{-1},g} = M^{g,h} \arrow[dl, "e_g \times e_h"'] \arrow[dr, "1"] & & \\
 & M^g \times M^h \arrow[dl, "\tau"'] \arrow[dr, "1"] & & M^{g,h} \arrow[dl, "e_g \times e_h"'] \arrow[dr, "e_{gh}"] & \\
M^h \times M^g & & M^g \times M^h & & M^{gh}
\end{tikzcd}$$

• $\mathsf{Eu}(\nu(g,h))\mathsf{Eu}(F_1)1$
where $\mathsf{Eu}(F_1) = \mathsf{Eu}\left(\frac{TM^g \times M^h|_{M^{g,h}}}{TM^g \times M^h|_{M^{g,h}} + TM^{g,h}|_{M^{g,h}}}\right) = \mathsf{Eu}(0) = 1.$
In K-theory

$$\langle 1\rangle + \langle g,h\rangle - \langle g\rangle - \langle h\rangle.$$

Then $\alpha_g(\beta) \star \alpha = i_{g,h}!\left(\mathsf{Eu}(\nu(g,h))\delta^*_{g,h}(\tau^*(\beta \times \alpha))\right)$ if and only if

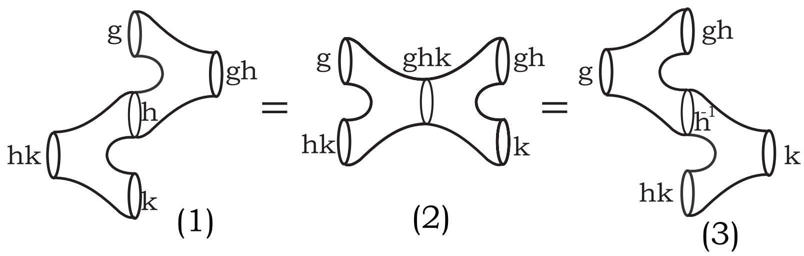
$$\langle h\rangle = \langle ghg^{-1}\rangle.$$

This is true because the bundles $TM^g|_{M^{g,h}}$ and $TM^{ghg^{-1}}|_{M^{g,h}}$ are isomorphic.
Now we need to understand $\tau^*(\beta \times \alpha)$.
Let be $\tau : M^g \times M^h \to M^h \times M^g$ the transposition, and $\pi_1 : M^g \times M^h \to M^g$, $\pi_2 : M^g \times M^h \to M^h$, $\pi'_1 : M^h \times M^g \to M^h$, $\pi'_2 : M^h \times M^g \to M^g$. Hence

$$\begin{aligned}
\tau^*(\beta \times \alpha) &= \tau^*(\pi_1'^*(\beta))\tau^*(\pi_2'^*(\alpha)) = (\pi'_1\tau)^*(\beta)(\pi'_2\tau)^*(\alpha)\\
&= \pi_2^*(\beta)\pi_1^*(\alpha) = (-1)^{|\alpha||\beta|}\pi_1^*(\alpha)\pi_2^*(\beta)\\
&= (-1)^{|\alpha||\beta|}\alpha \times \beta.
\end{aligned}$$

Then $\alpha_g(\beta) \star \alpha = (-1)^{|\alpha||\beta|} i_{g,h}!\left(\mathsf{Eu}(\nu(g,h))\delta^*_{g,h}(\alpha \times \beta)\right) = (-1)^{|\alpha||\beta|}\alpha \star \beta.$

5. **Abrams condition**



Remember that if $\alpha \in H^*(M^{gh})$ then

$$\Delta_{gh}^{g,h}(\alpha) := \left(e_g \boxtimes e_h\right)_* \left(e_{gh}^*(\alpha) \cdot \mathrm{Eu}\big(\mu(g,h)\big)\right)$$

where $\mathrm{Eu}(\mu(g,h)) = \mathrm{Eu}\left(\frac{TM|_{M^{g,h}}}{TM^{gh}|_{M^{g,h}}} + TM^{g,h}\right)$.

(2)

$$\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & & \\
 & M^{g,hk} \arrow[dl, "e_g \times e_{hk}"'] \arrow[dr, "e_{ghk}"] & & M^{gh,k} \arrow[dl, "e_{ghk}"'] \arrow[dr, "e_{gh} \times e_k"] & \\
M^g \times M^{hk} & & M^{ghk} & & M^{gh} \times M^k
\end{tikzcd}$$

• $e_2^*(\mathrm{Eu}(\mu(gh,k)))\mathrm{Eu}(F_1)e_1^*(\mathrm{Eu}(\nu(g,hk)))$,
where $\mathrm{Eu}(F_1) = \mathrm{Eu}\left(\frac{TM^{ghk}|_{M^{g,h,k}}}{TM^{g,hk}|_{M^{g,h,k}} + TM^{gh,k}|_{M^{g,h,k}}}\right)$.
Then $\langle 1\rangle + \langle gh,k\rangle - \langle ghk\rangle + \langle ghk\rangle + \langle g,h,k\rangle - \langle g,hk\rangle - \langle gh,k\rangle + \langle 1\rangle + \langle g,hk\rangle - \langle g\rangle - \langle hk\rangle = \langle 2\rangle + \langle g,h,k\rangle - \langle g\rangle - \langle hk\rangle$.

(1)

$$\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl, "e_1'"'] \arrow[dr, "e_2'"] & & \\
 & M^g \times M^{h,k} \arrow[dl, "1 \times e_{hk}"'] \arrow[dr, "1 \times e_h \times e_k"] & & M^{g,h} \times M^k \arrow[dl, "e_g \times e_h \times 1"'] \arrow[dr, "e_{gh} \times 1"] & \\
M^g \times M^{hk} & & M^g \times M^h \times M^k & & M^{gh} \times M^k
\end{tikzcd}$$

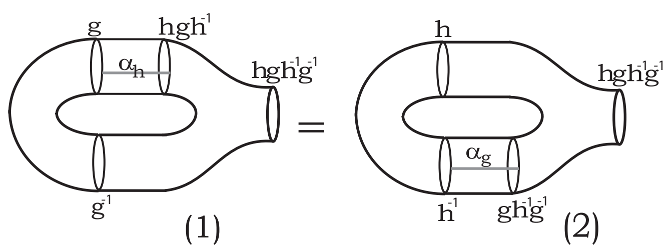
• $e_2'^*(\mathrm{Eu}(\nu(g,h)) \times 1)\mathrm{Eu}(F_2)e_1'^*(1 \times \mathrm{Eu}(\mu(h,k)))$,
where $\mathrm{Eu}(F_2) = \mathrm{Eu}\left(\frac{TM^g \times M^h \times M^k|_{M^{g,h,k}}}{TM^g \times M^{h,k}|_{M^{g,h,k}} + TM^{g,h} \times M^k|_{M^{g,h,k}}}\right)$.
Then $\langle 1\rangle + \langle g,h\rangle - \langle g\rangle - \langle h\rangle + \langle g\rangle + \langle h\rangle + \langle k\rangle + \langle g,h,k\rangle - \langle g\rangle - \langle h,k\rangle - \langle g,h\rangle - \langle k\rangle + \langle 1\rangle + \langle h,k\rangle - \langle hk\rangle = \langle 2\rangle + \langle g,h,k\rangle - \langle hk\rangle - \langle g\rangle$.

(3)

$$\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl, "e_1''"'] \arrow[dr, "e_2''"] & & \\
 & M^{gh,h^{-1}} \times M^{hk} \arrow[dl, "e_g \times 1"'] \arrow[dr, "e_{gh} \times e_{h^{-1}} \times 1"] & & M^{gh} \times M^{h^{-1},hk} \arrow[dl, "1 \times e_{h^{-1}} \times e_{hk}"'] \arrow[dr, "1 \times e_k"] & \\
M^g \times M^{hk} & & M^{gh} \times M^{h^{-1}} \times M^{hk} & & M^{gh} \times M^k
\end{tikzcd}$$

• $e_2''^*(1 \times \mathrm{Eu}(\nu(h^{-1},hk)))\mathrm{Eu}(F_3)e_1''^*(\mathrm{Eu}(\mu(gh,h^{-1})) \times 1)$,
where $\mathrm{Eu}(F_3) = \mathrm{Eu}\left(\frac{TM^{gh} \times M^{h^{-1}} \times M^{hk}|_{M^{g,h,k}}}{TM^{gh,h^{-1}} \times M^{hk}|_{M^{g,h,k}} + TM^{gh} \times M^{h^{-1},hk}|_{M^{g,h,k}}}\right)$.
Then $\langle 1\rangle + \langle h^{-1},hk\rangle - \langle h^{-1}\rangle - \langle hk\rangle + \langle gh\rangle + \langle h^{-1}\rangle + \langle hk\rangle + \langle g,h,k\rangle - \langle gh,h^{-1}\rangle - \langle hk\rangle - \langle gh\rangle - \langle h^{-1},hk\rangle + \langle 1\rangle + \langle gh,h^{-1}\rangle - \langle g\rangle = \langle 2\rangle + \langle g,h,k\rangle - \langle hk\rangle - \langle g\rangle$.
If we compare the three cases we have that the Abrams condition is satisfied.

6. **Torus axiom**



(1)

$$\begin{tikzcd}
 & & M^{g,h} \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & & \\
 & M^{g,g^{-1}} \arrow[dl, "e_{gg^{-1}}"'] \arrow[dr, "e_g \times e_{g^{-1}}"'] & & M^{hgh^{-1},g^{-1}} \arrow[dl, "(\alpha_h \times 1)\circ(e_{hgh^{-1}} \times e_{g^{-1}})"'] \arrow[dr, "e_{hgh^{-1}g^{-1}}"] & \\
M & & M^g \times M^{g^{-1}} & & M^{hgh^{-1}g^{-1}}
\end{tikzcd}$$

• $e_2^*(\mathrm{Eu}(\nu(hgh^{-1}, g^{-1}))\mathrm{Eu}(F_1)e_1^*(\mathrm{Eu}(\mu(g, g^{-1})))$,
where $\mathrm{Eu}(F_1) = \mathrm{Eu}\left(\frac{TM^g \times M^{g^{-1}}|_{M^{g,h}}}{TM^{g,g^{-1}}|_{M^{g,h}} + TM^{hgh^{-1},g^{-1}}|_{M^{g,h}}}\right)$ and $\mathrm{Eu}(\mu(g, g^{-1})) = \mathrm{Eu}\left(\frac{TM|_{M^{g,g^{-1}}}}{TM|_{M^{g,g^{-1}}}} + TM^{g,g^{-1}}\right) = \mathrm{Eu}(TM^g)$. Then

$$\langle 1\rangle + \langle hgh^{-1}, g^{-1}\rangle - \langle hgh^{-1}\rangle - \langle g^{-1}\rangle + \langle g\rangle + \langle g^{-1}\rangle + \langle g, h\rangle - \langle g, g^{-1}\rangle - \langle hgh^{-1}, g^{-1}\rangle + \langle g, g^{-1}\rangle$$
$$= \langle 1\rangle + \langle g, h\rangle - \langle hgh^{-1}\rangle + \langle g\rangle.$$

(2)

$$\begin{tikzcd}
 & & M^{g,h} \arrow[dl, "e_1'"'] \arrow[dr, "e_2'"] & & \\
 & M^{h,h^{-1}} \arrow[dl, "e_{hh^{-1}}"'] \arrow[dr, "e_h \times e_{h^{-1}}"'] & & M^{h,gh^{-1}g^{-1}} \arrow[dl, "e_h \times e_{gh^{-1}g^{-1}}"'] \arrow[dr, "e_{hgh^{-1}g^{-1}}"] & \\
M & & M^h \times M^{h^{-1}} & & M^{hgh^{-1}g^{-1}}
\end{tikzcd}$$

• $e_2'^*(\mathrm{Eu}(\nu(h, gh^{-1}g^{-1}))\mathrm{Eu}(F_2)e_1'^*(\mathrm{Eu}(\mu(h, h^{-1})))$,
where $\mathrm{Eu}(F_2) = \mathrm{Eu}\left(\frac{TM^h \times M^{h^{-1}}|_{M^{g,h}}}{TM^{h,h^{-1}}|_{M^{g,h}} + TM^{h,gh^{-1}g^{-1}}|_{M^{g,h}}}\right)$ and $\mathrm{Eu}(\mu(h, h^{-1})) = \mathrm{Eu}(TM^h)$. Then

$$\langle 1\rangle + \langle h, gh^{-1}g^{-1}\rangle - \langle h\rangle - \langle gh^{-1}g^{-1}\rangle + \langle h\rangle + \langle h^{-1}\rangle + \langle g, h\rangle - \langle h, h^{-1}\rangle - \langle h, gh^{-1}g^{-1}\rangle + \langle h, h^{-1}\rangle$$
$$= \langle 1\rangle + \langle g, h\rangle - \langle gh^{-1}g^{-1}\rangle + \langle h\rangle.$$

Using that $\langle g\rangle = \langle hgh^{-1}\rangle$ we finish the proof.

♣

Definition 10.3. We define the *orbifold virtual cohomology* as the G-invariant part of $H^*_{\text{virt}}(M; G)$. It is denoted by $H^*_{\text{virt}}([M/G]) = H^*_{\text{virt}}(M; G)^G$.

Corollary 10.4. *The orbifold virtual cohomology* $H^*_{\text{virt}}([M/G])$ *is a nearly Frobenius algebra.*

Let us explicitely calculate an example:

Example 10.1. *Consider the symmetric product of two copies of* $\mathbb{C}P^m$*, that is, consider* $M = \mathbb{C}P^m \times \mathbb{C}P^m$ *and* $G = \mathfrak{S}_2$ *acting on* M *permuting the coordinates. As an algebra*

$$H^*_{\text{virt}}(\mathbb{C}P^m \times \mathbb{C}P^m, \mathfrak{S}_2; \mathbb{Z}) \cong \mathbb{Z}[x, y, u]/\langle x^{m+1}, y^{m+1}, u^2 - (m+1)x^m y^m, u(x - y)\rangle$$

where x *and* y *are the generators of* $H^*(\mathbb{C}P^m \times \mathbb{C}P^m; \mathbb{Z})$ *labeled with* $1 \in \mathfrak{S}_2$*, and* u *is a generator of* $H^0((\mathbb{C}P^m \times \mathbb{C}P^m)^\tau; \mathbb{Z})$ *with label the non trivial transposition* $\tau \in \mathfrak{S}_2$*. The coalgebra structure is determined by the coproduct of the unit*

$$\Delta_{1,1}(1) = \sum_{j=1}^{m} x^j \otimes y^{m-j}, \quad \Delta_{\tau,\tau}(1) = (m+1)[(x^m u) \otimes (x^m u)].$$

If $m = 1$ *we have that*

$$H^*_{\text{virt}}(\mathbb{C}P^1 \times \mathbb{C}P^1, \mathfrak{S}_2; \mathbb{Z}) \cong \mathbb{Z}[x, y, u]/\langle x^2, y^2, u^2 - 2xy, u(x - y)\rangle$$

and therefore the $\mathfrak{S}_2$*-invariant subalgebra becomes*

$$H^*_{\text{virt}}([(\mathbb{C}P^1)^2/\mathfrak{S}_2]; \mathbb{R}) \cong \mathbb{R}[w, u]/\langle w^3, u^3, u^2 - 4w^2\rangle$$

where $2w = x + y$ *and the coalgebra structure is determined by*

$$\Delta(1) = 1 \otimes w + w \otimes 1 + 2wu \otimes wu.$$

Further examples can be seen in [RU08].

10.2 Open-closed Virtual Cohomology

Similarly as in the case of orbifold string topology, where we saw that it has the structure of a G-topological field theory with positive boundary, we will extend the virtual theory to an open-closed theory. The open part is the following: Let be

$\mathcal{B} = \{X \subset M\colon \text{G-invariant}\}$ such that, if $X, Y \in \mathcal{B}$ then $TX|_{(X\cap Y)^g} \cong TY|_{(X\cap Y)^g}$ for all $g \in G$. We define $\mathrm{Hom}_{\mathcal{B}}(X, Y) = H^*(X \cap Y)$, for $X, Y \in \mathcal{B}$.

Now we consider the diagram

$$\begin{tikzcd}
& X \cap Y \cap Z \arrow[dl, "(i^Z_{XY} \times i^X_{YZ})\circ\Delta"'] \arrow[dr, "i^Y_{XZ}"] & \\
(X\cap Y) \times (Y \cap Z) & & X \cap Z
\end{tikzcd}$$

where $i^Z_{XY} : X \cap Y \cap Z \hookrightarrow X \cap Y$ is the inclusion map.

We define the product $\eta^Y_{XZ} : H^*(X \cap Y) \otimes H^*(Y \cap Z) \to H^*(X \cap Z)$ by

$$\eta^Y_{XZ}(\alpha \otimes \beta) = i^Y_{XZ*}\left(E_{XYZ}((i^Z_{XY} \times i^X_{YZ}) \circ \Delta)^*(\alpha \otimes \beta)\right)$$

with

$$E_{XYZ} = e\left(\frac{TY|_{X\cap Y\cap Z}}{T(X\cap Y)|_{X\cap Y\cap Z} + T(Y\cap Z)|_{X\cap Y\cap Z}}\right).$$

In a similar way, we define the coproduct $\Delta^Y_{XZ} : H^*(X\cap Z) \to H^*(X\cap Y)\otimes H^*(Y\cap Z)$ by

$$\Delta^Y_{XZ}(\gamma) := \left((i^Z_{XY} \times i^X_{YZ}) \circ \Delta\right)_* \left(E(X, Y, Z) i^{Y*}_{XZ}(\gamma)\right)$$

where

$$E(X, Y, Z) = e\left(\frac{TM|_{X\cap Y\cap Z}}{TY|_{X\cap Y\cap Z} + T(X\cap Z)|_{X\cap Y\cap Z}}\right).$$

The next step consists in defining the connection maps. For this we consider the next diagram

$$\begin{tikzcd}
& X^g \arrow[dl, "j_g"'] \arrow[dr, "i_g"] & \\
X & & M^g
\end{tikzcd}$$

Then we define $\iota_{g,X} : H^*(M^g) \to H^*(X)$ as follows

$$\iota_{g,X}(\alpha) := j_{g*}\left(e(E_g) i^*_g(\alpha)\right)$$

where $E_g = \frac{TM|_{X^g}}{TX|_{X^g} + TM^g|_{X^g}}$. In the same way, the map $\iota^{g,X} : H^*(X) \to H^*(M^g)$ is defined by

$$\iota^{g,X}(\beta) := i_{g*}\left(e(F_g) j^*_g(\beta)\right)$$

with $F_g = TX^g$.

Theorem 10.5. *The virtual cohomology together with the category $\mathcal{B}$ as the D-branes is a* G*-OC-TFT with positive boundary.*

Proof. The proof follows the same lines as the one of the open part in the String Topology case. Simply note that the the closed and open theories associated to the Virtual product are obtained by looking at the theory that the String Topology G-OC-TFT with positive boundary induce on constant paths. We leave the details to the interested reader.

♣

Proposition 10.6. *There exist a natural open-closed TFT morphism between the open-closed Virtual orbifold cohomology and the open closed orbifold string topology induced by the inclusion of constant paths on all paths.*

Proof. We see the correspondence between the products. For this we consider the commutative diagram

$$\begin{tikzcd}
M^g \times M^h \arrow[d, "i"'] & M^{g,h} \arrow[l, "e_g \times e_h"'] \arrow[r, "e_{gh}"] \arrow[d, "i"] & M^{gh} \arrow[d, "(g,id)"'] \\
M \times M & M \arrow[l, "\Delta"'] \arrow[r, "\Delta"'] & M \times M \\
\mathcal{P}_g M \times \mathcal{P}_h M \arrow[u, "\epsilon_1 \times \epsilon_0"] & \mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \arrow[l, "j"'] \arrow[r, "\circledast"] \arrow[u, "\epsilon_\infty"] & \mathcal{P}_{gh} M \arrow[u, "\epsilon_0 g \times \epsilon_{\frac{1}{2}}"']
\end{tikzcd}$$

We consider the maps in homology:

$$
\begin{aligned}
&\left(\epsilon_0 g \times \epsilon_{\frac{1}{2}}\right)^*(g, id)_* e_{gh*}\left((e_g \times e_h)^*(\alpha \otimes \beta) \cap e(\nu(g,h))\right)\\
&= \left(\epsilon_0 g \times \epsilon_{\frac{1}{2}}\right)^* \Delta_* i_*\left((e_g \times e_h)^*(\alpha \otimes \beta) \cap e(\nu(g,h))\right)\\
&= \left(\epsilon_0 g \times \epsilon_{\frac{1}{2}}\right)^* \Delta_*\left(\Delta^* i_*(\alpha \times \beta)\right)\\
&= \circledast_*\left(e_\infty^*\left(\Delta^* i_*(\alpha \times \beta)\right) \cap e(F)\right)\\
&= \circledast_*\left(\left(\Delta e_\infty\right)^* i_*(\alpha \times \beta) \cap e(F)\right)\\
&= \circledast_*\left(\left((\epsilon_1 \times \epsilon_0) j\right)^* i_*(\alpha \times \beta) \cap e(F)\right)\\
&= \circledast_*\left(j^*\left((\epsilon_1 \times \epsilon_0)^* i_*\right)(\alpha \times \beta) \cap e(F)\right).
\end{aligned}
$$

To conclude the proof it is easy to observe that $e(F) = 0$, where F is the excess bundle of

$$
\begin{tikzcd}
\mathcal{P}_g M_{\epsilon_1} \times_{\epsilon_0} \mathcal{P}_h M \arrow[r, "\epsilon_\infty"] \arrow[d, "\circledast"] & \mathcal{P}_{gh} M \arrow[d, "\epsilon_0 g \times \epsilon_{\frac{1}{2}}"] \\
M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}
$$

♣

Notice that that fixing the closed string sector (commutative Frobenius algebra) of the theory the resulting extension into an open-closed theory is not rigid:

Proposition 10.7. *Let* $\left(\mathrm{H}^*_{\mathrm{virt}}(M; G), \mathcal{B}\right)$ *be the open closed virtual cohomology of* $[M/G]$. *If we change the correction clases of the open virtual coproduct and the closed map by*

$$
E_\varepsilon(X, Y, Z) = \left(TM + T(X \cap Y \cap Z) - T(X \cap Z) + \varepsilon TY\right)|_{X \cap Y \cap Z}
$$

and

$$
F_g^\varepsilon = \left(TX^g + \varepsilon TX\right)|_{X^g}
$$

we have a one parameter family of open closed TFT with positive boundary, where $\varepsilon \in \mathbb{C}$.

11 Chen-Ruan Cohomology and K-theory

11.1 Chern-Ruan K-theory

Stringy K-theory for compact complex orbifolds was introduced independently in [JKK07] and [ARZ07] as the K-theoretic analogue of Chen-Ruan Cohomology; both, the Chen-Ruan cohomology and the Stringy K-theory provide examples of Frobenius algebras. In this section we put forward an extension of these two Frobenius algebras to the context of G-Frobenius algebras.

Let M be a smooth possibly non-compact manifold, or in the algebraic case a quasi-projective variety, endowed with a holomorphic action of a finite group G. For each $g \in G$ we denote as in the previous sections the fixed locus of g in M by M^g, and we let

$$I_G(M) := \coprod_{g \in G} M^g \times g \subset M \times G$$

where $[I_G(M)/G]$ denotes the *inertia orbifold* of the global quotient $[M/G]$ where the action is given by

$$\begin{aligned} I_G(M) \times G &\to G \\ ((x, g), h) &\mapsto (xh, h^{-1}gh). \end{aligned}$$

The space $I_G(M)$ has a canonical G-equivariant involution $\sigma : I_G(M) \to I_G(M)$ which maps M^g to $M^{g^{-1}}$ via

$$\sigma : (x, g) \longmapsto \left(x, g^{-1}\right).$$

Definition 11.1. We define the *Chen-Ruan* K*-theory* $\mathcal{K}(M, G)$ of M, as a G-graded G-complex vector space, to be the complexified K-theory of the inertia orbifold, i.e.

$$\mathcal{K}(M, G) = \bigoplus_{g \in G} \mathcal{K}_g(M) = \bigoplus_{g \in G} K\left(M^g\right),$$

where $K\left(M^g\right) = K^*(M^g) \otimes_{\mathbb{Z}} \mathbb{C}$.

Note that the G-action on $\mathcal{K}(M, G)$ is induced by the map

$$\begin{array}{cccc} \alpha_g : & M^{ghg^{-1}} & \longrightarrow & M^h \\ & x & \longmapsto & xg \end{array}$$

that is, for $\mathcal{F}_h \in K(M^h)$

$$\begin{array}{ccc} G \times \mathcal{K}(M, G) & \longrightarrow & \mathcal{K}(M, G) \\ (g, \mathcal{F}_h) & \longmapsto & \alpha_g^*(\mathcal{F}_h) \end{array}$$

where $\alpha_g^* : K(M^h) \longrightarrow K(M^{ghg^{-1}})$.

The product structure for the Stringy K-theory is defined via the pull-push formalism as it was done for the virtual cohomology. The obstruction bundle that appears in the formula of the product is a version of an "equivariant holomorphic excess intersection bundle" which was constructed in [CR04a]; Chen and Ruan noted that the cohomology of the Inertia orbifold could be endowed with a product structure if one restricts the Quantum cohomology product on the orbifold to the information provided by constant maps from orbifold Riemann spheres.

In [JKK07] a simple procedure to construct the obstruction bundle was developed. We will follow this setup.

Definition 11.2. Define $\mathcal{S}$ in $K(M, G)$ to be such that for any $g \in G$, its restriction $\mathcal{S}_g$ in $K(M^g)$ is given by

$$\mathcal{S}_g := \mathcal{S}|_{M^g} := \bigoplus_{k=0}^{r-1} \frac{k}{r} W_{g,k},$$

where r is the order of g, and $W_{g,k}$ is the eigenbundle of $W_g := TM|_{M^g}$ where g acts with eigenvalue $\zeta^k = \exp(2\pi k i/r)$.

Remark 11.3 ([JKK07]). The G-equivariant involution $\sigma : M^g \to M^{g^{-1}}$ yields a G-equivariant isomorphism $\sigma^* : W_{g^{-1}} \to W_g$ for all $g \in G$. If g acts by multiplication by ζ^k, then g^{-1} acts by ζ^{r-k}, so we have

$$\sigma^* W_{g^{-1},0} = W_{g,0}$$

and

$$\sigma^* W_{g^{-1},k} = W_{g,r-k}$$

for all $k \in \{1, \ldots, r-1\}$. Consequently, the induced map $\sigma^* : K(X^{g^{-1}}) \to K(X^g)$ satisfies

$$\mathcal{S}_g \oplus \sigma^* \mathcal{S}_{g^{-1}} = N_g, \tag{43}$$

since the normal bundle, N_g, of M^g in M satisfies the equation $N_g = W_g \ominus W_{g,0}$.

For any two elements $g, h \in G$ we let $M^{g,h} = M^g \cap M^h$.

Definition 11.4. Define the element $\mathcal{R}(g,h)$ in $K^0\big(M^{g,h}\big)$ by

$$\begin{aligned}\mathcal{R}(g,h) &= \Big(TM^{g,h} \ominus TM \oplus \mathcal{S}_g \oplus \mathcal{S}_h \oplus \mathcal{S}_{(gh)^{-1}}\Big)|_{M^{g,h}} \\ &= \big(TM^{g,h} \ominus TM^{gh} \oplus \mathcal{S}_g \oplus \mathcal{S}_h \ominus \mathcal{S}_{(gh)}\big)|_{M^{g,h}}\end{aligned}$$

Note that if we define the bundles in rational K-theory

$$\overline{\mathcal{S}}_g := \overline{\mathcal{S}}|_{M^g} := \bigoplus_{k=1}^{r-1} \frac{r-k}{r} W_{g,k},$$

we have that $\mathcal{S}_g \oplus \overline{\mathcal{S}}_g = N_g$ and moreover we could define the bundle

$$\overline{\mathcal{R}}(g,h) := \Big(TM^{g,h} \ominus TM|_{M^{g,h}} \oplus \overline{\mathcal{S}}_g \oplus \overline{\mathcal{S}}_h \oplus \overline{\mathcal{S}}_{(gh)^{-1}}\Big)|_{M^{g,h}}$$

Let us see that

Lemma 11.5. *There is an isomorphism of bundles*

$$N_{g,h} = \overline{\mathcal{R}}(g,h) \oplus \mathcal{R}(g,h) \oplus N^g_{g,h} \oplus N^h_{g,h} \oplus N^{(gh)^{-1}}_{g,h}$$

where $N^{g_i}_{g_1,g_2}$ *denotes the normal bundle of the embbeding* $TM^{g_1,g_2} \to TM^{g_i}$ *and* N_{g_1,g_2} *denotes the normal bundle of the embbeding* $TM^{g_1,g_2} \to TM$.

Proof. Let us check the formula for the case on which g and h commute, the general case is similar. Denote $g_1 = g$, $g_2 = h$ and $g_3 = (gh)^{-1}$. Since we can simmultaneously daigonalize the action let us assume that we can split the bundle N_{g_1,g_2} into line bundles $N_{g_1,g_2} = \bigoplus_l W_l$ where g_j acts on W_l with eigenvalue $e^{2\pi i r_l(g_j)}$ with $0 \le r_l(g_j) < 1$.

Divide the line bundles W_l into three groups

$$\begin{aligned}\mathcal{O}_2^{g,h} &:= \bigoplus_{\{l \mid \sum_{j=1}^3 r_l(g_j) = 2\}} W_l \\ \mathcal{O}_1^{g,h} &:= \bigoplus_{\{l \mid \sum_{j=1}^3 r_l(g_j) = 1,\ \forall j, r_l(g_j) \neq 0\}} W_l \\ \mathcal{O}_0^{g,h} &:= \bigoplus_{\{l \mid \sum_{j=1}^3 r_l(g_j) = 1,\ \exists j, r_l(g_j) = 0\}} W_l\end{aligned}$$

we have $N_{g,h} = \mathcal{O}_2^{g,h} \oplus \mathcal{O}_1^{g,h} \oplus \mathcal{O}_0^{g,h}$. Moreover, it is easy to see that

$$\begin{aligned}
\mathcal{O}_2^{g,h} &= \mathcal{R}(g,h) \\
\mathcal{O}_1^{g,h} &= \overline{\mathcal{R}}(g,h) \\
\mathcal{O}_0^{g,h} &= N_{g,h}^{g} \oplus N_{g,h}^{h} \oplus N_{g,h}^{(gh)^{-1}}.
\end{aligned}$$

♣

We use $\mathcal{R}(g,h)$ to define the product in $\mathcal{K}(M,G)$ as follows. Denote by

$$e_g : M^{g,h} \to M^g, \quad e_h : M^{g,h} \to M^h, \quad e_{gh} : M^{g,h} \to M^{gh}$$

the canonical inclusions.

Definition 11.6. Given $g,h \in G$ and elements $\mathcal{F}_g \in \mathcal{K}(M^g)$ and $\mathcal{F}_h \in \mathcal{K}(M^h)$, we define the *string product* of $\mathcal{F}_g$ and $\mathcal{F}_h$ in $\mathcal{K}(M^{gh}) \subset \mathcal{K}(M,G)$ to be

$$\mathcal{F}_g * \mathcal{F}_h := (e_{gh})_*\big(e_g^*\mathcal{F}_g \otimes e_h^*\mathcal{F}_h \otimes \lambda_{-1}\big(\mathcal{R}(g,h)\big)\big)$$

and the product is extended linearly to every element in $\mathcal{K}(M,G)$.

Note that $\mathcal{R}(g,h)$ is a bona-fide complex bundle over $M^{g,h}$ and therefore its Euler class $\lambda_{-1}\big(\mathcal{R}(g,h)\big)$ is K-theory is well defined.

Definition 11.7. Define the element $\mathcal{R}'(g,h)$ in $K^0\big(M^{g,h}\big)$ by

$$\mathcal{R}'(g,h) := \Big(TM \oplus TM^{g,h} \ominus TM^g \ominus TM^h \ominus \mathcal{S}_g \ominus \mathcal{S}_h \oplus \mathcal{S}_{gh}\Big)|_{M^{g,h}}.$$

Note also that $\mathcal{R}'(g,h)$ is a bundle over $M^{g,h}$ and we will use it to define the coproduct in $\mathcal{K}(M,G)$ as follows.

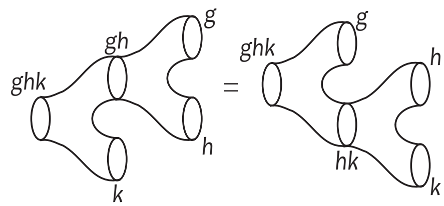
Definition 11.8. Given $g,h \in G$ and an element $\mathcal{F}_{gh} \in \mathcal{K}(M^{gh})$, we define the *string coproduct* of $\mathcal{F}_{gh}$ in $\mathcal{K}(M^g) \otimes \mathcal{K}(M^h)$ to be

$$\Delta_{gh}^{g,h}(\mathcal{F}_{gh}) = \big(e_g \boxtimes e_h\big)_*\big(e_{gh}^*(\mathcal{F}_{gh}) \otimes \lambda_{-1}\big(\mathcal{R}'(g,h)\big)\big)$$

where $e_g \boxtimes e_h$ denotes the map $e_g \boxtimes e_h : M^{g,h} \to M^g \times M^h, x \mapsto (x,x)$. If $gh = k$ then the total coproduct of $\mathcal{F}_k$ is $\Delta_k(\mathcal{F}_k) = \sum_{gh=k} \Delta_k^{g,h}(\mathcal{F}_k)$.

Theorem 11.9. $\mathcal{K}(M,G)$ *is a nearly* G*-Frobenius algebra.*

Proof. **Coassociativity**



(1) (2)

(1)

$$
\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[ddll, bend right, "\varphi"'] \arrow[ddrr, bend left, "\psi"] & & \\
 & M^{gh,k} \arrow[dl, "e_{ghk}"'] \arrow[dr, "e_{gh} \times e_k"] & & M^{g,h} \times M^k \arrow[dl, "e_{gh} \times 1"'] \arrow[dr, "e_g \times e_h \times 1"] & \\
M^{ghk} & & M^{gh} \times M^k & & M^g \times M^h \times M^k
\end{tikzcd}
$$

Let $E(g,h,k)$ be the excess intersection bundle of the square in the diagram (1), that is

$$
\begin{aligned}
E(g,h,k) &= TM^{gh}|_{M^{g,h,k}} \oplus TM^{k}|_{M^{g,h,k}} \oplus TM^{g,h,k} \ominus TM^{gh,k}|_{M^{g,h,k}} \ominus TM^{g,h}|_{M^{g,h,k}} \ominus TM^{k}|_{M^{g,h,k}} \\
&= TM^{gh}|_{M^{g,h,k}} \oplus TM^{g,h,k} \ominus TM^{gh,k}|_{M^{g,h,k}} \ominus TM^{g,h}|_{M^{g,h,k}}.
\end{aligned}
$$

$$\mathcal{R}'(g,h) = TM^{g,h}|_{M^{g,h,k}} \ominus TM^{g}|_{M^{g,h,k}} \ominus TM^{gh}|_{M^{g,h,k}} \ominus \mathcal{S}_g|_{M^{g,h,k}} \oplus \mathcal{S}_{h^{-1}}|_{M^{g,h,k}} \ominus \mathcal{S}_{(gh)^{-1}}|_{M^{g,h,k}}.$$

$$\mathcal{R}'(gh,k) = TM^{gh,k}|_{M^{g,h,k}} \ominus TM^{gh}|_{M^{g,h,k}} \ominus TM^{ghk}|_{M^{g,h,k}} \ominus \mathcal{S}_{gh}|_{M^{g,h,k}} \oplus \mathcal{S}_{k^{-1}}|_{M^{g,h,k}} \ominus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}.$$

The addition of these terms is

$$TM^{g,h,k} \ominus TM^{g}|_{M^{g,h,k}} \ominus TM|_{M^{g,h,k}} \ominus TM^{ghk}|_{M^{g,h,k}} \ominus \mathcal{S}_g|_{M^{g,h,k}} \oplus \mathcal{S}_{h^{-1}}|_{M^{g,h,k}} \oplus \mathcal{S}_{k^{-1}}|_{M^{g,h,k}} \ominus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}$$

(2)

$$\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl, "f'"'] \arrow[dr, "g'"] \arrow[ddll, bend right, "\varphi"'] \arrow[ddrr, bend left, "\psi"] & & \\
 & M^{g,hk} \arrow[dl, "e_{ghk}"'] \arrow[dr, "e_g \times e_{hk}"] & & M^g \times M^{h,k} \arrow[dl, "1 \times e_{hk}"'] \arrow[dr, "1 \times e_h \times e_k"] & \\
M^{ghk} & & M^g \times M^{hk} & & M^g \times M^h \times M^k
\end{tikzcd}$$

$$\begin{aligned}
E'(g,h,k) =& TM^g|_{M^{g,h,k}} \oplus TM^{hk}|_{M^{g,h,k}} \oplus TM^{g,h,k} \ominus TM^{g,hk}|_{M^{g,h,k}} \ominus TM^g|_{M^{g,h,k}} \ominus TM^{h,k}|_{M^{g,h,k}} \\
=& TM^{hk}|_{M^{g,h,k}} \oplus TM^{g,h,k} \ominus TM^{g,hk}|_{M^{g,h,k}} \ominus TM^{h,k}|_{M^{g,h,k}}.
\end{aligned}$$

$$\mathcal{R}'(h,k) = TM^{h,k}|_{M^{g,h,k}} \ominus TM^h|_{M^{g,h,k}} \ominus TM^{hk}|_{M^{g,h,k}} \ominus \mathcal{S}_h|_{M^{g,h,k}} \oplus \mathcal{S}_{k^{-1}}|_{M^{g,h,k}} \ominus \mathcal{S}_{(hk)^{-1}}|_{M^{g,h,k}}.$$

$$\mathcal{R}'(g,hk) = TM^{g,hk}|_{M^{g,h,k}} \ominus TM^g|_{M^{g,h,k}} \ominus TM^{ghk}|_{M^{g,h,k}} \ominus \mathcal{S}_g|_{M^{g,h,k}} \oplus \mathcal{S}_{(hk)^{-1}}|_{M^{g,h,k}} \ominus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}.$$

Then, the addition is

$$TM^{g,h,k} \ominus TM^h|_{M^{g,h,k}} \ominus TM^g|_{M^{g,h,k}} \ominus TM^{ghk}|_{M^{g,h,k}} \ominus \mathcal{S}_h|_{M^{g,h,k}} \oplus \mathcal{S}_{k^{-1}}|_{M^{g,h,k}} \ominus \mathcal{S}_g|_{M^{g,h,k}} \ominus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}$$

If we compare the two expression we only need to check that

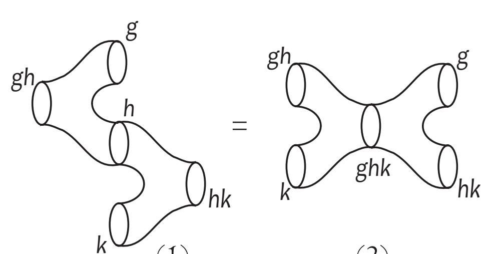
$$\mathcal{S}_{h^{-1}}|_{M^{g,h,k}} \ominus TM|_{M^{g,h,k}} = \ominus TM^h|_{M^{g,h,k}} \ominus \mathcal{S}_h|_{M^{g,h,k}}$$

or equivalently

$$\mathcal{S}_{h^{-1}}|_{M^{g,h,k}} \oplus \mathcal{S}_h|_{M^{g,h,k}} \oplus TM^h|_{M^{g,h,k}} = TM|_{M^{g,h,k}}$$

and this is a consequence of $\mathcal{S}_{h^{-1}}|_{M^{g,h,k}} \oplus \mathcal{S}_h|_{M^{g,h,k}} = N_h$, the normal bundle of M^h in M.

Abrams condition



(1) (2)

(1)

$$\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl] \arrow[dr] & & \\
 & M^{g,h}\times M^{k} \arrow[dl] \arrow[dr] & & M^{g}\times M^{h,k} \arrow[dl] \arrow[dr] & \\
M^{gh}\times M^{k} & & M^{g}\times M^{h}\times M^{k} & & M^{g}\times M^{hk}
\end{tikzcd}$$

$$\begin{aligned}E(g,h,k)=&\big(TM^{g}\oplus TM^{h}\oplus TM^{k}\oplus TM^{g,h,k}\ominus TM^{g,h}\ominus TM^{k}\ominus TM^{g}\ominus TM^{h,k}\big)|_{M^{g,h,k}}\\ =&TM^{h}|_{M^{g,h,k}}\oplus TM^{g,h,k}\ominus TM^{g,h}|_{M^{g,h,k}}\ominus TM^{h,k}|_{M^{g,h,k}}.\end{aligned}$$

$\mathcal{R}'(g,h)=TM^{g,h}|_{M^{g,h,k}}\ominus TM^{g}|_{M^{g,h,k}}\ominus TM^{gh}|_{M^{g,h,k}}\ominus \mathcal{S}_{g}|_{M^{g,h,k}}\oplus \mathcal{S}_{h^{-1}}|_{M^{g,h,k}}\ominus \mathcal{S}_{(gh)^{-1}}|_{M^{g,h,k}}.$

$\mathcal{R}(h,k)=TM^{h,k}|_{M^{g,h,k}}\ominus TM|_{M^{g,h,k}}\oplus \mathcal{S}_{h}|_{M^{g,h,k}}\oplus \mathcal{S}_{k}|_{M^{g,h,k}}\oplus \mathcal{S}_{(hk)^{-1}}|_{M^{g,h,k}}.$

Then the sum of these three formal bundles is

$TM^{g,h,k}\ominus TM^{g}|_{M^{g,h,k}}\ominus TM^{gh}|_{M^{g,h,k}}\ominus \mathcal{S}_{g}|_{M^{g,h,k}}\ominus \mathcal{S}_{(gh)^{-1}}|_{M^{g,h,k}}\oplus \mathcal{S}_{k}|_{M^{g,h,k}}\oplus \mathcal{S}_{(hk)^{-1}}|_{M^{g,h,k}}$

(2)

$$\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl] \arrow[dr] & & \\
 & M^{gh,k} \arrow[dl] \arrow[dr] & & M^{g,hk} \arrow[dl] \arrow[dr] & \\
M^{gh}\times M^{k} & & M^{ghk} & & M^{g}\times M^{hk}
\end{tikzcd}$$

$$E'(g,h,k)=TM^{ghk}|_{M^{g,h,k}}\oplus TM^{g,h,k}|_{M^{g,h,k}}\ominus TM^{gh,k}|_{M^{g,h,k}}\ominus TM^{g,hk}|_{M^{g,h,k}}$$

$\mathcal{R}(gh,k)=TM^{gh,k}|_{M^{g,h,k}}\ominus TM|_{M^{g,h,k}}\oplus \mathcal{S}_{gh}|_{M^{g,h,k}}\oplus \mathcal{S}_{k}|_{M^{g,h,k}}\oplus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}.$

$\mathcal{R}'(g,hk)=TM^{g,hk}|_{M^{g,h,k}}\ominus TM^{g}|_{M^{g,h,k}}\ominus TM^{ghk}|_{M^{g,h,k}}\ominus \mathcal{S}_{g}|_{M^{g,h,k}}\oplus \mathcal{S}_{(hk)^{-1}}|_{M^{g,h,k}}\ominus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}.$

Then the sum of these three formal bundles is

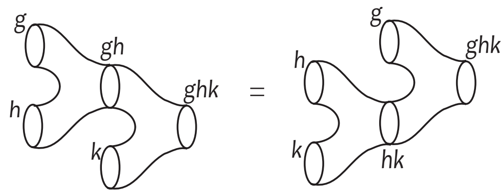
$TM^{g,h,k}\ominus TM^{g}|_{M^{g,h,k}}\ominus TM|_{M^{g,h,k}}\ominus \mathcal{S}_{g}|_{M^{g,h,k}}\oplus \mathcal{S}_{gh}|_{M^{g,h,k}}\oplus \mathcal{S}_{k}|_{M^{g,h,k}}\oplus \mathcal{S}_{(hk)^{-1}}|_{M^{g,h,k}}$

To get en equality of the tow terms, we are left with checking that

$$TM|_{M^{g,h,k}} \ominus TM^{gh}|_{M^{g,h,k}} = \mathcal{S}_{gh}|_{M^{g,h,k}} \oplus \mathcal{S}_{(gh)^{-1}}|_{M^{g,h,k}},$$

but this is true because both sides are the restrictions of the normal bundle of the inclusion $M^{gh} \to M$.

Associativity



(1) (2)

(1)

$$\begin{tikzcd}
 & & M^{g,h,k} \arrow[dl] \arrow[dr] & & \\
 & M^{g,h} \times M^k \arrow[dl] \arrow[dr] & & M^{gh,k} \arrow[dl] \arrow[dr] & \\
M^g \times M^h \times M^k & & M^{gh} \times M^k & & M^{ghk}
\end{tikzcd}$$

$$E(g,h,k) = TM^{gh}|_{M^{g,h,k}} \oplus TM^k|_{M^{g,h,k}} \oplus TM^{g,h,k} \ominus TM^{g,h}|_{M^{g,h,k}} \ominus TM^{gh,k}|_{M^{g,h,k}} \ominus TM^k|_{M^{g,h,k}}.$$

$$\mathcal{R}(g,h) = TM^{g,h}|_{M^{g,h,k}} \ominus TM|_{M^{g,h,k}} \oplus \mathcal{S}_g|_{M^{g,h,k}} \oplus \mathcal{S}_h|_{M^{g,h,k}} \oplus \mathcal{S}_{(gh)^{-1}}|_{M^{g,h,k}}.$$

$$\mathcal{R}(gh,k) = TM^{gh,k}|_{M^{g,h,k}} \ominus TM|_{M^{g,h,k}} \oplus \mathcal{S}_{gh}|_{M^{g,h,k}} \oplus \mathcal{S}_k|_{M^{g,h,k}} \oplus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}.$$

Then the sum of these three formal bundles is

$$TM^{g,h,k} \ominus TM|_{M^{g,h,k}} \oplus \mathcal{S}_g|_{M^{g,h,k}} \oplus \mathcal{S}_h|_{M^{g,h,k}} \oplus \mathcal{S}_k|_{M^{g,h,k}} \oplus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}.$$

(2)

$$\begin{tikzcd}
 & & M^{g,hk} \arrow[dl] \arrow[dr] & & \\
 & M^g \times M^{h,k} \arrow[dl] \arrow[dr] & & M^{g,hk} \arrow[dl] \arrow[dr] & \\
M^g \times M^h \times M^k & & M^g \times M^{hk} & & M^{ghk}
\end{tikzcd}$$

$$E'(g,h,k) = TM^{g}|_{M^{g,h,k}} \oplus TM^{hk}|_{M^{g,h,k}} \oplus TM^{g,h,k} \ominus TM^{g}|_{M^{g,h,k}} \ominus TM^{h,k}|_{M^{g,h,k}} \ominus TM^{g,hk}|_{M^{g,h,k}}.$$

$$\mathcal{R}(h,k) = TM^{h,k}|_{M^{g,h,k}} \ominus TM|_{M^{g,h,k}} \oplus \mathcal{S}_h|_{M^{g,h,k}} \oplus \mathcal{S}_k|_{M^{g,h,k}} \oplus \mathcal{S}_{(hk)^{-1}}|_{M^{g,h,k}}.$$

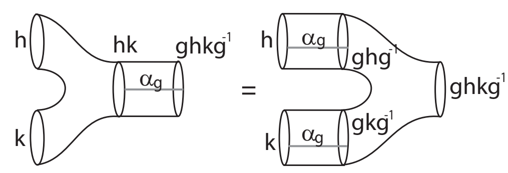
$$\mathcal{R}(g,hk) = TM^{g,hk}|_{M^{g,h,k}} \ominus TM|_{M^{g,h,k}} \oplus \mathcal{S}_g|_{M^{g,h,k}} \oplus \mathcal{S}_{hk}|_{M^{g,h,k}} \oplus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}.$$

Then the sum of these three formal bundles is

$$TM^{g,h,k} \ominus TM|_{M^{g,h,k}} \oplus \mathcal{S}_g|_{M^{g,h,k}} \oplus \mathcal{S}_h|_{M^{g,h,k}} \oplus \mathcal{S}_k|_{M^{g,h,k}} \oplus \mathcal{S}_{(ghk)^{-1}}|_{M^{g,h,k}}.$$

Therefore the two expressions agree.

The action is an algebra homomorphism



(1) (2)

(1)

$$\begin{tikzcd}
 & & M^{ghg^{-1},gkg^{-1}} \arrow[dl] \arrow[dr] & & \\
 & M^{h,k} \arrow[dl] \arrow[dr] & & M^{ghkg^{-1}} \arrow[dl] \arrow[dr] & \\
M^{h} \times M^{k} & & M^{hk} & & M^{ghkg^{-1}}
\end{tikzcd}$$

$$E(g,h,k) = TM^{hk}|_{M^{ghg^{-1},gkg^{-1}}} \oplus TM^{ghg^{-1},gkg^{-1}} \ominus TM^{h,k}|_{M^{ghg^{-1},gkg^{-1}}} \ominus TM^{ghkg^{-1}}|_{M^{ghg^{-1},gkg^{-1}}}.$$

$$\mathcal{R}(h,k) = TM^{h,k}|_{M^{ghg^{-1},gkg^{-1}}} \ominus TM|_{M^{ghg^{-1},gkg^{-1}}} \oplus \mathcal{S}_h|_{M^{ghg^{-1},gkg^{-1}}} \oplus \mathcal{S}_k|_{M^{ghg^{-1},gkg^{-1}}} \oplus \mathcal{S}_{(hk)^{-1}}|_{M^{ghg^{-1},gkg^{-1}}}.$$

(2)

$$\begin{tikzcd}
 & & M^{ghg^{-1},gkg^{-1}} \arrow[dl] \arrow[dr] & & \\
 & M^{ghg^{-1}} \times M^{gkg^{-1}} \arrow[dl] \arrow[dr] & & M^{ghg^{-1},ghg^{-1}} \arrow[dl] \arrow[dr] & \\
M^{h} \times M^{k} & & M^{ghg^{-1}} \times M^{gkg^{-1}} & & M^{ghkg^{-1}}
\end{tikzcd}$$

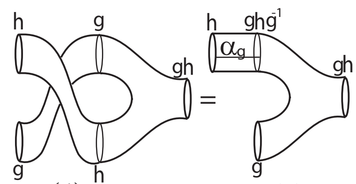
$\mathsf{E}'(g,h,k) = \big(TM^{ghg^{-1}} \oplus TM^{gkg^{-1}} \ominus TM^{ghg^{-1}} \ominus TM^{gkg^{-1}} \ominus TM^{ghg^{-1},gkg^{-1}} \oplus TM^{ghg^{-1},gkg^{-1}}\big)|_{M^{ghg^{-1},gkg^{-1}}}$.

$\mathcal{R}(ghg^{-1}, gkg^{-1}) = \big(TM^{ghg^{-1},gkg^{-1}} \ominus TM \oplus \mathcal{S}_{ghg^{-1}} \oplus \mathcal{S}_{gkg^{-1}} \oplus \mathcal{S}_{(ghkg^{-1})^{-1}}\big)|_{M^{ghg^{-1},gkg^{-1}}}$.

Pairing equal terms we get the desired equality

$$\mathsf{E}(g,h,k) \oplus \mathcal{R}(h,k) = \mathsf{E}'(g,h,k) \oplus \mathcal{R}(ghg^{-1}, gkg^{-1}).$$

Graded commutativity of the product



(1) (2)

(1)

$$\begin{tikzcd}
& & M^{g,h} \arrow[dl] \arrow[dr] & & \\
& M^g \times M^h \arrow[dl] \arrow[dr] & & M^{g,h} \arrow[dl] \arrow[dr] & \\
M^h \times M^g & & M^g \times M^h & & M^{gh}
\end{tikzcd}$$

$$\mathsf{E}(g,h) = TM^g|_{M^{g,h}} \oplus TM^h|_{M^{g,h}} \oplus TM^{g,h}|_{M^{g,h}} \ominus TM^g|_{M^{g,h}} \ominus TM^h|_{M^{g,h}} \ominus TM^{g,h}.$$

$$\mathcal{R}(g,h) = TM^{g,h} \ominus TM|_{M^{g,h}} \oplus \mathcal{S}_g|_{M^{g,h}} \oplus \mathcal{S}_h|_{M^{g,h}} \oplus \mathcal{S}_{(gh)^{-1}}|_{M^{g,h}}.$$

(2)

$$\begin{tikzcd}
& & M^{g,h} = M^{ghg^{-1},g} \arrow[dl] \arrow[dr] & & \\
& M^{ghg^{-1}} \times M^g \arrow[dl] \arrow[dr] & & M^{g,ghg^{-1}} \arrow[dl] \arrow[dr] & \\
M^h \times M^g & & M^{ghg^{-1}} \times M^g & & M^{gh}
\end{tikzcd}$$

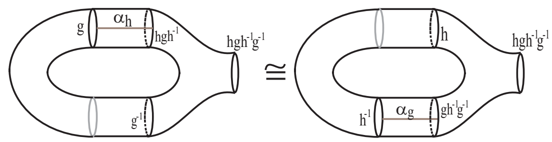
$$\mathsf{E}'(g,h) = TM^{ghg^{-1}}|_{M^{g,h}} \oplus TM^g|_{M^{g,h}} \oplus TM^{g,h}|_{M^{g,h}} \ominus TM^{ghg^{-1}}|_{M^{g,h}} \ominus TM^g|_{M^{g,h}} \ominus TM^{ghg^{-1},g}.$$

$$\mathcal{R}\big(g, ghg^{-1}\big) = TM^{g,ghg^{-1}} \ominus TM|_{M^{g,h}} \oplus \mathcal{S}_{ghg^{-1}}|_{M^{g,h}} \oplus \mathcal{S}_g|_{M^{g,h}} \oplus \mathcal{S}_{(gh)^{-1}}|_{M^{g,h}}.$$

Pairing equal terms we get theequality

$$E(g,h) \oplus \mathcal{R}(g,h) = E'(g,h) \oplus \mathcal{R}\big(g, ghg^{-1}\big).$$

Torus axiom



(1) (2)

It suffices to prove that maps associated to the next two diagrams coincide.
(1)

$$\begin{tikzcd}
 & & M^{g,h} \arrow[dl] \arrow[dr] & & \\
 & M^{g,g^{-1}} \arrow[dl] \arrow[dr] & & M^{hgh^{-1},g^{-1}} \arrow[dl] \arrow[dr] & \\
M & & M^{g} \times M^{g^{-1}} & & M^{hgh^{-1}g^{-1}}
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & M^{g,h} \arrow[dl] \arrow[dr] & & \\
 & M^{h,h^{-1}} \arrow[dl] \arrow[dr] & & M^{h,gh^{-1}g^{-1}} \arrow[dl] \arrow[dr] & \\
M & & M^{h} \times M^{h^{-1}} & & M^{hgh^{-1}g^{-1}}
\end{tikzcd}$$

Fort the diagram (1) we have the formal bundles

$$E(g,h) = TM^{g}|_{M^{g,h}} \oplus TM^{g^{-1}}|_{M^{g,h}} \ominus TM^{g,g^{-1}}|_{M^{g,h}} \ominus TM^{hgh^{-1},g^{-1}}|_{M^{g,h}} \oplus TM^{g,h}|_{M^{g,h}}$$

$$\mathcal{R}'\big(g, g^{-1}\big) = TM^{g,g^{-1}}|_{M^{g,h}} \ominus TM^{g}|_{M^{g,h}} \ominus TM|_{M^{g,h}} \ominus \mathcal{S}_g|_{M^{g,h}} \oplus \mathcal{S}_g|_{M^{g,h}} \ominus \mathcal{S}_e|_{M^{g,h}}$$

$$\mathcal{R}\big(hgh^{-1}, g^{-1}\big) = TM^{hgh^{-1},g^{-1}}|_{M^{g,h}} \ominus TM|_{M^{g,h}} \oplus \mathcal{S}_{hgh^{-1}}|_{M^{g,h}} \oplus \mathcal{S}_{g^{-1}}|_{M^{g,h}} \oplus \mathcal{S}_{(hgh^{-1}g^{-1})^{-1}}|_{M^{g,h}},$$

and their sum becomes

$$\mathsf{TM}^{g,h}|_{M^{g,h}} \ominus \mathsf{TM}|_{M^{g,h}} \oplus \mathcal{S}_{(hgh^{-1}g^{-1})^{-1}}|_{M^{g,h}} \ominus \mathcal{S}_e|_{M^{g,h}}.$$

For the diagram (2) we have the formal bundles

$$\mathsf{E}(g,h) = \mathsf{TM}^h|_{M^{g,h}} \oplus \mathsf{TM}^{h^{-1}}|_{M^{g,h}} \oplus \mathsf{TM}^{g,h}|_{M^{g,h}} \ominus \mathsf{TM}^{h,gh^{-1}g^{-1}}|_{M^{g,h}} \ominus \mathsf{TM}^{h,h^{-1}}|_{M^{g,h}}$$

$$\mathcal{R}'(h,h^{-1}) = \mathsf{TM}^{h,h^{-1}}|_{M^{g,h}} \ominus \mathsf{TM}^h|_{M^{g,h}} \ominus \mathsf{TM}|_{M^{g,h}} \ominus \mathcal{S}_h|_{M^{g,h}} \oplus \mathcal{S}_h|_{M^{g,h}} \ominus \mathcal{S}_e|_{M^{g,h}}$$

$$\mathcal{R}(h, gh^{-1}g^{-1}) = \mathsf{TM}^{h,gh^{-1}g^{-1}}|_{M^{g,h}} \ominus \mathsf{TM}|_{M^{g,h}} \oplus \mathcal{S}_h|_{M^{g,h}} \oplus \mathcal{S}_{gh^{-1}g^{-1}}|_{M^{g,h}} \oplus \mathcal{S}_{(hgh^{-1}g^{-1})^{-1}}|_{M^{g,h}}$$

whose sum becomes

$$\mathsf{TM}^{g,h}|_{M^{g,h}} \ominus \mathsf{TM}|_{M^{g,h}} \ominus \mathcal{S}_e|_{M^{g,h}} \oplus \mathcal{S}_{(hgh^{-1}g^{-1})^{-1}}|_{M^{g,h}},$$

which is equal to the sum associated to diagram (1).

We have just proved that $\mathcal{K}(M, G)$ is a nearly G-Frobenius algebra. Let us see now what happens in the case that M is compact.

♣

Theorem 11.10. *Whenever* M *is compact then* $\mathcal{K}(M, G)$ *is a* G*-Frobenius algebra, where* $\varepsilon : \mathcal{K}(M, G) \to \mathbb{C}$ *maps* $\mathcal{F} = \oplus_k \mathcal{F}_k$ *to* $p_*(\mathcal{F}_1)$ *where* $p_* : K(M) \to K(pt) = \mathbb{Z}$ *is the push-forward in* K*-theory of the map* $p : M \to pt$*.*

Proof. The manifold M is a complex manifold, therefore itself and all its submanifolds of fixed points are oriented in K-theory. Define the inner product $\langle , \rangle$ on $\mathcal{K}(M, G)$ by setting

$$\langle \mathcal{F}, \mathcal{G} \rangle := \varepsilon(\mathcal{F} * \mathcal{G}).$$

We will just prove that $\langle , \rangle$ is non degenerate since all the other properties will follow from its definition.

Consider a bundle $\mathcal{F}_g \in K(M^g)$ and take its Poincaré dual bundle $\mathcal{G}_g = PD(\mathcal{F}_g)$ in $K(M^g)$; this bundle exists since the K-theory of the complex manifold satisfies Poincaré duality. So we have that $(p^g)_*(\mathcal{F}_g \otimes \mathcal{G}_g) = 1$ where $(p^g)_*$ is the pushforward of the map $p^g : M^g \to pt$.

Take the bundle $\sigma^*(\mathcal{G}_g)$ in $K(M^{g^{-1}})$ and calculate

$$\begin{aligned}\langle \mathcal{F}_g, \sigma^*(\mathcal{G}_g)\rangle &= \varepsilon(\mathcal{F}_g * \sigma^*(\mathcal{G}_g))\\ &= p_*((e_g)_*(\mathcal{F}_g \otimes \mathcal{G}_g))\\ &= (p^g)_*(\mathcal{F}_g \otimes \mathcal{G}_g)\\ &= 1\end{aligned}$$

where we have used that $\mathcal{R}(g, g^{-1}) = 0$ and that $(p^g)_* = (p \circ e_g)_* = p_* \circ (e_g)_*$. We have then that $\langle , \rangle$ is non-degenerate and therefore $\mathcal{K}(M, G)$ becomes a G-Frobenius algebra.

♣

Remark 11.11. The G-invariant Frobenius algebra $\mathcal{K}(M, G)^G$ is usually called the "Stringy K-theory" of the complex orbifold $[M/G]$, see [JKK07, ARZ07, BU09].

11.2 Further Stringy Ring Structures in K-theory

The stringy K-theory modules $\mathcal{K}(M, G)$ could be endowed with other ring structures using the pull-push formalism provided we are given complex vector bundles

$$D_{g,h} \to M^{g,h}$$

over the fixed point sets of every pair of elements $g, h \in G$, such that they satisfy two conditions:

- Equivariantness: For every $k \in G$, we have that $\alpha_k^* D_{kgk^{-1},khk^{-1}} \cong D_{g,h}$ where $\alpha_k : M^{g,h} \to M^{k^{-1}gk,k^{-1}hk}$ is the map $\alpha_k(x) = xk$.
- Compatibility: For all triples of elements $g, h, k \in G$ the bundles satitisfy the equation

  $$D_{g,h}|_N \oplus D_{gh,k}|_N \oplus E(M^{gh}, M^{g,h}, M^{gh,k}) \cong D_{g,hk}|_N \oplus D_{h,k}|_N \oplus E(M^{hk}, M^{gh,k}, M^{h,k})$$

  for $N := M^{g,h,k}$ and $E(S; S_1, S_2)$ denoting the excess intersection bundle of the inclusions $S_1 \to S$ and $S_2 \to S$ which can be taken to be

  $$E(S; S_1, S_2) = TS|_V \ominus TS_1|_V \ominus TS_2|_V \oplus TV$$

  whenever $V = S_1 \cap S_2$ is a manifold.

If we denote by $D = \{D_{g,h}\}_{g,h}$ the collection of these bundles, then the K-theory module $\mathcal{K}(M, G)$ could be endowed with the ring structure:

Definition 11.12. Given $g, h \in G$ and elements $\mathcal{F}_g \in \mathcal{K}(M^g)$ and $\mathcal{F}_h \in \mathcal{K}(M^h)$, we define the D-*string product* of $\mathcal{F}_g$ and $\mathcal{F}_h$ in $\mathcal{K}(M^{gh}) \subset \mathcal{K}(M, G)$ to be

$$\mathcal{F}_g \star_D \mathcal{F}_h := (e_{gh})_* \big(e_g^* \mathcal{F}_g \otimes e_h^* \mathcal{F}_h \otimes \lambda_{-1}(D_{g_1,g_2})\big)$$

and the product is extended linearly to every element in $\mathcal{K}(M, G)$. The compatibility condition defined above implies that the product $\star_D$ is associative. Denote this ring by

$$\mathcal{K}(M, G; \lambda_{-1} D).$$

Lemma 11.13. *If M is compact and the bundles $D_{g,g^{-1}} = 0$ for all $g \in G$, then the inner product*

$$\langle \mathcal{F}, \mathcal{G} \rangle_D := p_*(\mathcal{F} \star_D \mathcal{G})$$

is nongenerate.
Therefore $\mathcal{K}(M, G; \lambda_{-1} D)$ together with $\langle , \rangle_D$ is a G-Frobenius algebra.

Proof. The proof is the same as the one in Theorem 11.10.

♣

11.2.1 Virtual K-theory

The bundles

$$\nu(g, h) = TM|_{M^{g,h}} \oplus TM^{g,h} \ominus TM^g|_{M^{g,h}} \ominus TM^h|_{M^{g,h}}$$

introduced in Chapter 10 to define the Virtual cohomology satisfy the compatibility and the equivariantness conditions. Therefore they define a ring structure in K-theory:

Definition 11.14. The Virtual K-theory of the G-manifold M is the G-graded G-vector space

$$\mathcal{K}(M, G; \lambda_{-1}(\nu))$$

endowed with the ring structure defined by the bundles $\nu = \{\nu(g, h)\}_{g,h}$.

The same proofs of chapter 10 hold for the Virtual K-theory since all of them are done at the level of bundles. Therefore we can indtroduce the virtual coproduct at the level of K-theory as the map

$$\Delta^{g,h}_{gh}(\mathcal{F}) := \big(e_g \boxtimes e_h\big)_* \big(e^*_{gh}(\mathcal{F}) \otimes \lambda_{-1}\big(\mu(g,h)\big)\big)$$

where $\mathcal{F} \in \mathsf{K}(M^g)$, $e_g \boxtimes e_h$ denotes the map $e_g \boxtimes e_h : M^{g,h} \to M^g \times M^h, x \mapsto (x,x)$, and

$$\mu(g,h) = e\left(\frac{TM|_{M^{g,h}}}{TM^{gh}|_{M^{g,h}}} \oplus TM^{g,h}\right)$$

is the sum of the normal bundle of the embedding $M^{gh} \to M$ restricted to $M^{g,h}$ together with the tangent bundle of $M^{g,h}$.

Theorem 11.15. *The graded ring $\mathcal{K}(M,G;\lambda_{-1}(\nu))$ together with the virtual coproduct is a nearly G-Frobenius algebra.*

11.2.2 New Structures From Old Ones

We will show in what follows two ways to modify a given nearly G-Frobenius algebra structure on the K-theory $\mathcal{K}\big(M,G\big)$, one by acting on the coefficients $\mathbb{C}$ of the complexified K-theory, and another by acting on the K-theory elements by tensoring with line bundles. Let us start with line bundles

• Recall that the automorphism group of the K-theory $\mathsf{K}(X)$ is isomorphic to the Picard group of line bundles $\mathsf{Pic}(X)$. Any line bundle $L \to X$ over X defines an automorphism $\mathsf{K}(X) \xrightarrow{\cong} \mathsf{K}(X)$, $E \mapsto L \otimes E$. With this in mind, let us consider line bundles $L_{g,h} \to M^{g,h}$ satisfying the equivariant condition under conjugation defined above, and the compatibility

$$L_{g,h}|_N \otimes L_{gh,k}|_N \cong L_{g,hk}|_N \otimes L_{h,k}|_N$$

with $N := M^{g,h,k}$; denote $L = \{L_{g,h}\}_{g,h}$ and the compatibility condition

$$(\delta L)_{g,h,k} = L_{h,k}|_N \otimes L^*_{gh,k}|_N \otimes L_{g,hk}|_N \otimes L_{g,h}|_N \cong 1.$$

If we are given vector bundles $D = \{D_{g,h}\}_{g,h}$, we can define a new ring structure by defining the product to be

$$\mathcal{F}_g \star^L_D \mathcal{F}_h := (e_{gh})_* \big(e^*_g \mathcal{F}_g \otimes e^*_h \mathcal{F}_h \otimes \lambda_{-1}\big(D_{g_1,g_2}\big) \otimes L_{g,h}\big).$$

Denote this ring structure by $\mathcal{K}\big(M,G;\lambda_{-1}D \otimes L\big)$ and call it the L-twisted structure of the stringy product of D.

Lemma 11.16. *If* M *is compact, the bundles* $D_{g,g^{-1}} = 0$ *for all* $g \in G$ *and line bundles* $L = \{L_{g,h}\}_{g,h}$ *with* $\delta L = 1$*, then the inner product*

$$\langle \mathcal{F}, \mathcal{G} \rangle_D^L := p_*(\mathcal{F} \star_D^L \mathcal{G})$$

is nongenerate. Therefore $\mathcal{K}\big(M, G; \lambda_{-1} D \otimes L\big)$ *together with* $\langle , \rangle_D^L$ *is a* G*-Frobenius algebra.*

Proof. Following the notation of Theorem 11.10 we see that

$$\begin{aligned}
\langle \mathcal{F}_g, \sigma^*(\mathcal{G}_g) \otimes L^*_{g,g^{-1}} \rangle_D^L &= \varepsilon(\mathcal{F}_g \star_D^L \sigma^*(\mathcal{G}_g)) \\
&= (p^g)_*(\mathcal{F}_g \otimes \mathcal{G}_g \otimes L^*_{g,g^{-1}} \otimes L_{g,g^{-1}}) \\
&= 1.
\end{aligned}$$

Therefore $\langle , \rangle_D^L$ is nondegenerate.

♣

• We could also act with automorphisms of the coefficients. If we choose elements $\tau_{g,h} \in \mathbb{C}^\times$ such that

$$(\delta\tau)_{g,h,k} = \tau_{h,k}(\tau_{gh,k})^{-1}\tau_{g,hk}(\tau_{g,h})^{-1} = 1$$

i.e. τ is a 2-cocyle of G with coefficients in $\mathbb{C}^\times$, $\tau \in Z^2(G, \mathbb{C}^\times)$, then we can define a a new ring structure by defining the product to be

$$\mathcal{F}_g \star_D^\tau \mathcal{F}_h := \tau_{g,h}(e_{gh})_*\big(e_g^*\mathcal{F}_g \otimes e_h^*\mathcal{F}_h \otimes \lambda_{-1}\big(D_{g_1,g_2}\big)\big).$$

Denote this ring structure by $\mathcal{K}\big(M, G; \lambda_{-1} D \otimes \tau\big)$ and call it the τ-twisted structure of the stringy product of D.

• Putting together the action of the line bundle $L = \{L_{g,h}\}_{g,h}$ and the action on the coefficients $\tau = \{\tau_{g,h}\}_{g,h}$, we can define a new ring structure to be

$$\mathcal{F}_g \star_D^{(\tau\otimes L)} \mathcal{F}_h := \tau_{g,h}(e_{gh})_*\big(e_g^*\mathcal{F}_g \otimes e_h^*\mathcal{F}_h \otimes \lambda_{-1}\big(D_{g_1,g_2}\big) \otimes L_{g,h}\big)$$

and we will denote it by $\mathcal{K}\big(M, G; \lambda_{-1} D \otimes (\tau \otimes L)\big)$.

11.2.3 Isomorphic Stringy Ring Structures

Given two ring structures defined by the stringy products through the bundles D and D' of equal rank (twisted or untwisted by line bundles), we say that the two rings $\mathcal{K}\big(M, G; \lambda_{-1}D\big)$ and $\mathcal{K}\big(M, G; \lambda_{-1}D'\big)$ are isomorphic provided we can find line bundles $\mathcal{L}_g \to M^g$ and complex numbers $\sigma_g \in \mathbb{C}^\times$, equivariant with respect to the conjugation action, such that

$$\frac{\sigma_g \sigma_h}{\sigma_{gh}} \lambda_{-1}(D_{g,h}) \otimes \mathcal{L}_g \otimes \mathcal{L}_h \otimes \mathcal{L}^*_{gh} = \lambda_{-1}(D'_{g,h})$$

If such equation is satisfied, it is easy to check that the map

$$\begin{aligned} \mathcal{K}\big(M, G; \lambda_{-1}D\big) &\xrightarrow{\sigma \otimes \mathcal{L}} \mathcal{K}\big(M, G; \lambda_{-1}D'\big) \\ \mathcal{F}_g \in K(M^g) &\mapsto \sigma_g(\mathcal{F}_g \otimes \mathcal{L}_g) \in K(M^g) \end{aligned}$$

gives the desired isomorphism.

For example, if we define the line bundles $L_{g,h} \to M^{g,h}$ to be

$$L_{g,h} := \mathcal{L}_g|_{M^{g,h}} \otimes \mathcal{L}_h|_{M^{g,h}} \otimes \mathcal{L}^*_{gh}|_{M^{g,h}},$$

i.e. $L = \delta\mathcal{L}$, then there is an isomorphism of rings

$$\otimes\mathcal{L} : \mathcal{K}\big(M, G; \lambda_{-1}D\big) \to \mathcal{K}\big(M, G; \lambda_{-1}D \otimes \delta\mathcal{L}\big)$$

In particular we can say that whenever the line bundles $\mathcal{L}_g \to M^g$ satisfy the equation

$$\mathcal{L}_g|_{M^{g,h}} \otimes \mathcal{L}_h|_{M^{g,h}} \cong \mathcal{L}_{gh}|_{M^{g,h}},$$

namely that $\delta\mathcal{L} = 1$, then the map $\mathcal{F}_g \mapsto \mathcal{F}_g \otimes \mathcal{L}_g$ produces an automorphism of the ring $\mathcal{K}\big(M, G; \lambda_{-1}D\big)$.

If we think in cohomological terms, we have that we obtain new ring structures provided we have line bundles $L = \{L_{g,h}\}_{g,h}$ satisfying $\delta L = 1$ where

$$(\delta L)_{g,h,k} := L_{g,h}|_N \otimes L_{gh,k}|_N \otimes L^*_{g,hk}|_N \otimes L^*_{h,k}|_N$$

for $N := M^{g,h,k}$, and two of these ring structures defined by L and L' are isomorphic provided there exists line bundles $\mathcal{L} = \{\mathcal{L}_g\}_g$ such that $\delta\mathcal{L} = L' \otimes L^*$, i.e.

$$\mathcal{L}_g|_{M^{g,h}} \otimes \mathcal{L}_h|_{M^{g,h}} \otimes \mathcal{L}^*_{gh}|_{M^{g,h}} = L^*_{g,h} \otimes L'_{g,h}.$$

Denoting this group by $\mathcal{H}^2(M/G, \mathsf{Pic}(\_)^G)$, since it is the second cohomology group of the complex

$$\mathsf{Pic}(M)^G \xrightarrow{\delta} \mathsf{Pic}(I_G M)^G \xrightarrow{\delta} \mathsf{Pic}((I_G)^2 M)^G \xrightarrow{\delta} \mathsf{Pic}((I_G)^3 M)^G ...$$

where $(I_G)^n M = \bigsqcup M^{g_1,\dots,g_n} \subset M \times G^n$, Pic denotes the Picard group of line bundles, the G invariant part denotes that we only consider line bundles with the equivariant condition, and the differential is the one that was defined above, then from a given ring structure defined by the line bundles D, we can obtain $|\mathcal{H}^2(M/G, \mathsf{Pic}(\_)^G)|$ non-isomorphic Stringy ring structures through tensor product of Line bundles.

The same argument applied to the action on coefficients tells us that we get $|H^2(G, \mathbb{C}^\times)|$ as many non-isomorphic String ring structures once we start from a fixed one. We will see in Section 12.2 that the procedure to act with a 2-cocycle τ is equivalent to tensoring the nearly G-Frobenius algebra structure with the G-Frobenius algebra structure obtained from a discrete torsion.

Finally note that the group $\mathcal{H}^1(M/G, \mathsf{Pic}(\_)^G)$ is the group of line bundle automorphisms of the Stringy K-theories $\mathcal{K}\big(M, G; \lambda_{-1} D\big)$, and that $H^1(G, \mathbb{C}^\times) = \mathrm{Hom}(G, \mathbb{C}^\times)$ is the group of coefficient automorphisms of the Stringy K-theories $\mathcal{K}\big(M, G; \lambda_{-1} D\big)$.

11.2.4 Relation among the Different Definitions of Stringy K-theory

In this work we have followed the approach described in [ARZ07] and [BU09] to define the K-theoretical version of the Chen-Ruan cohomology. There is another approach of the Stringy product taken in [JKK07] in order to deal with smooth projective algebraic orbifolds. In [JKK07] the ring structure in $\mathcal{K}\big(M, G\big)$, whenever M is a smooth projective algebraic variety and G is a finite group acting on M, is defined using the duals of the bundles $\mathcal{R}(g, h)$ defined previously. This is the case because the defining property of the push-forward maps that are used in [JKK07] contain the dual of the normal bundle, i.e. for the embedding of complex manifolds $i : M' \to M$ with normal bundle $N \to M'$, the pushforward map in algebraic geometry as defined in [FL85] satisfies

$$i^* i_! \mathcal{F} = \mathcal{F} \otimes \lambda_{-1}(N^*).$$

The reason for this choice is simple. For non smooth algebraic varieties the tangent bundle may not exist, whereas the cotangent bundle always does. So the pushforward map is defined through the conormal bundle, and not using the normal bundle.

In algebraic topology the push-forward map is defined using the Thom class, and in this case the defining formula is

$$i^* i_* \mathcal{F} = \mathcal{F} \otimes \lambda_{-1}(N).$$

Note that we have used different notation for the two push-forwards. The algebro-geometrical one will be with an exclamation sign, meanwhile the topological one will be denoted with an asterisque.

Fortunately, both push-forwards are related by the multiplication of a line bundle. Note that for a line bundle L we have that

$$\lambda_{-1}(L) = \mathbb{C} - L = -L \otimes (\mathbb{C} - L^*) = -L \otimes \lambda_{-1}(L^*)$$

and therefore this implies that

$$\lambda_{-1}(N) = (-1)^k \Lambda^k N \otimes \lambda_{-1} N^*$$

where k is the rank of the bundle N and $\Lambda^k N$ is the line bundle of top degree, also known as the determinant line bundle of N. Therefore we have that the two pushforward maps relate by the formula

$$i^* i_* \mathcal{F} = (-1)^k \Lambda^k N \otimes (i^* i_! \mathcal{F}).$$

In [JKK07] the Stringy product is defined by the formula

$$\mathcal{F}_g \star_{JKK} \mathcal{F}_h := (e_{gh})_! \big(e_g^* \mathcal{F}_g \otimes e_h^* \mathcal{F}_h \otimes \lambda_{-1}\big(\mathcal{R}(g,h)^*\big)\big)$$

where the push-forward is the algebro-geometric one. This formula is equivalent in topological terms to

$$\mathcal{F}_g \star_{JKK} \mathcal{F}_h = (e_{gh})_* \big(e_g^* \mathcal{F}_g \otimes e_h^* \mathcal{F}_h \otimes \lambda_{-1}\big(\mathcal{R}(g,h)^*\big) \otimes (-1)^{\dim(N^{gh}_{g,h})} \Lambda^{top}(N^{gh}_{g,h})^* \big)$$

where $N^{g_1 g_2}_{g_1,g_2}$ is the normal bundle of the embedding $M^{g_1,g_2} \to M^{g_1 g_2}$; the formula can be rewriten as

$$\mathcal{F}_g \star_{JKK} \mathcal{F}_h = (e_{gh})_* \big(e_g^* \mathcal{F}_g \otimes e_h^* \mathcal{F}_h \otimes \lambda_{-1}\big(\mathcal{R}(g,h)\big) \otimes (-1)^{\dim(\mathcal{R}(g,h) \oplus N^{gh}_{g,h})} \Lambda^{top}\big(\mathcal{R}(g,h)\big)^* \otimes \Lambda^{top}(N^{gh}_{g,h})^* \big).$$

Note that the line bundles

$$L_{g,h} := \Lambda^{top}\big(\mathcal{R}(g,h)\big)^* \otimes \Lambda^{top}(N^{gh}_{g,h})^*$$

satisfy the compatibility conditions, i.e. $\delta L = 1$: for $g, h, k \in G$. Sicne we already know that restricting each bundle to $M^{g,h,k}$,and denoting $\mathcal{R}_{g,h} := \mathcal{R}(g,h)$, we have

$$\mathcal{R}_{g,h} \oplus \mathcal{R}_{gh,k} \oplus E(M^{gh}, M^{g,h}, M^{gh,k}) \cong \mathcal{R}_{g,hk} \oplus \mathcal{R}_{h,k} \oplus E(M^{hk}, M^{gh,k}, M^{h,k})$$

which implies

$$\mathcal{R}_{g,h} \oplus \mathcal{R}_{gh,k} \oplus TM^{gh} \ominus TM^{g,h} \ominus TM^{gh,k} \oplus TM^{g,h,k} \cong \\ \mathcal{R}_{g,hk} \oplus \mathcal{R}_{h,k} \oplus TM^{hk} \ominus TM^{gh,k} \ominus TM^{h,k}) \oplus TM^{g,h,k}$$

and changing $TM^{g,h,k}$ by TM^{ghk} in each side and reorganizing we get

$$(\mathcal{R}_{g,h} \oplus TM^{gh} \ominus TM^{g,h}) \oplus (\mathcal{R}_{gh,k} \oplus TM^{ghk} \ominus TM^{gh,k}) \cong \\ (\mathcal{R}_{g,hk} \oplus TM^{ghk} \ominus TM^{gh,k}) \oplus (\mathcal{R}_{h,k} \oplus TM^{hk} \ominus TM^{h,k})$$

which implies

$$(\mathcal{R}_{g,h} \oplus N^{gh}_{g,h}) \oplus (\mathcal{R}_{gh,k} \oplus N^{ghk}_{gh,k}) \cong (\mathcal{R}_{g,hk} \oplus N^{ghk}_{gh,k}) \oplus (\mathcal{R}_{h,k} \oplus N^{hk}_{h,k}).$$

Dualizing and applying to both sides of the previous equation Λ^{top}, we obtain the desired equation $\delta L = 1$.

The equation $\delta L = 1$ also implies that the coefficients

$$\tau_{g,h} = (-1)^{\dim(\mathcal{R}(g,h) \oplus N^{gh}_{g,h})}$$

satisfy the cocycle condition $\delta\tau = 1$.

In general there might not exist line bundles $\mathcal{L} = \{\mathcal{L}_g\}_g$ and coefficients $\sigma = \{\sigma_g\}_g$ such that $\delta\mathcal{L} = L$ and $\delta\sigma = \tau$, and this would mean that the Chen-Ruan K-theoretical product $*$ defined at the begining of the chapter, and the product $\star_{JKK}$ defined in [JKK07] and explained above might in general endow the vector spaces $\mathcal{K}(M, G)$ with non-isomorphic ring structures. Summarizing:

Proposition 11.17. *If there exists virtual line bundles $\mathcal{L} = \{\mathcal{L}_g\}_g$ and coefficients $\sigma = \{\sigma_g\}_g$*

$$(\delta\sigma)_{g,h}(\delta\mathcal{L})_{g,h} = (-1)^{\dim(\mathcal{R}(g,h) \oplus N^{gh}_{g,h})} \Lambda^{top}\big(\mathcal{R}(g,h)\big)^* \otimes \Lambda^{top}(N^{gh}_{g,h})^*,$$

then $\sigma \otimes \mathcal{L} : (\mathcal{K}(M, G), *) \to (\mathcal{K}(M, G), \star_{JKK})$ induces an isomorphism of G-Frobenius algebras between the Chen-Ruan K-theory and the Stringy K-theory of [JKK07].

A famous example satisfying the hypothesis of the previous proposition is the symmetric product of even dimensional smooth projective varieties with trivial canonical divisor. Let us be more explicit:

Proposition 11.18. *Consider the symmetric product orbifold* $[\Sigma^n/\mathfrak{S}_n]$*, where* Σ *is a smooth projective algebraic variety of even dimension with trivial canonical class (or compact complex manifold with trivial cohomological Euler class) then the Chen-Ruan* K*-theory* $(\mathcal{K}(\Sigma^n, \mathfrak{S}_n), *)$ *is isomorphic Stringy* K*-theory* $(\mathcal{K}(\Sigma^n, \mathfrak{S}_n), \star_{JKK})$ *of [JKK07].*

Proof. The proposition follows from explicit calculations done in [Uri05] where it is shown that the obstruction bundles $\mathcal{R}(g, h)$ are isomorphic to a direct sum of copies of the tangent bundle $T\Sigma$; the explicit description of the bundle $\mathcal{R}(g, h)$ is carried out in section 13.4 and the relevant equation is (53). Then, since we know that Σ has trivial canonical class we get $\Lambda^{top}T\Sigma = 1$ and therefore $\Lambda^{top}\big(\mathcal{R}(g, h)\big)^* = 1$; and since $\dim(T\Sigma)$ is even we conclude that

$$(-1)^{\dim(\mathcal{R}_{g,h})}\Lambda^{top}\mathcal{R}_{g,h} = 1.$$

Moreover, since the normal bundles $N^{gh}_{g,h}$ are also isomorphic to a direct sum of copies of $T\Sigma$, then

$$(-1)^{\dim(N^{gh}_{g,h})}\Lambda^{top}N^{gh}_{g,h} = 1.$$

Therefore the identity map of vector spaces

$$\mathcal{K}(\Sigma^n, \mathfrak{S}_n) \xrightarrow{\mathrm{Id}} \mathcal{K}(\Sigma^n, \mathfrak{S}_n)$$

induces an isomorphism of $\mathfrak{S}_n$-Frobenius algebras

$$(\mathcal{K}(\Sigma^n, \mathfrak{S}_n), *) \xrightarrow{\cong} (\mathcal{K}(\Sigma^n, \mathfrak{S}_n), \star_{JKK}).$$

♣

11.3 Chen-Ruan Cohomology

Definition 11.19. For $[M/G]$ a complex orbifold we define the *Chen-Ruan cohomology* $H_{CR}(M, G)$ of M, as a G-graded G-complex vector space, to be the cohomology with complex coefficients of the inertia orbifold, i.e.

$$H_{CR}(M, G) := \bigoplus_{g \in G} H^*(M^g; \mathbb{C}).$$

We bring the bundles $\mathcal{R}(g, h)$ used to define the Chen-Ruan product in K-theory, in order to define the Chen-Ruan product in $H^*_{CR}(M, G)$. For $g, h \in G$, $\alpha \in H^*(M^g)$

and $\beta \in H^*(M^h)$, we define the *Chen-Ruan product* of α and β in $H^*(M^{gh}) \subset H^*_{CR}(M,G)$ to be

$$\alpha *_{CR} \beta := e_{gh*}\big(e_g^*\alpha \cdot e_h^*\beta \cdot \mathrm{Eu}(\mathcal{R}(g,h))\big)$$

and the product is extended linearly to all of $H^*_{CR}(M,G)$; here $\mathrm{Eu}(\mathcal{R}(g,h))$ denotes the Euler class of the bundle $\mathcal{R}(g,h)$.

The coproduct is also defined with the help of the bundles $\mathcal{R}'(g,h)$ used in the Chen-Ruan K-theory. For $g,h \in G$ and an element $\gamma_{gh} \in H^*(M^{gh})$, we define the *Chen Ruan coproduct* of γ_{gh} in $H^*(M^g) \otimes H^*(M^h)$ to be

$$\Delta_{gh}^{g,h}(\gamma_{gh}) = \big(e_g \boxtimes e_h\big)_*\big(e_{gh}^*(\gamma_{gh}) \cdot \mathrm{Eu}\big(\mathcal{R}'(g,h)\big)\big)$$

where $e_g \boxtimes e_h$ denotes the map $e_g \boxtimes e_h : M^{g,h} \to M^g \times M^h, x \mapsto (x,x)$. If $gh = k$ then the total coproduct of γ_k is $\Delta_{CR}(\gamma_k) = \sum_{gh=k} \Delta_k^{g,h}(\gamma_k)$.

Theorem 11.20. *$\big(H_{CR}(M,G), \star_{CR}, \Delta_{CR}\big)$ is a nearly G-Frobenius algebra. Moreover, if M is compact, then $\big(H_{CR}(M,G), \star_{CR}, \Delta_{CR}\big)$ is a G-Frobenius algebra.*

Proof. This theorem follows from the proofs of Theorems 11.9 and 11.10.

♣

The invariant Frobenius algebra $\big(H_{CR}(M,G), \star_{CR}, \Delta_{CR}\big)^G$ was the algebraic structure associated to complex orbifolds originally defined by Chen and Ruan in [CR04a].

Note that if we have bundles $D = \{D_{g,h}\}_{g,h}$ satisfying the equivariant and the compatibility condition defined at the begining of section 11.2 then we can define an associative ring structure in $H^*(M,G)$ by the formula

$$\alpha \star_D \beta := e_{gh*}\big(e_g^*\alpha \cdot e_h^*\beta \cdot \mathrm{Eu}(D_{g,h})\big).$$

Let us denote this ring structure on $H^*(M,G)$ by $H^*(M,G,\mathrm{Eu}(D))$.

11.4 Chen-Ruan K-theory (Cohomology) of the Cotangent Bundle and its Relation with Virtual K-theory (Cohomology)

Let $[M/G]$ be a complex orbifold and consider $[T^*M/G]$ the complex orbifold that the induced action of G on the cotangent bundle defines. For $A_g : T_mM \to T_{gm}M$ the induced action on the tangent bundle given by the element $g \in G$, then the induced action on the cotangent bundle that preserves the covariance is given by

$\overline{A_g} : T^*_m M \to T^*_{gm} M$, where as complex matrices $\overline{A_g} = (A_g^T)^{-1}$ is the conjugate matrix.

Since the zero section inclusion $j : M \to T^*M$ induces an isomorphism of modules

$$j^* : \mathcal{K}\big(T^*M, G\big) \xrightarrow{\cong} \mathcal{K}\big(M, G\big),$$

then we would like to find out whether the isomorphism j^* is compatible with the ring structures that can be endowed on each side: the Chen-Ruan product on the left hand side and the Virtual product on the right hand side.

The first step in order to find the explicit relation between the Chen-Ruan K-theory of the cotangent orbifold and the Virtal K-theory of the orbifold is to the ring structure that the Chen-Ruan product induce on $\mathcal{K}\big(M, G\big)$ via the isomorphism j^*. Let us do this first.

If we denote by $\mathcal{R}(T^*M, g, h)$ the obstruction bundle that the Chen-Ruan product defines on $\mathcal{K}\big(T^*M, G\big)$ for the pair of elements $g, h \in G$, then from the proof of Lemma 11.5 we deduce that

$$\mathcal{R}(T^*M, g, h)|_{M^{g,h}} \cong \mathcal{R}(g, h) \oplus \overline{\mathcal{R}}(g, h)^* \cong \mathcal{O}_2^{g,h} \oplus (\mathcal{O}_1^{g,h})^*$$

where we recall that $\overline{\mathcal{R}}(g, h)$ is equal to $\mathcal{R}(g, h)$ but with the conjugate action of g, h and gh.

Now what we need to calculate is the obstruction bundle in $\mathcal{K}\big(M, G\big)$ that the Chen-Ruan ring structure on $\mathcal{K}\big(T^*M, G\big)$ induces. The extra information that we need to add is the excess intersection formula for the inclusions

$$\begin{tikzcd} T^*M^{g,h} \arrow[r] & T^*M^{gh} \\ M^{g,h} \arrow[r] \arrow[u] & M^{gh} \arrow[u] \end{tikzcd}$$

which becomes a K-theory class on $K(M^{g,h})$ equal to

$$E(T^*M^{gh}, T^*M^{g,h}, M^{gh}) = T^*M^{gh} \ominus T^*M^{g,h}.$$

Therefore the obstruction class on $\mathcal{K}\big(M, G\big)$ defined by the Chen-Ruan product on $\mathcal{K}\big(T^*M, G\big)$ becomes

$$\begin{aligned} \kappa_{g,h} &:= \mathcal{R}(g, h) \oplus \overline{\mathcal{R}}(g, h)^* \oplus T^*M^{gh} \ominus T^*M^{g,h} \\ &= \mathcal{R}(g, h) \oplus \overline{\mathcal{R}}(g, h)^* \oplus (N_{g,h}^{gh})^* \end{aligned}$$

as an element in $K(M^{g_1,g_2})$.

Since we have the following set of equalities

$$\begin{aligned}\mathcal{R}(g,h)\oplus\overline{\mathcal{R}}(g,h)\oplus N^{gh}_{g,h} &= TM\ominus TM^{g,h}\ominus(N^{g}_{g,h}\oplus N^{g}_{g,h}\oplus N^{gh}_{g,h})\oplus(N^{gh}_{g,h})\\ &= TM\oplus TM^{g,h}\ominus TM^{g}\ominus TM^{h}\\ &= \nu(g,h)\end{aligned}$$

where $\nu(g,h)=TM\oplus TM^{g,h}\ominus TM^{g}\ominus TM^{h}$ is the obstruction bundle for the Virtual ring structure in K-theory; then we have the equation

$$\lambda_{-1}(\kappa_{g,h})=(-1)^{\dim(\overline{\mathcal{R}}(g,h))}\Lambda^{top}\overline{\mathcal{R}}(g,h)^*\otimes(-1)^{\dim(N^{gh}_{g,h})}\Lambda^{top}(N^{gh}_{g,h})^*\otimes\lambda_{-1}(\nu(g,h))$$

which relates the Euler class in K-theory of $\kappa_{g,h}$ with the Euler class in K-theory of $\nu(g,h)$.

The previous equation permit us to compare the ring structure $\mathcal{K}\big(M,G;\lambda_{-1}(\kappa)\big)$ induced by the Chen-Ruan ring structure on the cotangent orbifold, and the ring structure $\mathcal{K}\big(M,G;\lambda_{-1}(\nu)\big)$ defined by the virtual product.

Proposition 11.21. *Consider the cocycle of line bundles $L=\{L_{g,h}\}_{g,h}$ with*

$$L_{g,h}:=\Lambda^{top}\overline{\mathcal{R}}(g,h)^*\otimes\Lambda^{top}(N^{gh}_{g,h})^*,$$

and the 2-cocycle of coefficients $\tau=\{\tau_{g,h}\}_{g,h}$ with

$$\tau_{g,h}:=(-1)^{\dim(\overline{\mathcal{R}}(g,h)\oplus N^{gh}_{g,h})},$$

if there exist line bundles $\mathcal{L}=\{\mathcal{L}_g\}_g$ and coefficients $\sigma=\{\sigma_g\}_g$ such that $\delta\sigma\otimes\delta\mathcal{L}=\tau\otimes L$, then the Chen-Ruan product on the K-theory of the cotangent orbifold is isomorphic to the Virtual product on the K-theory of the orbifold

$$(\sigma\otimes\mathcal{L})^{-1}\circ j^*:\mathcal{K}\big(T^*M,G;\lambda_{-1}(\mathcal{R}(T^*M))\big)\xrightarrow{\cong}\mathcal{K}\big(M,G;\lambda_{-1}(\nu)\big).$$

Corollary 11.22. *Let Σ be an even dimensional smooth projective algebraic variety with trivial canonical class, or a compact even dimensional complex manifold with trivial determinant line bundle. Then the Chen-Ruan K-theory of the cotangent bundle of the symmetric product $T^*(\Sigma^n)/\mathfrak{S}_n$ is isomorphic to the virtual K-theory of the symmetric product $\Sigma^n/\mathfrak{S}_n$ via the map j^*,*

$$j^*:\mathcal{K}\big(T^*(\Sigma^n),\mathfrak{S}_n;\lambda_{-1}(\mathcal{R}(T^*\Sigma))\big)\xrightarrow{\cong}\mathcal{K}\big(\Sigma{}^{\cdot}\mathfrak{S}_n;\lambda_{-1}(\nu)\big).$$

Proof. The top Chern class of the normal bundles $N_{g,h}^{gh}$ and the bundles $\mathcal{R}(g,h)$ are zero since these bundles can be built out of direct sums of the bundle $T\Sigma$. Moreover, since Σ is even dimensional, then the bundles $N_{g,h}^{gh}$ and $\mathcal{R}(g,h)$ are all of even rank. Therefore in this case we have

$$\lambda_{-1}(\kappa_{g,h}) = \lambda_{-1}(\nu(g,h))$$

and the isomorphism follows.

♣

When comparing the Chen-Ruan cohomology ring of the cotangent orbifold and the Virtual cohomology ring of the orbifold itself, we can get a sharper result. The reason for this to happen is the following: for a complex vector bundle $E \to X$ over a manifold X the complex dual E^* and E are in general not isomorphic as complex bundles. Nevertheless, after picking a metric on the bundle E one can show that E and E^* become isomorphic as $\mathbb{R}$-bundles and their Euler classes are related by the equation

$$Eu(E) = (-1)^{\dim_{\mathbb{C}}(E)} Eu(E^*)$$

since $c_1(L) = -c_1(L^*)$ where L is a complex line bundle over E.

Therefore we have that for the obstruction classes κ and ν we obtain

$$Eu(\kappa_{g,h}) = (-1)^{\dim(\overline{\mathcal{R}}(g,h) \oplus N_{g,h}^{gh})} Eu(\nu(g,h)).$$

Theorem 11.23. *For an even dimensional complex orbifold* $[M/G]$ *such that all the fixed point sets* M^g *are even dimensional, then the Chen-Ruan cohomology of the cotangent orbifold is isomorphic to the virtual cohomology of the orbifold*

$$j^* : H^*_{CR}(T^*M, G) \xrightarrow{\cong} H^*_{virt}(M, G).$$

Proof. In this case all the bundles $\mathcal{R}(g,h)$ and $N_{g,h}^{gh}$ are of even dimension as complex bundles, and therefore their cohomological Euler classes are equal to the ones of their duals, therefore

$$Eu(\kappa_{g,h}) = Eu(\nu(g,h));$$

this implies that the identity map

$$H^*(M, G, Eu(\kappa)) \xrightarrow{\mathrm{Id}} H^*(M, G, Eu(\nu)) = H^*_{virt}(M, G)$$

is an isomorphism of rings.

Since the map $j^* : H^*(T^*M, G; Eu(\mathcal{R})) \xrightarrow{\cong} H^*(M, G; Eu(\kappa))$ induce the isomorphism of rings, then

$$j^* : H^*_{CR}(T^*M, G) = H^*(T^*M, G, Eu(\mathcal{R})) \xrightarrow{\cong} H^*(M, G, Eu(\mu))$$

and the theorem follows.

♣

In particular we have:

Corollary 11.24. *Let Σ be an even dimensional smooth projective algebraic variety, or a compact even dimensional complex manifold. Then the Chen-Ruan cohomology of the cotangent bundle of the symmetric product $T^*(\Sigma^n)/\mathfrak{S}_n$ is isomorphic to the Virtual cohomology of the symmetric product $\Sigma^n/\mathfrak{S}_n$ via the map j^*,*

$$j^* : H^*\big(T^*(\Sigma^n), \mathfrak{S}_n; Eu(\mathcal{R}(T^*\Sigma))\big) \xrightarrow{\cong} H^*(\Sigma^{\cdot}\mathfrak{S}_n; Eu(\nu)\big).$$

With the use of Chern character maps that will be developed in the next section, we will see that Corollary 11.24 will imply the equivalent isomorphism but at the level of the K-theories. In particular it will permit us to remove the condition of the triviality on the canonical class from Corollary 11.22, but the isomorphism will not be a priori obtained by the methods outlined in section 11.2.3.

11.5 Chern Characters

Let us suppose we have bundles $D = \{D_{g,h}\}_{g,h}$ satisfying the equivariantness and compatibility condition described in section 11.2. We show in this section sufficient conditions under which there exists a calibrated Chern character map from the stringy K-theory $\mathcal{K}\big(M, G; \lambda_{-1}(D)\big)$ and the stringy cohomology $H^*\big(M, G; Eu(D)\big)$ which is furthermore an isomorphism of rings.

The sufficient conditions are the following: Existence of elements $\mathcal{E}_g \in K^0(M^g) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mathcal{E} = \{\mathcal{E}_g\}_g$ is equivariant with respect to the G-action, and moreover the following equation

$$D_{g,h} \oplus TM^{gh}|_{M^{g,h}} \ominus TM^{g,h} = e_g^* \mathcal{E}_g \oplus s_h^* \mathcal{E}_h \ominus e_{gh}^* \mathcal{E}_{gh}$$

holds in $K(M^{g,h})$ for all $g, h \in G$.

But in order to define the calibrated Chern character map we will recall some properties of the Thom isomorphism in K-theory and in cohomology and their relation with the Chern character (see [AS68, section 2]).

Let X be a manifold and V a complex vector bundle over X. The Chern character of V is the cohomology class in X defined by the expresion

$$\mathrm{ch}(V) = \sum_i e^{x_i}$$

where the x_i denote the Chern roots of the bundle V. Denoting by

$$\phi : K^*(X) \to K^*(V)$$

$$\psi : H^*(X, \mathbb{Q}) \to H^*(V, \mathbb{Q})$$

the Thom isomorphisms in K-theory and in cohomology respectively, then for any $u \in K^*(X)$ one has

$$\mathrm{ch}(\phi(u)) = \psi(\mathrm{ch}(u) \cdot \mu(V))$$

where the cohomology class $\mu(V)$ is defined as

$$\mu(V) := \prod \frac{1 - e^{x_i}}{x_i}.$$

Moreover, the class $\mu(V)$ is multiplicative, i.e. $\mu(V \oplus F) = \mu(V) \cdot \mu(F)$ and measures the difference of the Chern character of the Euler class in K-theory with the one in cohomology, namely

$$\mathrm{ch}\lambda_{-1}(V) = \mathrm{eu}(V) \cdot \mu(V).$$

The inverse μ^{-1} is what is usually called the Thom class of V.

For an inclusion of manifolds $i : X \to Y$ with normal bundle V and pushforward maps

$$i_* : K^*(X) \to K^*(Y)$$

$$i_* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$$

then one has the equality

$$\mathrm{ch}(i_* u) = i_*(\mathrm{ch}(u) \cdot \mu(V)) \tag{44}$$

where in this case the cohomology class $\mu(V)$ has support on the normal bundle of X.

Definition 11.25. For $\mathcal{E} = \{\mathcal{E}_g\}_g$ where $\mathcal{E}_g \in K^0(M^g) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathcal{E}$ is equivariant with respect to the G-action, we can define the $\mathcal{E}$-calibrated Chern character map

$$\begin{aligned} \mathsf{Ch}_{\mathcal{E}} : \mathcal{K}(M, G) &\to H^*(M, G) \\ \mathcal{F}_g \in K(M^g) &\mapsto \mathsf{ch}(\mathcal{F}_g)\mu(\mathcal{E}_g) \in H^*(M^g) \end{aligned}$$

The calibrated Chern character map is a G-equivariant isomorphism of G-graded vector spaces since the Chern character is an isomorphism and the classes $\mu(\mathcal{E}_g)$ are invertible. We claim that the calibrated Chern character map becomes an isomorphism of rings if the conditions explained above hold.

Theorem 11.26. *Consider bundles $D = \{D_{g,h}\}_{g,h}$ and $\mathcal{E} = \{\mathcal{E}_g\}_g$ satisfying the equivariantness condition, an where D satisfy the compatibility condition described in section 11.2. Assume furthermore that*

$$D_{g,h} \oplus TM^{gh}|_{M^{g,h}} \ominus TM^{g,h} = e_g^*\mathcal{E}_g \oplus e_h^*\mathcal{E}_h \ominus e_{gh}^*\mathcal{E}_{gh}$$

holds in $K(M^{g,h})$ for all $g, h \in G$. Then the $\mathcal{E}$-calibrated Chern character map

$$\mathsf{Ch}_{\mathcal{E}} : \mathcal{K}(M, G; \lambda_{-1}(D)) \xrightarrow{\cong} H^*(M, G; Eu(D))$$

is an isomorphism of rings.

Proof. Take $\mathcal{F} \in K^*(M^g)$ and $\mathcal{H} \in K^*(M^h)$ and consider the following set of equalities:

$$\begin{aligned} \mathsf{Ch}_{\mathcal{E}}(\mathcal{F} \star_D \mathcal{H}) &= \mathsf{ch}(\mathcal{F} \star_D \mathcal{H})\mu(\mathcal{E}_{gh}) \\ &= \mathsf{ch}\left(e_{gh*}\left(e_g^*\mathcal{F} \otimes e_g^*\mathcal{H} \otimes \lambda_{-1}(D_{g,h})\right)\right)\mu(\mathcal{E}_{gh}) \\ &= e_{gh*}\Big(e_g^*\mathsf{ch}(\mathcal{F})\ e_h^*\mathsf{ch}(\mathcal{H})\ Eu(D_{g,h})\ \mu(D_{gh})\mu(TM^{gh}|_{M^{g,h}} \ominus TM^{g,h})\Big)\mu(\mathcal{E}_{gh}) \\ &= e_{gh*}\Big(e_g^*\mathsf{ch}(\mathcal{F})\ e_h^*\mathsf{ch}(\mathcal{H})\ Eu(D_{g,h})\ \mu(D_{g,h}) \\ &\qquad\qquad \mu(TM^{gh}|_{M^{g,h}} \ominus TM^{g,h})\ e_{gh}^*\mu(\mathcal{E}_{gh})\Big) \\ &= e_{gh*}\Big(e_g^*\mathsf{ch}(\mathcal{F})\ e_h^*\mathsf{ch}(\mathcal{H})\ Eu(D_{g,h})\ e_g^*\mu(\mathcal{E}_g)\ e_h^*\mu(\mathcal{E}_h)\Big) \\ &= e_{gh*}\Big(e_g^*\mathsf{Ch}_{\mathcal{E}}(\mathcal{F})\ e_h^*\mathsf{Ch}_{\mathcal{E}}(\mathcal{H})\ Eu(D_{g,h})\Big) \\ &= \mathsf{Ch}_{\mathcal{E}}(\mathcal{F}) \star_D \mathsf{Ch}_{\mathcal{E}}(\mathcal{H}). \end{aligned}$$

where in the third line we used formula (44), the fourth line follows from the properties of the pushforward and the fifth line is obtained by using the hypothesis of the Theorem once the map μ is applied.

♣

Chen-Ruan and Virtual K-theory are both endowed with calibrated Chern character isomorphisms. Let us see how.

11.5.1 Chen-Ruan's Calibrated Chern Character

Take the bundles $\mathcal{S} = \{\mathcal{S}_g\}_g$ defined in Definition 11.2 and note that by Definition 11.4 we have that $\mathcal{S}$ satisfies the hypothesis of Theorem 11.26 with respect to the bundles $\mathcal{R} = \{\mathcal{R}(g,h)\}_{g,h}$ which define the Chen-Ruan product in K-theory. Therefore we have

Theorem 11.27. *The $\mathcal{S}$-calibrated Chern character induces an isomorphism of G-Frobenius algebras*

$$\mathsf{Ch}_{\mathcal{S}} : \mathcal{K}(M, G; \lambda_{-1}(\mathcal{R})) \xrightarrow{\cong} H^*(M, G; Eu(\mathcal{R}))$$

between the Chen-Ruan K-theory and the Chen-Ruan cohomology.

11.5.2 Virtual's Calibrated Chern Character

Take the bundles $\mathcal{N} = \{N_g\}_g$ where N_g is the normal bundle of the embedding $M^g \to M$. Then we have that

$$\begin{aligned}
\nu(g,h) \oplus TM^{gh}|_{M^{g,h}} \ominus TM^{g,h} &= TM|_{M^{g,h}} \oplus TM^{g,h} \ominus TM^{g}|_{M^{g,h}} \ominus TM^{h}|_{M^{g,h}} \oplus TM^{gh}|_{M^{g,h}} \ominus TM^{g,h} \\
&= TM|_{M^{g,h}} \ominus TM^{g}|_{M^{g,h}} \ominus TM^{h}|_{M^{g,h}} \oplus TM^{gh}|_{M^{g,h}} \\
&= N_g|_{M^{g,h}} \oplus N_g|_{M^{g,h}} \ominus N_{gh}|_{M^{g,h}}
\end{aligned}$$

which implies that the hypothesis of Theorem 11.26 ais satisfied with respect to the clases $\nu = \{\nu(g,h)\}_{g,h}$ which define the virtual products. Therefore

Theorem 11.28. *The $\mathcal{N}$-calibrated Chern character induces an isomorphism of nearly G-Frobenius algebras*

$$\mathsf{Ch}_{\mathcal{N}} : \mathcal{K}(M, G; \lambda_{-1}(\nu)) \xrightarrow{\cong} H^*(M, G; Eu(\nu))$$

between the Virtual K-theory and the Virtual cohomology.

We can now use calibrated Chern character isomorphisms for Virtual and Chen-Ruan K-theory, together with Theorem 11.23 in order to obtain the following result:

Theorem 11.29. *For an even dimensional complex orbifold* $[M/G]$ *such that all the fixed point sets* M^g *are even dimensional, then the Chen-Ruan* K*-theory of the cotangent orbifold is isomorphic to the virtual* K*-theory of the orbifold*

$$\mathcal{K}(T^*M, G; \lambda_{-1}(\mathcal{R})) \xrightarrow{\cong} \mathcal{K}(M, G; \lambda_{-1}(\nu))$$

through the composition of the isomorphisms

$$\mathcal{K}(T^*M, G; \lambda_{-1}(\mathcal{R})) \stackrel{\mathcal{C}h_{\mathcal{S}}}{\to} H^*_{CR}(T^*M, G) \stackrel{j^*}{\to} H^*_{virt}(M, G) \stackrel{(\mathcal{C}h_{\mathcal{N}})^{-1}}{\longrightarrow} \mathcal{K}(M, G; \lambda_{-1}(\nu)).$$

In particular we obtain an improvement of Corollary 11.22:

Corollary 11.30. *Let* Σ *be an even dimensional smooth projective algebraic variety, or a compact even dimensional complex manifold. Then the Chen-Ruan* K*-theory of the cotangent bundle of the symmetric product* $T^*(\Sigma^n)/\mathfrak{S}_n$ *is isomorphic to the virtual* K*-theory of the symmetric product* $\Sigma^n/\mathfrak{S}_n$ *via the composition*

$$(\mathcal{C}h_{\mathcal{N}})^{-1} \circ j^* \circ \mathcal{C}h_{\mathcal{S}} : \mathcal{K}\big(T^*(\Sigma^n), \mathfrak{S}_n; \lambda_{-1}(\mathcal{R}(T^*\Sigma))\big) \stackrel{\cong}{\to} \mathcal{K}\big(\Sigma^{\cdot}\mathfrak{S}_n; \lambda_{-1}(\nu)\big).$$

We do not know whether the isomorphism of Theorem 11.29 might be obtained via tensorization with line bundles. A quick look at the composition of maps $(\mathcal{C}h_{\mathcal{N}})^{-1} \circ j^* \circ \mathcal{C}h_{\mathcal{S}}$ might lead one to think that this is not true in general; we do not know and we leave this question open.

Remark 11.31. Proposition 11.21 is the correct statement that replaces Theorem 5.4 of [GLS$^+$07] which has a mistake. Nevertheless, Theorem 5.4 of [GLS$^+$07] was correct at least for the cases specified by Corollary 11.22. In view of Theorem 11.23 we see that in order to avoid problems with signs we impose the hypothesis of Theorem 11.23 for Theorem 6.5 of [GLS$^+$07] to be correct. We thanks Tyler Jarvis for spotting the error.

Corollary 11.32. *In either one of the following cases the hypothesis of theorem 11.29 are satisfied:*

- $\mathcal{X} = [M/G]$ *is hyperkahler (e.g.* M *hyperkahler and* G *acting by hyperkahler isomorphisms.)*
- $\mathcal{X} = \mathcal{Y} \times \mathcal{Y}$ *for a complex orbifold* $\mathcal{Y}$*.*
- $\mathcal{X} = T\mathcal{Y}$ *for a complex orbifold* $\mathcal{Y}$*.*

and hence we have in those cases:

$$j^* : H^*_{CR}(T^*M, G) \xrightarrow{\cong} H^*_{virt}(M, G),$$

and

$$\mathcal{K}(T^*M, G; \lambda_{-1}(\mathcal{R})) \xrightarrow{\cong} \mathcal{K}(M, G; \lambda_{-1}(\nu))$$

Remark 11.33. Let s_g be the Chen-Ruan degree shifting number for a component of $I(T^*\mathcal{X})$ and σ_g the virtual degree shifting number for $I(\mathcal{X})$. Then it is a fun exercise to show that

$$s_g = \sigma_g.$$

Therefore the isomorphism of theorem 11.23 is a graded isomorphism. For more on gradings we refer the reader to [Hep10]. For related work we refer the reader to [EJK12b, EJK10, EJK12a] where there is alternative approaches to some very related results.

Remark 11.34. For ordinary manifolds Viterbo [Vit99], Salamon-Weber [SW06] and Abbondandolo-Schwarz [AS06] have constructed isomorphisms between a particular flavor of the Floer homology of the cotangent bundle T^*M and the ordinary homology of the free loop space

$$HF_*(T^*M) \simeq H_*(\mathcal{L}M).$$

Abbondandolo and Schwarz have proved that the pair of pants product in Floer cohomology of the cotangent corresponds to a product in the homology of the loop space, defined via Morse theory, which Antonio Ramirez and Ralph Cohen [CV] proved is the Chas-Sullivan product. One of the main conjectures in the field states that the symplectic field theory on the left-hand side corresponds to the string topology on the right-hand side. Here we should also mention that for a wide class of manifolds it has been shown that Floer cohomology is isomorphic to Quantum cohomology [PSS96].

The results of this chapter and more specifically Theorem 11.23 are in line with those conjectures.

It is routinary to generalize these results to non-global orbifolds.

12 Gerbes over Orbifolds and Discrete Torsion

12.1 Gerbes

12.1.1 Gerbes on Smooth Manifolds

We will start by explaining a well known example arising in electromagnetism as a motivation for the theory of gerbes. We will consider our space-time as canonically split as follows

$$M^4 = \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} = \{(x_1, x_2, x_3; t) \colon x \in \mathbb{R}^3, t \in \mathbb{R}\}.$$

We will consider a collection of differential forms as follows

- The electric field $E \in \Omega^1(\mathbb{R}^3)$.
- The magnetic field $B \in \Omega^2(\mathbb{R}^3)$.
- The electric current $J_E \in \Omega^2(\mathbb{R}^3)$.
- The electric charge density $\rho_E \in \Omega^3(\mathbb{R}^3)$.

We will assume that these differential forms *depend on* t (so to be fair $E \colon \mathbb{R} \to \Omega^1(\mathbb{R}^3)$, etc.).

We will *define* the *intensity of the electromagnetic field* by

$$F = B - dt \wedge E \in \Omega^2(M)$$

and the compactly supported *electric current* by

$$j_E = \rho_E - dt \wedge J_E \in \Omega^3_c(M).$$

We are ready to write the **Maxwell equations**. They are

$$dF = 0, \qquad d * F = j_E.$$

They are partial differential equations where the unknowns are the $3 + 3$ time-dependent components of the electric and the magnetic field.

If we would like them to look more symmetric we would need to introduce "magnetic monopoles", namely a compactly supported 3-form for the magnetic charge density

$$j_B \in \Omega^3_c(M)$$

and rewrite the equations as

$$dF = j_B, \qquad d * F = j_E.$$

Now we let $N_t = \mathbb{R}^3 \times \{t\}$ be a space-like slice. Then the instantaneous total electric magnetic charges are respectively

$$\int_{N_t} j_E \qquad \text{and} \qquad \int_{N_t} j_B.$$

But we prefer to consider the charges as elements in cohomology, namely

$$Q_E^t = [j_E|_{N_t}] \in H_c^3(N_t)$$

and

$$Q_B^t = [j_B|_{N_t}] \in H_c^3(N_t).$$

Now, quantum mechanics predicts that the charges above are quantized by the so-called *Dirac quantization condition*, namely Q_E^t is in the image of the homomorphism

$$H_c^3(N_t, \mathbb{Z}) \to H_c^3(N_t; \mathbb{R}).$$

We can give a geometric interpretation to this quantization condition. For this purpose we must introduce the concept of (abelian) *gauge field*.

Definition 12.1. Let M be a manifold. A U(1)-gauge field on M consists of a line bundle with a connection on M, to wit

i) A good Leray atlas $\mathcal{U} = \{U_i\}_i$ of M.

ii) Smooth transition maps $g_{ij}\colon U_{ij} := U_i \cap U_j \longrightarrow \mathrm{U}(1)$. (These are the gluing maps that define the line bundle).

iii) A collection $(A_i)_i$ of 1-forms $A_i \in \Omega^1(U_i)$ that together are referred to as the field potential.

iv) These forms must satisfy the following equations:

   a) g_{ij} is a cocycle (i.e. $g_{ij}g_{jk} = g_{ik}$ on U_{ijk})
   b) $dA_i = dA_j$ on $U_{ij} = U_i \cap U_j$.
   c) $A_j - A_i = -\sqrt{-1}\, g_{ij}^{-1} dg_{ij}$.

v) The 2-form $\omega = F = dA \in \Omega^2(M)$ is called the **curvature** of the connection A.

It is an immediate consequence of the definition that the *Bianchi identity* is satisfied, that is:

$$dF = 0$$

and therefore we have a de Rham cohomology class $-[F] \in H^2(M, \mathbb{R})$.

We can use the fact that g_{ij} is a cocycle and consider its Čech cohomology class $[g] \in H^1(M, \underline{U(1)})$ where $\underline{U(1)}$ is considered as a sheaf over M. The exponential sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \overset{\exp(2\pi i\_)}{\to} U(1) \longrightarrow 1$$

immediately implies an isomorphism

$$H^1(M, \underline{U(1)}) \cong H^2(M, \mathbb{Z})$$

The class of $[g]$ in $H^2(M, \mathbb{Z})$ is called *the Chern class* $c_1(L)$ *of* L.

It is a theorem of Weil [Wei52] that $-[F]$ is the image of the Chern class $c_1(L)$ under the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. The Chern class completely determines the isomorphism type of the line bundle L, but does not determine the isomorphism class of the connection.

We say that a line bundle with connection is flat if its curvature vanishes. We have therefore that if a line bundle with connection is flat then its Chern class is a torsion class.

To solve the Maxwell equations is therefore equivalent to finding a line bundle with connection that in addition satisfies the field equation $d * F = j_E$. Let us for a moment consider the equation in the vacuum, namely consider the case of the field equation of the form $d * F = 0$. We can write a rather elegant variational problem that solves the Maxwell equations in the vacuum (we learned this formulation from Dan Freed). Moreover, we can do so in a manner that exhibits fully the magnetic-electric duality of the problem. Let A' be a second connection so that $F' = *F$. The electromagnetic Lagrangian is

$$L(A, A') = \int_M \left(\frac{1}{4}|F|^2 + \frac{1}{4}|F'|^2 \right) dV$$

Clearly the equations in the vacuum are the Euler-Lagrange equation for $L(A, A')$, namely $\delta L = 0$.

To add charges to the previous Lagrangian we consider a electrically charged particle whose worldline is a mapping γ from a compact one-dimensional manifold to M. We consider the charge as an element $q \in H^0(\gamma, \mathbb{Z}) = \{q | q\colon \gamma \longrightarrow \mathbb{Z}\}$. To identify this with the charge as an element in $H^3_c(M, \mathbb{Z})$ we us the Gysin map in cohomology

$$i_!\colon H^0(\gamma, \mathbb{Z}) \longrightarrow H^3_c(M, \mathbb{Z})$$

given by the Thom-Pontrjagin collapse map and the Thom isomorphism. We can write the new Lagrangian that includes charges

$$L = \int_M \left(\frac{1}{4}|B|^2 + \frac{1}{4}|B'|^2 \right) dV + i \int_\gamma \frac{1}{2} qA$$

Several remarks are in order.

- We have switched notations. We call B what we used to call F. This is unfortunate but matches better the rest of the discussion.
- It is no longer true that $dB = 0$ (that is after all the whole point). In fact B is no longer a global form.
- Likewise A is not a global form an actually only $\exp\left(i \int_\gamma qA\right)$ is well defined. Nevertheless the Lagrangian does define the correct Euler-Lagrange equations.

This situation is no longer a form of a line bundle with a connection. In spite of this, there is a geometric interpretation of the previous situation. This can be seen as a motivation for the introduction of the concept of *gerbe* (cf. [Hit01]). (For more details on the physics see [FH00, Fre00].)

Definition 12.2. Let M be a manifold. A **gerbe** with connection on M is given by the following data:

i) A good Leray atlas $\mathcal{U} = \{U_i\}_i$ of M.

ii) Smooth maps $g_{ijk}\colon U_{ijk} \longrightarrow U(1)$.

iii) A collection (A_{ij}) of 1-forms $A_{ij} \in \Omega^1(U_{ij})$.

iv) A collection B_i of 2-forms $B_i \in \Omega^2(U_i)$

v) These forms must satisfy the following equations:

   a) g_{ijk} is a cocycle (i.e. $g_{ijk} g_{ijl}^{-1} g_{ikl} g_{jkl}^{-1} = 1$).

b) $A_{ij} + A_{jk} - A_{ik} = -\sqrt{-1}\, d \log g_{ijk}$

c) $B_j - B_i = dA_{ij}$

vi) The global 3-form $\omega = dB \in \Omega^3(M)$ is called the **curvature** of the gerbe with connection (g, A, B).

The class $[g_{ijk}] \in H^2(M, \underline{U(1)}) \cong H^3(M, \mathbb{Z})$ (where the isomorphism is induced by the exponential sequence of sheaves) is called the *Dixmier-Douady* class of the gerbe and is denoted by $dd(g)$. Just as before the class $[\omega] \in H^3(M, \mathbb{R})$ in de Rham cohomology is the real image of the Dixmier-Douady class $dd(g) \in H^3(M, \mathbb{Z})$.

Gerbes on M are classified up to isomorphism by their Dixmier-Douady class $dd(g) \in H^3(M, \mathbb{Z})$. This again ignores the connection altogether. In any case we have the following fact.

Proposition 12.3. *An isomorphism class of a gerbe on M is the same as an isomorphism class of an infinite-dimensional Hilbert projective bundle on M.*

Proof. We will use Kuiper's theorem that states that the group $U(\mathcal{H})$ of unitary operators in a Hilbert space $\mathcal{H}$ is contractible, and therefore one has

$$\mathbf{P}(\mathbb{C}^\infty) \simeq K(\mathbb{Z}, 2) \simeq BU(1) \simeq U(\mathcal{H})/U(1) = \mathbf{P}U(\mathcal{H}).$$

This fact immediately implies $K(\mathbb{Z}, 3) \simeq B\mathbf{P}U(\mathcal{H})$. Hence the class $dd(g) \in H^3(X, \mathbb{Z}) = [X, K(\mathbb{Z}, 3)] = [X, B\mathbf{P}U(\mathcal{H})]$ produces a Hilbert projective bundle $\mathbf{E}$.

♣

In fact more is true. The collection of all gerbes in M form a group under tensor product since U(1) is abelian (multiplication of the cocycles), and so do the set of all Hilbert projective bundles. One can prove that these two groups are isomorphic.

A gerbe with connection is said to be *flat* if its curvature vanishes. Notice the following consequence of this fact,

Proposition 12.4. *A gerbe with connection is flat if and only if $dd(g)$ is a torsion class in cohomology. This is the case if and only if the projective bundle $\mathbf{E}$ is finite dimensional.*

Proof. This is true because of a result of Serre [DK70] valid for any CW-complex M. It states that if a class $\alpha \in H^3(M, \mathbb{Z})$ is a torsion element then there exists a principal bundle $Z \to M$ with structure group $\mathbf{P}$U(n) so that when seen as an element

$\beta \in [M, B\mathbf{P}U(n)] \to [M, B\mathbf{P}U] = [M, BBU(1)] = [M, BK(\mathbb{Z}, 2)] = [M, K(\mathbb{Z}, 3)] = H^3(M, \mathbb{Z})$ then $\alpha = \beta$. In other words, the image of $[M, B\mathbf{P}U(n)] \to H^3(M, \mathbb{Z})$ is exactly the subgroup of torsion elements that are killed by multiplication by n.

♣

We refer the reader to the paper [CM] for gerbes from the point of view of bundle gerbes.

12.1.2 Gerbes over Orbifolds

In this section we discuss definitions and result first introduced in [LU04a].

Example 12.1. Let us recast the definition of gerbe over a manifold $(M, \mathcal{U})$, with Leray groupoid $M_{\mathcal{U}}$. Notice than in this case

- $(M_{\mathcal{U}})_0 = \coprod_i U_i$
- $(M_{\mathcal{U}})_1 = \coprod_{(i,j)} U_{ij}$
- $(M_{\mathcal{U}})_2 = \coprod_{(i,j,k)} U_{ijk}$

and so on.

To have a gerbe over an orbifold is the same as to have a map $g : (M_{\mathcal{U}})_2 \longrightarrow U(1)$ satisfying the cocycle condition. The data defining a gerbe with connection are in addition forms $A \in \Omega^1((M_{\mathcal{U}})_1)$ and $B \in \Omega^2((M_{\mathcal{U}})_0$, satisfying the equations of definition 12.2

Definition 12.5. A gerbe (with band U(1)) over an orbifold is a pair (G, g) where G is a groupoid representing the orbifold and g is a 2-cocycle $g\colon G_2 \to U(1)$. A gerbe with connection consists of a 1-form $A \in \Omega^1(G_1)$, a 2-form $B \in \Omega^2(G_0)$ satisfying:

- $t^*B - s^*B = dA$ and
- $\pi_1^*A + \pi_2^*A - m^*A = -\sqrt{-1}g^{-1}dg$

The G-invariant 3-form $\omega = dB \in \Omega^3(G_0)$ is called the curvature of the gerbe with connection (g, A, B). Here by G-invariant we mean that $s^*\omega = t^*\omega$.

The following theorem of [LU04a] describes the basic classification of gerbes over orbifold (without a connection).

Theorem 12.6. *The following holds.*

- *Every gerbe on an orbifold has a representative of the form* (G, g) *where* G *is a Leray groupoid.*
- *We define the characteristic class* $\ell(g)$ *of* g *to be the class in* $H^3(B\mathsf{G}, \mathbb{Z}) \simeq H^3(\mathsf{G}, \mathbb{Z}) \simeq H^2(\mathsf{G}, \underline{U(1)})$ *induced by the Čech cocycle* $g \in C^2(\mathsf{G}, \underline{U(1)})$*. Then isomorphism classes of gerbes over the orbifold* G *are in one to one correspondence with* $H^3(B\mathsf{G}, \mathbb{Z})$ *via the class* $\ell(g)$*.*

To classify gerbes with connection (g, A, B) up to isomorphism we need to introduce a new type of cohomology. We define now the so-called *Beilinson-Deligne cohomology* of G. We will be expository at this point and refer the reader to [LU02a, LU06a] for full details.

A G-sheaf is a sheaf over G on which G acts continuously. Let $\mathcal{A}^p_{\mathsf{G}}$ denote the G-sheaf of differential p-forms and $\mathbb{Z}_{\mathsf{G}}$ the constant $\mathbb{Z}$ valued G sheaf with $\mathbb{Z}_{\mathsf{G}} \to \mathcal{A}^0_{\mathsf{G}}$ the natural inclusion of constant into smooth functions.

Let's denote by $\check{C}^*(\mathsf{G}; U(1)(q))$ the total complex

$$\check{C}^0(\mathsf{G}; U(1)(q)) \xrightarrow{\delta - d} \check{C}^1(\mathsf{G}; U(1)(q)) \xrightarrow{\delta + d} \check{C}^2(\mathsf{G}; U(1)(q)) \xrightarrow{\delta - d} \cdots$$

induced by the double complex

$$\begin{tikzcd}
\vdots & \vdots & & \vdots \\
\Gamma(\mathsf{G}_2, U(1)_{\mathsf{G}}) \arrow[u, "\delta"] \arrow[r, "-\sqrt{-1}\,d\log"] & \Gamma(\mathsf{G}_2, \mathcal{A}^1_{\mathsf{G}}) \arrow[u, "\delta"] \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Gamma(\mathsf{G}_2, \mathcal{A}^{q-1}_{\mathsf{G}}) \arrow[u, "\delta"] \\
\Gamma(\mathsf{G}_1, U(1)_{\mathsf{G}}) \arrow[u, "\delta"] \arrow[r, "-\sqrt{-1}\,d\log"] & \Gamma(\mathsf{G}_1, \mathcal{A}^1_{\mathsf{G}}) \arrow[u, "\delta"] \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Gamma(\mathsf{G}_1, \mathcal{A}^{q-1}_{\mathsf{G}}) \arrow[u, "\delta"] \\
\Gamma(\mathsf{G}_0, U(1)_{\mathsf{G}}) \arrow[u, "\delta"] \arrow[r, "-\sqrt{-1}\,d\log"] & \Gamma(\mathsf{G}_0, \mathcal{A}^1_{\mathsf{G}}) \arrow[u, "\delta"] \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Gamma(\mathsf{G}_0, \mathcal{A}^{q-1}_{\mathsf{G}}) \arrow[u, "\delta"]
\end{tikzcd} \tag{45}$$

with $(\delta + (-1)^i d)$ as coboundary operator, where the δ's are the maps induced by the simplicial structure of the nerve of the category G and $\Gamma(\mathsf{G}_i, \mathcal{A}^j_{\mathsf{G}})$ stands for the global sections of the sheaf that induces $\mathcal{A}^j_{\mathsf{G}}$ over G_i (see [LU06a]). Then the Beilinson-Deligne cohomology is defined as as follows:

$$H^n(\mathsf{G}, \mathbb{Z}(q)) \cong H^{n-1}(\mathsf{G}, U(1)(q)) := H^{n-1}\check{C}(\mathsf{G}; U(1)(q)).$$

It is proved in [LU02a] that $\mathsf{H}^n(\mathsf{G}, \mathbb{Z}(q))$ only depends on the orbifold and not on the particular groupoid used to represent it, therefore we write $\mathsf{H}^n(\mathcal{X}, \mathbb{Z}(q)) := \mathsf{H}^n(\mathsf{G}, \mathbb{Z}(q))$. In the same paper the notation $\mathsf{H}^n(\mathsf{G}, \mathbb{Z}(q))$ (given by a refined version of the exponential sequence of sheaves for complexes of sheaves) is explained.

We have the following.

Proposition 12.7. *For* G *a Leray description of a smooth étale groupoid, a gerbe with connection is a 2-cocycle of the complex* $\check{\mathsf{C}}(\mathsf{G}, \mathrm{U}(1)(3))$*, that is, a triple* (h, A, B) *with* $B \in \Gamma(\mathsf{G}_0, \mathcal{A}^2_{\mathsf{G}})$*,* $A \in \Gamma(\mathsf{G}_1, \mathcal{A}^1_{\mathsf{G}})$ *and* $h \in \Gamma(\mathsf{G}_2, \mathrm{U}(1)_{\mathsf{G}})$ *that satisfies* $\delta B = dA$*,* $\delta A = -\sqrt{-1} d \log h$ *and* $\delta h = 1$*.*

Definition 12.8. An n-gerbe with connective structure over G is an $(n+1)$-cocycle of $\check{\mathsf{C}}^{n+1}(\mathsf{G}, \mathrm{U}(1)(n+2))$. Their isomorphism classes are classified by

$$\mathsf{H}^{n+1}(\mathsf{G}, \mathrm{U}(1)(n+2)) = \mathsf{H}^{n+2}(\mathsf{G}, \mathbb{Z}(n+2)).$$

The following theorems were proved in [LU02a, LU06a].

Proposition 12.9.

$$\mathsf{H}^p(\mathsf{G}, \mathbb{Z}(n)) \cong \mathsf{H}^{p-1}(\mathsf{G}, \mathrm{U}(1)(n)) = \begin{cases} \mathsf{H}^{p-1}(\mathsf{G}, \underline{\mathrm{U}(1)}) = \mathsf{H}^p(\mathsf{G}, \mathbb{Z}) & \text{for } p > n \\ \mathsf{H}^{p-1}(\mathsf{G}, \mathrm{U}(1)) & \text{for } p < n \end{cases}$$

where $\underline{\mathrm{U}(1)}$ *stands for the sheaf of* $\mathrm{U}(1)$ *valued functions.*

We have argued in [LU02a] that a B-field in the physics terminology for type II orbifold superstring theories is the same as a gerbe with connection on the orbifold.

The following theorem generalizes a result of Brylinski that he proved in the case of a smooth manifold M [Bry93].

Theorem 12.10. *We have the following classifications.*

- *The group of isomorphism classes of line orbibundles with connection on* G *is isomorphic to* $\mathsf{H}^2(M, \mathbb{Z}(2))$*.*
- *The group of isomorphism classes of gerbes with connection on* G *is isomorphic to* $\mathsf{H}^3(M, \mathbb{Z}(3))$*.*

Remark 12.11. It is quite interesting to point out that if $[g, A, B]$ is the BD-class of (g, A, B) then $\omega = dB$ is completely determined by $[g, A, B]$. We call the 3-from ω the curvature of the class $[g, A, B]$. An analogous definition can be made for n-gerbes yielding a $(n+2)$-form ω.

A *discrete torsion* on an orbifold $\mathcal{X} = [M/G]$ is a 2-cocycle $\theta\colon G \times G \to U(1)$ in the bar group cohomology complex of G [VW95] (cf. [Sha02]).

Proposition 12.12. *[LU02a] For a global orbifold* $[M/G]$ *the map* $\theta \mapsto (\theta, 0, 0)$ *injects the group of discrete torsions of an orbifold into the group of flat gerbes (=flat B-fields). In fact the induced map in cohomology* $H^3(G, \mathbb{Z}) \longrightarrow H^3(\mathcal{X}, \mathbb{Z}(3))$ *is injective.*

Remark 12.13. Let us remark that the gerbes coming from discrete torsion do not amount to all the flat gerbes. Consider the case in which $G = \{1\}$ and $H^2(M, U(1)) \neq 0$, then there is no discrete torsion but there are non trivial flat gerbes.

12.1.3 Holonomy

To warm up consider a line bundle with connection (L, g, A) over a manifold $(M, \mathcal{U})$. Classically the holonomy of (L, g, A) determines for every path $\gamma\colon [0, T] \longrightarrow M$ a linear mapping

$$\mathrm{hol}_{(L,g,A)}(\gamma)\colon L_{\gamma(0)} \longrightarrow L_{\gamma(T)}$$

that composes well with path concatenation. On a chart $\gamma\colon [0, T] \longrightarrow V \in \mathbb{R}^n$ of M where $L = V \times \mathbb{C}$ we can write such a map simply as an element in $U(1)$ by

$$\mathrm{hol}_{(L,g,A)}(\gamma) = \exp\left(2\pi i \int_\gamma A\right).$$

This formula is enough to completely define the holonomy for manifolds in general in view of the following.

Proposition 12.14. *Let* $\mathcal{S}^0(M)$ *be the 0-th Segal category of* M *having*

- *Objects: The points* $m \in M$.
- *Arrows: Paths* $\gamma\colon [0, T] \longrightarrow M$ *with composition given by concatenation of paths.*

Then the holonomy of a line bundle with connection defines a functor

$$\mathrm{hol}_{(g,A)}\colon \mathcal{S}^0(M) \to \mathsf{Vector\ Spaces}_1(\mathbb{C})$$

from $\mathcal{S}^0(M)$ *to the category of 1-dimensional vector spaces with linear isomorphisms.*

Notice that we can restrict our attention to the closed paths (automorphisms of $\mathcal{S}^0(M)$) to obtain a function on the loop space $\mathcal{L}M$ of M

$$\mathrm{hol}^{\circ}_{(g,A)} \colon \mathcal{L}M \to \mathrm{U}(1)$$

We consider this function as an element $\mathrm{hol}^{\circ}_{(g,A)} \in H^0(\mathcal{L}M, \underline{U(1)})$.

Definition 12.15. The *transgression map* $H^2(M;\mathbb{Z}) \to H^1(\mathcal{L}M;\mathbb{Z})$ is defined as the following composition. Let

$$S^1 \times \mathcal{L}M \longrightarrow M$$

be the *evaluation map* sending $(z,\gamma) \mapsto \gamma(z)$. We can use this map together with the Künneth theorem and the fact that $H^1(S^1;\mathbb{Z}) = \mathbb{Z}$ to get

$$H^2(M;\mathbb{Z}) \to H^2(S^1 \times \mathcal{L}M;\mathbb{Z}) \cong H^2(\mathcal{L}M;\mathbb{Z}) \oplus (H^1(\mathcal{L}M;\mathbb{Z}) \otimes H^1(S^1;\mathbb{Z}))$$

$$\xrightarrow{\cong} H^2(\mathcal{L}M;\mathbb{Z}) \oplus H^1(\mathcal{L}M;\mathbb{Z}) \to H^1(\mathcal{L}M;\mathbb{Z}) \cong H^0(\mathcal{L}M;\underline{U(1)})$$

(where the next to last map is projection into the second component, and the last is induced by the exponential sequence).

Proposition 12.16. *The element* $\mathrm{hol}^{\circ}_{(g,A)} \in H^0(\mathcal{L}M, \underline{U(1)})$ *is the image of* $c_1(g) \in H^2(M,\mathbb{Z})$ *under the transgression map.*

This implies that $\mathrm{hol}^{\circ}_{(g,A)}$ depends only on the Chern class (namely on the isomorphism class of (L,g) and not on the specific connection A. So the functor $\mathrm{hol}_{(g,A)}$ contains more information that $\mathrm{hol}^{\circ}_{(g,A)}$.

Example 12.2. Suppose that $\omega = dA = 0$, so the line bundle L is flat. Then $c_1(g)$ is a torsion class. In this case the holonomy induces a homomorphism $\rho \colon \pi_1(M) \longrightarrow \mathrm{U}(1)$ that determines the functor $\mathrm{hol}_{(g,A)}$ up to natural transformation.

Let us consider consider the holonomy as a map

$$\mathrm{hol}^{Z}_{(g,A)} \colon Z_1(M) \longrightarrow \mathrm{U}(1),$$

where $Z_1(M)$ are the closed smooth 1-chains on M. We define χ to be

$$\chi := -\frac{\sqrt{-1}}{2\pi} \log \mathrm{hol}^{Z}.$$

If we consider the curvature of L as a 2-form ω on M we have obtained a pair (χ,ω) with

$$\chi \colon Z_1(M) \longrightarrow \mathbb{R}/\mathbb{Z}$$

and

$$\chi(\partial c) = \int_c \omega \mod \mathbb{Z}$$

whenever c is a smooth 2-chain (the pair (χ, ω) is called a differential character).

Following Cheeger-Simons [CS85] we will denote by $\hat{H}^2_{cs}(M)$ the group of such differential characters of M.

If we substitute the line bundle by a $(q-2)$-gerbe with connection. The holonomy becomes now a homomorphism $Z_{q-1}(M) \to \mathrm{U}(1)$, then we can define in general $\hat{H}^q_{cs}(M)$.

The following theorem [Bry93, BCM+02] relates the CS-cohomology to the BD-cohomology of a manifold M:

Theorem 12.17.

$$H^q(M; \mathbb{Z}(q)) \cong \hat{H}^q_{cs}(M).$$

Actually the holonomy of a gerbe can also be seen as a functor.

Theorem 12.18. *Let* $\mathcal{S}^1(M)$ *be the 1-st Segal category of* M *having*

- *Objects: Maps* $\gamma \colon S^1 \coprod \ldots \coprod S^1 \longrightarrow M$.
- *Arrows: Maps* $\Sigma \colon F \longrightarrow M$ *from 2-dimensional compact manifolds* F *to* M *forming cobordisms between two objects, with composition given by concatenation of surfaces.*

Then the holonomy of a gerbe with connection (g, A, B) *defines a functor*

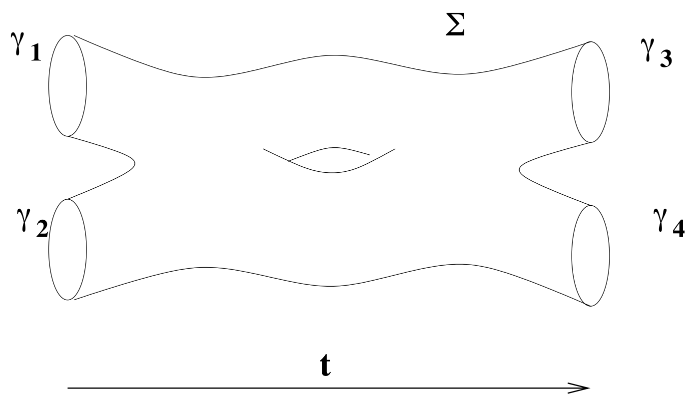
$$\mathrm{hol}_{(g,A,B)} \colon \mathcal{S}^1(M) \to \mathsf{Vector\ Spaces}_1(\mathbb{C})$$

from $\mathcal{S}^1(M)$ *to the category of 1-dimensional vector spaces with linear isomorphisms. Such a functor is called a* string connection*.*

For instance, in the picture below we have four maps $\gamma_i \colon S^1 \to M$ $(i = 1, 2)$ and a map $\Sigma \colon F \to M$ from a 2-dimensional manifold F into M. Such a configuration would produce a linear isomorphism

$$\mathrm{hol}_{(g,A,B)}(\Sigma) \colon L_{\gamma_1} \otimes L_{\gamma_2} \longrightarrow L_{\gamma_3} \otimes L_{\gamma_4}.$$

Where L is a line bundle on $\mathcal{L}M$ defined by the functor. The reader may imagine that these are two strings evolving and interacting in M if she prefers to do so.



(46)

Now consider in general an orbifold $\mathcal{X}$. We will describe now the results of [LU02b, LU06a, LU06b, LU] that refine the previous results to the case of orbifolds.

Recall that we have defined an infinite dimensional orbifold, *the loop orbifold* $\mathsf{L}\mathcal{X}$ associated to $\mathcal{X}$ by giving an explicit groupoid representation of it that we call the loop groupoid. Let Γ be a finite group.

As we have studied before an orbifold loop on $\mathcal{X} = [M/G]$ will consist of a map $\phi\colon Q \to M$ of a Γ-principal bundle Q over the circle S^1 together with a homomorphism $\phi_\#\colon \Gamma \to G$ such that ϕ is $\phi_\#$-equivariant. Let us denote this space of orbifold loops $(\phi, \phi_\#)$ by $\mathcal{L}[M/G]$. It has a natural action of the group G as follows. For $h \in G$ let $\psi := \phi \cdot h$ where $\psi(x) := \phi(x)h$ and $\psi_\#(\tau) = h^{-1}\phi_\#(\tau)h$, then $\psi\colon Q \to M$ and is $\psi_\#$ equivariant. We have called the (infinite dimensional) orbifold given by the groupoid $\mathcal{L}\mathcal{X}$ the *loop orbifold* in section 7.2.

We need to consider the equivalent definition for a morphism from a Riemann surface with boundary to the orbifold $[M/\Gamma]$. This will consist of a map $\Phi : P \to M$ of a Γ-principal bundle P over an oriented Riemann surface Σ (Γ finite) and a homomorphism $\Phi_\#\colon \Gamma \to G$ such that Φ is $\Phi_\#$-equivariant. Note that there is a natural action of the group G on Φ. It is defined in the same way as for the loop orbifold.

To define string connections in the case of orbifolds we must deal in one way or the other with 2-categories. Roughly speaking we *define* $\mathcal{S}^1(\mathcal{X})$ as a 2-category where the objects are orbifold loops $(\phi, \phi_\#)$, the arrows are orbifold surface maps as above. Then the boundary ∂P of P will consist of p incoming orbifold loops $\gamma_i\colon Q_i \to M$ $1 \le i \le p$ with the induced orientation, and q outgoing ones $\gamma_j\colon \overline{Q}_j \to M$, $p+1 \le j \le p+q$ with the opposite orientation so that $\partial P = \bigsqcup_i Q_i \sqcup \bigsqcup_j Q_j$. Here the Q_i's and the Q_j's are Γ-principal bundles over the circle. The 2-morphism of the

2-category are given by the natural action of G on the orbifold surface maps. We will define an *orbifold string connection* for $\mathcal{X} = [\mathsf{M}/\mathsf{G}]$ to be a 2-functor $\mathcal{S}^1(\mathcal{X}) \longrightarrow \mathsf{Vector\ Spaces}_1(\mathbb{C})$, namely a G-equivariant ordinary functor.

In [LU06a] we prove the following refined version of the transgression (for a general orbifold G).

Theorem 12.19. *There is a natural holonomy homomorphism*

$$\tau_2 : \check{\mathsf{C}}^2(\mathsf{G}, \mathrm{U}(1)(3)) \longrightarrow \check{\mathsf{C}}^1(\mathsf{LG}, \mathrm{U}(1)(2))$$

from the group of gerbes with connection over the orbifold G *to the group of line bundles with connection over the loop groupoid. Moreover this holonomy map commutes with the coboundary operator and therefore induces a map in orbifold Beilinson-Deligne cohomology*

$$\mathsf{H}^3(\mathsf{G}; \mathbb{Z}(3)) \longrightarrow \mathsf{H}^2(\mathsf{G}; \mathbb{Z}(2)).$$

In fact we give a proof for the corresponding statement in n-gerbes. So given a gerbe $\mathsf{L} = (g, \mathsf{A}, \mathsf{B})$ we obtain a line orbibundle E over the loop orbifold $\mathsf{L}\mathcal{X}$.

We remind the reader of two basic facts

Definition 12.20. The inertia groupoid $\mathrm{I}(\mathsf{G})$ is defined by:

- Objects $\mathrm{I}(\mathsf{G})_0$: Elements $v \in \mathsf{G}_1$ such that $s(v) = \mathsf{t}(v)$.
- Morphisms $\mathrm{I}(\mathsf{G})_1$: For $v, w \in \mathrm{I}(\mathsf{G})_0$ an arrow $v \xrightarrow{\alpha} w$ is an element $\alpha \in \mathsf{G}_1$ such that $v \cdot \alpha = \alpha \cdot w$

$$\begin{tikzcd} \circ \arrow[loop left, "v"] \arrow[r, bend left, "\alpha"] & \circ \arrow[l, bend left, "\alpha^{-1}"] \arrow[loop right, "w^{-1}"] \end{tikzcd}$$

Theorem 12.21. *The fixed suborbifold of* LG *under the natural* S^1*-action (rotating the loops) is*

$$\mathrm{I}(\mathsf{G}) = (\mathsf{LG})^{\mathsf{S}^1}$$

The following definition is due to Ruan [Rua02a, PRY08, Rua03]. He used this definition to obtain a twisted version of the Chen-Ruan cohomology [CR04b] that has revived the interest in the theory of orbifolds in the last few years.

Definition 12.22. An *inner local system* is a flat line bundle $\mathcal{L}$ over the inertia groupoid $\mathrm{I}(\mathsf{G})$ such that:

- $\mathcal{L}$ is trivial once restricted to $\mathsf{e}(\mathsf{G}_0) \subset \mathrm{I}(\mathsf{G})_1$ (i.e. $\mathcal{L}|_{\mathsf{e}(\mathsf{G}_0)} = 1$) and

- $i^*\mathcal{L} = \mathcal{L}^{-1}$ where $i : I(G) \to I(G)$ is the inverse map (i.e. $(i(\nu, \alpha) = (\alpha^{-1}\nu\alpha, \alpha^{-1}))$.

Theorem 12.23. *The restriction of the holonomy of a gerbe with connection over* $I(G)$ *(that is a line bundle with connection over* LG*) is an inner local system on* $I(G)$.

In the case in which we have a Lie group acting with finite stabilizers these line bundles are the coefficients Freed-Hopkins-Teleman [FHT08] used to twist the cohomology of the twisted sectors in order to get a Chern character isomorphism with the twisted K-theory of the orbifold. We have used gerbes in [LU04a] to obtain twisted versions of K-theory that act as recipients of the charges of D-branes in string theory [Wit01] generalizing the work of Adem and Ruan [AR03].

Returning to the subject of string connections we have the following result.

Theorem 12.24. *Take a global gerbe* ξ *with connection over* $\mathcal{X} = [M/G]$ *and let* E *be the line bundle with connection induced by it via transgression. Then* ξ *permits to define a string connection* hol *extending the line bundle* E *of the loop groupoid* $\mathcal{L}\mathcal{X}$.

The analogous result for a general orbifold is more subtle and we refer the reader to [LU] for details. There we use this theorem to generalize the results of Freed and Witten [FW99] on anomaly cancellation in string theory to the orbifold case.

To conclude let us mention that building on an idea of Hopkins and Singer [HS05] we have defined orbifold Chern-Simons cohomology. The main difficulty here is to make sense of what an orbifold differential character should be [LU06b]. We make a definition in such a way that we can prove the following result (see [LU06b])

Theorem 12.25. *The orbifold Beilinson-Deligne cohomology and the orbifold Cheeger-Simons cohomology are canonically isomorphic.*

12.2 Discrete Torsion

12.2.1 Geometric Interpretation of Discrete Torsion

Recall that a discrete torsion α can be interpreted as a flat gerbe on $\mathcal{B}G := [\bullet/G]$ by Proposition 12.12. Let us consider the meaning of theorem 12.18 for the orbifold $\mathcal{X} = \mathcal{B}G = [\bullet/G]$ taking the place of M.

Observe that a map $\Sigma : F \to \mathcal{B}G$ is the same as a pair of a principal G-bundle P over F together with an equivariant map $P \to \bullet$ (cf. Example 7.18). We can of course forget the map $P \to \bullet$ and we lose nothing.

Therefore the 1-st Segal category $\mathcal{S}^1(\mathcal{B}G)$ of $\mathcal{B}G$ is the category $2\mathrm{Cob}_G$ whose objects are circles provided with a principal G-bundles and whose morphisms are surfaces F together with a principal G-bundle on them (cf. Definition 8.1.)

From Theorem 12.18 and Proposition 12.12 we conclude that for every α we obtain a holonomy functor:

$$\mathrm{hol}_\alpha \colon \mathcal{S}^1(\mathcal{B}G) = 2\mathrm{Cob}_G \to \mathsf{Vector\ Spaces}_1(\mathbb{C}).$$

To wit, this gives us a G-TQFT for every α. Invoking the fact that it is the same to have a G-TQFT as it is to have a G-Frobenius algebra this procedure produces a G-Frobenius algebra for every α.

All this can be translated to pure algebra and we do so in the next paragraphs.

12.2.2 Algebraic Interpretation of Discrete Torsion

Discrete torsion can be interpreted as be a normalized co-cycle $\alpha : G \times G \longrightarrow \mathbb{C}^*$ with values in $\mathbb{C}^*$. This means that for all triples in G we have

$$\delta\alpha(g,h,k) = \frac{\alpha(h,k)\alpha(g,hk)}{\alpha(g,h)\alpha(gh,k)} = 1\,,$$

which is equivalent to

$$\alpha(h,k)\alpha(g,hk) = \alpha(g,h)\alpha(gh,k)\,,$$

and moreover that

$$\alpha(g,1) = 1 = \alpha(1,g)\,.$$

Define the G-Frobenius algebra $\mathbb{C}_\alpha[G] = \bigoplus_{g\in G}\mathbb{C}_g$ of elements in $\mathbb{C}$ labeled by elements in G, with the following structural operations on generators

- Product
$$m_\alpha(g,h) = g \cdot_\alpha h := \alpha(g,h)gh$$

- Coproduct
$$\begin{aligned}\Delta_\alpha(h) &= \sum_{k\in G} \alpha(gk,k^{-1})^{-1} gk \otimes k^{-1} \\ &= \sum_{k\in G} \alpha(k,k^{-1}g)^{-1} k \otimes k^{-1}g\end{aligned}$$

- Automorphism $\rho^\alpha : G \longrightarrow \mathrm{Aut}(\mathbb{C}_\alpha[G])$ which is generated by

$$\rho^\alpha_g(h) = \frac{\alpha(g,h)}{\alpha(ghg^{-1},g)} ghg^{-1}$$

for any $g, h \in G$.

Proposition 12.26. *Let $\alpha : G \times G \to \mathbb{C}^*$ be a normalized cocycle. Then the vector space $\mathbb{C}_\alpha[G]$ endowed with the structural operations $m_\alpha, \Delta_\alpha, \rho^\alpha$ becomes a G-Frobenius algebra.*

Proof.
- Associativity:

$$\begin{aligned} g \cdot_\alpha (h \cdot_\alpha k) &= \alpha(h,k) g \cdot_\alpha hk = \alpha(h,k)\alpha(g,hk)ghk \\ (g \cdot_\alpha h) \cdot_\alpha k &= \alpha(g,h) gh \cdot_\alpha k = \alpha(g,h)\alpha(gh,k)ghk \end{aligned}$$

- Coassociativity: on the one side we have

$$\begin{aligned} (1 \otimes \Delta_\alpha)\Delta_\alpha(g) &= (1 \otimes \Delta_\alpha) \sum_{k \in G} \alpha(gk, k^{-1}) gk \otimes k^{-1} \\ &= \sum_{k,l \in G} \alpha(gk,k^{-1})^{-1}\alpha(k^{-1}l, l^{-1})^{-1} gk \otimes k^{-1} \otimes l^{-1} \\ &= \sum_{k,l \in G} \alpha(gk^{-1},k)^{-1}\alpha(kl, l^{-1})^{-1} gk^{-1} \otimes kl \otimes l^{-1} \\ &= \sum_{k,l \in G} \alpha(k^{-1},kg)^{-1}\alpha(kgl, l^{-1})^{-1} k^{-1} \otimes kgl \otimes l^{-1} \end{aligned}$$

and on the other we have

$$\begin{aligned} (\Delta_\alpha \otimes 1)\Delta_\alpha(g) &= (\Delta_\alpha \otimes 1) \sum_{k \in G} \alpha(gl, l^{-1}) gl \otimes l^{-1} \\ &= \sum_{k,l \in G} \alpha(gl,l^{-1})^{-1}\alpha(k^{-1}, kgl)^{-1} k^{-1} \otimes kgl \otimes l^{-1}. \end{aligned}$$

Since $\delta\alpha(k^{-1}, kgl, l^{-1}) = 1$ we therefore we have

$$\alpha(kgl, l^{-1})\alpha(k^{-1}, kg) = \alpha(k^{-1}, kgl)\alpha(gl, l^{-1}).$$

The coassociativity follows.

- Unit

$$g \cdot_\alpha 1 = \alpha(g,1)g = g = \alpha(1,g)g = 1 \cdot_\alpha g$$

- Counit

$$(\varphi \otimes 1)\Delta_\alpha(g) = \sum_{k\in G} \alpha(gk, k^{-1})\varphi(gk)k^{-1} = \alpha(1, g)g = g$$

$$(1 \otimes \varphi)\Delta_\alpha(g) = \sum_{k\in G} \alpha(k^{-1}, kg)\varphi(kg)k^{-1} = \alpha(g, 1)g = g$$

- Frobenius identities

$$\Delta_\alpha(g \cdot_\alpha h) = \sum_{k\in G} \alpha(g, h)\alpha(ghk, k^{-1})^{-1} ghk \otimes k^{-1}$$

$$(m_\alpha \otimes 1) \circ (1 \otimes \Delta_\alpha)(g \otimes h) = \sum_{k\in G} \alpha(hk, k^{-1})^{-1}\alpha(g, hk)ghk \otimes k^{-1}$$

and $\delta\alpha(g, hk, k^{-1}) = 1$ then

$$\alpha(hk, k^{-1})\alpha(g, h) = \alpha(g, hk)\alpha(ghk, k^{-1})$$

$$\begin{aligned}(1 \otimes m_\alpha) \circ (\Delta_\alpha \otimes 1)(g \otimes h) &= \sum_{k\in G} \alpha(gk, k^{-1})^{-1}\alpha(k^{-1}, h)gk \otimes k^{-1}h \\ &= \sum_{k\in G} \alpha(ghk, k^{-1}h^{-1})^{-1}\alpha(k^{-1}h^{-1}, h)ghk \otimes k^{-1}\end{aligned}$$

and $\delta\alpha(ghk, k^{-1}h^{-1}, h) = 1$ then

$$\alpha(k^{-1}h^{-1}, h)\alpha(ghk, k^{-1}) = \alpha(ghk, k^{-1}h^{-1})\alpha(g, h)$$

- Twisted commutattivity of the product :

$$g \cdot_\alpha h = \alpha(g, h)gh = \frac{\alpha(g, h)}{\alpha(ghg^{-1}, g)}(ghg^{-1} \cdot_\alpha g) = \rho_g(h) \cdot_\alpha g$$

- The map ρ^α being a homomorphism: we have on the one hand

$$\rho^\alpha_k(\rho^\alpha_g(h)) = \frac{\alpha(g, h)}{\alpha(ghg^{-1}, g)}\rho^\alpha_k(ghg^{-1}) = \frac{\alpha(g, h)}{\alpha(ghg^{-1}, g)}\frac{\alpha(k, ghg^{-1})}{\alpha(kghg^{-1}k^{-1}, k)}kghg^{-1}k^{-1}$$

and on the other

$$\rho^\alpha_{kg}(h) = \frac{\alpha(kg, h)}{\alpha(kghg^{-1}k^{-1}, kg)}kghg^{-1}k^{-1}.$$

Now, note that

$$
\begin{aligned}
&\frac{\alpha(g,h)}{\alpha(ghg^{-1},g)}\frac{\alpha(k,ghg^{-1})}{\alpha(kghg^{-1}k^{-1},k)}\frac{\alpha(kghg^{-1}k^{-1},kg}{\alpha(kg,h)}\\
&=\delta\alpha(k,g,h)\frac{\alpha(k,g)}{\alpha(k,gh)}(\delta\alpha(k,ghg^{-1},g))^{-1}\frac{\alpha(k,gh)}{\alpha(kghg^{-1},g)}\delta\alpha(kghg^{-1}k^{-1},k,g)\quad\frac{\alpha(kghg^{-1},g)}{\alpha(k,g)}\\
&=1
\end{aligned}
$$

- The map ρ^α being an algebra map: on the one hand we have

$$\rho^\alpha_k(g\cdot_\alpha h)=\alpha(g,h)\rho^\alpha_k(gh)=\alpha(g,h)\frac{\alpha(k,gh)}{\alpha(kghk^{-1},k)}kghk^{-1}$$

and on the other

$$\rho^\alpha_k(g)\cdot_\alpha\rho^\alpha_k(h)=\alpha(kgk^{-1},khk^{-1})\frac{\alpha(k,g)}{\alpha(kgk^{-1},k)}\frac{\alpha(k,h)}{\alpha(khk^{-1},k)}kghk^{-1}.$$

A simple calculation shows that

$$
\begin{aligned}
\frac{\alpha(g,h)}{\alpha(kgk^{-1},khk^{-1})}\frac{\alpha(k,gh)}{\alpha(kghk^{-1},k)}\frac{\alpha(kgk^{-1},k)}{\alpha(k,g)}\frac{\alpha(khk^{-1},k)}{\alpha(k,h)}&\\
&=\delta\alpha(k,g,h)\cdot(\delta\alpha(kgk^{-1},k,h))^{-1}\cdot\delta\alpha(kgk^{-1},khk^{-1},k)\\
&=1
\end{aligned}
$$

- Torus axiom: we have to prove

$$m_\alpha\circ(\rho_h\otimes 1)(\Delta_\alpha(1)|_{A_g\otimes A_{g^{-1}}})=m_\alpha\circ(1\otimes\rho_g)(\Delta_\alpha(1)|_{A_h\otimes A_{h^{-1}}})$$

On the one hand we get

$$
\begin{aligned}
m_\alpha\circ(\rho_h\otimes 1)&(\Delta_\alpha(1)|_{A_g\otimes A_{g^{-1}}})\\
&=m_\alpha\circ(\rho_h\otimes 1)\left(\alpha(g,g^{-1})^{-1}g\otimes g^{-1}\right)\\
&=m_\alpha\left(\alpha(g,g^{-1})^{-1}\frac{\alpha(h,g)}{\alpha(hgh^{-1},h)}hgh^{-1}\otimes g^{-1}\right)\\
&=\frac{\alpha(hgh^{-1},g^{-1})}{\alpha(g,g^{-1})}\frac{\alpha(h,g)}{\alpha(hgh^{-1},h)}[h,g]
\end{aligned}
$$

and on the other

$$
\begin{aligned}
& m_\alpha \circ (1 \otimes \rho_g)(\Delta_\alpha(1)|_{A_h \otimes A_{h^{-1}}}) \\
&\qquad = m_\alpha \circ (1 \otimes \rho_g)\left(\alpha(h,h^{-1})^{-1} h \otimes h^{-1}\right) \\
&\qquad = m_\alpha \left(\alpha(h,h^{-1})^{-1} \frac{\alpha(g,h^{-1})}{\alpha(gh^{-1}g^{-1},g))} h \otimes gh^{-1}g^{-1}\right) \\
&\qquad = \frac{\alpha(h,gh^{-1}g^{-1})}{\alpha(h,h^{-1})} \frac{\alpha(g,h^{-1})}{\alpha(gh^{-1}g^{-1},g)} [h,g]
\end{aligned}
$$

since we have the identity

$$
\begin{aligned}
& \frac{\alpha(hgh^{-1},g^{-1})}{\alpha(g,g^{-1})} \frac{\alpha(h,g)}{\alpha(hgh^{-1},h)} \frac{\alpha(h,h^{-1})}{\alpha(h,gh^{-1}g^{-1})} \frac{\alpha(gh^{-1}g^{-1},g)}{\alpha(g,h^{-1})} \\
&\qquad = \frac{\delta\alpha(hg,h^{-1},h)}{\delta\alpha(gh^{-1},g^{-1},g)\delta\alpha(h,gh^{-1},g^{-1})\delta\alpha(h,g,h^{-1})} \\
&\qquad = 1
\end{aligned}
$$

it follows that the torus axiom is satisfied.

♣

12.2.3 The Tensor Product of nearly G-Frobenius Algebras

Given two G-nearly Frobenius algebras $(\mathcal{A},\alpha,\Delta^{\mathcal{A}})$ and $(\mathcal{B},\beta,\Delta^{\mathcal{B}})$ we can define a new G-nearly Frobenius algebra $(\mathcal{C},\gamma,\Delta)$ by

$$\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$$

where $\mathcal{C}_g = \mathcal{A}_g \otimes \mathcal{B}_g$ for all $g \in G$.

The homomorphism $\gamma : G \to \mathrm{Aut}(\mathcal{C})$ is defined by

$$
\begin{array}{rrcl}
\gamma_g = \alpha_g \otimes \beta_g : & \mathcal{C}_h = \mathcal{A}_h \otimes \mathcal{B}_h & \to & \mathcal{A}_{ghg^{-1}} \otimes \mathcal{B}_{ghg^{-1}} = \mathcal{C}_{ghg^{-1}} \\
& x_h \otimes y_h & \mapsto & \alpha_g(x_h) \otimes \beta_g(y_h)
\end{array}
$$

for all $g \in G$.

The coproduct $\Delta_{g,h} : \mathcal{C}_{gh} \to \mathcal{C}_g \otimes \mathcal{C}_h$ is defined by

$$\Delta_{g,h} = \left(1 \otimes \tau \otimes 1\right) \circ \left(\Delta^{\mathcal{A}}_{g,h} \otimes \Delta^{\mathcal{B}}_{g,h}\right).$$

These structural maps satisfy the next conditions:

1. $\gamma_g|_{\mathcal{C}_g} = \mathrm{Id}_{\mathcal{C}_g}$.

$$\gamma_g|_{\mathcal{C}_g} = (\alpha_g \otimes \beta_g)|_{\mathcal{A}_g \otimes \mathcal{B}_g} = \alpha_g|_{\mathcal{A}_g} \otimes \beta_g|_{\mathcal{B}_g} = \mathrm{Id}_{\mathcal{A}_g} \otimes \mathrm{Id}_{\mathcal{B}_g} = \mathrm{Id}_{\mathcal{C}_g}.$$

2. The product is twisted commutative, i.e. $uv = \gamma_g(v)u$, for all $u \in \mathcal{C}_g$ and $v \in \mathcal{C}_h$.

   Let $u = x_1 \otimes y_1 \in \mathcal{C}_g = \mathcal{A}_g \otimes \mathcal{B}_g$ and $v = x_2 \otimes y_2 \in \mathcal{C}_h = \mathcal{A}_h \otimes \mathcal{B}_h$:

$$\begin{array}{rcccl} uv & = & (x_1 \otimes y_1)(x_2 \otimes y_2) & = & x_1x_2 \otimes y_1y_2 \\ & = & \alpha_g(x_2)x_1 \otimes \beta_g(y_2)y_1 & = & (\alpha_g(x_2) \otimes \beta_g(y_2))(x_1 \otimes y_1) \\ & = & \gamma_g(x_2 \otimes y_2)(x_1 \otimes y_1) & = & \gamma_g(v)u. \end{array}$$

3. The family of coproducts verify that the diagrams

$$\begin{tikzcd} \mathcal{C}_g \otimes \mathcal{C}_{hk} \arrow[r, "m_{g,hk}"] \arrow[d, "1\otimes\Delta_{h,k}"'] & \mathcal{C}_{ghk} \arrow[d, "\Delta_{gh,k}"] \\ \mathcal{C}_g \otimes \mathcal{C}_h \otimes \mathcal{C}_k \arrow[r, "m_{g,h}\otimes 1"'] & \mathcal{C}_{gh} \otimes \mathcal{C}_k \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_{gh} \otimes \mathcal{C}_k \arrow[r, "m_{gh,k}"] \arrow[d, "\Delta_{g,h}\otimes 1"'] & \mathcal{C}_{ghk} \arrow[d, "\Delta_{g,hk}"] \\ \mathcal{C}_g \otimes \mathcal{C}_h \otimes \mathcal{C}_k \arrow[r, "1\otimes m_{h,k}"'] & \mathcal{C}_g \otimes \mathcal{C}_{hk} \end{tikzcd}$$

   commute.

   Remember that the coproducts $\Delta^{\mathcal{A}}$ and $\Delta^{\mathcal{B}}$ verify this property.

$$\begin{aligned} &\Delta_{gh,k} \circ m_{g,hk}\left((x_g \otimes y_g) \otimes (x_{hk} \otimes y_{hk})\right) \\ &= \Delta_{gh,k}(x_g x_{hk} \otimes y_g y_{hk}) \\ &= (1 \otimes \tau \otimes 1)\left(\Delta^{\mathcal{A}}_{gh,k}(x_g x_{hk}) \otimes \Delta^{\mathcal{B}}_{gh,k}(y_g y_{hk})\right) \\ &= (1 \otimes \tau \otimes 1)\left((m^{\mathcal{A}}_{g,h} \otimes 1)(x_g \otimes \Delta^{\mathcal{A}}_{h,k}(x_{hk})) \otimes (m^{\mathcal{B}}_{g,h} \otimes 1)(y_g \otimes \Delta^{\mathcal{B}}_{h,k}(y_{hk}))\right) \\ &= (1 \otimes \tau \otimes 1)(m^{\mathcal{A}}_{g,h} \otimes 1 \otimes m^{\mathcal{B}}_{g,h} \otimes 1)\left(x_g \otimes \Delta^{\mathcal{A}}_{h,k}(x_{hk}) \otimes y_g \otimes \Delta^{\mathcal{B}}_{h,k}(y_{hk})\right) \\ &= (m^{\mathcal{A}}_{g,h} \otimes m^{\mathcal{B}}_{g,h} \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1 \otimes 1 \otimes 1)\left(x_g \otimes y_g \otimes \Delta_{h,k}(x_{hk} \otimes y_{hk})\right) \\ &= (m_{g,h} \otimes 1) \circ (1 \otimes \Delta_{h,k})\left((x_g \otimes y_g) \otimes (x_{hk} \otimes y_{hk})\right) \end{aligned}$$

   The other case is similar.

4. The last condition is that the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "u"] \arrow[d, "u"'] & \mathcal{C}_e \arrow[r, "\Delta_{h,h^{-1}}"] & \mathcal{C}_h \otimes \mathcal{C}_{h^{-1}} \arrow[r, "1\otimes\gamma_g"] & \mathcal{C}_h \otimes \mathcal{C}_{gh^{-1}g^{-1}} \arrow[d, "m_{h,gh^{-1}g^{-1}}"] \\ \mathcal{C}_e \arrow[r, "\Delta_{g,g^{-1}}"'] & \mathcal{C}_g \otimes \mathcal{C}_{g^{-1}} \arrow[r, "\gamma_h\otimes 1"'] & \mathcal{C}_{hgh^{-1}} \otimes \mathcal{C}_{g^{-1}} \arrow[r, "m_{hgh^{-1},g^{-1}}"'] & \mathcal{C}_{hgh^{-1}g^{-1}} \end{tikzcd}$$

commutes.

$$1 \xmapsto{u} 1_{\mathcal{A}} \otimes 1_{\mathcal{B}} \xmapsto{\Delta_{h,h^{-1}}} 1_1^{\mathcal{A},h} \otimes 1_1^{\mathcal{B},h} \otimes 1_2^{\mathcal{A},h} \otimes 1_2^{\mathcal{B},h} \xmapsto{1\otimes\gamma_g}$$

$$1_1^{\mathcal{A},h} \otimes 1_1^{\mathcal{B},h} \otimes \alpha_g(1_2^{\mathcal{A},h}) \otimes \beta_g(1_2^{\mathcal{B},h}) \xmapsto{m_{h,ghg^{-1}}} 1_1^{\mathcal{A},h}\alpha_g(1_2^{\mathcal{A},h}) \otimes 1_1^{\mathcal{B},h}\beta_g(1_2^{\mathcal{B},h}) \ ,$$

on the other hand

$$1 \xmapsto{u} 1_{\mathcal{A}} \otimes 1_{\mathcal{B}} \xmapsto{\Delta_{g,g^{-1}}} 1_1^{\mathcal{A},g} \otimes 1_1^{\mathcal{B},g} \otimes 1_2^{\mathcal{A},g} \otimes 1_2^{\mathcal{B},g} \xmapsto{\gamma_h\otimes 1}$$

$$\alpha_h(1_1^{\mathcal{A},g}) \otimes \beta_h(1_1^{\mathcal{B},g}) \otimes 1_2^{\mathcal{A},g} \otimes 1_2^{\mathcal{B},g} \xmapsto{m_{hgh^{-1},g^{-1}}} \alpha_h(1_1^{\mathcal{A},g})1_2^{\mathcal{A},g} \otimes \beta_h(1_1^{\mathcal{B},g})1_2^{\mathcal{B},g} \ ,$$

using that $\Delta^{\mathcal{A}}_{g,h}$ and $\Delta^{\mathcal{B}}_{g,h}$ satisfy this property we have that

$$1_1^{\mathcal{A},h}\alpha_g(1_2^{\mathcal{A},h}) = \alpha_h(1_1^{\mathcal{A},g})1_2^{\mathcal{A},g}$$

and

$$1_1^{\mathcal{B},h}\beta_g(1_2^{\mathcal{B},h}) = \beta_h(1_1^{\mathcal{B},g})1_2^{\mathcal{B},g}.$$

Then the diagram commutes.

12.2.4 Twisting G-TQFTs by Discrete Torsion

To end this chapter we merely point out that given a nearly Frobenius algebra $\mathcal{A}$ and a discrete torsion $\alpha : G \times G \longrightarrow \mathbb{C}^*$ we can define the α-twisted Frobenius algebra $\mathcal{A}^\alpha$ by using the definition of the previous paragraph:

$$\mathcal{A}^\alpha := \mathcal{A} \otimes \mathbb{C}_\alpha[G].$$

This procedure allows us to twist any G-TQFT+ by a discrete torsion α.

13 Symmetric Products

The (naive) symmetric product of a space X is often defined as the *topological space*

$$X^n/\mathfrak{S}_n := X \times \cdots \times X/\mathfrak{S}_n.$$

We find that it is better to study instead the *orbispace*

$$[X^n/\mathfrak{S}_n] := [X \times \cdots \times X/\mathfrak{S}_n].$$

and we call it the *symmetric product of* X.

In this chapter we study the basic properties of the string topology of the symmetric product $[X^n/\mathfrak{S}_n]$, and also we give a description of the Virtual cohomology and of the Chen-Ruan cohomology associated to it.

13.1 Poincaré Polynomials

Let X be a topological space, we will denote by $\phi(X, y)$ its Poincaré polynomial

$$\phi(X, y) = \sum_i b^i(X) y^i$$

where $b^i(X)$ is the i-th Betti number of X.

Macdonald [Mac62] proved the formula:

$$\sum_{n=0}^{\infty} \phi(X^n/\mathfrak{S}_n, y) q^n = \frac{\prod_i \left(1 + qy^{2i+1}\right)^{b_{2i+1}(X)}}{\prod_i \left(1 - qy^{2i}\right)^{b_{2i}(X)}}.$$

Setting the variable $y = -1$ we get the famous formula for the Euler characteristic of the symmetric product:

$$\sum_{n=0}^{\infty} \chi(X^n/\mathfrak{S}_n) q^n = (1 - q)^{-\chi(X)}.$$

The previous formulæ are valid for topological spaces whose cohomology $H^i(X, \mathbb{R})$ is finitely generated for each $i \geq 0$, and there is no restriction on the homological dimension of X.

13.1.1 Equivariant Euler Characteristic

There is a similar formula associated to the equivariant Euler characteristic $\chi_{\mathfrak{S}_n}$ of the symmetric product, which is defined using the $\mathfrak{S}_n$-equivariant K-theory of X^n by the following expression,

$$\chi_{\mathfrak{S}_n}(X^n) := \text{Rank } K^0_{\mathfrak{S}_n}(X^n) - \text{Rank } K^1_{\mathfrak{S}_n}(X^n)$$

and can also be calculated using generating functions by the following formula

$$\sum_{n=0}^{\infty} \chi_{\mathfrak{S}_n}(X^n) q^n = \prod_{j>0} \left(1 - q^j\right)^{-\chi(X)}. \tag{47}$$

This last equation is obtained by using a formula due to Segal [Seg68b] that allows to calculate the torsion free part of $K^*_G(Y)$ (where G acts on Y and G is a finite group) by localizing on the prime ideals of $R(G)$, the representation ring of G; namely

$$K^*_G(Y) \otimes \mathbb{C} \cong \bigoplus_{(g)} K^*(Y^g)^{C(g)} \otimes \mathbb{C}$$

where (g) runs over the conjugacy classes of elements in G, Y^g are the fixed point loci of g and $C(g)$ is the centralizer of g in G.

For the symmetric group $\mathfrak{S}_n$, its conjugacy classes are in one-to-one correspondence with partitions of n. Given $\tau \in \mathfrak{S}_n$ we will write $\sum_j jn_j = n$ to denote the partition corresponding to its cojugacy class. Here n_j stands for the number of cycles of size j that appear in the τ. Then we have that the fixed point set $(X^n)^\tau$ is isomorphic to $X^{\sum_j n_j}$ and $C(\tau) \cong \prod_j \mathfrak{S}_{n_j} \ltimes (\mathbb{Z}/j)^{n_j}$. As the cylic groups $\mathbb{Z}/j$ act trivially in $K^*(X^{\sum_j n_j})$ the following decomposition holds

$$K^*_{\mathfrak{S}_n}(X^n) \otimes \mathbb{C} \cong \bigoplus_{(\tau)} K^*((X^n)^\tau)^{C(\tau)} \otimes \mathbb{C} \cong \bigoplus_{\sum jn_j = n} \otimes_j K^*(X^{n_j})^{\mathfrak{S}_{n_j}} \otimes \mathbb{C}.$$

Since the equivariant Euler characteristic can also be obtained via the Orbifold Cohomology, we postpone the proof of Formula (47) to the following section.

13.1.2 Orbifold Cohomology

For an orbifold $[Y/G]$ its *orbifold cohomology* is $H^*_{orb}([Y/G]) := H^*(Y, G)^G$, and therefore $H^*_{orb}([Y/G]) \cong \oplus_{(g)} H^*(Y^g)^{C(g)}$ where (g) runs over the conjugacy classes and

$C(g)$ is the centralizer of g in G. By the chern character isomorphism we have then $K^*_G(Y) \otimes \mathbb{C} \cong H^*_{orb}([Y/G])$.

We can define the Poincaré orbifold polynomial $\phi_{orb}([Y/G], y) = \sum b^i_{orb}([Y/G])y^i$ where the orbifold Betti number $b^i_{orb}([Y/G])$ is the rank of $H^i_{orb}([Y/G]))$.

For the symmetric product we get that

$$H^*_{orb}([X^n/\mathfrak{S}_n]) \cong \bigoplus_{\sum jn_j = n} \bigotimes_j H^*(X^{n_j})^{\mathfrak{S}_{n_j}} \tag{48}$$

and calculating the orbifold Poincaré polynomial one gets

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi_{orb}([X^n/\mathfrak{S}_n], y) q^n &= \sum_{n=0}^{\infty} q^n \left(\sum_{\sum jn_j = n} \prod_j \phi(X^{n_j}/\mathfrak{S}_{n_j}, y) \right) &(49)\\
&= \sum_{n=0}^{\infty} \left(\sum_{\sum jn_j = n} \prod_j \phi(X^{n_j}/\mathfrak{S}_{n_j}, y)(q^j)^{n_j} \right) &(50)\\
&= \prod_{j>0} \left(\sum_{n=0}^{\infty} \phi(X^n/\mathfrak{S}_n, y) q^{jn} \right) &(51)\\
&= \prod_{j>0} \frac{\prod_i (1 + q^j y^{2i+1})^{b^{2i+1}(X)}}{\prod_i (1 - q^j y^{2i})^{b^{2i}(X)}} &(52)
\end{aligned}$$

that when $y = -1$, yields the formula (47) for the equivariant Euler characteristic.

Again, for the previous formulæ to be valid one only needs that the cohomology of X is finitely generated at each degree.

13.1.3 Loop Orbifold of the Symmetric Product

For an orbifold [Y/G] the loop orbifold $L[Y/G]$ has been defined in [LU02b, LUX08] and for the case of a global quotient it has a very simple description: $L[Y/G] = [\mathcal{P}_G Y/G]$ where $\mathcal{P}_G Y = \sqcup_{g \in G} \mathcal{P}_g Y \times \{g\}$ with $\mathcal{P}_g Y = \{f \colon [0, 1] \to Y | f(0)g = f(1)\}$ and the G action is given by

$$\begin{aligned}
G \times \sqcup_{g \in G} \mathcal{P}_g Y \times \{g\} &\to \sqcup_{g \in G} \mathcal{P}_g Y \times \{g\}\\
(h, (f, g)) &\mapsto (f \cdot h, h^{-1} g h)
\end{aligned}$$

with $f \cdot h(t) := f(t)h$. The loop orbifold has another presentation (Morita equivalent) given by

$$L[Y/G] \cong \bigsqcup_{(g)} [\mathcal{P}_g Y/C(g)]$$

where $\mathsf{C}(g)$ acts on $\mathcal{P}_g Y$ in the natural way. It is a theorem proved in [LUX08] that $\mathsf{BL}[Y/G] \simeq \mathcal{L}\mathsf{B}[Y/G]$, i.e. the geometrical realization of the loop orbifold is homotopically equivalent to the free loop space of the geometrical realization of the orbifold, which in terms of the Borel construction states:

$$\bigsqcup_{(g)} \left(\mathcal{P}_g Y \times_{C(g)} EC(g)\right) \simeq \mathrm{Maps}(S^1, Y \times_G EG).$$

For the case of the symmetric product, one gets

$$L[X^n/\mathfrak{S}_n] \cong \bigsqcup_{(\tau)} [\mathcal{P}_\tau X^n/C(\tau)].$$

But there is a better presentation of this orbifold, namely, .

Lemma 13.1. *The orbifold* $[\mathcal{P}_\tau X^n/C(\tau)]$ *is isomorphic to* $\prod_j [(\mathcal{L}X)^{n_j}/\mathfrak{S}_{n_j} \ltimes (\mathbb{Z}/j)^{n_j}]$ *where the action of* $\mathbb{Z}/j$ *is given by rotation by the angles* $2\pi k/j$ *on* $\mathcal{L}X$*, the free loop space of* X*.*

Proof. When (τ) is represented by the product $\tau_1^1 \dots \tau_1^{n_1} \tau_2^1 \dots \tau_2^{n_2} \dots$ of disjoint cycles, with τ_j^i the i-th cycle of size j, and $\sum j n_j = n$, then

$$\mathcal{P}_\tau X^n \cong \prod_j \prod_{i=1}^{n_j} \mathcal{P}_{\tau_j^i} X^j \cong \prod_j (\mathcal{P}_{\sigma_j} X^j)^{n_j}$$

where σ_j is the cycle $(1, 2, \dots, j)$. Now, the space $\mathcal{P}_{\sigma_j} X^j$ consists of j-tuples $f = (f_1, \dots, f_j)$ of paths $f_i \colon [0,1] \to X$ such that $f(0)\sigma_j = f(1)$, i.e. $f_i(0) = f_{\sigma_j(i)}(1)$, which imply that the paths f_i could be concatenated into a loop $\tilde{f}$ which belongs to $\mathcal{L}X$. The map $\mathcal{P}_{\sigma_j} X^j \to \mathcal{L}X$, $f \mapsto \tilde{f}$ is clearly a homeomorphism.

We have then,

$$[\mathcal{P}_\tau X^n/C(\tau)] \cong \prod_j [(\mathcal{P}_{\sigma_j} X^j)^{n_j}/\mathfrak{S}_{n_j} \ltimes (\mathbb{Z}/j)^{n_j}] \cong \prod_j [(\mathcal{L}X)^{n_j}/\mathfrak{S}_{n_j} \ltimes (\mathbb{Z}/j)^{n_j}]$$

where the action of $\mathbb{Z}/j$ on an element $f = (f_1, \dots, f_j) \in \mathcal{P}_{\sigma_j} X^j$ is generated by the action of σ_j, namely $f \cdot \sigma_j = (f_j, f_1, \dots, f_{j-1})$. As $f_j(0) = f_1(1)$, then the cyclic action rotates the loop $\tilde{f}$ by an angle of $2\pi/j$.

♣

Since the action of $\mathbb{Z}/j$ in $\mathcal{L}X$ factors through the rotation action of the circle S^1 in $\mathcal{L}X$, then the action of the group $\mathbb{Z}/j$ is trivial in $H^*(\mathcal{L}X)$, therefore

Corollary 13.2.

$$H^*_{orb}(L[X^n/\mathfrak{S}_n]) \cong \bigoplus_{(\tau)} H^*(\mathcal{P}_\tau X^n)^{C(\tau)} \cong \bigoplus_{\sum jn_j = n} \prod_j H^*((\mathcal{L}X)^{n_j})^{\mathfrak{S}_{n_j}}$$

At this point we can see some similarities between the loop orbifold of the symmetric product of X, and the inertia orbifold of the symmetric product of $\mathcal{L}X$, namely that their rational cohomologies agree even though the orbifolds cannot be isomorphic

Proposition 13.3. *The orbifolds* $L[X^n/\mathfrak{S}_n]$ *and* $I[(\mathcal{L}X)^n/\mathfrak{S}_n]$ *cannot be naturally isomorphic unless* $n = 1$*, but their cohomologies with real coefficients agree.*

Proof. By formula (48) we have

$$H^*_{orb}([(\mathcal{L}X)^n/\mathfrak{S}_n]) \cong \bigoplus_{\sum jn_j = n} \prod_j H^*((\mathcal{L}X)^{n_j})^{\mathfrak{S}_{n_j}}$$

which is isomorphic by the previous corollary to $H^*(L[X^n/\mathfrak{S}_n])$.

But the orbifolds $L[X^n/\mathfrak{S}_n]$ and $I[(\mathcal{L}X)^n/\mathfrak{S}_n]$ cannot be naturally isomorphic because the actions of the cyclic groups $\mathbb{Z}/j$ are different. On the one hand, for $L[X^n/\mathfrak{S}_n]$, we just argued that the action of the cyclic groups are by rotation on $\mathcal{L}X$ (coming from the action of σ_j into $\mathcal{P}_{\sigma_j}X^j$), and on the other, for $I[(\mathcal{L}X)^n/\mathfrak{S}_n]$, the action of the cyclic groups are trivial, because the copies of $\mathcal{L}X$ come from the fixed point loci of the group action generated by the cycle σ_j into $(\mathcal{L}X)^j$. Therefore on the one hand one has the orbifold $[\mathcal{L}X/(\mathbb{Z}/j)]$ with the rotation action, and in the other one has the orbifold $[\mathcal{L}X/(\mathbb{Z}/j)]$ with the trivial action. These orbifolds cannot be naturally isomorphic. In the case that $n = 1$ both orbifolds are the same.

Let us see the case when $X = S^1$ and $n = 2$. Then $L[(S^1)^2/\mathfrak{S}_2] = [(\mathcal{L}S^1)^2/\mathfrak{S}_2] \sqcup [\mathcal{L}S^1/(\mathbb{Z}/2)]$ where the action of $\mathbb{Z}/2$ in the second component is by rotation, and $I[(\mathcal{L}S^1)^2/\mathfrak{S}_2] = [(\mathcal{L}S^1)^2/\mathfrak{S}_2] \sqcup [\mathcal{L}S^1/\mathbb{Z}/2]$ where the action of $\mathbb{Z}/2$ is the trivial one. As $\mathcal{L}S^1 \simeq \mathbb{Z} \times S^1$ it is easy to see that in the first case the geometrical realization of $[\mathcal{L}S^1/(\mathbb{Z}/2)]$ is homotopically equivalent to $(\mathbb{Z} \times S^1) \sqcup (\mathbb{Z} \times S^1 \times \mathbb{R}P^\infty)$ and in the second case is just $\mathbb{Z} \times S^1 \times \mathbb{R}P^\infty$.

♣

Using the previous result and formula (49), we get

Corollary 13.4. *Let* X *be such that* $H^i(\mathcal{L}X;\mathbb{R})$ *is finitely generated. Then*

$$\sum_{n=0}^{\infty} \phi(L[X^n/\mathfrak{S}_n], y)q^n = \prod_{j>0} \frac{\prod_i (1+q^j y^{2i+1})^{b^{2i+1}(\mathcal{L}X)}}{\prod_i (1-q^j y^{2i})^{b^{2i}(\mathcal{L}X)}}$$

where $b_i(\mathcal{L}X)$ *is the* i*-th Betti number of* $\mathcal{L}X$*. And via the chern character map we get*

$$K^*_{\mathfrak{S}_n}((\mathcal{L}X)^n) \otimes \mathbb{C} \cong H^*(L[X^n/\mathfrak{S}_n]).$$

Remark 13.5. The fact that the cohomologies of $I[\mathcal{L}X^n/\mathfrak{S}_n]$ and $L[X^n/\mathfrak{S}_n]$ agree is a feature of the symmetric product. In general, for any orbifold $[Y/G]$, the cohomologies of $I[\mathcal{L}Y/G]$ and $L[Y/G]$ do not have to agree. Take for example the $\mathbb{Z}/2$ action on S^2 by rotating π radians along the z-axis. $I[\mathcal{L}S^2/\mathbb{Z}/2] = [\mathcal{L}S^2/\mathbb{Z}/2] \sqcup [\mathcal{L}(S^2)^{\xi}/\mathbb{Z}/2]$ where ξ generates the group $\mathbb{Z}/2$, and therefore $\mathcal{L}(S^2)^{\xi}$ is the set of two points, the north and the south pole. Hence $H^*(I[\mathcal{L}S^2/\mathbb{Z}/2];\mathbb{R}) \cong H^*(\mathcal{L}S^2;\mathbb{R}) \oplus \mathbb{R}^{\oplus 2}$. On the other hand $L[S^2/\mathbb{Z}/2] = [\mathcal{L}S^2/\mathbb{Z}/2] \sqcup [\mathcal{P}_{\xi}S^2/\mathbb{Z}/2]$ with cohomology $H^*(L[S^2/\mathbb{Z}/2];\mathbb{R}) \cong H^*(\mathcal{L}S^2;\mathbb{R}) \oplus H^*(\mathcal{L}S^2;\mathbb{R})$ (this is shown in the examples of [LUX08]).

13.2 String Topology for the Symmetric Product

In this section we will study the ring structure of of the String Topology $H_*(\mathcal{P}_{\mathfrak{S}_n}M^n, \mathfrak{S}_n)$ as it was defined in Chapter 9, and we will show that it induces a ring structure in the homology

$$H_*(M^n, \mathfrak{S}_n) := \bigoplus_{\tau} H_*((M^n)^{\tau})$$

in such a way that $H_*(M^n, \mathfrak{S}_n)$ becomes a sub ring of $H_*(\mathcal{P}_{\mathfrak{S}_n}M^n, \mathfrak{S}_n)$.

Let us start by showing the previous statement for M itself

Lemma 13.6. *The natural inclusion* $i : M \to \mathcal{L}M$ *of constant loops and the evaluation at* 0*,* $ev : \mathcal{L}M \to M$ *induce ring maps in homology* $i_* : H_*(M) \to H_*(\mathcal{L}M)$ *and* $ev_* : H_*(\mathcal{L}M) \to H_*(M)$ *such that* $ev_* \circ i_* = id$*, in paticular as* i_* *is injective,* $H_*(M)$ *can be seen as a subring of* $H_*(\mathcal{L}M)$*.*

Proof. One just need to check that the following diagram is commutative

$$\begin{tikzcd}
\mathcal{L}M \times_M \mathcal{L}M \arrow[r] \arrow[d, "ev_\infty"] & \mathcal{L}M \times \mathcal{L}M \arrow[d, "ev \times ev"] \\
M \arrow[r, "diag"] \arrow[u, bend left, "i"] & M \times M. \arrow[u, bend left, "i \times i"]
\end{tikzcd}$$

This induces the following diagram relating the Thom-Pontryagin construction of the top row with the bottom row (recall that the normal bundle of the diagonal inclusion is isomorphic to the tangent bundle, and the subindex 0 means that we are taking everything ouside the zero section)

$$\begin{tikzcd}
\mathcal{L}M \times \mathcal{L}M \arrow[r] \arrow[d, "ev\times ev"] & (ev_\infty^* TM, (ev_\infty^* TM)_0) \arrow[d, "ev"] \\
M \times M \arrow[r] \arrow[u, bend left, "i\times i"] & (TM, TM_0) \arrow[u, bend left, "i"]
\end{tikzcd}$$

that at the level of homology gives

$$\begin{tikzcd}
H_*(\mathcal{L}M \times \mathcal{L}M) \arrow[r] \arrow[d, "ev_*\times ev_*"] & H_*(ev_\infty^* TM, (ev_\infty^* TM)_0) \arrow[r, "\cong"] \arrow[d, "ev_*"] & H_{*-d}(\mathcal{L}M) \arrow[d, "ev_*"] \\
H_*(M \times M) \arrow[r] \arrow[u, bend left, "i_*\times i_*"] & H_*(TM, TM_0) \arrow[r, "="] & H_{*-d}(M) \arrow[u, bend left, "i_*"]
\end{tikzcd}$$

where $d = \dim(M)$. Then one has that i_* and ev_* are ring homomorphism, and as $ev \circ i = id$ then i_* is injective

♣

For the case of orbifold of the symmetric product, the String Topology had a similar setup. Since the following diagram is a pull-back square

$$\begin{tikzcd}
\mathcal{P}_\tau {M^n}_1 \times_0 \mathcal{P}_\sigma M^n \arrow[r] \arrow[d, "ev_\infty"] & \mathcal{P}_\tau M^n \times \mathcal{P}_\sigma M^n \arrow[d, "ev_1 \times ev_0"] \\
M^n \arrow[r] & M^n \times M^n,
\end{tikzcd}$$

one can do the Thom-Pontryagin construction, defining a homomorphism

$$H_*(\mathcal{P}_\tau M^n \times \mathcal{P}_\sigma M^n) \to H_{*-nd}(\mathcal{P}_{\tau\sigma} M^n)$$

where the map $H_*(\mathcal{P}_\tau {M^n}_1 \times_0 \mathcal{P}_\sigma M^n) \to H_*(\mathcal{P}_{\tau\sigma} M^n)$ is induced by the natural concatenation of paths $\circledast : \mathcal{P}_\tau {M^n}_1 \times_0 \mathcal{P}_\sigma M^n \to \mathcal{P}_{\tau\sigma} M^n$.

Then we have a product

$$\begin{array}{rcl}
H_p(\mathcal{P}_\tau M^n) \times H_q(\mathcal{P}_\sigma M^n) & \to & H_{p+q-nd}(\mathcal{P}_{\tau\sigma} M^n) \\
(\alpha, \beta) & \mapsto & \alpha \cdot \beta
\end{array}$$

that is graded (shifted by $-nd$) associative, and thus defines a product in

$$H_*(\mathcal{P}_{\mathfrak{S}_n} M^n, \mathfrak{S}_n)$$

which is what we have called the Strin Topology product.

By taking the $\mathfrak{S}_n$ invariant part

$$(H_*(\mathcal{P}_{\mathfrak{S}_n} M^n, \mathfrak{S}_n))^{\mathfrak{S}_n} \cong H_*(L[M^n/\mathfrak{S}_n])$$

we have a ring structure on the homology of the loop orbifold of the symmetric product.

Now let us study what is the behavior of the evaluation and inclusion of constant maps. So consider the following commutative diagram

$$\begin{tikzcd} \mathcal{P}_\tau M^n \arrow[r, "ev"] & M^n \\ (M^n)^\tau \arrow[u, "i^\tau"] \arrow[ur, "f^\tau"'] & \end{tikzcd}$$

where f^τ is the lnclusion of fixed point set, i^τ is the inclusion of constant loops, and ev is the evaluation at 0, we have the following

Lemma 13.7. *The image in homology of* ev_* *is equal to the image in homology of* f^τ_*.

Proof. Restricting the previous diagram to one of the cycles σ of size l that defines τ, the diagram becomes

$$\begin{tikzcd} \mathcal{P}_\sigma M^l = \mathcal{L}M \arrow[r, "ev"] & M^l \\ (M^l)^\sigma = M \arrow[u, "i^\sigma"] \arrow[ur, "f^\sigma"'] & \end{tikzcd}$$

where f^σ becomes the diagonal inclusion $M \to M^l$ and the evaluation map ev takes a loop $\alpha : S^1 \to M$ and maps it to $ev(\alpha) = (\alpha(0), \alpha(\frac{2\pi}{l}), \ldots, \alpha(\frac{2(l-1)\pi}{l}))$. Defining the homotopy $ev^t(\alpha) = (\alpha(0), \alpha(\frac{2\pi t}{l}), \ldots, \alpha(\frac{2(l-1)\pi t}{l}))$ one sees that $ev^1 = ev$ and ev^0 are homotopic, and as $ev^0(\alpha) = f^\sigma(\alpha(0))$, the lemma follows.

♣

Since the inclusion maps f^τ induce injective homomorphisms $f^\tau_* : H_*((M^n)^\tau) \to H_*(M^n)$, we define the groups $H^\tau_*(M^n) := \mathrm{image}(f^\tau_*) \subset H_*(M^n)$ and therefore we get

$$\begin{tikzcd} H_*(\mathcal{P}_\tau M^n) \arrow[r, "ev_*"] & H^\tau_*(M^n) \\ H_*((M^n)^\tau) \arrow[u, "i^\tau_*"] \arrow[ur, "\cong", "f^\tau_*"'] & \end{tikzcd}$$

So we can define a ring structure in $H_*(M^n, \mathfrak{S}_n)$ in the following way

$$\begin{aligned} \bullet : H^\tau_*(M^n) \times H^\sigma_*(M^n) &\to H^{\tau\sigma}_{*-nd}(M^n) \\ (\alpha, \beta) &\mapsto \alpha \bullet \beta \end{aligned}$$

where

$$\alpha \bullet \beta = ev_* \left(\left(i^\tau_* \circ (f^\tau_*)^{-1} \alpha \right) \cdot \left(i^\sigma_* \circ (f^\sigma_*)^{-1} \beta \right) \right)$$

and $\cdot$ is the product structure of String Topology. Using the isomorphisms f^τ_* we define the ring structure in $H_*(M^n, \mathfrak{S}_n)$ that we will also denote by $\bullet$.

Then we have the compatibility of all the products

$$\begin{tikzcd} H_*((M^n)^\tau) \times H_*((M^n)^\sigma) \arrow[r, "i^\tau_* \times i^\sigma_*"'] \arrow[rr, bend left, "\cong"] \arrow[d, "\bullet"] & H_*(\mathcal{P}_\tau M^n) \times H_*(\mathcal{P}_\sigma M^n) \arrow[r, "ev_* \times ev_*"'] \arrow[d, "\cdot"] & H^\tau_*(M^n) \times H^\sigma_*(M^n) \arrow[d, "\bullet"] \\ H_*((M^n)^{\tau\sigma}) \arrow[r, "i^{\tau\sigma}_*"] \arrow[rr, bend right, "\cong"'] & H_*(\mathcal{P}_{\tau\sigma} M^n) \arrow[r, "ev_*"] & H^{\tau\sigma}_*(M^n) \end{tikzcd}$$

so we can conclude

Proposition 13.8. *The homology* $H_*(M^n, \mathfrak{S}_n)$ *becomes a* $\mathfrak{S}_n$*-graded ring. Moreover, the inclusion of constant loops* $i : (M^n)^\tau \to \mathcal{P}_\tau M^n$ *and the evaluation maps induce ring homomorphisms that makes the following diagram commute*

$$\begin{tikzcd} & H_*(\mathcal{P}_{\mathfrak{S}_n} M^n, \mathfrak{S}_n) \arrow[dr, "ev_*"] & \\ H_*(M^n, \mathfrak{S}_n) \arrow[ur, "i_*"] \arrow[rr, "\cong"'] & & (\bigoplus_\tau H^\tau_*(M^n) \times \{\tau\}) \end{tikzcd} .$$

Remark 13.9. The inclusion of the inertia orbifold into the loop orbifold, in general does not induce an injective homomorphism in homology. Take the example of remark 13.5, namely the action of $\mathbb{Z}/2$ in S^2 by rotation along the z-axis. If the generator of $\mathbb{Z}/2$ is ξ, then the fixed point set $(S^2)^\xi$ consist of two points, the north and the south pole. The inclusion of the inertia orbifold into the loop orbifold is then $(S^2)^\xi \to \mathcal{P}_\xi S^2$, where $\mathcal{P}_\xi S^2 = \{f : [0,1] \to S^2 | f(0)\xi = f(1)\}$. It is clear that $\mathcal{P}_\xi S^2 \simeq \mathcal{L}S^2$ which is connected, then the homomorphism $H_*((S^2)^\xi) \to H_*(\mathcal{P}_\xi S^2)$ is not injective.

Remark 13.10. We have seen how to define a ring structure in the homology of $I[M^n/\mathfrak{S}_n]$ using the structure of the homology of the loop orbifold. It is easy to see that the homology product we have defined boils down to intersection of cycles in M^n. Namely, for cycles in $(M^n)^\tau$ and $(M^n)^\sigma$ (say $\alpha \in H_*^\tau(M^n)$ and $\beta \in H_*^\sigma(M^n)$), their transversal intersection in M^n is a cycle in $(M^n)^{\langle\tau,\sigma\rangle}$ ($\alpha \cap \beta \in H^{\tau,\sigma}_{*-nd}(M^n)$), and therefore could be pushforwarded to a cycle in $(M^n)^{\tau\sigma}$ ($\alpha \cap \beta \in H^{\tau\sigma}_{*-nd}(M^n)$). The associativity follows directly from the fact that transversal intersection is associative in homology.

13.3 The Virtual Intersection Product

We would like to compare the product structure that we have defined in the previous section to the Virtual product of Chapter 10

In the symmetric product, it is easy to see that the Virtual product $\star$ defined in the cohomology of the inertia orbifold is just the Poincaré dual of the product $\bullet$ in homology we defined previously. Using the isomorphisms $f_*^\tau : H_*((M^n)^\tau) \cong H_*^\tau(M^n)$ we get the following commutative diagram:

$$\begin{tikzcd}
H^\tau_p(M^n) \times H^\sigma_q(M^n) \arrow[r, leftrightarrow, "PD"] \arrow[dd, "\cap"] \arrow[ddd, bend right, "\bullet"'] & H^{d|\mathcal{O}(\langle\tau\rangle)|-p}((M^n)^\tau) \times H^{d|\mathcal{O}(\langle\sigma\rangle)|-q}((M^n)^\sigma) \arrow[d, "e_\tau^*(_)\cup e_\sigma^*(_)"'] \arrow[ddd, bend left, "\star"] \\
 & H^{d|\mathcal{O}(\langle\tau\rangle)|+d|\mathcal{O}(\langle\sigma\rangle)|-p-q}((M^n)^{\tau,\sigma}) \arrow[d, "\cup Eu(\nu_{\tau,\sigma})"'] \\
H^{\tau,\sigma}_{p+q-nd}(M^n) \arrow[r, leftrightarrow, "PD"] \arrow[d, "\text{inclusion}"] & H^{dn+d|\mathcal{O}(\langle\tau,\sigma\rangle)|-p-q}((M^n)^{\tau,\sigma}) \arrow[d, "e_{\tau\sigma*}"'] \\
H^{\tau\sigma}_{p+q-nd}(M^n) \arrow[r, leftrightarrow, "PD"] & H^{dn+d|\mathcal{O}(\langle\tau\sigma\rangle)|-p-q}((M^n)^{\tau,\sigma})
\end{tikzcd}$$

where the horizontal maps are Poincaré duality maps, $d = \dim_{\mathbb{R}}(M)$, $\mathcal{O}(H)$ is the set of orbits of the action of $H \subset \mathfrak{S}_n$ on $\{1, 2, \ldots, n\}$ and $|\mathcal{O}(H)|$ is its cardinality. The commutativity of the diagram permit us to conclude

Proposition 13.11. *The Poincaré duality maps induce an isomorphism of rings*

$$H^*_{virt}(M^n, \mathfrak{S}_n) = (H^*(M^n, \mathfrak{S}_n), \star) \xrightarrow{\cong} (H_*(M^n, \mathfrak{S}_n), \bullet)\,.$$

Therefore the Virtual cohomology ring is isomorphic to a subring of the String Topology ring

$$H^*_{virt}(M^n, \mathfrak{S}_n) \subset H_*(\mathcal{P}_{\mathfrak{S}_n} M^n, \mathfrak{S}_n).$$

The same theorems are valid also in K-theory, the proofs are the same.

13.4 Chen-Ruan Cohomology

In this chapter we will study the obstruction bundle $\mathcal{R}(\tau, \sigma)$ (see Definition 11.4) associated to the Chen-Ruan product in the particular case of the symmetric product, and we will show a simple description of this bundle. This description is the key ingredient needed in Section 11.1 to prove Proposition 11.18 and Corollary 11.22. Let us start with some notation.

For two elements $\tau, \sigma \in \mathfrak{S}_n$ let $\mathcal{O}(\tau, \sigma) = \{\Gamma_1, ..., \Gamma_k\}$ be the set of orbits of the action of the group generated by τ and σ on $\{1, 2, \dots, n\}$. Let $n_i = |\Gamma_i|$ and without loss of generality assume that the orbit Γ_i consists of the numbers

$$\Gamma_i = \{n_1 + \cdots + n_{i-1} + 1, n_1 + \cdots + n_{i-1} + 2, ..., , n_1 + \cdots + n_i\}.$$

Denote by τ_i and σ_i the elements in $\mathfrak{S}_{n_i}$ which encode the restricted action of τ and σ on the set orbit Γ_i; in particular we have that $(\tau\sigma)_i = \tau_i\sigma_i$ and the action of the group $\langle \tau_i, \sigma_i \rangle$ is transitive on Γ_i.

If we denote by $\mathcal{R}(\tau_i, \sigma_i)$ the obstrcution bundle of the action of τ_i and σ_i on M^{n_i} then we have that

$$\mathcal{R}(\tau, \sigma) \cong \prod_{i=1}^{k} \mathcal{R}(\tau_i, \sigma_i).$$

Since tha action of $\langle \tau_i, \sigma_i \rangle$ on Γ_i is transitive, we have that $\Delta_i(M) = (M^{n_i})^{\tau_i, \sigma_i}$ where $\Delta_i : M \to M^{n_i}$ is the diagonal inclusion. By Definition 11.4 we have that

$$\mathcal{R}(\tau_i, \sigma_i) = \Big(T\Delta_i(M) \ominus TM^{n_i} \oplus \mathcal{S}_{\tau_i} \oplus \mathcal{S}_{\sigma_i} \oplus \mathcal{S}_{(\tau_i\sigma_i)^{-1}} \Big)|_{\Delta_i(M)}$$

where we have that $TM^{n_i}|_{\Delta_i(M)} \cong n_i T\Delta_i(M)$ and by a simple linear algebra a we get

$$\mathcal{S}_{\tau_i}|_{\Delta_i(M)} \cong \frac{1}{2}(n_i - |\mathcal{O}(\tau_i)|)T\Delta_i(M)$$
$$\mathcal{S}_{\sigma_i}|_{\Delta_i(M)} \cong \frac{1}{2}(n_i - |\mathcal{O}(\sigma_i)|)T\Delta_i(M)$$
$$\mathcal{S}_{(\tau_i\sigma_i)^{-1}}|_{\Delta_i(M)} \cong \mathcal{S}_{\tau_i\sigma_i}|_{\Delta_i(M)} \cong \frac{1}{2}(n_i - |\mathcal{O}(\tau_i\sigma_i)|)T\Delta_i(M)$$

where $\mathcal{O}(\tau_i)$ denotes the set of orbits of the action of τ_i on Γ_i; and hence

$$\begin{aligned}\mathcal{R}(\tau_i, \sigma_i) &= \frac{1}{2}\left(2 - 2n_i + n_i - |\mathcal{O}(\tau_i)| + n_i - |\mathcal{O}(\sigma_i)| + n_i - |\mathcal{O}(\tau_i\sigma_i)|\right) T\Delta_i(M)\\ &= \frac{1}{2}\left(2 + n_i - |\mathcal{O}(\tau_i)| - |\mathcal{O}(\sigma_i)| - |\mathcal{O}(\tau_i\sigma_i)|\right) T\Delta_i(M).\end{aligned}$$

Denoting the natural number

$$g(\tau_i, \sigma_i) = \frac{1}{2}\left(2 + n_i - |\mathcal{O}(\tau_i)| - |\mathcal{O}(\sigma_i)| - |\mathcal{O}(\tau_i\sigma_i)|\right)$$

we have then that the Euler class of the obstruction bundle $\mathcal{R}(\tau_i, \sigma_i)$ is a multiple of the Euler class of the manifold $M \cong \Delta_i(M)$, i.e.

$$Eu(\mathcal{R}(\tau_i, \sigma_i)) = Eu(\Delta_i(M))^{g(\tau_i,\sigma_i)}$$

and therefore if $g(\tau_i, \sigma_i) > 1$ then we get that $Eu(\mathcal{R}(\tau_i, \sigma_i)) = 0$.

The total obstruction bundle is then equal to

$$\mathcal{R}(\tau, \sigma) = \prod_{i=1}^{k} (T\Delta_i(M))^{\oplus g(\tau_i,\sigma_i)} \tag{53}$$

and its Euler class becomes

$$Eu(\mathcal{R}(\tau, \sigma)) = \prod_{i=1}^{k} Eu\left(\Delta_i(M)\right)^{g(\tau_i,\sigma_i)}.$$

This explicit description of the obstruction bundle was the key fact that led Uribe [Uri05] and Fantechi and Göttsche [FG03] to prove independently the following result that we quote:

Theorem 13.12. *Let Σ be a smooth projective surface with trivial canonical divisor. Then there is an isomorphism of graded rings between the orbfiodl Chen-Ruan cohomology of the symmetric product $[\Sigma^n/\mathfrak{S}_n]$ and the cohomology of the n-th Hilbert scheme $\Sigma^{[n]}$ of the surface Σ:*

$$(\mathsf{H}^*_{\mathsf{CR}}(\Sigma^n, \mathfrak{S}_n))^{\mathfrak{S}_n} \cong \mathsf{H}^*(\Sigma^{[n]}; \mathbb{C}).$$

This result led Ruan to state what is known as the *Crepant Resolution Conjecture* which basically states that the Chen-Ruan orbifold cohomology of a Gorenstein orbifold is isomorphic to a semiclassical limit of the quantum cohomology of a crepant resolution of the underlying quotient variety (see the original conjecture in [Rua06]). This dovetails nicely with the discussion of the McKay correspondence of Appendix 19.

14 Final Comments

To end the book we merely point towards further reading that you may find interesting.

The interested reader should first look at the excellent book [ALR07] where she will find a complementary point of view on the theory of orbifolds.

The name Calabi-Yau category can be better understood by noticing that given a Calabi-Yau manifold its $\mathbb{B}$-model is a 2-dimensional open-closed TQFT. We refer the reader to [CW10] for details. From this fact we conjecture that for a *non-compact* Calabi-Yau orbifold we should obtain a nearly G-Frobenius structure as we defined it in this book. This would be a sort of generalized Serre duality for non-compact orbifolds. We will return to this issue elsewhere.

In this book we only considered the connected component of the moduli spaces of curves, but the full cohomology of the moduli space can be made to act on the state spaces of the theories described, for a first approximation to this we recommend [God07].

This points towards the fact that to have a fuller picture in string theory we must work at the level of chains rather than at the homological level that we have worked at in this book. For the concept of a Calabi-Yau category at the level of chains see [Cos07]. For string topology at the level of chains see [BCT09].

Throughout this book we worked over a field $\Bbbk$ usually the rational, real or complex numbers. But orbifold string topology can be done over the integers, see [ÁBU12].

For generalizations of the structures developed here to the case in which G is a Lie group see [GW12b] for extensions of Chen-Ruan theory and [BGNX07] for extensions of string topology. Also [FHLT09] is very interesting. It is reasonable to conjecture that a version of the relation between virtual cohomology (ghost string topology) and Chen-Ruan theory of the cotangent bundle will still hold when the groups are no longer finite.

For some interesting explicit calculations see [GW12a], [GS08], [Pod03],[Pod02], and [CH06], [Per07], [Jia07], [JK02].

For the crepant resolution conjecture of Ruan see [Rua01], [CR07], [BG06], [CCIT07], [Coa09], [BMP09], [BG09], and [Ito94].

The McKay correspondence has a truly vast literature, for example: [Rei02], [BKR01],[BD96], [Kal02], [Kos84], [DL02a], [LP04], [AP01], [BL05] and [IN00].

The classification of topological field theories has been developed in [Bae01],

[BD95], [GMTW09] and [Lur09].

Finally for conformal field theories from the point of view described in this book look at the classic [Seg02].

15 Appendix: Categories and Functors

15.1 Categories

Category theory was discover by Eilenberg and MacLane in the 50's [ML98] and ever since has pervaded all fields of mathematics.

You may want to think of the category of sets as you read the following definition. The objects of the category of sets are all sets and the arrows are all mappings between them. You may also want to think of an object as a sort of dot and an arrow as something with a direction joining the dots.

Definition 15.1. A **category** consists of:

- A class $\mathcal{O}\mathsf{bj}(\mathcal{C})$, that we will denote by $\mathcal{C}_0$, of objects of $\mathcal{C}$.
- A class $\mathcal{A}\mathsf{rr}(\mathcal{C})$, that we will denote by $\mathcal{C}_1$, of arrows of $\mathcal{C}$. For each pair of objects a and b the class of all arrows from a to b is denoted by $\mathcal{C}(a, b)$.
- Two assignments $s_\mathcal{C}, t_\mathcal{C} : \mathcal{A}\mathsf{rr}(\mathcal{C}) \to \mathcal{O}\mathsf{bj}(\mathcal{C})$ called source and target respectively.
- *Unit.* An assignment $u_\mathcal{C} : \mathcal{O}\mathsf{bj}(\mathcal{C}) \to \mathcal{A}\mathsf{rr}(\mathcal{C})$ such that:

  $$s_\mathcal{C}(u_\mathcal{C}(a)) = t_\mathcal{C}(u_\mathcal{C}(a)) = a,$$

  for every $a \in \mathcal{O}\mathsf{bj}(\mathcal{C})$.
- *Composition Law.* For each triple a, b and c of objects of $\mathcal{C}$ an assignment $m_{(a,b,c)} : \mathcal{C}(a, b) \times \mathcal{C}(b, c) \to \mathcal{C}(a, c)$, where its image on $(\alpha, \beta) \in \mathcal{C}(a, b) \times \mathcal{C}(b, c)$ well be denoted by $\beta \circ \alpha$, satisfying the following properties:

  1. For every $a \in \mathcal{O}\mathsf{bj}(\mathcal{C})$

     $$s_\mathcal{C}(u_\mathcal{C}(a)) = t_\mathcal{C}(u_\mathcal{C}(a)) = a,$$

     $$\begin{tikzcd} \mathcal{O}\mathsf{bj}(\mathcal{C}) \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, "\mathrm{Id}"] & \mathcal{A}\mathsf{rr}(\mathcal{C}) \arrow[d, "s"] \\ \mathcal{A}\mathsf{rr}(\mathcal{C}) \arrow[r, "t"] & \mathcal{O}\mathsf{bj}(\mathcal{C}) \end{tikzcd}$$

     in other words the source and target of $u_\mathcal{C}(a) = a$ for every a.

2. *Associativity.* For all $\alpha, \beta, \gamma \in \mathcal{A}rr(\mathcal{C})$ it holds that $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$, formally for every elements a, b, c, d fixed in $\mathcal{O}bj(\mathcal{C})$ we have

$$m_{(a,c,d)} \circ (m_{(a,b,c)} \times \mathrm{Id}_{\mathcal{C}(c,d)}) = m_{(a,b,d)}(\mathrm{Id}_{\mathcal{C}(a,b)} \times m_{(b,c,d)}),$$

$$\begin{tikzcd}
\mathcal{C}(a,b) \times \mathcal{C}(b,c) \times \mathcal{C}(c,d) \arrow[r, "m_{a,b,c} \times \mathrm{Id}_{\mathcal{C}(c,d)}"] \arrow[d, "\mathrm{Id}_{\mathcal{C}(a,b)} \times m_{b,c,d}"'] & \mathcal{C}(a,c) \times \mathcal{C}(c,d) \arrow[d, "m_{(a,c,d)}"] \\
\mathcal{C}(a,b) \times \mathcal{C}(b,d) \arrow[r, "m_{(a,b,d)}"] & \mathcal{C}(a,d)
\end{tikzcd}$$

3. *Unity.* For every $a, b \in \mathcal{O}bj(\mathcal{C})$ and $\alpha \in \mathcal{C}(a,b)$ $\alpha = u_{\mathcal{C}}(b) \circ \alpha = \alpha \circ u_{\mathcal{C}}(a)$ holds, formally

$$m_{(a,b,b)}(\alpha, u_{\mathcal{C}}(b)) = m_{(a,a,b)}(u_{\mathcal{C}}(a), \alpha) = \alpha.$$

Example 15.1. Let us define **Sets** the category with objects the class of all spaces (proper class) and arrows the class of function of sets. The unity of this category assigns to each set X the usual identity function of sets over X and the function $m_{\mathcal{C}}$ the composition of functions, when it is defined.

Example 15.2. The category **Ab** the subclass of **Sets** whose objects are all abelian groups and arrows the class of morphism of groups with the same unity and rule of composition as **Sets**. In he same manner are defined the categories **Mod**, **Ring**, **Groups**, etc.

Example 15.3. The category **Top** of topological spaces and continuous functions.

Example 15.4. Let us consider the category **Corr** of correspondences [CV] whose objects are topological spaces and whose arrows (from X to Y) are diagrams of continuous mappings of the form

$$\begin{tikzcd}
 & Z \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
X & & Y
\end{tikzcd}$$

for Z some topological space. We define the composition of arrows by

$$(X \xleftarrow{\alpha} V \xrightarrow{\beta} Y) \circ (Y \xleftarrow{\gamma} W \xrightarrow{\delta} Z) = X \xleftarrow{\alpha} U \xrightarrow{\delta} Z$$

where U is defined as the fiber product

$$U = V \times_Y W = \{(v, w) | \beta(v) = \gamma(w)\}.$$

Observe that the ordinary category of topological spaces can be embedded as a subcategory of **Corr** since a continuous map $f\colon X \to Y$ can be interpreted as the correspondence

$$X \overset{\pi_X}{\leftarrow} \mathcal{G}_f \overset{\pi_Y}{\to} Y,$$

where $\mathcal{G}_f = \{(x, y)|y = f(x)\}$ is the graph of f. This is functorial for we have

$$\mathcal{G}_f \times_Y \mathcal{G}_h = \mathcal{G}_{h \circ f}.$$

Unfortunately homology is not a functor from **Corr** to graded abelian groups. Nevertheless suppose that we have a correspondence $X \overset{\alpha}{\leftarrow} Z \overset{\beta}{\to} Y$ where

- X, Y and Z are manifolds (possibly infinite dimensional).
- α is a regular embedding of finite codimension d.

In this case we say that $X \overset{\alpha}{\leftarrow} Z \overset{\beta}{\to} Y$ is a smooth correspondence of degree $-d$. In any case using the Gysin map we can produce the composition

$$H_*(X) \overset{\alpha_!}{\to} H_{*-d}(Z) \overset{\beta_*}{\to} H_{*-d}(Y)$$

which is the induced homomorphism of degree $-d$ in homology.

Definition 15.2. A *Groupoid* is a category in which each arrow has an inverse, namely for each pair $a, b \in \mathcal{O}bj\mathcal{C}$ and each $\alpha \in \mathcal{C}(a, b)$ there exist an arrow $\alpha^{-1} \in \mathcal{C}(b, a)$ in such a way that $\alpha^{-1} \circ \alpha = u(a)$ y $\alpha \circ \alpha^{-1} = u(b)$. In this case we will denote by $i : \mathcal{C}(a, b) \to \mathcal{C}(b, a)$ the map that assigns to each arrows its inverse.

Example 15.5. Let G be a group acting on a set M. Let $G \ltimes M$ be the groupoid whose objects are the set M, and arrows $g : x \to y$ such that $y = gx$, this set can be seen as the set $G \times M$. Here the composition is defined of natural manner $gg' : x \to z$ where $g' : x \to y$ and $g : y \to z$. For each object x the unit map associates the unit e of G. The structure maps are defined in the obvious way as $s : G \times M \to M$ the projection and $t : G \times M \to M$ the action.

15.2 Natural Transformations as Homotopies.

Definition 15.3. A (covariant) *functor* F from $\mathcal{C}$ to $\mathcal{B}$ is an assignment so that to every object $a \in \mathcal{O}bj(\mathcal{C})$ associates an object $F(a) \in \mathcal{O}bj(\mathcal{B})$ and to every arrow $\alpha \in \mathcal{A}rr(\mathcal{C})$, $\alpha\colon a \to b$ associates an arrow $F(\alpha) \in \mathcal{A}rr(\mathcal{B})$, $F(\alpha)\colon F(a) \to f(b)$, sending identities to identities and satisfying:

$$F(\alpha \circ \beta) = F(\alpha) \circ F(\beta).$$

The category **Top** of topological spaces with continuous mappings has an interesting additional structure. Homotopies of smooth mappings. This endows $\mathbf{Top}(X, Y)$ with the structure of a category. We will call a category with this additional structure a *bicategory*.

The category **Cat** of all categories is also a bicategory. Let us define the homotopies between functors. Let F and D functors from $\mathcal{C}$ to $\mathcal{B}$, a *homotopy* of functors is a functor $H\colon \mathcal{C} \times \mathcal{I} \to \mathcal{B}$ where $\mathcal{I}$ is a category with two objects and one arrow going between them, and the restrictions of H to the two copies of $\mathcal{C}$ above, coincide with F and D respectively. The reader can verify that to have a homotopy between functors is the same as having a natural transformation.

Definition 15.4. A natural transformation of functors is a map $\Phi : \mathcal{C}_0 \to \mathcal{B}_1$ in such a way that

- For every $a \in \mathcal{C}_0, \Phi(a) \in \mathcal{B}(F(a), D(a))$, and
- For each $\alpha \in \mathcal{C}(a, b)$

$$\Phi(b) \circ F(\alpha) = D(\alpha) \circ \Phi(a)$$

$$\begin{tikzcd} F(a) \arrow[r, "F(\alpha)"] \arrow[d, "\Phi(a)"'] & F(b) \arrow[d, "\Phi(b)"] \\ D(a) \arrow[r, "D(\alpha)"] & D(b) \end{tikzcd}$$

16 Appendix: Monoidal Categories

16.1 Definitions

Definition 16.1. A monoidal category (or tensor category) consists of the following data: a category $\mathcal{C}$, a covariant functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, called the monoidal product(or tensor product), an object $u \in \mathrm{Ob}(\mathcal{C})$, called the unit and natural isomorphisms

- $\alpha_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z$,
- $\lambda_x : u \otimes x \longrightarrow x$,
- $\rho_x : x \otimes u \longrightarrow x$,

called *associativity*, *left unit* and *right unit.* This natural isomorphisms satisfy the following axioms:

$$\begin{tikzcd}
x \otimes (y \otimes (w \otimes z)) \arrow[r, "\alpha_{x,y,w\otimes z}"] \arrow[d, "1\otimes\alpha_{y,w,z}"'] & (x \otimes y) \otimes (w \otimes z) \arrow[r, "\alpha_{x\otimes y,w,z}"] & ((x \otimes y) \otimes w) \otimes z \\
x \otimes ((y \otimes w) \otimes z) \arrow[rr, "\alpha_{x,y\otimes w,z}"'] & & (x \otimes (y \otimes w)) \otimes z \arrow[u, "\alpha_{x,y,w}\otimes 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes (u \otimes y) \arrow[rr, "\alpha_{x,u,y}"] \arrow[dr, "1\otimes\lambda_y"'] & & (x \otimes u) \otimes y \arrow[dl, "\rho_x\otimes 1"] \\
& x \otimes y &
\end{tikzcd}$$

for $x, y, w, z \in \mathrm{Ob}(\mathcal{C})$, and also

$$\lambda_u = \rho_u : u \otimes u \longrightarrow u.$$

A monoidal category is called *strict monoidal category* if the morphisms α, λ, ρ are the identity morphisms.

16.2 Monoidal Functors

Definition 16.2. Let $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \otimes)$ be monoidal categories. A *monoidal functor* is a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ together with natural isomorphisms

- $\xi_{x,y} : F(x) \otimes F(y) \longrightarrow F(x \otimes y)$
- $\xi_0 : u_{\mathcal{D}} \longrightarrow F(u_{\mathcal{D}})$

which satisfy the following commutative diagrams:

$$\begin{tikzcd}
F(x) \otimes (F(y) \otimes F(z)) \arrow[r, "1\otimes\xi"] \arrow[d, "\alpha"'] & F(x) \otimes F(y \otimes z) \arrow[r, "\xi"] & F(x \otimes (y \otimes z)) \arrow[d, "F(\alpha)"] \\
(F(x) \otimes F(y)) \otimes F(z) \arrow[r, "\xi\otimes 1"'] & F((x \otimes y) \otimes F(z) \arrow[r, "\xi"'] & F((x \otimes y) \otimes z)
\end{tikzcd}$$

$$\begin{tikzcd}
u \otimes F(x) \arrow[r, "\xi_0\otimes 1"] \arrow[dr, "\lambda"'] & F(u) \otimes F(x) \arrow[r, "\xi"] & F(u \otimes x) \arrow[dl, "F(\lambda)"] \\
& F(x) &
\end{tikzcd}$$

$$\begin{tikzcd}
F(x) \otimes u \arrow[r, "1\otimes\xi_0"] \arrow[dr, "\rho"'] & F(x) \otimes F(u) \arrow[r, "\xi"] & F(x \otimes u) \arrow[dl, "F(\rho)"] \\
& F(x) &
\end{tikzcd}$$

A monoidal functor is called *stric monoidal functor* if ξ and ξ_0 are the identity morphisms.

Remark 16.3. For any monoidal functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$. Let (ξ, ξ_0) and (ξ', ξ_0') the natural isomorphisms of F and G, respectively. The natural isomorphisms (ξ'', ξ_0'') for the composition $F \circ G : \mathcal{C} \longrightarrow \mathcal{E}$ are defined by

$$\begin{tikzcd}
G \circ F(x) \otimes G \circ F(y) \arrow[r, "\xi'"] \arrow[rr, bend right, "\xi''"'] & G(F(x) \otimes F(y)) \arrow[r, "G(\xi)"] & G \circ F(x \otimes y)
\end{tikzcd}$$

$$\begin{tikzcd}
u_{\mathcal{E}} \arrow[r, "\xi_0'"] \arrow[rr, bend right, "\xi_0''"'] & G(u_{\mathcal{D}}) \arrow[r, "G(\xi_0)"] & G \circ F(u_{\mathcal{C}})
\end{tikzcd}$$

Example 16.1. The most important ones are

$(Set, \times, \{*\})$,	the category of sets with the cross product.
$(Set, \sqcup, \emptyset)$,	the category of sets with the disjoint union.
$(\mathrm{Vect}_{\Bbbk}, \otimes, \Bbbk)$,	the category of vector spaces with the tensor product over $\Bbbk$.
$(\mathrm{Top}, \times, *)$,	the category of topological spaces with the cross product.
$(\mathrm{Ab}, \otimes, \mathbb{Z})$,	the category of abelian groups with the usual tensor product over $\mathbb{Z}$.
$(\mathrm{nCob}, \sqcup, \emptyset)$,	the category of n-cobordisms with the disjoint union.

16.3 Monoidal Natural Transformations

Definition 16.4. A natural transformation $\sigma : F \longrightarrow F'$ between two monoidal functors is called a *monoidal natural transformation* if the diagrams

$$\begin{tikzcd} F(x) \otimes F(y) \arrow[r, "\xi"] \arrow[d, "\sigma\otimes\sigma"'] & F(x \otimes y) \arrow[d, "\sigma"] \\ F'(x) \otimes F'(y) \arrow[r, "\xi"'] & F'(x \otimes y) \end{tikzcd}$$

$$\begin{tikzcd} u \arrow[r, "\xi_0"] \arrow[dr, "\xi_0'"'] & F(u) \arrow[d, "\sigma"] \\ & F'(u) \end{tikzcd}$$

commute.

Let $\mathcal{C}$ and $\mathcal{D}$ monoidal categories. A monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called a *monoidal equivalence* if there exists a monoidal functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ and monoidal natural isomorphisms $\varphi : G \circ F \cong 1_{\mathcal{C}}$ and $\psi : F \circ G \cong 1_{\mathcal{D}}$.

16.4 Braided Monoidal Categories

A *braided monoidal category* consists of a monoidal category $\mathcal{M}$ together with a *braiding*, which is defined by a family of isomorphims

$$\sigma_{x,y} : x \otimes y \longrightarrow y \otimes x.$$

They are natural for x and y in $\mathcal{M}$, and satisfy for the unit u the commutative diagram

$$\begin{tikzcd} x \otimes u \arrow[rr, "\sigma"] \arrow[dr, "\rho"'] & & u \otimes x \arrow[dl, "\lambda"] \\ & x, & \end{tikzcd}$$

Moreover the maps $\sigma_{x,y}$, together with the associativity α make commutative the following hexagonal diagrams:

$$\begin{tikzcd} & (x \otimes y) \otimes z \arrow[r, "\sigma"] \arrow[dl, "\alpha^{-1}"'] & z \otimes (x \otimes y) \arrow[dr, "\alpha"] & \\ x \otimes (y \otimes z) \arrow[dr, "1\otimes\sigma"'] & & & (z \otimes x) \otimes y \arrow[dl, "\sigma\otimes 1"] \\ & x \otimes (z \otimes y) \arrow[r, "\alpha"'] & (x \otimes z) \otimes y, & \end{tikzcd}$$

$$\begin{tikzcd} & x \otimes (y \otimes z) \arrow[r, "\sigma"] \arrow[dl, "\alpha"'] & (y \otimes z) \otimes x \arrow[dr, "\alpha^{-1}"] & \\ (x \otimes y) \otimes z \arrow[dr, "\sigma\otimes 1"'] & & & y \otimes (z \otimes x) \arrow[dl, "1\otimes\sigma"] \\ & (y \otimes x) \otimes z \arrow[r, "\alpha^{-1}"'] & y \otimes (x \otimes z). & \end{tikzcd}$$

16.5 Symmetric Monoidal Categories

A *symmetric monoidal category* is a monoidal category with a braiding, which satisfies the identity

$$\sigma_{y,x} \circ \sigma_{x,y} = 1.$$

Proposition 16.5. *For $\mathcal{M}$ a symmetric monoidal category we have the identity*

$$(1 \otimes \sigma) \circ \sigma \circ \alpha^{-1} = \alpha \circ \sigma \circ (1 \otimes \sigma).$$

Proof.

$$\begin{tikzcd}
& \cdot \arrow[r, "\alpha"] \arrow[dl, swap, "\sigma"] & \cdot \arrow[dr, "\sigma\otimes 1"] & & & \\
\cdot \arrow[dr, swap, "\alpha^{-1}"] & & & \cdot \arrow[r, "\sigma"] \arrow[dl, "\alpha^{-1}"] & \cdot \arrow[dr, "\alpha"] & \\
& \cdot \arrow[r, swap, "1\otimes\sigma"] & \cdot \arrow[dr, swap, "1\otimes\sigma"] & & & \cdot \arrow[dl, "\sigma\otimes 1"] \\
& & & \cdot \arrow[r, swap, "\alpha"] & \cdot\,, &
\end{tikzcd}$$

then

$$\begin{aligned}
\sigma &= (\sigma \otimes 1) \circ \alpha \circ \sigma \cdot (\sigma \otimes 1) \cdot \alpha, \\
\Rightarrow \quad \alpha^{-1} \cdot (\sigma \otimes 1) \cdot \sigma &= \sigma \cdot (\sigma \otimes 1) \cdot \alpha, \\
\Rightarrow (1 \otimes \sigma) \circ \sigma \circ \alpha^{-1} &= \alpha \circ \sigma \circ (1 \otimes \sigma).
\end{aligned}$$

♣

17 Appendix: Classifying Spaces

Let G be a topological group. The *classifying space* of G is defined as the unique (up to homotopy) space BG so that the set $\mathrm{Bun}_G(Y)$ of isomorphism classes of G-principal bundles on Y is in one-to-one correspondence

$$\mathrm{Bun}_G(Y) \cong [Y, BG],$$

where $[Y, Z]$ denotes the set of continuous maps $Y \to Z$ modulo homotopies. This is of course closely related to the orbifold $\mathcal{B}G := [\bullet/G]$. Recall that whenever Y is a manifold we have that

$$C^\infty(Y, \mathcal{B}G) = \{P \to Y \colon P \text{ is a } G-\text{principal bundle}\},$$

namely $C^\infty(Y, \mathcal{B}G)$ is the *groupoid* of principal G-bundles on Y. Notice that $C^\infty(Y, \mathcal{B}G)$ is a discrete groupoid, not merely a set (cf. Example 7.18). Notice that

$$[Y, BG] = \mathrm{Bun}_G(Y) = \{P \to Y \colon P \text{ is a } G-\text{principal bundle}\}/\text{iso } = C^\infty(Y, \mathcal{B}G)/\text{hom}.$$

From this we can clearly see that BG carries strictly less information that $\mathcal{B}G$ which sometimes is good and sometimes is bad. Morally speaking BG (which is defined only up to homotopy) is the homotopy type of $\mathcal{B}G$.

There is various ways of understanding the space BG, the most common one is to construct a space EG that is a contractible space with a *free* action of G. Then one can define $BG := EG/G$. This is unique up to homotopy and has the desired properties [Ste99]. The space BG thus defined classifies bundles up to isomorphism. The quotient map $EG \to BG$ is called the universal G-principal bundle. The reason is that for any G-principal bundle $P \to Y$ there is a map (unique up to homotopy) $f : Y \to BG$ so that $P \cong f^*EG$.

Example 17.1. Consider the group $G = \mathbb{Z}_2$. The infinite dimensional sphere S^∞ is contractible and has a free action of $\mathbb{Z}_2$ given by the antipodal map $x \mapsto -x$. Therefore we can take $E\mathbb{Z}_2 := S^\infty$ and then

$$B\mathbb{Z}_2 = S^\infty/\mathbb{Z}_2 = \mathbb{R}P^\infty.$$

In other words: the classifying space of $\mathbb{Z}_2$ is the infinite dimensional real projective space. The space $\mathbb{R}P^\infty$ can be interpreted as the space of real lines l in $\mathbb{R}^\infty$ passing through the origin. There is a canonical universal bundle $\Gamma \to \mathbb{R}P^\infty$ called the *tautological bundle*. It is called tautological because the fiber over $l \in \mathbb{R}P^\infty$ is $\Gamma_l = l$. To have a double cover over Y is the same as to have a real line bundle over Y, for by taking the *sphere bundle* of unit vectors of a line bundle we obtain a double cover. So $\mathbb{R}P^\infty$ classifies both double covers and real line bundles.

Example 17.2. Consider the circle group $G = S^1$ of complex numbers of modulus 1. The infinite even dimensional sphere $S^\infty \subset \mathbb{C}^\infty$ is contractible and has a free action of S^1 given by the map $x \mapsto zx$. Therefore we can take $ES^1 := S^\infty$ and then

$$BS^1 = S^\infty/S^1 = \mathbb{C}P^\infty.$$

In other words: the classifying space of S^1 is the infinite dimensional complex projective space. The space $\mathbb{C}P^\infty$ can be interpreted as the space of complex lines l in $\mathbb{C}^\infty$ passing through the origin. There is a canonical universal bundle $\Gamma \to \mathbb{C}P^\infty$ called the *tautological bundle*. It is called tautological because the fiber over the complex line $l \in \mathbb{C}P^\infty$ is $\Gamma_l = l$. To have a circle bundle over Y is the same as to have a complex line bundle over Y, for by taking the *sphere bundle* of unit vectors of a line bundle we obtain a circle bundle. So $\mathbb{C}P^\infty$ classifies both circle bundles and complex line bundles.

Example 17.3. Consider the group $G = U(n)$ of unitary $n \times n$ matrices. By considering $EU(n)$ to be the space of orthonormal frames on $\mathbb{C}^\infty$ it is proved that $BU(n)$ is the grassmannian of n-dimensional complex subspaces of $\mathbb{C}^\infty$

Example 17.4. Consider any finite group G. Notice that by Cayley's theorem G can be though of as a group of permutations, and this in turn realizes G as a subgroup of U_n. The classifying space of BG can then be constructed by the space of frames in infinite dimensional space modulo the appropriate permutations.

While the geometric constructions of BG are quite useful to study more structural properties of BG a combinatorial approach is very convenient.

Recall that a group G can be thought of as a category with one object $\bullet$ and as many morphisms $g : \bullet \to \bullet$ as elements of G. Composition of morphisms is given by group multiplication.

Definition 17.1. A *(semi-)simplicial set (resp. group, space, scheme)* $X_\bullet$ is a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ (resp. groups, spaces, schemes) together with maps

$$X_0 \leftrightarrows X_1 \leftrightarrows X_2 \leftrightarrows \cdots \leftrightarrows X_m \leftrightarrows \cdots$$

$$\partial_i \colon X_m \to X_{m-1}, \quad s_j \colon X_m \to X_{m+1}, \quad 0 \le i, j \le m.$$

called *boundary* and *degeneracy* maps, satisfying

$$\begin{aligned}
\partial_i \partial_j &= \partial_{j-1}\partial_i \quad \text{if } i < j \\
s_i s_j &= s_{j+1} s_i \quad \text{if } i < j \\
\partial_i s_j &= \begin{cases} s_{j-1}\partial_i & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ s_j \partial_{i-1} & \text{if } i > j+1 \end{cases}
\end{aligned}$$

The nerve of a category (following Segal [Seg68a]) is a semi-simplicial set $N\mathcal{C}$ where the objects of $\mathcal{C}$ are the vertices, the morphisms the 1-simplices, the triangular commutative diagrams the 2-simplices, and so on.

We can consider small categories $\mathcal{C}$ that are *topological categories* in Segal's sense. What this means is that both the set of objects and the set of morphisms are topological spaces and all the structural maps that define the category are continuous.

We can define the boundary maps $\partial_i : \mathcal{X}^{(n)} \to \mathcal{X}^{(n-1)}$ by:

$$\partial_i(\gamma_1, \dots, \gamma_n) = \begin{cases} (\gamma_2, \dots, \gamma_n) & \text{if } i = 0 \\ (\gamma_1, \dots, m(\gamma_i, \gamma_{i+1}), \dots, \gamma_n) & \text{if } 1 \le i \le n-1 \\ (\gamma_1, \dots, \gamma_{n-1}) & \text{if } i = n \end{cases}$$

and the degeneracy maps by

$$s_j(\gamma_1, \dots, \gamma_n) = \begin{cases} (e(s(\gamma_1)), \gamma_1, \dots, \gamma_n) & \text{for } j = 0 \\ (\gamma_1, \dots, \gamma_j, e(t(\gamma_j)), \gamma_{j+1}, \dots, \gamma_n) & \text{for } j \ge 1 \end{cases}$$

We will write Δ^n to denote the standard n-simplex in $\mathbb{R}^n$. Let $\delta_i \colon \Delta^{n-1} \to \Delta^n$ be the linear embedding of Δ^{n-1} into Δ^n as the i-th face, and let $\sigma_j \colon \Delta^{n+1} \to \Delta^n$ be the linear projection of Δ^{n+1} onto its j-th face.

Definition 17.2. The *geometric realization* $|X_\bullet|$ of the simplicial object $X_\bullet$ is the space

$$|X_\bullet| = \left(\coprod_{n \in \mathbb{N}} \Delta^n \times X_n \right) \Big/ \begin{array}{l} (z, \partial_i(x)) \sim (\delta_i(z), x) \\ (z, s_j(x)) \sim (\sigma_j(z), x) \end{array}$$

Notice that the topologies of X_n are relevant to this definition.

The simplicial object $N\mathcal{C}$ determines $\mathcal{C}$ and its topological realization is called $B\mathcal{C}$, the *classifying space of the category.*

Observe that B is actually a functor

$$B \colon \mathbf{Cat} \to h\mathbf{Top},$$

where $h\mathbf{Top}$ is the category of topological spaces modulo homotopy. It sends categories to spaces, functors to continuous maps, and natural transformations to homotopies. It also satisfies the less evident property

$$B(\mathcal{C} \times \mathcal{D}) = B\mathcal{C} \times B\mathcal{D}.$$

For a nice proof of this we refer the reader to [Dri04, Bes03, Crh01]. Also look at the classical reference [May93].

Example 17.5. Consider a finite group G. This produces a category $\mathcal{C}_G$ with one object $\bullet$ and arrows $g\colon \bullet \to \bullet$. So we have that X_0 is a one element set, $X_1 = G$ is the set of arrows, $X_2 = G \times G$ is the set of commutative triangles with edges (g, h, gh) (completely determined by the pair (g, h), and so on. Therefore $X_k = G^k$. It turns out that

$$BG \simeq B\mathcal{C}_G = |X_\bullet|.$$

Computing the cellular homology of BG given by this model one recovers the algebraic definition of group cohomology. We refer the readers to Segal's paper for a very elegant proof [Seg68a].

Example 17.6. Let G be a group acting on a set M. Let $\mathcal{C}_{G\ltimes M}$ be the category (groupoid) whose objects are the set M, and arrows $g : x \to y$ such that $y = gx$, this set can be seen as the set $G \times M$. Here the composition is defined of natural manner $gg' : x \to z$ where $g' : x \to y$ and $g : y \to z$. We have in this case:

$$B\mathcal{C}_{G\ltimes M} = (M \times EG)/G = M \times_G EG,$$

called the *Borel construction* or *homotopy quotient* of the group action. See [Seg68a].

Example 17.7. Let M be a smooth manifold. Consider an atlas $\mathcal{U} = (U_i)_{i\in I}$. To have a pair $(M, \mathcal{U})$ is the same thing as to have a small topological category $M_\mathcal{U}$ defined as follows.

- Objects: Pairs (m, i) so that $m \in U_i$. We endow the space of objects with the topology
$$\coprod_i U_i.$$
- Arrows: Triples (m, i, j) so that $m \in U_i \cap U_j = U_{ij}$. An arrow acts according to the following diagram.
$$(x, i) \overset{(x,i,j)}{\to} (x, j).$$
- The composition of arrows is given by
$$(x, i, j) \circ (x, j, k) = (x, i, k)$$

  The topology of the space of arrows in this case is
$$\coprod_{(i,j)} U_{ij}.$$

The category M is actually a **groupoid**, in fact

$$(x, i, j) \circ (x, j, i) = (x, i, i) = \mathtt{Id}_{(x,i)}.$$

The classifying space can be computed in this case to be homotopy equivalent to the original manifold:

$$\mathsf{BM} \simeq \mathsf{M}.$$

See [Seg68a, Section 4].

18 Appendix: K-theory

18.1 Basic Concepts

Ordinary cohomology satisfies a collection of axioms known as the Eilenberg-Steenrod axioms. We define an *extraordinary cohomology theory* to be an integer indexed sequence of contravariant functors h^i that take a pair of spaces (X, A) and deliver a sequence of abelian groups $h^i(X, A)$, together with a natural transformation $\delta\colon h^{i-1}(A) \to h^i(X, A)$ satisfying:

- *Homotopy invariance*: Homotopic maps $f, g\colon (X, A) \to (Y, B)$, $f \simeq g$ induce the same map after applying the functor h^i, to wit $f^* = g^*\colon h^i(Y, B) \to h^i(X, A)$.

- *Excision*: Whenever the closure of U is contained in the interior of A then the inclusion map $j\colon (X - U, A - U) \to (X, A)$ induces an isomorphism $j^*\colon h^i(X, A) \cong h^i(X - U, A - U)$.

- *Additivity*: If $X = \coprod_j X_j$ then $h^i(X) = \bigoplus^j h_i(X_j)$.

- *Exacness*: Given inclusions $i : A \to X$ and $j : X \to (X, A)$ we get the long exact sequence $\cdots \to h^{k-1}(A) \xrightarrow{\delta} h^k(X, A) \xrightarrow{j^*} h^k(X) \xrightarrow{i^*} h^k(A) \to \cdots$

Surprisingly such a theory is entirely determined by the value of the functor evaluated at a point $h^*(\bullet)$ called the coefficient group. For ordinary cohomology $H^*(\bullet) = \mathbb{Z}$.

For excellent accounts of K-theory we refer the reader to [Ati89, Kar08].

Let X be a compact topological space. We denote by $\mathrm{Vect}(X)$ the category of all complex vector bundles on X, and by $\mathbb{Z}[\mathrm{Vect}(X)]$ the free abelian group generated by the objects of $\mathrm{Vect}(X)$. Write N to denote the subgroup of $\mathbb{Z}[\mathrm{Vect}(X)]$ generated by all those elements of the form

$$[E] + [F] - ([E] \oplus [F]),$$

then we can define the K-*group of* X as

$$K(X) := \mathbb{Z}[\mathrm{Vect}(X)]/N.$$

We can endow $K(X)$ naturally with the structure of a ring by defining the product as the pull-back under the diagonal map of the exterior tensor product of bundles on $X \times X$.

The group K(X) satisfies the universal property for abelian homomorphisms $\mathbb{Z}[\text{Vect}(X)] \to G$, namely every such homomorphism factorizes through the canonical homomorphism $\mathbb{Z}[\text{Vect}(X)] \to K(X)$. From this it is an easy exercise to show that every element of $K(X)$ can be written in the form

$$[E] - [F],$$

for vector bundles E,F over X. This is used throughout this book.

Using partitions of unity we can show that every bundle E can be realized as a sub-bundle of a trivial bundle (of large dimension M). We write $M \cdot \epsilon$ to denote such trivial bundle over X. By using the Gram-Schmidt process we can construct then a complementary bundle $E^{\perp}$ so that $E \oplus E^{\perp} = M \cdot \epsilon$, hence we can improve the previous statement to say that every element in $K(X)$ can be written in the form

$$[E] - M \cdot [\epsilon],$$

for some bundle E and some integer M.

We define the *reduced* $\tilde{K}$*-ring* by making $\tilde{K}(X)$ to be the kernel of the map $K(X) \to K(\bullet)$. In turn we define

$$K(X,Y) := \tilde{K}(X/Y),$$

and

$$K^{-i}(X,Y) := \tilde{K}(\Sigma^{i}(X/Y)),$$

where Σ is the reduced suspension $\Sigma(X) := S^1 \wedge X$. Using this definitions Atiyah and Hirzebruch proved that $K^*(X,Y)$ defines an extraordinary cohomology theory [AH59, AH61].

Ordinary cohomology and K-theory do not coincide, and the coefficient group of K-theory is computed by the ring homomorphisms established by the Bott periodicity theorem:

$$K^{-*}(\bullet) = \mathbb{Z}[\beta].$$

The Bott periodicity theorem implies that K-theory is $\mathbb{Z}_2$ graded and periodic:

$$K^{*+2}(X) \simeq K^*(X).$$

There is a very convenient isomorphism between $K(X) \otimes \mathbb{Q}$ and $H^*(X,\mathbb{Q})$ given by the Chern character

$$\text{ch} : K(X) \otimes \mathbb{Q} \to H^*(X,\mathbb{Q}),$$

and completely determined by the requirement that $ch(L) = e^x$ for line bundles, where $x := c_1(L)$ denotes the first Chern class. This completely determines the

Chern character because of the *splitting principle*. The splitting principle states that whenever we have a bundle $E \to X$ we can find a space Y and a map $p: Y \to X$ so that $p^*(E)$ is a direct sum of line bundles $p^*(E) = L_1 \oplus \cdots \oplus L_k$ over Y, and p^* is injective in cohomology. This allows us to pretend that every bundle in the sum of line bundles in calculations. Such calculations are performed in the variables $x_1, \ldots, x_k$, where $x_i := c_1(L_i)$ are known as the *Chern roots* of E.

There is analogous concepts in K-theory of most concepts in cohomology. In particular one has the K-theoretic Euler class. This can be defined as follows. Consider the homomorphism $\lambda_t\colon K(X) \to K(X)[[t]]$ given by the generating series

$$\lambda_t(E) = \sum_{k \geq 0} [\Lambda^k E] t^k,$$

satisfying

$$\lambda_t(E \oplus F) = \lambda_t(E)\lambda_t(F).$$

Then we define the K-theoretic Euler class by defining it on generators of $\mathbb{Z}[\mathrm{Vect}(X)]$ by evaluation at $t = -1$:

$$\lambda_{-1}(E) := \sum_k (-1)^k [\Lambda^k E].$$

That this is the Euler class can be justified by noticing that $ch(\lambda_{-1}(E)) = e(E) \cdot \mu(E)$ where $e(E)$ is the cohomological Euler class and $\mu(E)$ is invertible in cohomology. Whenever we use the letter e for other purposes in our calculations we denote the cohomological Euler class by $Eu(E) := e(E)$.

Example 18.1. Let us take a bundle E and add a trivial bundle or rank M to obtain $E \oplus M\epsilon$. Then $\lambda_{-1}(E \oplus M\epsilon) = \lambda_{-1}(E)\lambda_{-1}(M\epsilon)$. And we compute:

$$\lambda_{-1}(M\epsilon) = \sum_k (-1)^k [\Lambda^k(M\epsilon)] = \sum_k (-1)^k \binom{M}{k} [\epsilon] = ((-1)+1)^M \epsilon = 0 \cdot \epsilon = 0,$$

and hence $\lambda_{-1}(E \oplus M\epsilon) = 0$. Using the Chern character we conclude immediately that $e(E \oplus M\epsilon) = 0$.

Let us mention the *Brown representability theorem*. Is states that for every reduced cohomology functor $\tilde{h}$ there is sequence of spaces P^n such that we have a natural isomorphism $\tilde{h}^k(X) \simeq [X, P^k]$. Here $[X, P]$ is the set of homotopy classes of maps $X \to P$. Moreover the spaces P^n are not quite independent but they form what is known as an Ω-prespectrum. What this means is that the spaces P^n come equipped with homotopy equivalences $q_n : P^n \simeq \Omega P^{n+1}$ where ΩP is the space of based loops $S^1 \to P$ on P.

For ordinary cohomology the spaces $P^n = K(\mathbb{Z}, n)$ are known as the Eilenberg-McLane spaces.

For K-theory the space $\mathbb{Z} \times BU$ in the zeroth space of its associated spectrum, thus determining the whole spectrum. Here BU is the classifying space of the infinite unitary group $U = \lim U(n)$.

This extends the definition of $K(X) := [X, \mathbb{Z} \times BU]$ to non-compact spaces X.

A remarkable fact relating K-theory and functional analysis is the fact that the space $\mathcal{F}$ of Fredholm operators on a separable Hilbert space $\mathcal{H}$ satisfies the following homotopy equivalence:

$$\mathcal{F} \simeq \mathbb{Z} \times BU.$$

This is the starting point for index theory.

For excellent accounts of K-theory we refer the reader to [Ati89, Kar08].

18.2 Orbifold K-Theory

In their seminal paper [DHVW86], Dixon, Harvey, Vafa, and Witten defined the orbifold Euler characteristic of an orbifold $X = [M/G]$ by the formula

$$\chi_{\mathbf{Orb}}(X) = \frac{1}{|G|} \sum_{gh=hg} \chi(M^{g,h}), \tag{54}$$

where (g, h) runs through all the pairs of commuting elements of G and $M^{g,h}$ is the set of points in M that are fixed both by g and by h. They obtained this formula by considering a supersymmetric string sigma model on the target space M/G and noting that in the known case in which $G = \{1\}$ the Euler characteristic of $X = M$ is a limiting case (over the worldsheet metric) of the partition function on the 2-dimensional torus.

In essentially every interesting example, the *stringy orbifold Euler characteristic* $\chi_{\mathbf{Orb}}(X)$ is not equal to the ordinary Euler characteristic of the quotient space $\chi(X)$. More interestingly, $\chi_{\mathbf{Orb}}(X)$ is truly independent of the particular groupoid representation, namely if $X = [M/G] \cong [N/H]$ then it does not matter which representation one uses to compute $\chi_{\mathbf{Orb}}(X)$. In other words, this is a truly physical quantity independent of the choice of coordinates. This last remark, which can be readily verified by the reader, is quite telling, since *a priori* the sigma model depends on the particular groupoid representation. But as the theory is indeed physical, the final partition function is independent of the choice of coordinates.

Moreover, since the partition function of the theory is physical, one may expect a stronger sort of invariance. Should there be a well-behaved (smooth) resolution of X defining the same quantum theory, then one should have that the Euler characteristic of the resolution is the same as that of the original orbifold. Here we are shifting our point of view, thinking of an orbifold as the quotient space with a mild type of singularities. It is a remarkable fact in algebraic geometry [Car57] that in good cases, remembering X plus some additional algebraic data (for example the structure sheaf), one can recover X. This point of view has proved extremely fruitful as we shall see. In any case, it often happens that there are resolutions of X, the *crepant resolutions*, for which the quantum theory is the same as that for X. We will come back to this later.

There is, of course, a far more classical interpretation of the Euler characteristic, the topological interpretation. The classical interpretation of the Euler characteristic in terms of triangulations tells us that the Euler characteristic is the alternating sum of the Betti numbers, namely, the ranks of the cohomologies of the space in question. Thus, a natural question is whether there is a cohomology theory for an orbifold that is physical and that simultaneously produces the appropriate Euler characteristic of Formula (54). One is first tempted to consider equivariant cohomology $\mathsf{H}^*_{\mathsf{G}}(\mathsf{M}) = \mathsf{H}^*(\mathsf{M} \times_{\mathsf{G}} \mathsf{EG})$ but unfortunately the relation between cohomology and Euler characteristic breaks down, for the expression (54) is not recovered.

Considering the orbifold $\mathsf{X} = [*/\mathsf{G}]$ consisting of a finite group acting on a single point gives us a clue into the right answer. In this case, $\chi_{\mathbf{Orb}}([*/\mathsf{G}])$ becomes the number of pairs of commuting elements in G divided by $|\mathsf{G}|$. An amusing exercise in finite group theory readily shows that this is the same as the number of conjugacy classes of elements in G. Given a finite group there are two basic quantities that we can consider, its group cohomology $\mathsf{H}^*(\mathsf{BG})$ and its representation ring $\mathsf{R}(\mathsf{G})$. While equivariant cohomology is akin to group cohomology, it is *equivariant* K*-theory* $\mathsf{K}_{\mathsf{G}}(\mathsf{M})$ that is intimately related to representation theory. For a start, $\mathsf{K}_{\mathsf{G}}(*) = \mathsf{R}(\mathsf{G})$.

As a first test, we consider an orbifold $\mathsf{X} = [\mathsf{M}/\mathsf{G}] \cong [\mathsf{N}/\mathsf{H}]$ and see whether the theory is invariant under the representation. This is not too hard (see for example [LU04a, AR03]), and hence it fully deserves the name of *orbifold* K*-theory* and can unambiguously be written as $\mathsf{K}_{\mathbf{Orb}}(\mathsf{X}) = \mathsf{K}_{\mathsf{G}}(\mathsf{M}) \cong \mathsf{K}_{\mathsf{H}}(\mathsf{M})$.

The second test is to see whether we can recover Formula (54). That this is possible was first observed by Atiyah and Segal [AS89]. The idea is to use the Segal character of an equivariant vector bundle. Let us remember that the basic cocycles of equivariant K-theory are G-equivariant vector bundles [Ati67], namely bundles $\mathsf{p} \colon \mathsf{E} \to \mathsf{M}$ over the G-manifold M with a G-action by bundle automorphisms on all of E that extends the action on M (considered as the zero section) and that is fiberwise linear. Should there be a fixed point $m \in M$, then $E_m := p^{-1}(m)$ becomes a representation of G; in particular, if the space M is a point then a G-equivariant vector bundle over M is the same as a representation of G (by choosing a basis we get a matrix for every $g \in G$).

The (Segal) character of an equivariant vector bundle is an *isomorphism* [Seg, Moe02] of the form

$$K_G(M) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{(g)} K(M^g)^{C(g)} \otimes \mathbb{C}, \tag{55}$$

where the sum is over all conjugacy classes (g) of elements $g \in G$.

The character isomorphism is explicitly given by the expression

$$\begin{array}{rcl} K_G(M) \otimes \mathbb{C} & \to & K(M^g)^{C(g)} \otimes \mathbb{C} \\ E \otimes 1 & \mapsto & \mathrm{char}(E)(g) = \sum_{\zeta} (E|_{M^g})_\zeta \otimes \zeta. \end{array}$$

Here the sum is over all roots of unity ζ, the symbol $(\ )_\zeta$ denotes the ζ-eigenspace of g, and finally M^g is the subspace of fixed point under g of M. We call this isomorphism the *Segal localization formula* (for it localizes equivariant K-theory to ordinary K-theory of the fixed point sets). Clearly, in the case in which M is a point, this recovers the usual theory of characters for the finite-dimensional representations of a finite group. Remarkably enough this is indeed related to the localization of equivariant K-theory as an $R(G)$-module with respect to prime ideals [Seg68b].

From Segal's isomorphism (55) we conclude immediately that [AS89, BC88, Uri]

$$\mathrm{rank} K^0_G(M) - \mathrm{rank} K^1_G(M) = \sum_{(g)} \chi(M^g/C(g)) = \frac{1}{|G|} \sum_{gh=hg} \chi(M^{g,h}) = \chi_{\mathbf{Orb}}(X).$$

Here we have applied the algebraic equality

$$\chi_{\mathbf{Orb}}(X) = \sum_{(g)} \chi(M^g/C(g)),$$

which follows by an inclusion-exclusion argument [HH90]; in the next section we talk about a geometric explanation for this algebraic fact.

For now let us mention that the theory described in this section can be generalized to orbifolds that are not necessarily global quotients [LU04a, AR03]. This is done as follows. We will denote by X_0 and X_1 the set of objects and morphism of our orbifold groupoid respectively, and the structure maps by

$$X_1 {}_t\times_s X_1 \xrightarrow{m} X_1 \xrightarrow{i} X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0 \xrightarrow{e} X_1,$$

where $X_1 {}_t\times_s X_1$ is the subspace of $X_1 \times X_1$ such that whenever $(\alpha, \beta) \in X_1 {}_t\times_s X_1$ then the target of α equals the source β; s and t are the source and the target maps on morphisms, m is the composition arrows, i gives us the inverse morphism, and e assigns the identity arrow to every object.

We define a *vector orbibundle* over X to be a pair (E, τ) where E is an ordinary vector bundle over X_0 and $\tau\colon s^*E \xrightarrow{\cong} t^*E$ is an isomorphism of vector bundles over X_1.

The set of isomorphism classes of such orbibundles is denoted by $\mathsf{Orbvect}(X)$ and its Grothendieck group by $K^0_{\mathrm{orb}}(X)$ [LU04a].

This coincides with equivariant K-theory if the orbifold happens to be of the form $[M/G]$.

19 Appendix: The McKay Correspondence

In this appendix we will assume that the reader is comfortable with the language of algebraic geometry. Let us consider a classical example. Let G be a finite subgroup of $SL_2(\mathbb{C})$; then $X = \mathbb{C}^2/G$ is called a *Kleinian quotient singularity*; see [Slo80, DLHS79] for more details and historical discussion. In the second half of the 19th century, Klein classified the possible groups G as either cyclic, dihedral or binary dihedral and gave equations for these singularities in $\mathbb{C}^3$. Let us consider the simplest case in which $G \cong \mathbb{Z}/r\mathbb{Z}$. We can realize X as a subvariety of $\mathbb{C}^3$ by

$$X\colon z^r = xy$$

or, in parametric form,

$$\begin{aligned} x &= u^r \\ y &= v^r \\ z &= uv \end{aligned} \tag{56}$$

as the image of a map $\mathbb{C}^2 \to \mathbb{C}^3$ by G-invariant polynomials. We can resolve the singularity very easily in this case by taking $(r-1)$-blow ups to obtain

$$Y \xrightarrow{\phi} X$$

where the exceptional divisor is

$$\phi^{-1}(0) = E_1 \cup E_2 \cup \cdots \cup E_{r-1}$$

whose incidence graph is A_{r-1}.

On the other hand, G clearly has $r-1$ nontrivial irreducible representations.

The *McKay correspondence* establishes (among other things) a one-to-one correspondence between the number of components of the exceptional divisor in a minimal resolution of the singularity and the number of nontrivial irreducible representations of G. Notice that in our example this is equivalent to the statement that *the orbifold Euler characteristic of* X *is the same as the ordinary Euler characteristic of* Y. So one may expect that some functional integral argument may be provided to prove the McKay correspondence.

There is in fact a rigorous version of the functional integration method in algebraic geometry discovered by Kontsevich [Kon] and known as *motivic integration.* We now briefly outline the construction of this method.

Given a smooth complex variety Y, one can define its *arc space* JY. This is a scheme whose $\mathbb{C}$-points are arcs $\gamma\colon \mathrm{Spec}(\mathbb{C}[[t]]) \to Y$. The scheme JY is obtained as the inverse limit of the *jet schemes* $J_m Y$, whose $\mathbb{C}$-points are jets $\gamma_m\colon \mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to Y$. The morphisms $J_p Y \to J_m Y$, for $0 \le m \le p \le \infty$, are given by truncation. For any effective divisor $D \subset Y$, one can define an order function

$$\mathrm{ord}_D : JY \to \mathbb{Z}_{\geq 0} \cup \{\infty\},$$

which to each arc γ associates its order of contact $\mathrm{ord}_D(\gamma)$ along D. The idea is then to "integrate these functions," in some reasonable sense. But first one needs to introduce the algebra of measurable sets and the measure. The first is easily defined as the algebra generated by *cylinder sets* in JY, namely, inverse images of constructible sets on finite levels $J_m Y$. The measure will then take values in the so-called *motivic ring*.

The motivic ring is constructed as follows: we fix a complex variety X and assume that Y is an X-variety (that is, a complex algebraic variety of finite type over X). Let $K_0(\mathcal{V}ar_X)$ be the ring generated by X-isomorphism classes of X-varieties subjected to the relation

$$\{V\} = \{V \setminus W\} + \{W\}$$

whenever W is a closed variety of a X-variety V. The product is defined by

$$\{V\} \cdot \{W\} = \{V \times_X W\}.$$

The zero of this ring is $\{\varnothing\}$, and the identity is $\{X\}$. We let

$$\mathcal{M}_X = K_0(\mathcal{V}ar_X)[\mathbf{L}_X^{-1}],$$

where $\mathbf{L}_X$ is the class of the affine line over X. Finally the motivic ring is the completion $\hat{\mathcal{M}}_X$ of $\mathcal{M}_X$ under a certain natural dimension filtration [Loo02, DL02b, DFLNU07].

Via composition, every subvariety of the jet schemes of Y can be viewed as an X-variety. Thus, one can define the *motivic measure* of a cylinder $C \subseteq JY$ by fixing a large enough integer m such that C is the inverse image of a constructible set $C_m \subseteq J_m Y$ and then setting

$$\mu(C) = \{C_m\} \cdot \mathbf{L}_X^{-m \dim Y} \in \hat{\mathcal{M}}_X.$$

Then, by suitable stratification, one defines the *motivic integral*

$$\int_{JY} \mathbf{L}_X^{-\mathrm{ord}_D}\, d\mu \in \hat{\mathcal{M}}_X.$$

For instance, if $D = \sum a_j D_j$ is a simple normal crossing divisor and we define

$$D_J^\circ = \bigcap_{j \in J} D_j \setminus \bigcup_{i \notin J} D_i,$$

then one has

$$\int_{JY} \mathbf{L}_X^{-\mathrm{ord}_D} d\mu = \sum_{J \subseteq I} \{D_J^\circ\} \prod_{j \in J} \frac{\mathbf{L}_X - 1}{\mathbf{L}_X^{a_j+1} - 1}.$$

The power of this theory is a *change of variable formula*; this allows us to reduce to computing integrals for divisors with simple normal crossings (hence apply the above formula) by replacing any effective divisor D on Y by $D' = K_{Y'/Y} + g^* D$, where $g : Y' \to Y$ is a simple normal crossing resolution of the pair (Y, D). The theory can be also extended to singular varieties (under suitable conditions): in this case the measure itself needs to be opportunely "twisted" to make the change of variable formula work. The resulting measure is called *Gorenstein measure* and denoted by μ^{Gor}.

We can now review the *motivic McKay correspondence* [DL02b, Loo02, Rei02]. To give a formulation of this correspondence that better fits with the localization principle of this book, we need to further quotient the ring $K_0(\mathcal{V}ar_X)$ by identifying X-varieties that become isomorphic after some étale base change $X_k' \to X_k \subseteq X$ of each piece X_k of a suitable stratification $X = \bigsqcup X_k$ of X. We obtain in this way a new ring: $K_0(\mathcal{V}ar_X)^{\mathrm{et}}$. This leads to the definition of a different motivic ring, which we denote by $\widehat{\mathcal{M}}_X^{\mathrm{et}}$ (the reader will notice that, if X is a point, then we are not changing anything).

Let $X = [M/G]$, where M is a quasiprojective variety and G is a finite group, let $X = M/G$, and assume that X is Gorenstein. We can find a resolution of singularities $Y \to X$ with relative canonical divisor $K_{Y/X}$ having simple normal crossings. Write $K_{Y/X} = \sum a_j D_j$. Then the McKay correspondence is given by the identity

$$\sum_{J \subseteq I} \{D_J^\circ\} \prod_{j \in J} \frac{\mathbf{L}_X - 1}{\mathbf{L}_X^{a_j+1} - 1} = \sum_{(g)} \{M^g / C(g)\} \mathbf{L}_X^{w(g)} \quad \text{in } \widehat{\mathcal{M}}_X^{\mathrm{et}}, \tag{57}$$

where the sum in the left side runs over conjugacy classes (g) in G and $w(g)$ are integers depending on the local action of g on the normal bundle of M^g in M.

For instance, by noticing that the Euler characteristic defines a ring homomorphism

$$\chi \colon K_0(\mathcal{V}ar_X)^{\mathrm{et}} \to \mathbb{Z},$$

it is easy to see that Formula (57) implies the classical McKay correspondence,[3] which in particular says that the orbifold Euler characteristic is equal to the ordinary Euler characteristic of the resolution if the latter is crepant.

The proof of Formula (57) breaks into three parts. By the change of variable formula, one has

$$\int_{JX} \mathbf{L}_X^0 d\mu^{\mathrm{Gor}} = \sum_{J\subseteq I}\{D_J^\circ\}\prod_{j\in J}\frac{\mathbf{L}-1}{\mathbf{L}^{a_j+1}-1} \quad \text{in } \widehat{\mathcal{M}}_X.$$

Then, by an accurate study of lifts of the arcs of X to arcs on M, one proves that

$$\int_{JX} \mathbf{L}_X^0 d\mu^{\mathrm{Gor}} = \sum_{(H)}\{X^H\}\sum_{(h)}\mathbf{L}_X^{w(h)} \quad \text{in } \widehat{\mathcal{M}}_X.$$

Here the first sum runs over conjugacy classes (H) of subgroups of G, $X^H \subseteq X$ is the image of the set of points on M whose stabilizer is H, and the last sum is taken over conjugacy classes in H. The above identity is the core of the proof. Finally, one shows that

$$\sum_{(H)}\{X^H\}\sum_{(h)}\mathbf{L}_X^{w(h)} = \sum_{(g)}\{M^g/C(g)\}\mathbf{L}_X^{w(g)} \quad \text{in } \widehat{\mathcal{M}}_X^{\mathrm{et}}.$$

Here is where we need to pass to the ring $\widehat{\mathcal{M}}_X^{\mathrm{et}}$. This last part can be easily verified using certain properties of Deligne-Mumford stacks (see [DFLNU07]). In general, if we do not perform the additional localization in the relative motivic ring, but instead work with the ring $\widehat{\mathcal{M}}_X$, we do not expect the last identity to hold.

These results have been extended to general (not necessarily global quotient) orbifolds independently by Yasuda [Yas04] and by Lupercio-Poddar [LP04].

In [DFLNU07], we used a natural homomorphism from $K_0(\mathcal{V}ar_X)$ to the ring of constructible functions $F(X)$ on X to associate to any motivic integral an element in $F(X)_{\mathbb{Q}}$, that is, a rational-valued constructible function on X. In fact, one observes that this construction factors through $K_0(\mathcal{V}ar_X)^{\mathrm{et}}$.[4] The result is the following localization formula for constructible functions:

$$\sum_{J\subseteq I}\frac{(f|_{D_J^\circ})_*\mathbf{1}_{D_J^\circ}}{\prod_{j\in J}(a_j+1)} = \sum_{(g)}(\pi_g)_*\mathbf{1}_{M^g/C(g)} \quad \text{in } F(X), \tag{58}$$

[3]Here we are referring only to the counting statement, and not that we recover the full incidence graph of $\phi^{-1}(0)$ from the representation theory of G, as the classical correspondence establishes.

[4]In particular, this tells us that the identification performed to define $\widehat{\mathcal{M}}_X^{\mathrm{et}}$ does not trivialize the ring too much, as we can still recover all the information in $F(X)$.

where $\pi_g : M^g/C(g) \to X$ is the morphism naturally induced by the quotient map $\pi : M \to X$ [DFLNU07, Theorem 6.1].

Motivic integration was used in [DFLNU07] to define the *stringy Chern class* $c_{\mathrm{str}}(X)$ of X. In the case at hand, we use the MacPherson transformation [Mac74] to deduce from (58) the following localization formula for the stringy Chern class of a quotient [DFLNU07, Theorem 6.3]:

$$c_{\mathrm{str}}(X) = \sum_{(g)} (\pi_g)_* c_{\mathrm{SM}}(M^g/C(g)) \quad \text{in } A_*(X),$$

where $c_{\mathrm{SM}}(M^g/C(g))$ is the Chern-Schwartz-MacPherson class of $M^g/C(g)$ [Mac74]. This generalizes and implies Batyrev's formula for the Euler characteristic [Bat99].

20 Appendix: Orbifold Index Theory

20.1 Orbifolding Atiyah-Singer

Suppose that we have a compact symplectic $2m$-dimensional manifold N with symplectic form ω and that $H\colon N \to \mathbb{R}$ is the Hamiltonian function of a Hamiltonian circle action. Let F_α be the critical manifolds of H (namely the fixed points of the action) with critical values H_α. The Liouville volume form on N is $\omega^m/m!$. The Duistermaat-Heckman formula reads [AB84, DH82]

$$\int_N e^{\hbar H} \frac{\omega^m}{m!} = \sum_\alpha e^{\hbar H_\alpha} \int_{F_\alpha} \frac{e^\omega}{E_\alpha},$$

where E_α is the equivariant Euler class of the normal bundle of F_α in N. If $\hbar$ is taken as purely imaginary, the integral over N is oscillatory, the submanifolds F_α are the stationary points of H, and the right-hand side of this formula is given by stationary phase approximation.

Witten [Ati85] had the idea of using the Duistermaat-Heckman formula in the case $N = \mathcal{L}M$, the free loop space of a manifold M, with Hamiltonian

$$H(\gamma) = \frac{1}{2} \oint_{S^1} |\gamma'(t)|^2 dt.$$

In this case Atiyah defines a symplectic form on $\mathcal{L}M$ whenever M is compact and orientable. Then he goes on to show that *when M is a Spin manifold, $\mathcal{L}M$ is orientable*. Moreover, he shows that the left-hand side of the corresponding Duistermaat-Heckman formula is the heat kernel expression for the index of the Dirac operator while the right-hand side is the $\hat{A}$-genus, thus giving the Atiyah-Singer index theorem.

We do the same now for the loop groupoid. In order to simplify the calculation, we will consider the case of a global quotient $X = [M/G]$, but everything that we will say generalizes to general (non–global-quotient) orbifolds. We will suppose thus that M is a compact, even-dimensional spin manifold such that for every $g \in G$ the map $g : M \to M$ given by the action is a spin-structure-preserving isometry. We will argue that applying stationary phase approximation to the integral[5]

$$\int_{\mathcal{P}_g} e^{-tE(\phi)} \{\mathrm{Tr}\ S^+(T_\phi) - \mathrm{Tr}\ S^-(T_\phi)\} \mathcal{D}\phi \tag{59}$$

[5] In [Ati85] it is explained how to make sense of this integral.

one obtains

$$\mathsf{Spin}(M,g) := \mathsf{ind}_g(D^+) = \mathrm{tr}(g|_{\ker D^+}) - \mathrm{tr}(g|_{\mathrm{coker} D^+}),$$

the value of the g-index of the Dirac operator D^+ over M. Here E is the energy of the path (Hamiltonian)

$$E(\phi) := \frac{1}{2}\int_0^{2\pi} |\phi'(t)|^2 dt,$$

$\mathcal{D}\phi$ denotes the formal part of the Weiner measure on $\mathcal{P}_g$, T_ϕ is the tangent space at $\phi \in \mathcal{P}_g$, and S^+, S^- denote the two half-spin representations of $\mathsf{Spin}(2m)$ ($2m = \dim M$).

The real numbers act on $\mathcal{P}_g$ by shifting the path

$$\begin{array}{rcl} \mathcal{P}_g \times \mathbb{R} & \to & \mathcal{P}_g \\ (f,s) & \mapsto & f_s : \mathbb{R} \to M \\ & & f_s(t) := f(t-s) \end{array}$$

and the fixed point set of this action on $\mathcal{P}_g$ consists of the constant maps to M^g (the fixed point set of the action of g in M), that is,

$$(\mathcal{P}_g)^{\mathbb{R}} \cong M^g.$$

Applying the stationary phase approximation (see [Ati85, Formula 2.2]) to the integral (59), we get

$$\int_{(\mathcal{P}_g)^{\mathbb{R}}} \frac{e^{-tE(\phi)}}{\prod_j (tm_j - i\alpha_j)} = \int_{M^g} \frac{1}{\prod_j (tm_j - i\alpha_j)}, \tag{60}$$

where the energy of the constant paths is zero, the m_j are rotation numbers normal to M^g, and the α_j are the Chern roots, so that the total Chern class of the normal bundle N to M^g is given by

$$\prod_j (1+\alpha_j).$$

20.1.1 The Normal Bundle

For $f \in \mathcal{P}_g$, the tangent space T_f at f can be seen as the space of maps

$$\sigma : \mathbb{R} \to f^* TM$$

such that $\sigma(t)dg_{\sigma(t)} = \sigma(2\pi + t)$, so for the constant map at $x \in M^g$, its tangent space is equal to the space of maps

$$\sigma : \mathbb{R} \to T_x M$$

with $\sigma(t)dg_x = \sigma(2\pi + t)$.

We can split the vector space $T_x M$ into subspaces $N(\theta)$ that consist of 2-dimensional spaces on which dg_x rotates every vector by θ (see [LM89]):

$$T_x M \cong N(0) \oplus \bigoplus_{\theta} N(\theta).$$

It is clear that the number of θ is finite, that we could choose them in the interval[6] $0 < \theta < \pi$, and that $N(0) \cong T_x M^g$.

The constant functions

$$\{\sigma : \mathbb{R} \to T_x M^g \cong N(0) \mid \ \sigma \text{ is constant}\} \subset T_x \mathcal{P}_g$$

give the directions along M^g. We are interested in finding a description of the normal directions of M^g in $T_x \mathcal{P}_g$.

Let $2s(\theta) := \dim_{\mathbb{R}} N(\theta)$ and, for $l = 1, \ldots, s(\theta)$, let $N_l(\theta)$ be the 2-dimensional subspaces fixed by dg_x through the rotation of θ. Then any $\sigma \in T_x \mathcal{P}_g$ can be seen as

$$\sigma = \sum_{l,\theta} \sigma_l^\theta \qquad \text{with} \qquad \sigma_l^\theta : \mathbb{R} \to N_l(\theta).$$

Let $N_l(\theta)^{\mathbb{C}}$ be the complexification $N_l(\theta) \otimes \mathbb{C}$. Then

$$N_l(\theta)^{\mathbb{C}} \cong L_l \oplus \overline{L_l},$$

where L_l is a complex line bundle, the action of dg_x on L_l is by multiplication by $e^{i\theta}$, and $\overline{L_l}$ is the conjugate bundle of L_l (see [LM89, p. 226]). The map

$$\sigma_l^\theta : \mathbb{R} \to N_l(\theta) \subset N_l(\theta)^{\mathbb{C}}$$

can be seen in $L_l \oplus \overline{L_l}$ via a Fourier expansion as

$$\sigma_l^\theta(t) = \sum_{k \in \mathbb{Z}} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \begin{pmatrix} e^{itk} e^{it\frac{\theta}{2\pi}} & 0 \\ 0 & e^{itk} e^{-it\frac{\theta}{2\pi}} \end{pmatrix} \tag{61}$$

[6]For simplicity we will assume that the eigenvalue π is not included, in order to avoid the use of Pontrjagin classes. The result still holds with π as rotation number.

with $a_k \in L_l$, $b_k \in \overline{L_l}$, $a_k = \overline{a_{-k}}$ and $b_k = \overline{b_{-k}}$ (the last two equations hold because $\sum_k a_k e^{itk}$ and $\sum_k b_k e^{itk}$ are real for all t; in particular $a_0, b_0 \in \mathbb{R}$).

Then the tangent bundle to $T_x\mathcal{P}_g$ can be decomposed as an infinite direct sum

$$T_xM \oplus (T_xM^{\mathbb{C}})_1 \oplus (T_xM^{\mathbb{C}})_2 \oplus \cdots$$

with

$$(T_xM^{\mathbb{C}})_n \cong (N(0)^{\mathbb{C}})_n \oplus \bigoplus_{\theta} (N(\theta)^{\mathbb{C}})_n$$

where the circle acts in each $(N(\theta)^{\mathbb{C}})_n$ by rotation number n. The coefficients (a_k, b_k) of the Fourier expansion of (61) take values in $(N(\theta)^{\mathbb{C}})_k$ for $k > 0$, $(a_0, b_0) \in N(\theta)$, and $(a_k, b_k) = (\overline{a_{-k}}, \overline{b_{-k}})$ for $k < 0$.

As $T_xM \cong N(0) \oplus \bigoplus_\theta N(\theta)$ and $N(0) \cong T_xM^g$ represent the directions along M^g, the normal bundle to M^g in $\mathcal{P}_g$ can be represented as

$$\left\{(N(0)^{\mathbb{C}})_1 \oplus (N(0)^{\mathbb{C}})_2 \oplus \cdots\right\} \oplus \bigoplus_{\theta} \left\{N(\theta) \oplus (N(\theta)^{\mathbb{C}})_1 \oplus (N(\theta)^{\mathbb{C}})_2 \oplus \cdots\right\}.$$

Let the Chern class of $N(\theta)$ be

$$\prod_{k=1}^{s(\theta)} (1 + y_k^\theta),$$

so its g-Chern character is

$$ch_g(N(\theta)) = \sum_{k=1}^{s(\theta)} ch(N_k(\theta))\chi(g) = \sum_{k=1}^{s(\theta)} e^{y_k^\theta + i\theta};$$

then the g-Chern class of the complexification of $N(\theta)$ is

$$\prod_{k=1}^{s(\theta)} (1 + y_k^\theta + i\theta)(1 - y_k^\theta - i\theta).$$

If we let x_k denote the Chern classes of M^g, then the denominator in (60) with $t = 1$ becomes

$$\prod_{j=1}^{s(0)} \prod_{p=1}^{\infty} \left(p^2 + x_j^2\right) \prod_{\theta} \left\{ \prod_{k=1}^{s(\theta)} (y_k^\theta + i\theta) \prod_{p=1}^{\infty} \left(p^2 + (y_k^\theta + i\theta)\right) \right\},$$

which is formally

$$\prod_{j=1}^{s(0)} \left(\prod_{p=1}^{\infty} p^2 \right) \frac{\sinh(\pi x_j)}{\pi x_j} \prod_{\theta} \left\{ \prod_{k=1}^{s(\theta)} \left(\prod_{p=1}^{\infty} p^2 \right) (y_k^\theta + i\theta) \frac{\sinh(\pi(y_k^\theta + i\theta))}{\pi(y_k^\theta + i\theta)} \right\}.$$

Replacing the infinite product of the p^2 by its renormalized factor 2π, we get

$$\prod_j \frac{2\sinh(\pi x_j)}{x_j} \prod_\theta \left\{ \prod_k 2\sinh(\pi(y_k^\theta + i\theta)) \right\},$$

which is the same as

$$\prod_j \frac{\sinh(x_j/2)}{x_j/2} \prod_\theta \left\{ \prod_k \frac{\sinh((y_k^\theta + i\theta)/2)}{1/2} \right\} \tag{62}$$

provided we interpret $\prod_{p=1}^{\infty} t$ as $t^{\zeta(0)}$ where $\zeta(s)$ is the Riemann zeta function. As $\zeta(0) = -\frac{1}{2}$, in each component we get a factor of t which cancels with the factor t^{-1} that arises from replacing x_j by x_j/t and $y_k^\theta + i\theta$ by $(y_k^\theta + i\theta)/t$. Our use of the stationary phase approximation is independent of t, and setting $t = 2\pi$ we get formula (62).

In the notation of [LM89, p. 267] formula (62) is equivalent to

$$\left(\hat{A}(M^g) \prod_\theta \hat{A}(N(\theta)) \right)^{-1},$$

which after replacing it in the denominator of (60) and integrating over M^g matches the formula for $\mathsf{Spin}(M, g)$ [LM89, Th. 14.11]:

$$\mathsf{Spin}(M, g) = (-1)^{\tau_g} \hat{A}(M^g) \left\{ \prod_\theta \hat{A}(N(\theta)) \right\} [M^g].$$

We conclude that after applying the stationary phase approximation to (59), we obtain the g-index of the Dirac operator.

Proposition 20.1. *The path integral*

$$\int_{\mathcal{P}_g} e^{-tE(\phi)} \{\mathrm{Tr}\ S^+(T_\phi) - \mathrm{Tr}\ S^-(T_\phi)\} d\phi = \mathsf{Spin}(M, g)$$

equals $\mathrm{ind}_g(D^+)$*, the g-index of the Dirac operator over* M*.*

20.1.2 The G-index and Kawasaki's Formula

The G-index of the Dirac operator is an element of $R(G)$, the representation ring of G. Using localization, its dimension is equal to

$$\mathrm{ind}_G(D^+) = \frac{1}{|G|}\sum_{g\in G}\mathrm{ind}_g(D^+) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Spin}(M,g).$$

But instead of summing over all the elements g in G, we could sum over the conjugacy classes of G. It is clear that $\mathrm{Spin}(M,g) = \mathrm{Spin}(M,h^{-1}gh)$. The size of the conjugacy class (g) of g is $\frac{|G|}{|C(g)|}$ where $C(g)$ is the centralizer of G, that is, the set of elements which commute with g (equivalently, the fixed point set of the action of G in g via conjugation). Thus, we obtain

$$\mathrm{ind}_G(D^+) = \sum_{(g)}\frac{1}{|C(g)|}\mathrm{Spin}(M,g).$$

We would like to derive a formula that depends on the twisted sectors (inertia groupoid) of the orbifold $X=[M/G]$, and this clearly matches our previous description. In [LU02b] it was argued that the fixed point set of the action of $\mathbb{R}$ in the loop groupoid LX was precisely $I(X)$ the inertia groupoid of X; then, applying stationary phase approximation to

$$\int_{LX} e^{-tE(\phi)}\{\mathrm{Tr}\ S^+(T_\phi) - \mathrm{Tr}\ S^-(T_\phi)\}\mathcal{D}\phi,$$

which can be rewritten as

$$\sum_{(g)}\frac{1}{|C(g)|}\int_{\mathcal{P}_g} e^{-tE(\phi)}\{\mathrm{Tr}\ S^+(T_\phi) - \mathrm{Tr}\ S^-(T_\phi)\}\mathcal{D}\phi,$$

we get the G-index of the Dirac operator,

$$\mathrm{ind}_G(D^+) = \sum_{(g)}\frac{(-1)^{\tau_g}}{|C(g)|}\int_{M^g}\hat{A}(M^g)\prod_{\theta}\hat{A}(N(\theta)_g).$$

Which can be shown to coincide with the formula given by Kawasaki [Kaw81, p. 139] for the index theorem for V-manifolds. Thus, the localization principle applies in this case.

20.2 The Elliptic Genus

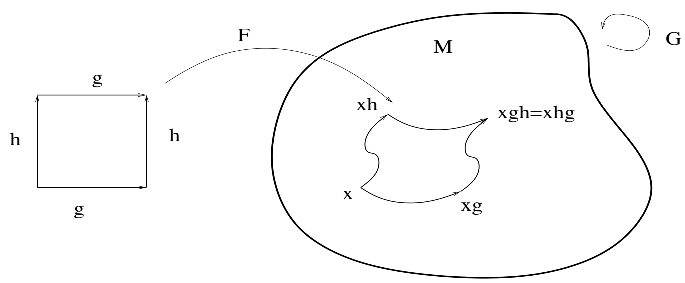
We move on now to localizing functional integrals in the double loopspace $\mathcal{L}^2\mathsf{M} = \mathcal{L}\mathcal{L}\mathsf{M} = \mathrm{Maps}(\mathsf{T}, \mathrm{Maps}(\mathsf{T}, \mathsf{M})) = \mathrm{Maps}(\mathsf{T}^2, \mathsf{M})$, where $\mathsf{T} = \mathsf{S}^1$ and T^2 is the 2-torus. By performing the corresponding functional integral over $\mathcal{L}^2\mathsf{M}$, we should obtain the index of the dirac operator over $\mathcal{L}\mathsf{M}$ considered by Witten in [Wit88] and known as the elliptic genus [Seg88b]. This has been verified by Ando and Morava [AM01]. We want to perform the calculation in the orbifold case (cf. [AF07]).

Let the groupoid X be $[\mathsf{M}/\mathsf{G}]$, and let the torus T be represented by the groupoid $[\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}]$. The *double loop groupoid* $\mathsf{L}^2\mathsf{X}$ is the category with smooth functors $\mathsf{T} \to \mathsf{X}$ as objects and natural transformations between functors as morphisms.

A morphism in $\mathsf{L}^2\mathsf{X}$ can be seen as

$$\begin{array}{ccc} \mathbb{R}^2 \times (\mathbb{Z} \oplus \mathbb{Z}) & \longrightarrow & \mathsf{M} \times \mathsf{G} \\ \downdownarrows & & \downdownarrows \\ \mathbb{R}^2 & \longrightarrow & \mathsf{M}, \end{array}$$

that is, as a map $\mathsf{F} \colon \mathbb{R}^2 \to \mathsf{M}$ together with a homomorphism $\mathsf{H} \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathsf{G}$ such that F is equivariant with respect to H. This is equivalent to choosing a pair of commuting elements $g, h \in \mathsf{G}$ such that $\mathsf{F}(1, 0) = \mathsf{F}(0, 0)g$, $\mathsf{F}(0, 1) = \mathsf{F}(0, 0)h$ and in general $\mathsf{F}(n, m) = \mathsf{F}(0, 0)g^n h^m$.



The group $\mathbb{R}^2$ acts naturally by translations on the double loop groupoid. This action factors through $\mathbb{R}^2/\{|\mathsf{G}|\mathbb{Z} \oplus |\mathsf{G}|\mathbb{Z}\}$ because every orbifold loop can be closed in M/G.

The fixed points under the action of $\mathbb{R}^2$ are the constant double loops; they are uniquely determined by a choice of a point in M and two commuting elements in G.

The groupoid of *ghost double loops* is the groupoid whose objects is the set of functors

$$\mathsf{Funct}([*/\mathbb{Z} \oplus \mathbb{Z}], [\mathsf{M}/\mathsf{G}])$$

and whose morphisms are natural transformations (i.e., it is a groupoid $[(\mathsf{Funct}([*/\mathbb{Z}\oplus \mathbb{Z}], [M/G]))/G]$ with G acting by conjugation on the functors).

Here we will apply the stationary phase approximation formula to the double loop groupoid, which we have shown above to be endowed with an action of the torus.

We will use an alternative description of the double loop groupoid. Its elements will be smooth maps

$$\phi \colon [0,1]^2 \to M$$

together with commuting elements $g, h \in G$ such that $\phi(1,0) = \phi(0,0)g$, $\phi(0,1) = \phi(0,0)h$. Call this set $\mathcal{L}^2_{\langle g,h\rangle}M$ and take

$$\mathcal{L}^2M := \bigsqcup_{\{(g,h)\in G^2 | gh=hg\}} \mathcal{L}^2_{\langle g,h\rangle}M.$$

The natural action of conjugation by elements in G gives us the description: $L^2X \cong [(\mathcal{L}^2M)/G]$.

We consider the functional of double loops

$$\mathcal{H}(\phi) := \int_{[0,1]^2} (||\frac{d\phi}{ds}||^2 + ||\frac{d\phi}{dt}||^2) ds\, dt;$$

we will apply stationary phase approximation à la Witten-Atiyah to the Feynman integral

$$\int_{L^2X} e^{-i\mathcal{H}(\phi)}\mathcal{D}\phi.$$

We need to find the equivariant normal bundle on L^2X to the fixed points of the action of $\mathbb{R}^2$, namely the ghost double loops.

For commuting $g, h \in G$, take the part of the groupoid of ghost double loops parameterized by $M^{\langle g,h\rangle}$, the fixed point set of the group generated by g and h. Call $\iota : M^{\langle g,h\rangle} \hookrightarrow M$ the inclusion, and suppose the the orbifold X is a complex orbifold (the pullback bundle ι^*TM can be locally simultaneously diagonalized with respect to the actions of g and h). Then one can write the total Chern class of ι^*TM as $\prod_j(1 + x_j)$ such that the line bundle x_j comes provided with the action of the group $\langle g, h\rangle$ parameterized by the irreducible representation λ_j.

We are using the following fact about equivariant complex K-theory. If a group Γ acts trivially on a space Y, then

$$K^*_\Gamma(Y) \cong K^*(Y) \otimes R(\Gamma),$$

that is, the equivariant K-theory of Y is isomorphic to the ordinary K-theory of Y tensored with the representation ring of Γ. Then the equivariant Chern character associated to the $\langle g,h\rangle$ equivariant line bundle x_j is $\mathrm{ch}_{\langle g,h\rangle}(x_j) = e^{x_j} \otimes \chi_{\lambda_j}$, where e^{x_j} is the Chern character of the line bundle and χ_{λ_j} is the character of the λ_i representation. As we have simultaneously diagonalized the actions of g and h, the character of an irreducible representation is determined by a root of unity associated to each g and h. So let $\sigma_j\colon \langle g,h\rangle \to [0,1)$ be such that $\chi_{\lambda_j}(g) = e^{2\pi i \sigma_j(g)}$; then one can consider $(1 + x_j + 2\pi i \sigma_j)$ as the Chern class of the equivariant bundle x_j.

The equivariant Euler class of the normal bundle of the embedding of ghost double loops

$$M^{\langle g,h\rangle} \to \mathcal{L}^2_{\langle g,h\rangle} M$$

is then

$$\left\{ \prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} \frac{1}{x_j} \right\} \left\{ \prod_j \prod_{(k,l)\in\mathbb{Z}^2} (x_j + l\hat{p} + k\hat{q} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q}) \right\},$$

where $\hat{p}$ and $\hat{q}$ are formal variables that keep track of the fractional periods of each of the circles of the torus.

Applying the fixed point formula (3.2.1) of Ando-Morava [AM01], one obtains

$$p^{\mathcal{L}^2_{\langle g,h\rangle} M}(1) = p^{M^{\langle g,h\rangle}} \left\{ \prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j \right\} \left\{ \prod_j \prod_{(k,l)\in\mathbb{Z}^2} \frac{1}{x_j + l\hat{p} + k\hat{q} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q}} \right\}.$$

Rearranging the expression in the second parenthesis by factoring $k\hat{q}$ and keeping the l fixed, the second parenthesis becomes:

$$\prod_{l\in\mathbb{Z}} \left(\prod_{k>0} \frac{1}{k^2\hat{q}^2} \right) \left((x_j + l\hat{p} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q}) \prod_{k>0} \left(1 - \frac{(x_j + l\hat{p} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q})^2}{k^2\hat{q}^2} \right) \right)^{-1}.$$

Renormalization (see [Ati85, AM01]) gives

$$\prod_{k>0} \frac{1}{k^2\hat{q}^2} = \frac{\hat{q}}{2\pi},$$

$$\prod_{k>0} \left(1 - \frac{(x_j + l\hat{p} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q})^2}{k^2\hat{q}^2} \right)^{-1} = \frac{\hat{q}}{2\pi} \frac{\frac{\pi}{\hat{q}}}{\sin\left(\frac{\pi}{\hat{q}} (x_j + l\hat{p} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q}) \right)}.$$

Replacing the variable $\hat{q}$ by its holonomy $2\pi i$, our push forward $p^{\mathcal{L}^2_{\langle g,h\rangle}M}(1)$ becomes

$$p^{M^{\langle g,h\rangle}}\left\{\prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j\right\}\left\{\prod_j\prod_{l\in\mathbb{Z}}\frac{\frac{1}{2}}{\sinh\frac{1}{2}(x_j+l\hat{p}+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h))}\right\};$$

pairing the hyperbolic sines of l and $-l$ one gets that

$$2\sinh\frac{(x_j+l\hat{p}+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h))}{2}2\sinh\frac{(x_j-l\hat{p}+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h))}{2}=$$

$$\frac{1-e^{-x_j-\sigma_j(g)\hat{p}-2\pi i\sigma_j(h)-l\hat{p}}}{e^{-\frac{1}{2}(x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h))}e^{-\frac{1}{2}\hat{p}}}\frac{e^{x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h)-l\hat{p}}-1}{e^{\frac{1}{2}(x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h))}e^{-\frac{1}{2}\hat{p}}}.$$

As a result,

$$p^{\mathcal{L}^2_{\langle g,h\rangle}M}(1)=p^{M^{\langle g,h\rangle}}\left\{\prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j\right\}\times$$

$$\left\{\prod_j\frac{\frac{1}{2}}{\sinh\frac{1}{2}(x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h))}\prod_{l>0}\frac{-e^{-\hat{p}l}}{(1-e^{-x_j-\sigma_j(g)\hat{p}-2\pi i\sigma_j(h)-l\hat{p}})(1-e^{x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h)-l\hat{p}})}\right\}$$

$$=p^{M^{\langle g,h\rangle}}\left\{\prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j\right\}(-e^{\hat{p}})^{\frac{1}{12}}\times$$

$$\left\{\prod_{l>0,j}\frac{e^{\frac{1}{2}(-x_j-\sigma_j(g)\hat{p}-2\pi i\sigma_j(h))}}{(1-e^{-x_j-\sigma_j(g)\hat{p}-2\pi i\sigma_j(h)-(l-1)\hat{p}})(1-e^{x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h)-l\hat{p}})}\right\}.$$

Making the change of variables $p=e^{-\hat{p}}$, assuming that the first Chern class of M satisfies $c_1(M)=0$, i.e. $\prod_j e^{x_j}=1$, and integrating over $M^{\langle g,h\rangle}$, we have that

$$p^{\mathcal{L}^2_{\langle g,h\rangle}M}(1)=\frac{p^{\left(-\frac{\dim(M)}{12}+i\pi+\frac{\mathrm{age}(g)}{2}\right)}e^{-\pi i\,\mathrm{age}(h)}\left\{\prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}}x_j\right\}}{\prod_{l>0,j}(1-p^{l-1+\sigma_j(g)}e^{-x_j-2\pi i\sigma_j(h)})(1-p^{l-\sigma_j(g)}e^{x_j+2\pi i\sigma_j(h)})}[M^{\langle g,h\rangle}].$$

Adding all the fixed point data and averaging, one gets the orbifold elliptic genus:

$$\mathrm{Ell}_{orb}([M/G])=$$

$$\frac{1}{|G|}\sum_{gh=hg}\frac{p^{\left(-\frac{\dim(M)}{12}+i\pi+\frac{\mathrm{age}(g)}{2}\right)}e^{-\pi i\,\mathrm{age}(h)}\left\{\prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}}x_j\right\}}{\prod_{l>0,j}(1-p^{l-1+\sigma_j(g)}e^{-x_j-2\pi i\sigma_j(h)})(1-p^{l-\sigma_j(g)}e^{x_j+2\pi i\sigma_j(h)})}[M^{\langle g,h\rangle}].$$

This coincides with the constant term in the y-expansion of the formula of Borisov-Libgober [BL03, DMVV97, DLM02] except for a renormalization factor. One could use a device like that of Hirzebruch [HBJ92] to recover the full formula. In any case the localization principle holds in this case.

21 Appendix: Loop Groups and nearly Frobenius algebras

Let us start by some general abstract considerations concerning nearly Frobenius algebras useful from the point of view of Morse theory.

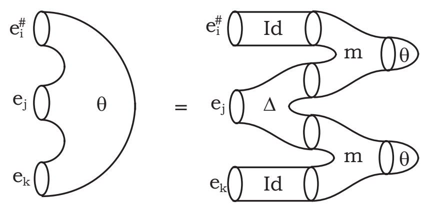
Proposition 21.1. *Let* A *be a Frobenius algebra with trace* θ *and let* Δ *be its natural nearly Frobenius structure. Let* $\{e_i\}$ *be a basis for* A *and* $\{e_i^\#\}$ *be its dual basis with respect to* θ. *If the structure constants of* Δ *are*

$$\Delta(e_i) = \sum_{i,j,k} \lambda_{ijk} e_j \otimes e_k^\#,$$

then

$$\lambda_{ijk} = \theta(e_i^\# e_j e_k).$$

Proof. By looking at the picture:



(63)

we compute:

$$\theta(e_i^\# e_j e_k) = \theta \otimes \theta(m \otimes m(1 \otimes \Delta \otimes 1(e_i^\# e_j e_k))) =$$

$$\theta\otimes\theta(m\otimes m(\sum_{l,r} \lambda_{jlr} e_l \otimes e_r^\# e_k)) = \theta\otimes\theta(\sum_{l,r} \lambda_{jlr} e_i^\# e_l \otimes e_r^\# e_k) = \sum_{l,r} \lambda_{jlr}\theta(e_i^\# e_l)\theta(e_r^\# e_k) = \lambda_{ijk}.$$

♣

From this we conclude that the structural coefficients of Δ serve as substitutes in TQFT+ of 3-point functions in TQFT.

The theory of loop groups [PS86] provides some highly non-trivial examples of nearly Frobenius algebras using infinite dimensional Morse theory. In that theory infinite dimensional manifolds with natural Morse functions naturally appear. These are not arbitrary manifolds but rather they posses what G. Segal calls a polarisation of the tangent bundle inducing a semi-infinite structure on their topology. We shall not go too far afield by simply stating three results from [GLU12].

Let us start by describing two semi-infinite dimensional manifolds that appear in [PS86], [CLS99] and [GO01].

The first one is $\mathsf{Gr}^{(n)}$ the *Grassmannian model for the loop group* ΩSU_n defined in pages 125 and 127 of [PS86]. Roughly speaking if we consider the Hilbert space $H = H^{(n)} := L^2(S^1, \mathbb{C}^n)$ of vector valued functions $f(z) \in \mathbb{C}^n$. We must consider the natural polarisation

$$H = H_+ \oplus H_-$$

defined by

$$f \in H_+ \Leftrightarrow f(z) = \sum_{k \geq 0} A_k z^k,$$

and

$$f \in H_- \Leftrightarrow f(z) = \sum_{k \leq 0} A_k z^k,$$

Then by definition a subspace W of H is an element of $\mathsf{Gr}^{(n)}$ if and only if:

- The orthogonal projection $pr_+ : W \to H_+$ is Fredholm,
- the orthogonal projection $pr_- : W \to H_-$ if Hilbert-Schmidt,
- for some k we have $z^k H_+ \subseteq W \subseteq z^{-k} H_+$,
- and finally $zW \subseteq W$.

In page 118 of [PS86] it is proved that there is an energy functional

$$\mathcal{E} : \mathsf{Gr}^{(n)} \to \mathbb{R},$$

which is a Palais-Smale Morse Function on $\mathsf{Gr}^{(n)}$. The stable and unstable cell decompositions for this Morse function are called the Bruhat and Birkhoff cell decompositions of $\mathsf{Gr}^{(n)}$. Every Bruhat cell C_i is finite dimensional and their closures $e_i = \bar{C}_i$ form a basis for the homology $H_*(\mathsf{Gr}^{(n)})$. Dually every Birkhoff cell Σ_j is finite co-dimensional and their closures $e_j^{\#} = \bar{\Sigma}_i$ form a basis for the cohomology of $H^*(\mathsf{Gr}^{(n)})$. We have that e_i and $e_j^{\#}$ are either disjoint or they intersect transversally and in fact

$$e_i \cap e_j^{\#} = \delta_{ij}.$$

Notice that if there was a trace θ (Poincaré duality) in $H_*(Gr^{(n)})$ with its natural intersection product we would have

$$\theta(e_i \cdot e_j \cdot e_k^{\#}) = \langle e_i \cdot e_j | e_k^{\#} \rangle = e_i \cap e_j \cap e_k^{\#}.$$

The infinite dimensionality of $Gr^{(n)}$ implies that there is no such θ for there is no Poincaré duality. Nevertheless we can define

$$\lambda_{ijk} := e_i \cap e_j \cap e_k^{\#},$$

and

$$\Delta(e_i) := \sum_{i,j,k} (e_i \cap e_j \cap e_k^{\#}) e_j \otimes e_k^{\#},$$

and we can prove [GLU12] that:

Theorem 21.2. *The homology of the loop group* $H_*(\Omega SU_n) \cong H_*(Gr^{(n)})$ *has a natural structure of nearly Frobenius algebra.*

There is a closely related space $Fl^{(n)}$ called the *periodic Flag manifold* (page 145 of [PS86]). An element of $Fl^{(n)}$ is a sequence of subspaces $\{W_k\}_k$ so that

- each W_k belongs to $Gr(H^{(n)})$,
- $W_{k+1} \subset W_k$, and $\dim(W_k/W_{k+1}) = 1$,
- $zW_k = W_{k+n}$.

Using the same methods and the results of [GO01] we get [GLU12]:

Theorem 21.3. *The cohomology* $H^*(Fl^{(n)})$ *of the periodic flag manifold and the quantum cohomology* $QH^{\#}(Fl^{(n)})$ *have natural structures of nearly Frobenius algebras.*

22 Appendix: The Calculus of Obstruction Classes

In this appendix we develop the technical machinery of obstruction classes for our computations of virtual fundamental classes. For now on we shall assume that all the manifolds are almost complex manifolds, this is not essential but rather allows us to forget the signs in the calculations.

Let Y, Z be closed submanifolds of X which intersect *cleanly*, that is, $W = Y \cap Z$ is a submanifold of X and at each point x of W the tangent space of W at x is the intersection of the tangent spaces of Y and Z. Let F be the *excess* bundle of the intersection, i.e., the vector bundle over W which is the quotient of the tangent bundle of X by the sum of the tangent bundles of Y and Z restricted to W. Sometimes F is called an *obstruction bundle*. Thus $F = 0$ if and only if Y and Z intersect transversally. If the relevant inclusion maps are denoted

$$\begin{tikzcd} W \arrow[r, "j'"] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ Y \arrow[r, "j"'] & X \end{tikzcd}$$

then F fits into an exact sequence

$$0 \longrightarrow \nu_{i'} \longrightarrow {j'}^{*}\nu_i \longrightarrow F \longrightarrow 0$$

where ν_i denotes the normal bundle of the embedding i.

We call this square a *Quillen square*. We have the following result by Quillen [Qui71]:

Proposition 22.1. *If* $z \in H^*(Z)$*, then*

$$j^* i_*(z) = i'_*(e(F) \cdot {j'}^{*}(z))$$

in $H^{*+a}(Y)$*, where* a *is the rank of* ν_i*.*

This result also holds in K-theory:

Proposition 22.2. *If* $\zeta \in K^*(Z)$*, then*

$$j^* i_*(\zeta) = i'_*(\lambda_{-1}(F) \cdot {j'}^{*}(\zeta))$$

in $K^*(Y)$*.*

In this appendix we work in cohomology and leave the corresponding results in K-theory to the interested reader. The proofs are formal and are the same.

Proposition 22.3. *For the diagram*

$$\begin{tikzcd}
 & & Z \arrow[dl, "\varepsilon_i"'] \arrow[dr, "\xi_i"] \arrow[ddll, bend right, "\varphi"'] \arrow[ddrr, bend left, "\psi"] & & \\
 & B_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\beta_i"] & & D_i \arrow[dl, "\gamma_i"'] \arrow[dr, "\delta_i"] & \\
A & & C_i & & E
\end{tikzcd}$$

classes $e_{B_i} \in H^*(B_i)$ *and* $e_{D_i} \in H^*(D_i)$, *let be* $\varphi = \alpha_i\varepsilon_i$, $\psi = \delta_i\xi_i$ *and*

$$e_{B_i,D_i} = \xi_i^*(e_{D_i})e(F_i)\varepsilon_i^*(e_{B_i})$$

where $e(F_i)$ *is the excess intersection class of the Quillen square. If* $e_{B_{i_1},D_{i_1}} = e_{B_{i_2},D_{i_2}}$ *then for* $z \in H^*(A)$ *we have the identity*

$$\delta_{i_1}!\left(e_{D_{i_1}}\gamma_{i_1}^*\left(\beta_{i_1}!\left(e_{B_{i_1}}\alpha_{i_1}^*(z)\right)\right)\right) = \delta_{i_2}!\left(e_{D_{i_2}}\gamma_{i_2}^*\left(\beta_{i_2}!\left(e_{B_{i_2}}\alpha_{i_2}^*(z)\right)\right)\right)$$

Proof. We use the projection formula $f!(x)y = f!(xf^*(y))$. Then the Quillen formula is

$$\begin{aligned}
\delta_i!\,(e_{D_i}\gamma_i^*\,(\beta_i!\,(e_{B_i}\alpha_i^*(z)))) &= \delta_i!\,(e_{D_i}\xi_i!\,(e(F_i)\varepsilon_i^*\,(e_{B_i}\alpha_i^*(z)))) \\
&= \delta_i!\,(e_{D_i}\xi_i!\,(e(F_i)\varepsilon_i^*(e_{B_i})\varphi^*(z))) \\
&= \delta_i!\xi_i!\,(\xi_i^*(e_{D_i}e(F_i)\varepsilon_i(e_{B_i})\varphi^*(z))) \\
&= \psi!\,(\xi_i^*(e_{D_i})e(F_i)\varepsilon_i(e_{B_i})\varphi^*(z))
\end{aligned}$$

♣

22.1 Homological formulæ

In this section we describe the analogous result to that of the last section, but this time in homology. Let us start by recalling the definition of the umkehr map in homology.

Let $f : A \to X$ be an inclusion such that there is a tubular neighborhood around A isomorphic to a bundle over A. The umkehr map $f_!$ is defined by the next steps

Step 1: We consider the projection map

$$\tau_f : X \to \frac{X}{X - \eta_f(A)}$$

where η_f is the tubular neighborhood of f.

Step 2: We use the exponential function $(E(\varepsilon), E_0(\varepsilon)) \to (\eta_f, \eta_f - A) \subset (X, X - A)$ and by excision we have the next isomorphisms

$$H_*(X, X - A) \cong H_*(E(\varepsilon), E_0(\varepsilon)) \cong H_*(E, E_0)$$

then

$$H_*(X/(X - A)) \cong H_*(E/E_0) \xrightarrow{\mathrm{Thom}} H_{*-k}(A)$$

Finally, the next diagram gives the umkehr map

$$\begin{tikzcd} H_*(X) \arrow[r, "(\tau_f)_*"] \arrow[rrr, bend right, "f_!"'] & H_*(X/(X-A)) \arrow[r, "\cong"] & H_*(E/E_0) \arrow[r, "\mathrm{Thom}"] & H_{*-k}(A) \end{tikzcd}$$

Lemma 22.4. *Let* $i : Z \hookrightarrow X$ *an inclusion of manifolds with* $k = \dim X - \dim Z$. *Then, for* $z \in H_*(Z)$

$$i!i_*(z) = e(\nu_i) \cap z,$$

where ν_i *is the normal bundle of the inclusion* i.

Proof.

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[rr, bend right, "s"'] & X \arrow[r, "\tau"] & \frac{X}{X-\eta_i} \arrow[r, "\pi"] & Z \end{tikzcd}$$

In homology is

$$\begin{tikzcd} i! : H_*(Z) \arrow[r, "i_*"] \arrow[rr, bend left, "s_*"] & H_*(X) \arrow[r, "\tau_*"] & H_*(\nu_i, \nu_0) \arrow[r, "\phi"] & H_{*-k}(Z) \end{tikzcd}$$

$$z \longmapsto i_*(z) \longmapsto s_*(z) \longmapsto \pi_*(\mathrm{Th} \cap s_*(z))$$

Note that we can give another expression for $\pi_*(\mathrm{Th} \cap s_*(z))$, that is
$\pi_*(\mathrm{Th} \cap s_*(z)) = \pi_* s_*(s^*(\mathrm{Th}) \cap z) = (\pi \circ s)_*(e(\nu_i) \cap z) = (\mathrm{Id})_*(e(\nu_i) \cap z) = e(\nu_i) \cap z$,
where $e(\nu_i) = s^*(\mathrm{Th})$ because in cohomology the umkehr map is

$$\begin{array}{ccccc} i_* : H^*(Z) & \xrightarrow{\Phi} & H^{*+k}(\nu_i, \nu_0) & \xrightarrow{\tau^*} & H^{*+k}(X) \\ \\ \alpha & \longmapsto & \mathrm{Th} \cup \alpha & \longmapsto & \tau^*(\mathrm{Th} \cup \alpha) \end{array}$$

Then $i^*(\tau^* \Phi(1)) = i^* i_*(1) = e(\nu_i)$, by Quillen's result. In the other hand, $i^*(\tau^* \Phi(1)) = (\tau \circ i)^*(\Phi(1)) = s^*(\Phi(1)) = s^*(\mathrm{Th})$.
Finally we obtain that $i! i_*(z) = e(\nu_i) \cap z$, for $z \in H_*(Z)$.

♣

Proposition 22.5. *Let* Y, Z *be closed submanifolds of* X *which intersect cleanly and* $W = Y \cap Z$ *is a submanifold of* X *such that at each point of* W *the tangent space of* W *at* x *is the intersection of the tangent spaces of* Y *and* Z.

$$\begin{tikzcd} W \arrow[r, "j'"] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ Y \arrow[r, "j"'] & X \end{tikzcd} \tag{64}$$

and $z \in H_*(Z)$*, then*

$$j! i_*(z) = i'_*(e(F) \cap j'!(z))$$

where

$$0 \longrightarrow \nu_{i'} \longrightarrow j'^* \nu_i \longrightarrow F \longrightarrow 0$$

is an exact sequence.

Proof. We can replace X by a tubular neighborhood of W. Thus we may suppose that (64) is of the form

$$\begin{tikzcd} W \arrow[r, "j'"] \arrow[d, "i'"'] & E_1 \arrow[d, "i"] \\ E_2 \arrow[r, "j"'] & E_1 \oplus E_2 \oplus F \end{tikzcd}$$

where E_1 is a complex vector bundle over W with zero section j', E_2 is a complex vector bundle with zero section i', and i and j are the obvious inclusions. Let

$i_\epsilon : E_\epsilon \to E_1 \oplus E_2$, $\epsilon = 1,2$ and $k : E_1 \oplus E_2 \to E_1 \oplus E_2 \oplus F$ be the inclusion map. Hence

$$
\begin{array}{rclcll}
j!i_*(z) & = & i_2!k!k_*i_{1*}(z) & = & i_2!(e(\nu_k) \cap i_{1*}(z)) & \text{by the lemma 22.4} \\
& = & i_2!i_{1*}(i_1^*e(\nu_k) \cap z) & = & i'_*j'!(i_1^*e(\nu_k) \cap z) & \text{by Claim 1} \\
& = & i'_*j'!(\pi^*(e(F)) \cap z) & & & \text{by Claim 2} \\
& = & i'_*(e(F) \cap j'!(z)) & & & \text{by Claim 3}
\end{array}
$$

- *Claim 1:* We consider the next commutative diagram

$$
\begin{tikzcd}
W \arrow[r, "i'"] \arrow[d, "j'"'] & E_2 \arrow[d, "i_2"'] \arrow[dr, "i"] & \\
E_1 \arrow[r, "i_1"'] \arrow[rr, bend right, "j"'] & E_1 \oplus E_2 \arrow[r, "k"'] & E_1 \oplus E_2 \oplus F
\end{tikzcd}
$$

  Then $i_2!i_{1*} = i'_*j'!$. To prove this we check that the next diagrams commute in homology.

$$
\begin{tikzcd}
E_1 \arrow[r, "\tau_1"] \arrow[d, "i_1"'] & \frac{E_1}{E_1 - \eta_{j'}} \arrow[r, "\pi_1"] \arrow[d, "l"] & W \arrow[d, "i'"] \\
E_1 \oplus E_2 \arrow[r, "\tau_2"'] & \frac{E_1 \oplus E_2}{E_1 \oplus E_2 - \eta_{i_2}} \cong \frac{E_1}{E_1 - \eta_{j'}} \oplus E_2 \arrow[r, "\pi_2"'] & E_2
\end{tikzcd}
$$

  The first commutes by definition of the maps, and the second commutes by the following:
  Let $x \in H_*\left(\frac{E_1}{E_1 - \eta_{j'}}\right)$, then $\pi_{2*}(\mathrm{Th}_2 \cap l_*(x)) = \pi_{2*}l_*(l^*(\mathrm{Th}_2) \cap x) = i'_*\pi_{1*}(l^*(\mathrm{Th}_2) \cap x) = i'_*\pi_{1*}(\mathrm{Th}_1 \cap x)$.
  Finally, if $x \in H_*(E_1)$: $\pi_{2*}(\mathrm{Th}_2 \cap \tau_{2*}i_{1*}(x)) = \pi_{2*}(\mathrm{Th}_2 \cap l_*\tau_{1*}(x))$
  $= \pi_{2*}l_*(l^*(\mathrm{Th}_2) \cap \tau_{1*}(x)) = i'_*\pi_{1*}(\mathrm{Th}_1 \cap \tau_{1*}(x))$. Then, $i_2!i_*(x) = i'_*j'!(x)$.

- *Claim 2:* The bundles $i_1^*(\nu_k)$ and $\pi^*(F)$ coincide, in particular $i_1^*(e(\nu_k)) = \pi^*(e(F))$.
  To prove this, we consider the pullback square

$$
\begin{tikzcd}
\pi^*(F) \arrow[r] \arrow[d] & F \arrow[d, "\pi_F"] \\
E_1 \arrow[r, "\pi"'] & W
\end{tikzcd}
$$

where $\pi^*(F) = \{(x, z) \in E_1 \times F : \ \pi(x) = \pi_F(z)\} = E_1 \oplus F$ bundle over E_1.
Hence it is enough to prove $i_1^*(\nu_k) = E_1 \oplus F$. First we note that the next diagram commutes

$$\begin{tikzcd}
E_1 \oplus F \arrow[r, "j"] \arrow[d, "\pi_1"'] & \nu_k \arrow[d, "\pi_k"] \\
E_1 \arrow[r, "i_1"'] & E_1 \oplus E_2
\end{tikzcd}$$

where $k : E_1 \oplus E_2 \to E_1 \oplus E_2 \oplus F$, $\pi_1 : E_1 \oplus F \to E_1$ is the projection and $j : E_1 \oplus F \to \nu_k$ is given by $j(x, y) = (x, 0, y) \in \nu_k$.
This square commutes by $i_1 \circ \pi_1(x, y) = i_1(x) = (x, 0)$ and $\pi_k \circ j(x, y) = \pi_k(x, 0, y) = (x, 0)$.
To finish we need to check that $E_1 \oplus F$ is the pullback square of the maps

$$\begin{tikzcd}
 & \nu_k \arrow[d, "\pi_k"] \\
E_1 \arrow[r, "i_1"'] & E_1 \oplus E_2
\end{tikzcd}$$

Let Z be a manifold such that

$$\begin{tikzcd}
Z \arrow[drr, "g"] \arrow[dr, "f"'] & & \\
 & E_1 \oplus F \arrow[r, "j"] \arrow[d, "\pi_1"] & \nu_k \arrow[d, "\pi_k"] \\
 & E_1 \arrow[r, "i_1"'] & E_1 \oplus E_2
\end{tikzcd}$$

$\pi_k \circ g = i_1 \circ f$. We define $h : Z \to E_1 \oplus F$ by $h(z) = (f(z), \pi_3 \circ g(z))$.
Note that $\pi_k \circ g(z) = (f(z), 0)$ since $\pi_k \circ g = i_1 \circ f$. Then $j \circ h(z) = j(f(z), \pi_3 \circ g(z)) = (f(z), 0, \pi_3 \circ g(z)) = (\pi_k(g(z)z), \pi_3(g(z))) = g(z)$, and $\pi_1(h(z)) = \pi_1(f(z, \pi_3(g(z)))) = f(z)$.

- *Claim 3:* For $\varphi \in H^*(W)$ and $z \in H_*(E_1)$ then $j'!(\pi^*(\varphi) \cap z) = \varphi \cap j'!(z)$. This is an immediately consequence of the definition of the umkehr map, that is: $j' : W \to E_1$ and $\begin{tikzcd} E_1 \arrow[r, "\tau"] \arrow[rr, bend right, "\pi"'] & \frac{E_1}{E_1 - \eta_{j'}} \arrow[r, "p"] & W \end{tikzcd}$.

Then

$$\begin{array}{rcccl}
\varphi \cap j'!(z) & = & \varphi \cap p_*(\mathrm{Th} \cap \tau_*(z)) & = & p_*(p^*(\varphi) \cap \mathrm{Th} \cap \tau_*(z)) \\
 & = & p_*(\mathrm{Th} \cap \tau_*(\tau^* p^*(\varphi) \cap z)) & = & p_*(\mathrm{Th} \cap \tau_*(\pi^*(\varphi) \cap z)) \\
 & = & j'!(\pi^*(\varphi) \cap z) & &
\end{array}$$

♣

Proposition 22.6. *For the diagram*

$$\begin{tikzcd}
& & Z \arrow[dl, "\varepsilon_i"'] \arrow[dr, "\xi_i"] \arrow[ddll, bend right, "\varphi_i"'] \arrow[ddrr, bend left, "\psi_i"] & & \\
& B_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\beta_i"] & & D_i \arrow[dl, "\gamma_i"'] \arrow[dr, "\delta_i"] & \\
A & & C_i & & E
\end{tikzcd}$$

with classes $e_{B_i} \in H_*(B_i)$ *and* $e_{D_i} \in H_*(D_i)$, *let* $\varphi_i = \alpha_i \varepsilon_i$, $\psi_i = \delta_i \xi_i$ *such that* $(\varphi_1)! = (\varphi_2)!$ *and* $(\psi_1)_* = (\psi_2)_*$. *Let*

$$e_i = \xi_i!(e_{D_i})(e(F_i) \cap \varepsilon_i!(e_{B_i}))$$

where $e(F_i)$ *is the excess intersection class of the Quillen square. If* $e_1 = e_2$, *then for* $z \in H_*(A)$ *we have the identity*

$$\delta_{1*}(e_{D_1}\gamma_1!(\beta_{1*}(e_{B_1}\alpha_1!(z)))) = \delta_{2*}(e_{D_2}\gamma_2!(\beta_{2*}(e_{B_2}\alpha_2!(z))))$$

Proof. We use the Quillen's formula and the projection formula: $f_*(x)y = f_*(xf!(y))$, then

$$\begin{aligned}
\delta_{1*}(e_{D_1}\gamma_1!(\beta_{1*}(e_{B_1}\alpha_1!(z)))) &= (e_{D_1}\xi_{1*}(e(F_1) \cap \varepsilon_1!(e_{B_1}\alpha_1!(z)))) \\
&= \delta_{1*}\xi_{1*}(\xi_{1*}(e_{D_1})(e(F_1) \cap \varepsilon_1!(e_{B_1}\alpha_1!(z)))) \\
&= (\psi_1)_*(\xi_1!(e_{D_1})(e(F_1) \cap \varepsilon_1!(e_{B_1}))\varphi_1!(z)) \\
&= (\psi_2)_*(\xi_1!(e_{D_2})(e(F_2) \cap \varepsilon_2!(e_{B_2}))\varphi_2!(z)) \\
&= \delta_{2*}(e_{D_2}\gamma_2!(\beta_{2*}(e_{B_2}\alpha_2!(z))))
\end{aligned}$$

♣

Corollary 22.7. *The next diagrams,* $i = 0, 1$.

$$\begin{tikzcd}
 & & Z_i \arrow[dl, "\varepsilon_i"'] \arrow[dr, "\xi_i"] \arrow[ddll, bend right, "\varphi_i"'] \arrow[ddrr, bend left, "\psi_i"] & & \\
 & B_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\beta_i"] & & D_i \arrow[dl, "\gamma_i"'] \arrow[dr, "\delta_i"] & \\
A & & C_i & & E
\end{tikzcd}$$

where the squares are Quillen's squares, i.e. the intersection of B_i *and* D_i *is clean and the spaces* Z_1 *and* Z_2 *are homotopically equivalents,*

$$\begin{tikzcd}
Z_1 \arrow[r, shift left, "f_1"] & Z_2 \arrow[l, shift left, "f_2"]
\end{tikzcd}$$

such that

$$\begin{tikzcd}
 & Z_2 \arrow[d, shift left, "f_2"] \arrow[ddl, bend right, "\varphi_2"'] \arrow[ddr, bend left, "\psi_2"] & \\
 & Z_1 \arrow[u, shift left, "f_1"] \arrow[dl, "\varphi_1"'] \arrow[dr, "\psi_1"] & \\
A & & E
\end{tikzcd}$$

commutes up to homotopy. Then, if $f_2^*(e(\nu_{f_1} \oplus F_1)) = e(F_2)$, *for* $z \in H_*(A)$ *we have*

$$\delta_{1*} \circ \gamma_1! \circ \beta_{1*} \circ \alpha_1!(z) = \delta_{2*} \circ \gamma_2! \circ \beta_{2*} \circ \alpha_2!(z).$$

Proof.

$$\begin{aligned}
\delta_{1*} \circ \gamma_1! \circ \beta_{1*} \circ \alpha_1!(z) &= \delta_{1*}\xi_{1*}\left(e(F_1) \cap \varepsilon_1!(\alpha_1!(z))\right) && \text{property of Quillen} \\
&= \psi_{1*}\left(e(F_1) \cap \varphi_1!(z)\right) && \text{by } \delta_1\xi_1 \simeq \psi_1, \alpha_1 \circ \varepsilon_1 \simeq \varphi_1 \\
&= \psi_{2*}f_{1*}\left(e(F_1) \cap f_1!\varphi_2!(z)\right) && \text{by } \psi_1 \simeq \psi_2 \circ f_1, \varphi_1 \simeq \varphi_2 \circ f_1 \\
&= \psi_{2*}f_{1*}\left(f_1^* f_2^* e(F_1) \cap f_1!\varphi_2!(z)\right) && \text{by } f_2 \circ f_1 \simeq 1 \\
&= \psi_{2*}\left(f_{1*}(f_2^*(e(F_1)) \cap f_1!(\varphi_2!(z)))\right) && \text{by the projection formula} \\
&= \psi_{2*}\left(f_2^*(e(F_1)) \cap f_{1*}f_1!(\varphi_2!(z))\right)
\end{aligned}$$

Now we need to understand the map $f_* \circ f! : H_*(B) \to H_*(A) \to H_*(B)$, where $f : A \to B$. First we consider the next Quillen's diagram.

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\ A \arrow[dr, "f"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

For Quillen's property we have $f!f_*(z) = e(\nu_f) \cap z$, where $z \in H_*(A)$ and ν_f is the normal bundle of the map $f : A \to B$.

$$\begin{aligned}
&f_1!f_{1*}f_{2*}(z) = f_1!(z) && \text{by } f_{1*}f_{2*} = \mathrm{Id},\\
&e(\nu_{f_1}) \cap f_{2*}(z) = f_1!(z) && \text{using that } f_1!f_{1*}(z) = z \cap e(\nu_{f_1}),\\
&f_{1*}(e(\nu_{f_1}) \cap f_{2*}(z)) = f_{1*}f_1!(z) && \text{composition with } f_{1*},\\
&f_{1*}(f_1^*f_2^*(e(\nu_{f_1})) \cap f_{2*}(z)) = f_{1*}f_1!(z) && \text{using that } f_1^*f_2^* = \mathrm{Id},\\
&f_2^*(e(\nu_{f_1})) \cap f_{1*}f_{2*}(z) = f_{1*}f_1!(z) && \text{by the projection formula},\\
&f_2^*(e(\nu_{f_1})) \cap z = f_{1*}f_1!(z) && \text{using that } f_{1*}f_{2*} = \mathrm{Id}\,.
\end{aligned}$$

Then $f_{1*}f_1!(z) = f_2^*(e(\nu_{f_1})) \cap z$, for all $z \in H_*(B)$.

Finally, returning to the calculations, we have

$$\begin{aligned}
\psi_{2*}\left(f_2^*(e(F_1)) \cap f_{1*}f_1!(\varphi_2!(z))\right) &= \psi_{2*}\left(f_2^*(e(F_1)) \cap f_2^*(e(\nu_{f_1})) \cap \varphi_2!(z)\right)\\
&= \psi_{2*}\left((f_2^*(e(F_1)) \cup f_2^*(e(\nu_{f_1}))) \cap \varphi_2!(z)\right)\\
&= \psi_{2*}\left(f_2^*(e(F_1) \cup e(\nu_{f_1})) \cap \varphi_2!(z)\right)\\
&= \psi_{2*}\left(f_2^*(e(\nu_{f_1} \oplus F_1)) \cap \varphi_2!(z)\right)
\end{aligned}$$

Since $f_2^*(e(\nu_{f_1} \oplus F_1)) = e(F_2)$ then

$$\psi_{2*}\left(\varphi_2!(z) \cap f_2^*(e(\nu_{f_1}\ F_1))\right) = \psi_{2*}\left(\varphi_2!(z) \cap e(F_2)\right) = \delta_{2*} \circ \gamma_2! \circ \beta_{2*} \circ \alpha_2!(z).$$

♣

In particular we have the next result.

Corollary 22.8. *In the hypothesis of the last corollary, if* Z_1 *and* Z_2 *are diffeomorphic spaces, where* $f_1 : Z_1 \to Z_2$ *is the diffeomorphism between them, then the identity* $e(F_1) = e(F_2)$*, implies*

$$\delta_{1*} \circ \gamma_1! \circ \beta_{1*} \circ \alpha_1!(z) = \delta_{2*} \circ \gamma_2! \circ \beta_{2*} \circ \alpha_2!(z).$$

Proof. This is because if f_1 is a diffeomorphism then $\nu_{f_1} = 0$.

♣

Theorem 22.9. *Let* $f, g : A \to X$ *be cofibration maps, and* $H : A \times I \to X$ *an homotopy between them, i.e* $H(x, 0) = f(x)$ *and* $H(x, 1) = g(x)$ *for* $x \in A$*. Then*

$$f_! = g_! : H_*(X) \to H_*(A)$$

Proof. Note that $(X, f(A))$ and $(X, g(A))$ are good pairs, $f(A) \hookrightarrow X$ and $g(A) \hookrightarrow X$ are cofibrations. Then the homotopy

$$H' : f(A) \times I \to X$$

given by $H'(f(x), t) = H(x, t)$ extends to X such that

$$H'|_{\eta_f \times \{1\}} = \eta_g$$

and

$$H'|_{A \times \{0\}} = f, \quad H'|_{A \times \{1\}} = g$$

Set by $f' := H'(-, 0)$ and $g' := H'(-, 1)$.

Let $\alpha \in H_*(X)$ with $\alpha = f'_{\#}(\beta) + \gamma$, where $\beta \in C_{\#}(\eta_f)$, $\gamma \in C_{\#}(X - A)$ and $f_{\#}$ is the map induced in the chain complexes. This is posible by the using of the barycentric subdivision $C_{\#}(\eta_f + (X - A)) \xrightarrow{\cong} C_{\#}(X)$. Since we have the Quillen diagram

$$\begin{tikzcd} & \eta_f \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\ \eta_f \arrow[dr, "f'"'] & & \eta_f \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

then $f'_! f'_*(\beta) = \beta\, e(\nu_{f'})$.
Finally
$$f'_!(\alpha) = f'_!(f'_*(\beta) + \gamma) = \beta\, e(\nu_{f'}) + f'_!(\gamma)$$
where we note that $f'_!(\gamma) = 0$ because $\gamma \in C_\#(X - A)$. Then $f'_!(\alpha) = \beta\, e(\nu_{f'})$.
In other hand, using the homotopy $H' : X \times I \to X$ we can find a new representant of α in $C_\#(\eta_g + (X - g(A)))$ of the form $g'_\#(\beta') + \gamma'$ with $\beta' \in C_\#(\eta_g)$ and $\gamma' \in C_\#(X - g(A))$. Then
$$g'_!(g'_*(\beta') + \gamma') = g'_! g'_*(\beta') + g'_!(\gamma') = g'_! g'_*(\beta') = \beta'\, e(\nu_{g'}) = \beta\, e(\nu_{f'}).$$
Therefore $f'_!(\alpha) = g'_!(\alpha)$, and in particular $f_!(\alpha) = g_!(\alpha)$.

♣

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