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Nearly Frobenius Structures

A dissertation presented by

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Resumen

Las álgebras de Frobenius fueron introducidas en álgebra en teoría de grupos y la teoría de sus representaciones. Posteriormente aparecieron de forma inesperada en topología. En la década pasada las álgebras de Frobenius han aparecido en una variedad de contextos topológicos, en física y en computación. En física las álgebras de Frobenius aparecen en el contexto de Teorías Topológicas de Campos.

Las álgebras de Frobenius fueron estudiadas por primera vez por Frobenius en [Fro03]. En los 30' Brauer y Nesbitt retomaron el estudio de las mismas. Nakayama en [Nak39] y [Nak41] descubrió la dualidad que estas tienen y Lawvere en [Law69] las caracterizó en términos de coproductos.

En este trabajo estamos interesados en la relación entre las álgebras de Frobenius y las Teorías Topológicas de Campo. Donde una descripción de las últimas fue dada por Atiyah en [Ati88]. Una n-Teoría Topológica de Campo ($nTTC$) es una regla Z donde a cada $(n-1)$-variedad cerrada orientada M le asocia un espacio vectorial $Z(M)$ y a cada n-variedad cuya frontera es M le asocia un vector de $Z(M)$. Esta regla está sujeta a una colección de axiomas, entre los cuales está el que pide que variedades difeomorfas tengan asociados espacios vectoriales isomorfos y que la unión disjunta de variedades tengan asociado el producto tensorial de los espacios correspondientes a cada variedad. En dimensión dos tenemos el siguiente teorema.

Theorem 0.0.1. *Existe una equivalencia de categorías*

$$2TTC \cong cAF$$

donde cAF es la categoría de álgeras de Frobenius conmutativas. El funtor está definido de la siguiente manera, el espacio vectorial subyacente al álgebra de Frobenius es el espacio asociado, mediante la 2*-Teoría Topológica, al círculo* S^1.

En el trabajo de Moore y Segal, [MS06] se estudia la posibilidad de abrir estas teorías y considerar la acción de un grupo finito en ellas. En este trabajo daremos una generalización al caso no compacto.

Una familia de ejemplos de álgebras de Frobenius es la cohomología de una variedad compacta cerrada M. El hecho que $H^*(M)$ sea un álgebra de Frobenius

es equivalente a la dualidad de Poincaré. Nos podemos preguntar cuanta de esta estructura podemos recuperar en el caso en que la variedad M sea no compacta. En este trabajo damos una posible respuesta. Probamos que $H^*(M)$ es una casi-álgebra de Frobenius cuando M es no compacta. La noción de casi-álgebra de Frobenius es una noción más débil que la de álgebra de Frobenius pero ambas son más fuertes que la noción de álgebra. En este trabajo también veremos que esta construcción está lejos de ser trivial, para ello daremos una familia infinita de ejemplos usando teoría de cuerdas y orbifolds.

Abstract

Frobenius algebras were introduced in algebra motivated by group theory and the theory of their representations. It was somewhat unexpected to find them later appearing in topology. During the past decade, Frobenius algebras have shown up in a variety of topological contexts, in theoretical physics and in computer science. In physics, the appearance of Frobenius algebras occurs in the context of topological quantum field theories (TQFTs), which in their axiomatization amount to a precise mathematical theory.

Frobenius algebras were first studied by Frobenius [Fro03] around 1900. During the 1930's Brauer and Nesbitt in their classical paper *On the regular representations of algebras* took up again the study of these structures. It is only then that they are christened as *Frobenius algebras*. Nakayama discovered a rich duality theory in [Nak39] and [Nak41]. Dieudonné used this duality to characterize Frobenius algebras, in [Die58] where he called this property of an algebra a *perfect duality*. A very important characterization of Frobenius algebras in terms of coproducts goes back at least to Lawvere [Law69] (1967), rediscovered by Quinn [Qui95] and by Abrams [Abr96] in the 1990's, in the then brand new context of topological quantum field theories.

In this work we are interesting in the relation between Frobenius algebras and Topological Quantum Field Theory, where an axiomatic formulation of the last was described by M. Atiyah in [Ati88]. An n-dimensional topological quantum field theory (TQFT) is a rule Z which to each closed oriented $(n-1)$-manifold M associates a vector space ZM, and to each oriented n-manifold whose boundary is M associates a vector in ZM. This rule is subject to a collection of axioms which express that topologically equivalent manifolds have isomorphic vector spaces associated to them, and that disjoint unions of manifolds go to tensor products of vector spaces.

In dimension 2 we have the following important theorem:

Theorem 0.0.2. *There is an equivalence of categories*

$$2TQFT \simeq cFA.$$

The underlying vector space of the Frobenius algebra is the vector space that the

TQFT (seen as a functor $2Cob \to Vect$) associates to the circle (seen as an object in $2Cob$.)

In their seminal paper Moore and Segal [MS06] studies some of these generalizations. In particular they studied the possibility of open strings and of gauging under the action of a finite group. The work we present here is a further generalization to the case of non-compact background space-time in terms of this over-simplified toy model for string theory.

A family of examples of Frobenius algebras is the cohomology of a compact closed manifold M. In fact the statement that $H^*(M)$ is a Frobenius algebra is equivalent to Poincaré duality. We can ask then what if anything of this information can be encoded in some sort of algebraic entity for the cohomology of a non-compact closed manifold. We provide in this work one possible answer. We prove that $H^*(M)$ is a nearly Frobenius algebra even when M in non-compact, the notion of a nearly Frobenius algebra being weaker than that of a Frobenius algebra but of course stronger than that of an algebra. Isolating the definition of a nearly Frobenius algebra is not hard once one is inspired in TQFTs. In this way we isolate the corresponding algebraic generalizations for the various notions of Moore and Segal of a Frobenius structure in the non-compact framework. These definitions are one of the main contributions of this work. The second main contribution is to prove that the definitions a very far from trivial, for we construct infinite families of examples using string topology and orbifolds.

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Chapter 1
Introduction

Frobenius algebras were introduced in algebra motivated by group theory and the theory of their representations. It was somewhat unexpected to find them later appearing in topology. During the past decade, Frobenius algebras have shown up in a variety of topological contexts, in theoretical physics and in computer science. In physics, the appearance of Frobenius algebras occurs in the context of topological quantum field theories (TQFTs), which in their axiomatization amount to a precise mathematical theory. In computer science, Frobenius algebras arise in the study of flowcharts, proof nets, and circuit diagrams.

Frobenius algebras were first studied by Frobenius [Fro03] around 1900. During the 1930's Brauer and Nesbitt in their classical paper *On the regular representations of algebras* took up again the study of these structures. It is only then that they are christened as *Frobenius algebras*. Nakayama discovered a rich duality theory in [Nak39] and [Nak41]. Dieudonné used this duality to characterize Frobenius algebras, in [Die58] where he called this property of an algebra a *perfect duality*. A very important characterization of Frobenius algebras in terms of coproducts goes back at least to Lawvere [Law69] (1967), rediscovered by Quinn [Qui95] and by Abrams [Abr96] in the 1990's, in the then brand new context of topological quantum field theories. Indeed, a Frobenius algebra $\mathcal{A}$ can be defined as an algebra with a coproduct which is a map of $\mathcal{A}$-modules.

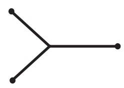
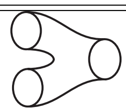
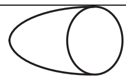
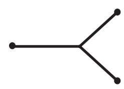
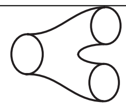
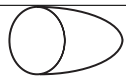
In the axiomatic formulation described by M. Atiyah in [Ati88] an n-dimensional topological quantum field theory (TQFT) is a rule Z which to each closed oriented $(n-1)$-manifold M associates a vector space ZM, and to each oriented n-manifold whose boundary is M associates a vector in ZM. This rule is subject to a collection of axioms which express that topologically equivalent manifolds have isomorphic vector spaces associated to them, and that disjoint unions of manifolds go to tensor products of vector spaces, etc.

In our opinion the clearest formulation of a TQFT is in terms of category theory, as introduced by G. Segal: first one defines the category of cobordisms $nCob$ the objects of which are closed oriented $(n-1)$-manifolds (up to diffeomorphism), and an arrow from Σ_1 to Σ_2 is an oriented n-manifold M whose incoming boundary is

Σ_1 and whose outgoing boundary is Σ_2. The composition of cobordisms is defined by gluing together the underlying manifolds along common boundary components. The cylinder $\Sigma \times I$ seen as a cobordism is the identity arrow of Σ. The operation of taking disjoint unions of manifolds and cobordisms gives this category a monoidal structure. On the other hand, the category $Vect_{\Bbbk}$ of vector spaces is monoidal under tensor products. Roughly speaking the Atiyah axioms amount to saying that a $TQFT$ is a (symmetric) monoidal functor from $nCob$ to $Vect_{\Bbbk}$. This is also called a linear representation of $nCob$.

In dimension 2, these structures are classified: since surfaces are completely classified, one can also describe completely the cobordism category. Every cobordism is obtained by composing the following four basic cobordisms: the disc with an outgoing circle, the pair of pants, the cylinder and the disc with an incoming circle. Two cobordisms are equivalent if they have the same genus and the same number of incoming and outgoing boundaries. This gives a set of relations, and a complete description of the monoidal category $2Cob$ in terms of generators and relations.

The generating bordisms for $2Cob$ are "creation", "merging", "splitting up" and "annihilation". These bordisms transform under the symmetric monoidal functor as algebraic operations:

Principle	Feynman diagram	2D cobordism	Algebraic operation in a $\Bbbk$-algebra $\mathcal{A}$	
merging			product	$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$
creation			unit	$\Bbbk \to \mathcal{A}$
splitting			coproduct	$\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$
annihilation			counit	$\mathcal{A} \to \Bbbk$

Finally, the relations that hold in $2Cob$ correspond precisely to the axioms of a commutative Frobenius algebra. This is encoded in the following important theorem:

Theorem 1.0.3. *There is an equivalence of categories*

$$2TQFT \simeq cFA.$$

The underlying vector space of the Frobenius algebra is the vector space that the $TQFT$ (seen as a functor $2Cob \to Vect$) associates to the circle (seen as an object in $2Cob$.)

All of the above can be thought of as an extremely simplified version of string theory. And all of the above talks only about closed strings. It is quite natural to consider several possible generalizations.

In their seminal paper Moore and Segal [MS06] studies some of these generalizations. In particular they studied the possibility of open strings and of gauging under the action of a finite group. The work we present here is a further generalization to the case of non-compact background space-time in terms of this over-simplified toy model for string theory.

What this means for a topologist is the following. A natural source of examples of Frobenius algebras in topology is the cohomology (or the homology) of a compact closed manifold M. In fact the statement that $H^*(M)$ is a Frobenius algebra is equivalent to Poincaré duality. We can ask then what if anything of this information can be encoded in some sort of algebraic entity for the cohomology of a *non-compact* closed manifold. We provide in this work one possible answer. We prove that $H^*(M)$ is a *nearly Frobenius algebra* even when M in non-compact, the notion of a nearly Frobenius algebra being weaker than that of a Frobenius algebra but of course stronger than that of an algebra. Isolating the definition of a nearly Frobenius algebra is not hard once one is inspired in TQFTs. In this way we isolate the corresponding algebraic generalizations for the various notions of Moore and Segal of a Frobenius structure in the non-compact framework. These definitions are one of the main contributions of this work. The second main contribution is to prove that the definitions a very far from trivial, for we construct infinite families of examples using string topology and orbifolds.

We will describe now the contents of this work.

The beginning of each chapter contains a more detailed summary of its contents. Here we shall just give a brief overview of each chapter together with the statements of the major results.

Chapter 2. In this chapter we review the standard definitions of Frobenius algebras and we give a list of examples that illustrate the theory. We recall some basic algebraic results due to Lowell Abrams [Abr96] and Aaron D. Lauda [Lau08]. They give two additional equivalent definitions of a Frobenius algebra using the coalgebra structure.

These results are important to us because they admit the "non-compact" generalizations that we are looking for in a very natural manner. Indeed, while it is difficult to guess what the correct definition would be using the traditional definitions using traces, the equivalent definition of Abrams is very easy to modify for our purposes.

So following Cohen and Godin [CG04] we define the concept of a *nearly Frobenius algebras*. A nearly Frobenius algebra consists of an algebra $\mathcal{A}$ with a map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that the following diagrams commute:

• The coalgebra axioms

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "\Delta\otimes 1"] \\ \mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes\Delta"'] & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \end{tikzcd}$$

i.e. $(\Delta\otimes 1)\Delta(x) = (1\otimes\Delta)\Delta(x)$ for all $x\in\mathcal{A}$.

• The Frobenius identities

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] \arrow[d, "\Delta\otimes 1"'] & \mathcal{A} \arrow[d, "\Delta"] \\ \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes m"'] & \mathcal{A}\otimes\mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] \arrow[d, "1\otimes\Delta"'] & \mathcal{A} \arrow[d, "\Delta"] \\ \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "m\otimes 1"'] & \mathcal{A}\otimes\mathcal{A} \end{tikzcd}$$

i.e. $\Delta(x)y = \Delta(xy) = x\Delta(y)$, for all $x, y\in\mathcal{A}$.
Or equivalently, a nearly Frobenius algebra consists of an algebra $\mathcal{A}$ with a map $\theta : \mathbb{k}\to\mathcal{A}\otimes\mathcal{A}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "1\otimes\theta"] \arrow[d, "\theta\otimes 1"'] \arrow[dr, "\Delta"] & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[d, "m\otimes 1"] \\ \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes m"'] & \mathcal{A}\otimes\mathcal{A} \end{tikzcd}$$

If we note $\theta(1) = \sum \xi_i\otimes\xi^j$ to say that the last diagram commutes is equivalent to say that $\sum \xi_i\otimes\xi^j x = \Delta(x) = \sum x\xi_i\otimes\xi^j$, for all $x\in\mathcal{A}$.

Note that this new concept is a generalization of the concept of Frobenius algebra where we remove the counit, in particular there is no trace involved in the definition. All of the above is essentially well known, but we have set to clarify the details in the literature.

The first new result of this thesis is the proof that a natural example of a nearly Frobenius algebra is the Poincaré algebra of an oriented, connected not necessarily compact smooth manifold, which in turn proves that there in infinitely many non-trivial examples.

Chapter 3. We review in this chapter the notion of Topological Field Theory. Although not strictly necessary in the logical sense for the results of this work, it provides a great source of clarification for our definitions.

In the first section we review the basic definition of a nD-Topological Field Theory given by Michael Atiyah in [Ati88] and [Ati90]. In the section 2 we study a formulation in categorical terms. Graeme Segal woked out very carefully this characterization in his Lecture Notes form the Workshop on Geometry and Physics.

The main result in this chapter is the so called Folk theorem which classifies such theories. This result was obtained for example by Dijkgraaf in [Dij89] and Voronov in [Vor94], further details of the proof having been provided by Quinn in [Qui95], Dubrovin in [Dub96], Abrams in [Abr96], Kock in [Koc03] and Moore-Segal in [MS06].

Theorem 3.3.1 *There is a canonical equivalence of categories*

$$\text{2D-TFT}_{\Bbbk} \simeq \text{cFA}_{\Bbbk}$$

where $\text{cFA}_{\Bbbk}$ *is the category of commutative Frobenius algebras over* $\Bbbk$ *(field of characteristic zero).*

In the last section we introduce a new structure in topology in analogy with the new structure defined in the chapter 1. This definition is motivated by the Folk theorem and Cohen and Godin [CG04] decided to call this a *Topological field theory with positive boundary* TFT_+. It is defined in the same way as TFT but with the difference that we can write the maps linear maps associated to a surface only when this surface has non-empty outgoing boundary.

In this new context we have an analogous result to the Folklore theorem.

Theorem 1.0.4. *The category of nearly Frobenius algebras is equivalent to the category of 2D-TFT with positive boundary.*

This result was obtained in collaboration with Ernesto Lupercio, Carlos Segovia and Bernardo Uribe, and it appears in [GLSU]. We shall not use this result to obtain the theorems of this thesis, so we will not include a proof here.

Finally we recover a theorem of Cohen and Godin. We prove that the homology of free loop spaces of a compact, oriented, closed manifold with the algebra structure of Chas and Sullivan (discovered in their seminar work from 1999 [CS99]), naturally has the structure of a nearly Frobenius algebra. While this result is not new we think our proof is sufficiently different to interest the reader.

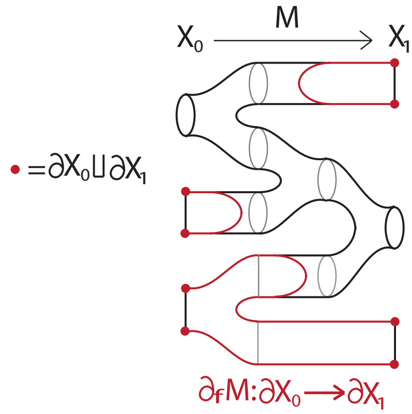
Chapter 4. One way to extend the notion of a 2-dimensional topological field theory is to assign a vector space $Z(X)$ to each compact oriented 1-dimensional manifold, with or without boundary, and each boundary component is labeled with an element of a fixed set $\mathcal{B}_0$, called the set of *boundary conditions.* One still requires

$$Z(X_1 \amalg X_2) = Z(X_1) \otimes Z(X_2).$$

Any compact 1-manifold is a union of circles and intervals, so this part of the data amounts to two vector spaces

$$\mathscr{C} = Z(S^1) \quad \text{and} \quad \mathcal{A} = Z(I).$$

Cobordisms $M : X_0 \to X_1$ must now be taken to be surfaces M whose boundary ∂M is the union of $X_0 \amalg X_1$ with a free part $\partial_f M$, which is itself a cobordism $\partial_f M : \partial X_0 \to \partial X_1$.



A theory of this type is called 2D open-closed topological fiel theory.
Moore and Segal in his work [MS06] proved the next classification theorem:

Theorem 1.0.5. *To given an open-closed topological field theory is the same as to give a Frobenius structure, where a Frobenius structure consists of the following algebraic data:*

1. $(\mathcal{A}, \Delta_{\mathcal{A}}, 1_{\mathcal{A}})$ *is a commutative Frobenius algebra.*

2. *A* $\mathbb{C}$*-linear category* $\mathcal{B}$*, where* $\mathrm{Obj}(\mathcal{B}) = \mathcal{B}_0$ *and* $\mathscr{O}_{ab} = \mathrm{Hom}(a, b)$ *for* $a, b \in \mathscr{B}_0$.

2a. *With associative linear maps* η^b_{ac} *and units* u_a

$$\eta^b_{ac} : \mathscr{O}_{ab} \otimes \mathscr{O}_{bc} \to \mathscr{O}_{ac}, \tag{1.1}$$

$$u_a : \mathbb{C} \to \mathscr{O}_{aa}, \tag{1.2}$$

2b. *The spaces* $\mathscr{O}_{aa}$ *have nondegenerate traces*

$$\Theta_a : \mathscr{O}_{aa} \to \mathbb{C} \tag{1.3}$$

In particular, each $\mathscr{O}_{aa}$ *is not necessarily a commutative Frobenius algebra.*

2c. *Moreover,*

$$\begin{array}{ccccc} \mathcal{O}_{ab}\otimes\mathcal{O}_{ba} & \longrightarrow & \mathcal{O}_{aa} & \xrightarrow{\Theta_a} & \mathbb{C} \\ \mathcal{O}_{ba}\otimes\mathcal{O}_{ab} & \longrightarrow & \mathcal{O}_{bb} & \xrightarrow{\Theta_b} & \mathbb{C} \end{array} \tag{1.4}$$

are perfect pairings with

$$\Theta_a(\psi_1\psi_2) = \Theta_b(\psi_2\psi_1) \tag{1.5}$$

for $\psi_1 \in \mathcal{O}_{ab}$*, and* $\psi_2 \in \mathcal{O}_{ba}$.

3. *There are linear maps:*

$$\iota_a : \mathcal{A} \to \mathcal{O}_{aa}, \quad \iota^a : \mathcal{O}_{aa} \to \mathcal{A} \tag{1.6}$$

such that

3a. ι_a *is an algebra homomorphism*

$$\iota_a(\phi_1\phi_2) = \iota_a(\phi_1)\iota_a(\phi_2), \tag{1.7}$$

3b. *the identity is preserved*

$$\iota_a(1_{\mathcal{A}}) = 1_a. \tag{1.8}$$

3c. *Moreover,* ι_a *is central in the sense that*

$$\iota_a(\phi)\psi = \psi\iota_b(\phi), \tag{1.9}$$

for all $\phi \in \mathcal{A}$ *and* $\psi \in \mathcal{O}_{ab}$.

3d. ι_a *and* ι^a *are adjoint*

$$\Theta_{\mathcal{A}}(\iota^a(\psi)\phi) = \Theta_a(\psi\iota_a(\phi)).$$

3e. *We define the map* $\pi_b^a : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$ *as follows. Since* $\mathcal{O}_{ab}$ *and* $\mathcal{O}_{ba}$ *are in duality, if we let* ψ_μ *be a basis for* $\mathcal{O}_{ba}$ *then there is a dual basis* ψ^μ *for* $\mathcal{O}_{ab}$*. Then we define*

$$\pi_b^a(\xi) = \sum_\mu \psi_\mu\xi\psi^\mu.$$

We require the Cardy conditions*:*

$$\pi_b^a = \iota_b \circ \iota^a. \tag{1.10}$$

We will call axioms 2b and 2c in the above definition the *trace axioms.*

The central result that we prove in this chapter is the next purely algebraic theorem.

Theorem 4.4.7 *The trace axioms in the definition of Frobenius structure are equivalent to the following coproduct axiom:*

Coproducts Axiom: There exist a family of coassociative linear maps $\Delta^c_{ab} : \mathscr{O}_{ab} \to \mathscr{O}_{ac} \otimes \mathscr{O}_{cb}$ *which are* $\mathscr{O}_{aa} \times \mathscr{O}_{bb}$*-bimodule morphisms and linear maps* $\Theta_a : \mathscr{O}_{aa} \to \mathbb{C}$ *such that*

$$\begin{tikzcd} \mathscr{O}_{ab} \arrow[r, "\Delta^b_{ab}"] \arrow[d, "\cong"'] & \mathscr{O}_{ab} \otimes \mathscr{O}_{bb} \arrow[dl, "1\otimes\Theta_b"] \\ \mathscr{O}_{ab} \otimes \Bbbk & \end{tikzcd} \qquad \begin{tikzcd} \mathscr{O}_{ab} \arrow[r, "\Delta^a_{ab}"] \arrow[d, "\cong"'] & \mathscr{O}_{aa} \otimes \mathscr{O}_{ab} \arrow[dl, "\Theta_a\otimes 1"] \\ \Bbbk \otimes \mathscr{O}_{ab} & \end{tikzcd}$$

commute.

This theorem permits us to reconstruct the definition of Frobenius categories in terms of the coproducts rather than traces, and motivated by this new presentation we define a new algebraic structure. We have decided to call this structure *nearly Frobenius structure.*

A nearly Frobenius category is given by the following algebraic data:

1. $(\mathcal{A}, \Delta_{\mathcal{A}}, 1_{\mathcal{A}})$ is a commutative nearly Frobenius algebra.

2. A $\mathbb{C}$-linear category $\mathcal{B}$, where $\mathscr{O}_{ab} = \mathrm{Hom}(a, b)$ for $a, b \in \mathscr{B}_0$.

2a. With associative linear maps

$$\eta^b_{ac} : \mathscr{O}_{ab} \otimes \mathscr{O}_{bc} \to \mathscr{O}_{ac} \tag{1.11}$$

2b. With co-associative linear maps

$$\Delta^c_{ab} : \mathscr{O}_{ab} \to \mathscr{O}_{ac} \otimes \mathscr{O}_{cb}. \tag{1.12}$$

2c. where Δ^c_{ab} is a morphism of $\mathscr{O}_{da} \times \mathscr{O}_{be}$-bimodule, i.e. the diagrams:

$$\begin{tikzcd} \mathscr{O}_{da} \otimes \mathscr{O}_{ab} \arrow[r, "\eta^a_{db}"] \arrow[d, "1\otimes\Delta^c_{ab}"'] & \mathscr{O}_{db} \arrow[d, "\Delta^c_{db}"] \\ \mathscr{O}_{da} \otimes \mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \arrow[r, "\eta^a_{dc}\otimes 1"'] & \mathscr{O}_{dc} \otimes \mathscr{O}_{cb} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{O}_{ab} \otimes \mathscr{O}_{bb} \arrow[r, "\eta^b_{ae}"] \arrow[d, "\Delta^c_{ab}\otimes 1"'] & \mathscr{O}_{ae} \arrow[d, "\Delta^c_{ae}"] \\ \mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \otimes \mathscr{O}_{be} \arrow[r, "1\otimes\eta^b_{ce}"'] & \mathscr{O}_{ac} \otimes \mathscr{O}_{ce} \end{tikzcd}$$

commute.

3. There are linear maps:

$$\iota_a : \mathcal{A} \to \mathscr{O}_{aa}, \iota^a : \mathscr{O}_{aa} \to \mathcal{A} \tag{1.13}$$

such that

3a. ι_a is an algebra homomorphism

$$\iota_a(\phi_1\phi_2) = \iota_a(\phi_1)\iota_a(\phi_2) \tag{1.14}$$

3b. The identity is preserved

$$\iota_a(1_{\mathcal{A}}) = 1_a \tag{1.15}$$

3c. Moreover, ι_a is central in the sense that

$$\iota_a(\phi)\psi = \psi\iota_b(\phi) \tag{1.16}$$

for all $\phi \in \mathcal{A}$ and $\psi \in \mathscr{O}_{ab}$.

3d. We define the map

$$\pi^a_b := \eta^a_{bb} \circ \tau \circ \Delta^b_{aa} : \mathscr{O}_{aa} \to \mathscr{O}_{bb},$$

where $\tau : \mathscr{O}_{ab} \otimes \mathscr{O}_{ba} \to \mathscr{O}_{ba} \otimes \mathscr{O}_{ab}$ is the transposition map. We require the *Cardy condition*:

$$\pi^a_b = \iota_b \circ \iota^a. \tag{1.17}$$

We dedicate the last section of this chapter to prove that *open-closed string topology* satisfies all the axoms of nearly Frobenius structure. This example was studied by Dennis Sullivan in [Sul04], and by Cohen, Hess and Voronov in the book [CHV06]. In this situations our background manifold comes equipped with a collection of submanifolds,

$$\mathcal{B} = \{D_i \subset M\},$$

and the morphisms are given by the homology of the path spaces

$$\mathcal{P}_M(D_i, D_j) = \{\gamma : [0,1] \to M \text{ picewise smoooth: } \gamma(0) \in D_i,\ \gamma(1) \in D_j\}.$$

Chapter 5. In this chapter we gauge (or orbifold) Frobenius structures by the action of a finite group.

The first algebraic structure that we consider is that of a *G-Frobenius algebra*, namely an algebra $\mathscr{C} = \oplus_{g\in G}\mathscr{C}_g$, where for $g \in G$, $\mathscr{C}_g$ is a vector space of finite dimension such that

1. There is a homomorphism $\alpha : G \to \mathrm{Aut}(\mathscr{C})$, where $\mathrm{Aut}(\mathscr{C})$ are the algebra homomorphisms of $\mathscr{C}$, such that
$$\alpha_h : \mathscr{C}_g \to \mathscr{C}_{hgh^{-1}},$$
and for every $g \in G$ we have
$$\alpha_g|_{\mathscr{C}_g} = 1_{\mathscr{C}_g}.$$
2. There is a G-invariant trace or counit $\varepsilon : \mathscr{C}_e \to \mathscr{C}$ which induce nondegenerate pairings
$$\theta_g : \mathscr{C}_g \otimes \mathscr{C}_{g^{-1}} \to \mathscr{C}.$$
3. For all $x \in \mathscr{C}_g$ and $y \in \mathscr{C}_h$ we have that the product is twisted commutative, i.e.
$$xy = \alpha_g(y)x.$$
4. Let $\Delta_g = \sum_i \xi_i^g \otimes \xi_i^{g^{-1}} \in \mathscr{C}_g \otimes \mathscr{C}_{g^{-1}}$ the Euler element, where $\{\xi_i^g\}$ is a basis of $\mathscr{C}_g$ and $\left\{\xi_i^{g^{-1}}\right\}$ is the dual basis of $\mathscr{C}_{g^{-1}}$. We have that for all $g, h \in G$
$$\sum_i \alpha_h(\xi_i^g)\xi_i^{g^{-1}} = \sum_i \xi_i^h \alpha_g(\xi_i^{h^{-1}}).$$

This definition is due to Moore and Segal, [MS06].

We show in this chapter that the G-invariant part of a G-Frobenius algebra is a Frobenius algebra. Just as before we give a new presentation of G-Frobenius algebras using coproducts rather than traces. Then, motivated by this new presentation, we define a new algebraic object, that we have call *nearly G-Frobenius algebra*, which consists of an algebra $\mathscr{C} = \oplus_{g\in G}\mathscr{C}_g$, where $\mathscr{C}_g$ is a vector space for $g \in G$ such that

1. There is a homomorphism $\alpha : G \to \mathrm{Aut}(\mathscr{C})$, where $\mathrm{Aut}(\mathscr{C})$ is the algebra of homomorphisms of $\mathscr{C}$, such that
$$\alpha_h : \mathscr{C}_g \to \mathscr{C}_{hgh^{-1}},$$
for every $g \in G$ we have
$$\alpha_g|_{\mathscr{C}_g} = \mathrm{Id}_{\mathscr{C}_g}.$$
2. For all $x \in \mathscr{C}_g$ and $y \in \mathscr{C}_h$ we have that the product is twisted commutative, i.e.
$$xy = \alpha_g(y)x.$$

3. There are coproducts $\Delta_{g,h} : \mathscr{C}_{gh} \to \mathscr{C}_g \otimes \mathscr{C}_h$ such that the following diagrams commute.

$$\begin{tikzcd}
\mathscr{C}_g \otimes \mathscr{C}_{hf} \arrow[r, "m_{g,hf}"] \arrow[d, "1\otimes\Delta_{h,f}"'] & \mathscr{C}_{ghf} \arrow[d, "\Delta_{gh,f}"] \\
\mathscr{C}_g \otimes \mathscr{C}_h \otimes \mathscr{C}_f \arrow[r, "m_{g,h}\otimes 1"'] & \mathscr{C}_{gh} \otimes \mathscr{C}_f
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C}_g \otimes \mathscr{C}_{hf} \arrow[r, "m_{g,hf}"] \arrow[d, "\Delta_{gh,h^{-1}}\otimes 1"'] & \mathscr{C}_{ghf} \arrow[d, "\Delta_{gh,f}"] \\
\mathscr{C}_{gh} \otimes \mathscr{C}_{h^{-1}} \otimes \mathscr{C}_{hf} \arrow[r, "1\otimes m_{h^{-1},hf}"'] & \mathscr{C}_{gh} \otimes \mathscr{C}_f
\end{tikzcd}$$

   that is, for all $x \in \mathscr{C}_g$ and $y \in \mathscr{C}_{hf}$ we have $x\Delta_{hf}(y) = \Delta_{gh,f}(xy) = \Delta_{gh,h^{-1}}(x)y$.

4. These coproducts have the next property: for every $g, h \in G$ the next diagram commutes

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "u"] \arrow[d, "u"'] & \mathscr{C}_e \arrow[r, "\Delta_h"] & \mathscr{C}_h \otimes \mathscr{C}_{h^{-1}} \arrow[r, "1\otimes\alpha_g"] & \mathscr{C}_h \otimes \mathscr{C}_{gh^{-1}g^{-1}} \arrow[d, "m_{h,gh^{-1}g^{-1}}"] \\
\mathscr{C}_e \arrow[r, "\Delta_g"'] & \mathscr{C}_g \otimes \mathscr{C}_{g^{-1}} \arrow[r, "\alpha_h\otimes 1"'] & \mathscr{C}_{hgh^{-1}} \otimes \mathscr{C}_{g^{-1}} \arrow[r, "m_{hgh^{-1},g^{-1}}"'] & \mathscr{C}_{hgh^{-1}g^{-1}}
\end{tikzcd}$$

In other words the trace has been removed.

The main result in this section is the next algebraic theorem.

Theorem 5.3.3 If $\mathscr{A}$ is a nearly G-Frobenius algebra then its G-invariant part is a nearly Frobenius algebra.

We proof the major results of this thesis in the last section of this chapter. They consist of two examples of nearly G-Frobenius algebras. The first example called *virtual cohomology*, was introduced by Lupercio, Uribe and Xicoténcatl in [LUX07]. We consider the complex global quotient orbifold $[M/G]$, where M is a complex manifold and G is a finite group acting holomorphically on M. We define the *virtual cohomology*

$$\mathrm{H}^*(M,G) := \bigoplus_{g\in G} \mathrm{H}^*(M^g),$$

where M^g is the fixed point set of the element g. The group G acts in the natural way on the cohomologies by conjugation of the labels.

If $\alpha \in \mathrm{H}^*(M^g)$ and $\beta \in \mathrm{H}^*(M^h)$, we define the *virtual product* as

$$\alpha \star \beta := i_{g,h}!\left((\nu(g,h)\delta^*_{g,h}(\alpha\times\beta)\right),$$

where $\nu(g,h) = e(M; M^g, M^h)$ is the Euler class of the excess bundle $\frac{TM|_{M^{g,h}}}{TM^g|_{M^{g,h}}+TM^h|_{M^{g,h}}}$, $i_{g,h} : M^{g,h} = M^g \cap M^h \hookrightarrow M^{gh}$ and $\delta_{g,h} : M^{g,h} \hookrightarrow M^g \times M^h$ is the diagonal map. As the same form, if $\alpha \in \mathrm{H}^*(M^{gh})$ we can define the *virtual coproduct* as

$$\Delta_{g,h}(\alpha) := \delta_{g,h}!\left(\mu(gh,g,h)i^*_{g,h}(\alpha)\right),$$

where $\mu(g,h) = e\left(\frac{TM|_{M^{g,h}}}{TM^{gh}|_{M^{g,h}}} \oplus TM^{g,h}\right)$.

The second example *orbifold string topology*, was introduced again by Lupercio, Uribe and Xicoténcatl in [LUX08].
As the same as before we consider the complex orbifold $[M/G]$, and we construct the space

$$\mathcal{P}_G(M) := \bigsqcup_{g\in G} \mathcal{P}_g(M) \times \{g\}$$

where

$$\mathcal{P}_g(M) = \{\gamma : [0,1] \to Y : \gamma(0)g = \gamma(1)\},$$

together with the G-action given by

$$G \times \bigsqcup_{g\in G} \mathcal{P}_g(M) \times \{g\} \to \bigsqcup_{g\in G} \mathcal{P}_g(M) \times \{g\}$$

$$(h, (\gamma, g)) \mapsto (\gamma_h, h^{-1}gh)$$

where $\gamma_h(t) := \gamma(t)h$.
We define the product as the composition

$$\eta_{g,h} : \mathrm{H}_p(\mathcal{P}_g(M)) \otimes \mathrm{H}_q(\mathcal{P}_h(M)) \xrightarrow{j!} \mathrm{H}_{p+q-d}(\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M)) \xrightarrow{\circledast_*} \mathrm{H}_{p+q-d}(\mathcal{P}_{gh}(M)),$$

where $\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) = \{(\gamma_0,\gamma_1) : \gamma_0(1) = \gamma_1(0)\}$, $j : \mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) \to \mathcal{P}_g(M) \times \mathcal{P}_h(M)$ is the inclusion and $\circledast : \mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) \to \mathcal{P}_{gh}(M)$ is the composition of paths.
The coproducts are defined in a similar manner by the compositions

$$\Delta_{g,h} : \mathrm{H}_{p+q+d}(\mathcal{P}_{gh}(M)) \xrightarrow{\circledast!} \mathrm{H}_{p+q}(\mathcal{P}_g(M)_{\epsilon_i}\times_{\epsilon_0}\mathcal{P}_h(M)) \xrightarrow{j_*} \mathrm{H}_p(\mathcal{P}_g(M)) \otimes \mathrm{H}_q(\mathcal{P}_h(M)).$$

Chapter 6. In this chapter we study the case of finite groups acting on Frobenius categories. The first theorem that we prove is quite natural:
Theorem 6.0.6 *The G-invariant part of a G-Frobenius category is a Frobenius category.*

Again we extend the concept of G-Frobenius category to G-*nearly Frobenius category*, and we prove the analogous theorem.

The main results of this chapter are the fact that the virtual cohomology and the loop orbifold admit extensions to G-nearly Frobenius categories. In the first case the category of boundary conditions is the following:

$$\mathcal{B} = \{X \subset M \ G\text{-invariante}\}$$

such that, if $X, Y \in \mathcal{B}$ then $TX|_{(X\cap Y)^g} \cong TY|_{(X\cap Y)^g}$ for all $g \in G$, and $\mathrm{Hom}_{\mathcal{B}}(X, Y) = \mathrm{H}^*(X \cap Y)$, for $X, Y \in \mathcal{B}$.
In the second case the category of branes is the following:

$$\mathscr{B} = \{X \subset M \ G\text{-invariant submanifold with } X \pitchfork Y \text{ transverse for } X \neq Y \text{ and } X \cap Y \neq \emptyset\}$$

Chapter 7. We will wind down with an expository chapter in which we present an example of a Frobenius category studied by Caldararu and Willerton in [CW07]. This example is a very interesting example because the category of D-branes or boundary conditions is the derived category of a compact Calabi-Yau manifold. This example shows an interaction between these TQFTs and algebraic geometry.

We finish presenting a conjecture that is quite natural from our point of view. Namely that the derived category of a non-compact Calabi-Yau manifold satisfies the axioms of a nearly Frobenius category. This is would include a very nice generalization of Serre duality to the non-compact case. I shall return to this issue elsewhere in collaboration with Ernesto Lupercio.

Chapter 2

Frobenius structures

In this chapter we will give some equivalent definitions of Frobenius algebras and a serie de examples that let us to understand in a better way this concept. A fundamental example of Frobenius algebra is the Poncaré algebra associated to every compact closed manifold M, this is provided by its cohomology algebra $\mathcal{A} = \mathrm{H}^*(M)$ with trace

$$\Theta(X) = \int_M X.$$

It is not very hard to see that the Frobenius property is equivalent to Poincaré duality.
When we consider the case of a non-compact manifold M its cohomology algebra is no longer a Frobenius algebra, but we may ask ourselves what structure remains.

To get an idea of a possible answer to this question we have to recall that the concept of Frobenius algebra is strongly related to the concept of a 2D-Topological Field Theory. A natural generalization of this concepts are not to consider the trace as a component of the algebraic structure in the topological structure. This will motivate us to introduce the new concepts of *nearly Frobenius algebra* and *2D-Topological Fiel Theory with positive boundary.* This structures are strongly related as we will see.

A Frobenius algebra is a finite dimensional unital associative algebra with a special kind of bilinear form which gives an isomorphism to the dual.
The concept of Frobenius algebras was first studied in the 1930s by Brauer and Nesbitt [BN38] where they were named after Frobenius. Nakayama discovered the beginnings of a rich duality theory in [Nak39] and in [Nak41]. Dieudonné used this to characterize Frobenius algebras in [Die58] where he called this property of Frobenius algebras a perfect duality. The characterization of Frobenius algebras in terms of coproducts goes back at least to Lawvere [Law69], it has been rediscovered by Quinn [Qui95] and by Abrams [Abr97].

2.1 Frobenius algebras

Fix a fiel $\Bbbk$ of characteristic zero. A $\Bbbk$-algebra is a $\Bbbk$-vector space $\mathcal{A}$ together with two $\Bbbk$-linear maps

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \quad u : \Bbbk \to \mathcal{A}$$

called *multiplication* and *unit* such that μ is associative and u is the unit $(u(1) = 1_{\mathcal{A}})$.

We start by giving a serie of equivalent definitions of Frobenius algebras.

Definition 2.1.1. A *Frobenius algebra* is a $\Bbbk$-algebra $\mathcal{A}$ of finite dimension whit a non-degenerate bilinear form $\langle\, , \rangle : \mathcal{A} \otimes \mathcal{A} \to \Bbbk$ which is associative, in the sense $\langle ab, c\rangle = \langle a, bc\rangle$.

Definition 2.1.2. A *Frobenius algebra* is a $\Bbbk$-algebra $\mathcal{A}$ of finite dimension with a linear function $\varepsilon : \mathcal{A} \to \Bbbk$ called counit, such that the $\ker(\varepsilon)$ do not have non trivial ideals.

Definition 2.1.3. A *Frobenius algebra* is a $\Bbbk$-algebra $\mathcal{A}$ of finite dimension with an $\mathcal{A}$-module isomorphism $\lambda : \mathcal{A} \to \mathcal{A}^*$, where the dual space $\mathcal{A}^*$ is an $\mathcal{A}$-module with the action $a \cdot \varphi = \varphi \circ \overline{m}(a)$, where $\overline{m} : \mathcal{A} \to \mathrm{End}(\mathcal{A})$ is the multiplication by $a \in \mathcal{A}$.

Proposition 2.1.4. *These definitions are equivalent.*

Proof. (1) $\Rightarrow$ (2) Given a pairing $\langle\, , \rangle : \mathcal{A} \otimes \mathcal{A} \to \Bbbk$ we define the counit as follows

$$\begin{array}{rccc} \varepsilon : & \mathcal{A} & \to & \Bbbk \\ & a & \mapsto & \langle 1_{\mathcal{A}}, a\rangle. \end{array} \tag{2.1}$$

(2) $\Rightarrow$ (3) If we have the counit $\varepsilon : \mathcal{A} \to \Bbbk$ we define the isomorphism λ as follows

$$\begin{array}{rccccc} \lambda : & \mathcal{A} & \to & \mathcal{A}^* & & \\ & a & \mapsto & \lambda(a) : & \mathcal{A} & \to & \Bbbk \\ & & & & b & \mapsto & \varepsilon(ab) \end{array} \tag{2.2}$$

(3) $\Rightarrow$ (1) Finally, given $\lambda : \mathcal{A} \to \mathcal{A}^*$ we define the pairing as follows

$$\begin{array}{rccc} \langle\, , \rangle : & \mathcal{A} \otimes \mathcal{A} & \to & \Bbbk \\ & a \otimes b & \mapsto & \lambda(1_{\mathcal{A}})(ab) \end{array} \tag{2.3}$$

♣

The next theorem is due to Lowell Abrams [Abr96] and Aaron D. Lauda in [Lau08]. They give two additional definitions of a Frobenius algebra.

Theorem 2.1.5. *A commutative algebra $\mathcal{A}$ of finite dimension with product $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and unit $u : \Bbbk \to \mathcal{A}$ is a Frobenius algebra if and only if it satisfies one of the next conditions*

i) (Abrams) There is a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, with a counit $\varepsilon : \mathcal{A} \to \Bbbk$ satisfying the Frobenius identities which define a coalgebra structure on $\mathcal{A}$. Explicitly the following diagrams commute:

- *The coalgebra axioms*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\Delta\otimes 1"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "1\otimes\Delta"'] & \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \otimes \Bbbk & \mathcal{A} \otimes \mathcal{A} \arrow[l, "1\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes 1"] & \Bbbk \otimes \mathcal{A} \\ & \mathcal{A} \arrow[ul, "\cong"] \arrow[u, "\Delta"'] \arrow[ur, "\cong"'] & \end{tikzcd}$$

If we note $\Delta(x) = \sum x_1 \otimes x_2$, then for $x \in \mathcal{A}$ the coalgebra axioms are given by the next relations

$$(\Delta \otimes 1)(\Delta(x)) = \sum x_{11} \otimes x_{12} \otimes x_2 = \sum x_1 \otimes x_{21} \otimes x_{22} = (1 \otimes \Delta)(\Delta(x))$$

$$(1 \otimes \varepsilon)(\Delta(x)) = \sum x_1 \varepsilon(x_2) = x = \sum \varepsilon(x_1) x_2 = (\varepsilon \otimes 1)(\Delta(x)).$$

- *The Frobenius identities*

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \arrow[r, "m"] \arrow[d, "1\otimes\Delta"'] & \mathcal{A} \arrow[d, "\Delta"] \\ \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "m\otimes 1"'] & \mathcal{A} \otimes \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \arrow[r, "m"] \arrow[d, "\Delta\otimes 1"'] & \mathcal{A} \arrow[d, "\Delta"] \\ \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "1\otimes m"'] & \mathcal{A} \otimes \mathcal{A} \end{tikzcd}$$

i.e. $\sum xy_1 \otimes y_2 = \sum (xy)_1 \otimes (xy)_2 = \sum x_1 \otimes x_2 y$, *for* $x, y \in \mathcal{A}$.

ii) (Lauda) There exists a co-pairing $\theta : \Bbbk \to \mathcal{A} \otimes \mathcal{A}$ such that the following diagrams commute:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "1\otimes\theta"] \arrow[d, "\theta\otimes 1"'] \arrow[dr, "\Delta"] & \mathcal{A} \otimes A \otimes A \arrow[d, "m\otimes 1"] \\ \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "1\otimes m"'] & \mathcal{A} \otimes \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "\theta"] \arrow[d, "\theta"'] \arrow[dr, "u"] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\varepsilon\otimes 1"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "1\otimes\varepsilon"'] & \mathcal{A} \end{tikzcd}$$

Let $x \in \mathcal{A}$, if we denote $\theta(1) = \sum \xi_i \otimes \xi^j$ then the Lauda condition is the following:

$$\sum x\xi_1 \otimes \xi_2 = \sum x_1 \otimes x_2 = \sum \xi_1 \otimes \xi_2 x,$$

and

$$\sum \varepsilon(\xi_1)\xi_2 = 1_{\mathcal{A}} = \sum \xi_1 \varepsilon(\xi_2).$$

Proof. i) We define the coproduct as follows

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\Delta"] \arrow[d, "\lambda"'] & \mathcal{A}\otimes\mathcal{A} \\
\mathcal{A}^* \arrow[r, "m^*"'] & \mathcal{A}^*\otimes\mathcal{A}^* \arrow[u, "\lambda^{-1}\otimes\lambda^{-1}"']
\end{tikzcd}$$

that is $\Delta := (\lambda^{-1}\otimes\lambda^{-1})\circ m^*\lambda$, where λ is as defined in equation 9.1.
Using that m is a commutative and an associative map we have that Δ is a cocommutative and a coassociative map. We need to check that Δ is an $\mathcal{A}$-module morphism, for this we construct the next map

$$\begin{array}{rcccccc}
\overline{m}: & \mathcal{A} & \longrightarrow & \mathrm{End}(\mathcal{A}) & \cong & \mathcal{A}\otimes\mathcal{A}^* \\
& a & \longmapsto & a\cdot & \mapsto & a\sum_i e_i\otimes e_i^*
\end{array}$$

where $\{e_1,\ldots,e_n\}$ is a basis of $\mathcal{A}$ and $\{e_1^*,\ldots,e_n^*\}$ is the dual basis.
It is easy to prove that the next diagrams commute.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\lambda"] \arrow[d, "\Delta"'] & \mathcal{A}^* \arrow[r, "\lambda^{-1}"] \arrow[d, "m^*"'] & \mathcal{A} \arrow[d, "\overline{m}"] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "\lambda^{-1}\otimes\lambda^{-1}"'] & \mathcal{A}^*\otimes\mathcal{A}^* \arrow[r, "\lambda\otimes 1^*"'] & \mathcal{A}\otimes\mathcal{A}^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[d, "\Delta"'] \arrow[dr, "\overline{m}"] & \\
\mathcal{A}\otimes\mathcal{A} & \mathcal{A}\otimes\mathcal{A}^* \arrow[l, "1\otimes\lambda^{-1}"]
\end{tikzcd}$$

We consider the next diagram.

$$\begin{tikzcd}
& \mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] \arrow[dl, "1\otimes\overline{m}"'] \arrow[d, "1\otimes\Delta"] \arrow[dr, phantom, "1"] & \mathcal{A} \arrow[d, "\Delta"'] \arrow[dr, "\overline{m}"] \arrow[dr, phantom, "2"'] & \\
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}^* \arrow[r, "1\otimes 1\otimes\lambda^{-1}"'] \arrow[dr, "m\otimes 1^*"'] \arrow[ur, phantom, "5"'] & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "m\otimes 1"'] \arrow[dr, phantom, "4"] & \mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes\lambda^{-1}"] \arrow[d, "1\otimes\lambda"'] \arrow[dr, phantom, "3"] & \mathcal{A}\otimes\mathcal{A}^* \arrow[dl, "1\otimes 1^*"] \\
& \mathcal{A}\otimes\mathcal{A}^* \arrow[r, "1\otimes 1^*"'] & \mathcal{A}\otimes\mathcal{A}^* &
\end{tikzcd}$$

Note that 2 and 5 commute by definition of $\overline{m}$, 3 and 4 clearly commute. The external diagram commute because

$$\begin{tikzcd}
& x\otimes y \arrow[r, mapsto, "m"] \arrow[dl, mapsto, "1\otimes\overline{m}"'] & xy \arrow[dr, mapsto, "\overline{m}"] & \\
x\otimes\sum_i ye_i\otimes e_i^* \arrow[dr, mapsto, "m\otimes 1^*"'] & & & \sum_i xye_i\otimes e_i^* \arrow[dl, mapsto, "1\otimes 1^*"] \\
& x\sum_i ye_i\otimes e_i^* \arrow[r, mapsto, "1\otimes 1^*"'] & xy\sum_i e_i\otimes e_i^* &
\end{tikzcd}$$

Then the diagram (1) commutes and Δ is a morphism of $\mathcal{A}$-modules.
Reciprocally, we define $\langle\, , \rangle : \mathcal{A}\otimes\mathcal{A} \to \Bbbk$ by $\langle\, , \rangle = \varepsilon \circ m$. Using that m and ε are linear maps we have that $\langle\, , \rangle$ is also linear. The associativity is a consequence of the associativity of the product. Finally, to prove that the pairing is non-degenerate, we use that the next diagram commutes since Δ is a $\mathcal{A}$-module morphism.

$$\begin{tikzcd} & & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[dr, "1\otimes m"] & & \\ \Bbbk\otimes\mathcal{A} \arrow[r, "u\otimes 1"] & \mathcal{A}\otimes\mathcal{A} \arrow[ur, "\Delta\otimes 1"] \arrow[dr, "m"'] & & \mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes\varepsilon"] & \mathcal{A}\otimes\Bbbk \\ & & \mathcal{A} \arrow[ur, "\Delta"'] & & \end{tikzcd}$$

The top composition gives

$$1\otimes x \mapsto 1_{\mathcal{A}}\otimes x \mapsto (\sum_j u_j\otimes e_j)\otimes x \mapsto \sum_j u_j \otimes e_j x \mapsto \sum_j \langle e_j, x\rangle u_j \otimes 1.$$

and the bottom composition gives

$$1\otimes x \mapsto 1_{\mathcal{A}}\otimes x \mapsto x \mapsto \Delta(x) \mapsto (1\otimes\varepsilon)\Delta(x) = x\otimes 1$$

Then $x = \sum_j \langle e_j, x\rangle u_j$, therefore $\{u_j\}$ is a basis of $\mathcal{A}$. In particular if we take $x = u_i$ we have $\langle e_j, u_i\rangle = \delta_{ij}$.
Now we take k_i such that $\langle \sum_i k_i e_i, x\rangle = 0$ for all $x \in \mathcal{A}$. If $x = u_j$ we have $\sum_i k_i\langle e_i, u_j\rangle = 0$, then $k_i = 0$ for all $i = 1, \dots n$. Therefore $\sum_i k_i e_i = 0$ and the pairing $\langle\ ,\ \rangle$ is non-degenerate.

ii) It is easy to see that this condition is equivalent to the Abrams condition. Given the coproduct Δ we define $\theta : \Bbbk \to \mathcal{A}\otimes\mathcal{A}$ by $\theta = \Delta\circ u$. We deduce the commutativity of the diagrams using the $\mathcal{A}$-module properties. If we consider the co-pairing $\theta : \Bbbk \to \mathcal{A}\otimes\mathcal{A}$ we define $\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ as follows

$$\Delta = (1\otimes m)\circ(\theta\otimes 1) = (m\otimes 1)\circ(1\otimes\theta)$$

♣

Definition 2.1.6. A Frobenius algebra $\mathcal{A}$ is called a *symmetric Frobenius algebra* if one (and hence all) of the following equivalent conditions holds.

(i) The Frobenius form $\varepsilon : \mathcal{A} \to \Bbbk$ is *central*; this means that $\varepsilon(ab) = \varepsilon(ba)$ for all $a, b \in \mathcal{A}$.

(ii) The pairing $\langle\ ,\ \rangle$ is symmetric (i.e. $\langle a, b\rangle = \langle b, a\rangle$ for all $a, b \in \mathcal{A}$).

(iii) The left $\mathcal{A}$-isomorphism $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^*$ is also right $\mathcal{A}$-linear.

(iv) The right $\mathcal{A}$-isomorphism $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^*$ is also left $\mathcal{A}$-linear.

Definition 2.1.7. A *Frobenius algebra homomorphism* $\phi : (\mathcal{A}, \varepsilon) \longrightarrow (\mathcal{A}', \varepsilon')$ between two Frobenius algebras is an algebra homomorphism which is at the same time a coalgebra homomorphism. In particular it preserves the Frobenius form, in the sense that $\varepsilon = \phi\varepsilon'$.

Let $\mathrm{FA}_{\Bbbk}$ denote the category of Frobenius algebras, and let $\mathrm{cFA}_{\Bbbk}$ denote the full subcategory of all commutative Frobenius algebras.

Lemma 2.1.8. *If a $\Bbbk$-algebra homomorphism ϕ between two Frobenius algebras $(\mathcal{A}, \varepsilon)$ and $(\mathcal{A}', \varepsilon')$ is compatible with the forms in the sense that the diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\phi"] \arrow[dr, "\varepsilon"'] & & \mathcal{A}' \arrow[dl, "\varepsilon'"] \\ & \Bbbk & \end{tikzcd}$$

commutes, then ϕ is injective.

Proof. The kernel of ϕ is an ideal and it is clearly contained in $\ker(\varepsilon)$. But $\ker(\varepsilon)$ contains no nontrivial ideals, so $\ker(\phi) = 0$ and thus ϕ is injective.

♣

Lemma 2.1.9. *A Frobenius algebra homomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ is always invertible. In other worlds, the category $\mathrm{FA}_{\Bbbk}$ is a groupoid and so is $\mathrm{cFA}_{\Bbbk}$.*

Proof. Since ϕ is comultiplicative and respects the counits ε and ε' (as well as the units η and η'), the dual map $\phi^* : \mathcal{A}'^* \to \mathcal{A}^*$ is multiplicative and respects the units and counits. But then the preceding lemma applies and shows that ϕ^* is injective. Since $\mathcal{A}$ is a finite-dimensional vector space this implies that ϕ is surjective. We already know it is injective, hence it is invertible.

♣

2.2 Basic examples

In this section we will present a collection of examples of Frobenius algebras. A good reference for this is [Koc03]. Our main example is the Poincaré algebra, it is the principal motivation for the definition presented in the next section, this is because if we consider M a manifold not necessarily compact we do not necessarily have the trace but all the other structures are preserved.

2.2.1 The trivial Frobenius algebra

Let $\mathcal{A} = \Bbbk$, and $\varepsilon : \mathcal{A} \to \Bbbk$ be the identity map of $\Bbbk$. Clearly there are no ideals in the kernel of this map, so we have a Frobenius algebra.

2.2.2 A concrete example

The field of complex number $\mathbb{C}$ is a Frobenius algebra over $\mathbb{R}$: an obvious Frobenius form is taking the real part

$$\begin{aligned} \mathbb{C} &\to \mathbb{R} \\ a + ib &\mapsto a. \end{aligned}$$

2.2.3 Skew-fields

Let $\mathcal{A}$ be a skew-field (also called division algebra) of finite dimension over $\Bbbk$. Since just like a field, a skew-field has no nontrivial left ideals (or right ideals), any nonzero linear form $\mathcal{A} \to \Bbbk$ will make $\mathcal{A}$ into a Frobenius algebra over $\Bbbk$, for example the quaternions $\mathbb{H}$ form a Frobenius algebra over $\mathbb{R}$.

2.2.4 Matrix algebras

Let $\mathcal{A}$ be the space $\mathrm{Mat}_n(\Bbbk)$ of all $n \times n$ matrices over $\Bbbk$, this is a Frobenius algebra with the usual trace map

$$\begin{aligned} \mathrm{Tr} : \mathrm{Mat}_n(\Bbbk) &\to \quad \Bbbk \\ (a_{ij}) \quad &\mapsto \sum_i a_{ii} \end{aligned}$$

To see that the bilinear pairing resulting from Tr is nondegenerate, take the linear basis of $\mathrm{Mat}_n(\Bbbk)$ consisting of E_{ij} with only one nonzero entry $e_{ij} = 1$. Clearly E_{ji} is the dual basis element to E_{ij} under this pairing. Note that this is a symmetric Frobenius algebra since two matrix products AB and BA have the same trace. If we twist the Frobenius form by multiplication with a noncentral invertible matrix we obtain a nonsymmetric Frobenius algebra.

As a concrete example, consider $\mathrm{Mat}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ with the usual trace map

$$\begin{array}{rccc} \mathrm{Tr}: & \mathrm{Map}_2(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & a + d \end{array}$$

Now twist and take as Frobenius form the composition

$$\begin{array}{ccccc} \mathrm{Mat}_2(\mathbb{R}) & \longrightarrow & \mathrm{Mat}_2(\mathbb{R}) & \xrightarrow{\mathrm{Tr}} & \mathbb{R} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \longmapsto & b+c \end{array}$$

This composition is not a central function, for example if we take $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ then $AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ and finally the map gives, in the first case 1 and in the second 2.

2.2.5 Finite group algebras

Let $G = \{e, g_1, \dots, g_n\}$ be a finite group, the *group algebra* $\mathbb{C}[G]$ is defined as the set of formal linear combinations $\sum_{i=0}^n c_i g_i$, where $c_i \in \mathbb{C}$, with multiplication given by the multiplication of G. It can be made into a Frobenius algebra by taking the Frobenius form to be the functional

$$\begin{array}{rccl} \varepsilon : & \mathbb{C}G & \longrightarrow & \mathbb{C} \\ & e & \longmapsto & 1 \\ & g_i & \longmapsto & 0 \quad \text{for } i \neq 0. \end{array}$$

Indeed, the corresponding pairing $g \otimes h \mapsto \varepsilon(gh)$ is nondegenerate since $g \otimes h \mapsto 1$ if and only if $h = g^{-1}$.

2.2.6 The ring of group characters

Assume the group field is $\Bbbk = \mathbb{C}$. Let G be a finite group of order n. A *class function* on G is a function $G \to \mathbb{C}$ which is constant on each conjugacy class; the class functions form a ring denoted $R(G)$. In particular, the characters (traces of representations) are class functions, and in fact every class function is a linear combination of characters. There is a bilinear pairing on $R(G)$ defined by

$$\langle \phi, \psi \rangle := \frac{1}{n} \sum_{t \in G} \phi(t)\psi(t^{-1}).$$

The characters form an orthonormal basis of $R(G)$with respect to this bilinear pairing, so in particular the pairing is nondegenerate and provides a Frobenius algebra structure on $R(G)$.

2.2.7 The Poincaré Algebra

Let M be a compact, closed, connected, oriented manifold of finite dimension n. We can define a counit map $\varepsilon : \mathrm{H}^*(M) \to \Bbbk$ by

$$\varepsilon(\varphi) = \varphi([M]),$$

where $[M]$ is the fundamental class of M in homology. This map induce the pairing

$$\langle\, , \rangle : \mathrm{H}^*(M) \otimes \mathrm{H}^*(M) \to \Bbbk$$

defined by $\langle \varphi, \psi \rangle = \varepsilon(\varphi \smile \psi) = (\varphi \smile \psi)([M]) = \varphi([M] \frown \psi)$. Remember that we have the next isomorphism induced by Poincaré duality

$$\Phi : \mathrm{H}^{n-k}(M) \stackrel{h}{\to} \mathrm{Hom}_{\Bbbk}(\mathrm{H}_{n-k}(M), \Bbbk) \stackrel{D^*}{\to} \mathrm{Hom}_{\Bbbk}(\mathrm{H}^k(M), \Bbbk)$$

where h is the map induced by the evaluation of cochains on chains, and D^* is the dual of Poincaré duality. Then $\Phi(\varphi)(\psi) = \varphi([M] \frown \psi)$, this proves that the pairing is nondegenerate.

2.3 Nearly Frobenius algebras

In this section we develop the central concept of study in this work. That is the structure of *nearly Frobenius algebra*. In the next chapters we will give a serie of interesting examples of this new concept. String topology is the first example that we will develop, but is not the only one.

Definition 2.3.1. A commutative algebra $\mathcal{A}$ with product $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a *nearly Frobenius algebra* if and only if it satisfies one of the following conditions:

(i) There exists a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ that makes the following diagrams commutative:
• The coalgebra axioms

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\Delta\otimes 1"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "1\otimes\Delta"'] & \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \end{tikzcd}$$

i.e. $\sum x_{11} \otimes x_{12} \otimes x_2 = \sum x_1 \otimes x_{21} \otimes x_{22}$ for all $x \in \mathcal{A}$, with the notation $\Delta(x) = \sum x_1 \otimes x_2$.

- The Frobenius identities

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] \arrow[d, "\Delta\otimes 1"'] & \mathcal{A} \arrow[d, "\Delta"] \\ \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes m"'] & \mathcal{A}\otimes\mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] \arrow[d, "1\otimes\Delta"'] & \mathcal{A} \arrow[d, "\Delta"] \\ \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "m\otimes 1"'] & \mathcal{A}\otimes\mathcal{A} \end{tikzcd}$$

i.e. $\sum xy_1 \otimes y_2 = \sum (xy)_1 \otimes (xy)_2 = \sum x_1 \otimes x_2 y$, for $x, y \in \mathcal{A}$.

(ii) There exists a copairing $\theta : \mathbb{k} \to \mathcal{A}\otimes\mathcal{A}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "1\otimes\theta"] \arrow[d, "\theta\otimes 1"'] \arrow[dr, "\Delta"] & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[d, "m\otimes 1"] \\ \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes m"'] & \mathcal{A}\otimes\mathcal{A} \end{tikzcd}$$

with the same notation as before the commutativity of the last diagram is equivalent to

$$\sum x\xi_i \otimes \xi^j = \sum x_1 \otimes x_2 = \sum \xi_i \otimes \xi^j x.$$

A nearly Frobenius structure can be drastically different from a Frobenius structure: there is nothing akin to definition, as the following example illustrates.

Example 2.3.1. Every Frobenius algebra is also a nearly Frobenius algebra.

Example 2.3.2. Let $\mathcal{A}$ be the truncated polynomial algebra in one variable $\mathbb{C}[x]/x^{n+1}$, where x is of degree 2, together with the coproduct given by

$$\Delta_1(x^i) = \Sigma_{k+l=i+n} x^k \otimes x^l.$$

- *The coalgebra axiom*: Let $x^i \in \mathcal{A}$ with $i \geq 0$.

$$\begin{array}{rcl} (\Delta_1\otimes 1)(\Delta_1(x^i)) & = & (\Delta_1\otimes 1)\left(\sum_{k+l=i+n} x^k\otimes x^l\right) & = & \sum_{k+l=i+n}\sum_{r+s=k+n} x^r\otimes x^s\otimes x^l \\ & & & = & \sum_{r+s+l=i+2n} x^r\otimes x^s\otimes x^l \end{array}$$

$$\begin{array}{rcl} (1\otimes\Delta_1)(\Delta_1(x^i)) & = & (1\otimes\Delta_1)\left(\sum_{k+l=i+n} x^k\otimes x^l\right) & = & \sum_{k+l=i+n}\sum_{t+u=l+n} x^k\otimes x^t\otimes x^u \\ & & & = & \sum_{k+t+u=i+2n} x^k\otimes x^t\otimes x^u \end{array}$$

- *The Frobenius identities:* Let $x^i, x^j \in \mathcal{A}$.

$$\Delta_1(x^i x^j) = \Delta_1(x^{i+j}) = \sum_{k+l=i+j+n} x^k\otimes x^l = x^i \sum_{r+l=j+n} x^r\otimes x^l = x^i\Delta_1(x^j)$$

this is because $k \geq i$. For the other hand

$$\Delta_1(x^i x^j) = \Delta_1(x^{i+j}) = \sum_{k+l=i+j+n} x^k \otimes x^l = \left(\sum_{k+s=i+n} x^k \otimes x^s \right) x^j = \Delta_1(x^i)x^j$$

because $l \geq j$.
Then $\mathcal{A}$ is a nearly Frobenius algebra. This structure comes from a Frobenius algebra, because in this case we have a trace map $\varepsilon : \mathcal{A} \to \mathbb{C}$ given by $\varepsilon(x^i) = \delta_{i,n}$.

Not every nearly Frobenius algebra structure comes from a Frobenius algebra structure.

Example 2.3.3. Let $\mathcal{A}$ be the truncated polynomial algebra in one variable $\mathbb{C}[x]/x^{n+1}$, where x is of degree 2, together with the coproduct given by:

$$\Delta_2(x^i) = \Sigma_{k+l=i+n+1} x^k \otimes x^l.$$

As the same before we can prove the coalgebra axiom and the Frobenius identities. But in this case $\mathcal{A}$ has not counit. If we have a counit map $\varepsilon : \mathcal{A} \to \mathbb{C}$ then it satisfies the axiom of counit $m(\varepsilon \otimes 1)(\Delta_2(x^i)) = x^i$ for all $x^i \in \mathcal{A}$. But

$$m(\varepsilon \otimes 1)(\Delta_2(x^i)) = \sum_{k+l=i+n+1} \varepsilon(x^k)x^l,$$

with $l > i$, so $m(\varepsilon \otimes 1)(\Delta_2(x^i)) \neq x^i$. Then this structure does not come from a Frobenius algebra structure.

Example 2.3.4. The Poicaré algebra is a nearly Frobenius algebra when M is a non-compact smooth manifold. Consider the diagram:

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & M \times M \arrow[d, "1\times\Delta"] \\ M \times M \arrow[r, "\Delta\times 1"'] & M \times M \times M \end{tikzcd}$$

Using transversality we have that:

$$(\Delta \times 1)^*(1 \times \Delta)^! = \Delta^! \Delta^*,$$

where $\Delta^* : \mathrm{H}^*(M) \otimes \mathrm{H}^*(M) = \mathrm{H}^*(M \times M) \to \mathrm{H}^*(M)$ is the map induced by the diagonal map in cohomology, and $\Delta^! : \mathrm{H}^*(M) \to \mathrm{H}^*(M) \otimes \mathrm{H}^*(M)$ is the Gysin map of the diagonal map. Therefore

$$(\Delta^* \otimes 1)(1 \otimes \Delta^!) = \Delta^! \Delta^*.$$

Then $\mathrm{H}^*(M)$ is an algebra with a coproduct which is a module homomorphism.
But in this case we can not define a trace because we can not guarantee the existence of the fundamental class $[M]$.

Chapter 3

2D-Topological Field Theory

Topological Field Theories (TFT) are a somewhat recent development in the interface between physics and mathematics. The mathematical interest in them comes from the hope that they will disclose new phenomena, or at least offer efficient organization of invariants like the Jones polynomials, or the Donaldson invariants of 4-manifolds. The interest in physics comes from their value as examples in which extensive calculations are possible. They also shed light on mathematical structures involved in more realistic theories. It seems fair to say that from both points of view this is still an exploration- in a state of flux with the best applications still to come. As mathematicians understand them they were essentially discovered by Witten.

In this work, I am going to concentrate in the most elementary case, that of a topological field theory in dimension two. I will give first an axiomatic approach, valid for all dimensions, following [Ati88]. Then I will specialize this down to dimension two and show that in this case the theory is entirely equivalent to a Frobenius algebra. Subsequently I will give a natural generalization of this theory, called *Topological Field Theory with positive boundary* (due to Cohen and Godin), and show that this theory is equivalent to a nearly Frobenius algebra.

Vaguely speaking, Segal interprets a topological field theory as a functor from a geometric category to a linear category, where we choose the geometric category to be the category whose objects are closed, oriented $(d-1)$-manifolds, and whose morphisms are oriented cobordisms (two such cobordisms being identified if they are diffeomorphic by a diffeomorphism which is the identity on the incoming and outgoing boundaries). The linear category in this case is just the category of complex vector spaces and linear maps, and the only property we require of the functor is that (on objects and morphisms) it takes disjoint unions to tensor products.

3.1 Atiyah's definition of nD-Topological Field Theory

Sir Michael Atiyah in [Ati88] and [Ati90] defined *nD-Topological Field Theory* (nD-TFT) Z^A, a set of the following data:

1. A vector space $\mathrm{Z}^A(\Sigma)$ associated to each $(n-1)$-dimensional closed manifold Σ.

2. A vector $\mathrm{Z}^A(M) \in \mathrm{Z}^A(\partial M)$ associated to each oriented n-dimensional manifold M with boundary ∂M.

3. An isomorphism $\mathrm{Z}(f) : \mathrm{Z}(\Sigma_1) \to \mathrm{Z}(\Sigma_2)$, where $f : \Sigma_1 \to \Sigma_2$ is an orientation preserving diffeomorphism.

This data is subject to the following axioms:

(i) Z^A is *functorial* with respect to orientation-preserving diffeomorphisms of Σ and M.

(ii) Z^A is *involutory*, i.e. $\mathrm{Z}^A(\Sigma^*) = \mathrm{Z}^A(\Sigma)^*$ where Σ^* is Σ with opposite orientation and $\mathrm{Z}^A(\Sigma)^*$ is the dual vector space of $\mathrm{Z}^A(\Sigma)$.

(iii) Z^A is *multiplicative*

$$\mathrm{Z}^A(\Sigma_1 \sqcup \Sigma_2) = \mathrm{Z}^A(\Sigma_1) \otimes \mathrm{Z}^A(\Sigma_2).$$

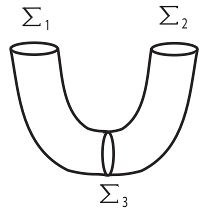
(iv) $\mathrm{Z}^A(\emptyset) = \Bbbk$, where $\emptyset$ is interpreted as the empty $(n-1)$-dimensional closed manifold.

(v) $\mathrm{Z}^A(\emptyset) = 1$, where $\emptyset$ is interpreted as the empty n-dimensional manifold.

(vi) If $f : \Sigma_1 \to \Sigma_2$ is an orientation-preserving diffeomorphism, then $\mathrm{Z}(f) : \mathrm{Z}(\Sigma_1) \to \mathrm{Z}(\Sigma_2)$ is an isomorphism.

These axioms are meant to be understood as follows. The functoriality axiom means first that an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma'$ induces an isomorphism $\mathrm{Z}^A(f) : \mathrm{Z}^A(\Sigma) \to \mathrm{Z}^A(\Sigma')$ and that $\mathrm{Z}^A(gf) = \mathrm{Z}^A(g)\,\mathrm{Z}^A(f)$ for $g : \Sigma' \to \Sigma''$. Also if f extends to an orientation-preserving diffeomorphism $M \to M'$, with $\partial M = \Sigma$ and $\partial M' = \Sigma'$, then $\mathrm{Z}^A(f)$ takes the element $\mathrm{Z}^A(M)$ to $\mathrm{Z}^A(M')$. The multiplicative axiom is clear. Moreover if $\partial M_1 = \Sigma_1 \sqcup \Sigma_3^*$, $\partial M_2 = \Sigma_3 \sqcup \Sigma_2$

and $M = M_1 \sqcup_{\Sigma_3} M_2$ is the manifold obtained by gluing together the common Σ_3-component:



Then we require:

$$Z^A(M) = \langle Z^A(M_1), Z^A(M_2)\rangle$$

where $\langle\, ,\rangle$ denotes the natural pairing from the duality map,

$$Z^A(\Sigma_1) \otimes Z^A(\Sigma_3)^* \otimes Z^A(\Sigma_3) \otimes Z^A(\Sigma_2) \to Z^A(\Sigma_1) \otimes Z^A(\Sigma_2)$$

defined by $a \otimes \varphi \otimes b \otimes c \longmapsto \varphi(b)a \otimes c$. This is a very powerful axiom which implies that $Z^A(M)$ can be computed (in many different ways) by "cutting M in half" along Σ_3.

3.2 Categorical definition of nD-Topological Field Theory

The first step is to define the category of cobordisms that permits us to give a categorical concept of nD-TFT.

Definition 3.2.1. Let Σ_0 and Σ_1 two compact, connected, oriented $(n-1)$-manifolds, we say that they are *cobordat* if there is a n-manifold M, with boundary $\Sigma_1^* \sqcup \Sigma_2$, in this case we say that M is a *n-cobordism* of Σ_1 to Σ_2.

If we fix a positive entire n, we can construct a category $n\widetilde{Cob}$ where the objects are the closed smooth $(n-1)$-dimensional manifolds, and the morphisms are the oriented smooth n-dimensional manifolds(n-cobordism). An obliged question is if the composition of two cobordism of the same dimension is a smooth manifold, the answer is yes up to a smooth process (for reference see [Koc03]).

Let be $nCob' = n\widetilde{Cob}/\sim$ where $\sim$ is the relation up to diffeomorphisms.

Let Σ be a closed submanifold of M of codimension 1. Assume both are oriented. At a point $x \in \Sigma$, let $[v_1, \dots, v_{n-1}]$ be a positive basis for $T_x\Sigma$. A vector $w \in T_xM$ is called a *positive normal* if $[v_1, \dots, v_{n-1}, w]$ is a positive basis for T_xM. Now suppose Σ is a connected component of the boundary of M with an specific orientation; then it makes sense to ask if the positive normal w points inward or it points outward compared to M. Locally the situation is the following, a vector

in $\mathbb{R}^n$ for which we ask if it points inward or outward compared to the half-space $\mathbb{H}^n$ ($\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n \geq 0)\}$). If a positive normal points inward we call Σ an *in-boundary*, and if it points outward we call it an *out-boundary.* To see that this makes sense we have to check that this does not depend on the choice of positive normal (neither the choice of the point $x \in \Sigma$). If some positive normal points inward, it is a fact that every other positive normal at any other point $y \in \Sigma$ points inward as well. This follows from the fact that the normal bundle is a trivial line bundle on Σ. This in turn is a consequence of the assumption that both M and Σ are orientable (see Hirsch [Hir95], theorem 4.4.2.).
Thus the boundary of a manifold M is the union of various in-boundaries and out-boundaries. The in-boundary of M may be empty, and the out-boundary may also be empty. Note that if we reverse the orientation of both M and its boundary Σ, then the notion of what is in-boundary or out-boundary is still the same. We will denote nCob the category $nCob'$ giving an orientation to every object (therefore any cobordism has a direction).

In the next definition we asume that the reader is familiar with the concept of monoidal category, if this is not the case you can read the Appendix 1.

Definition 3.2.2. An *n-dimensional topological field theory* is a symmetric monoidal functor Z^C, from (nCob,$\sqcup$,$\emptyset$,T) to (Vect$_{\Bbbk}$,$\otimes$,$\Bbbk$,σ).

Proposition 3.2.3. *The Atiyah and the categorical definition of TFT coincide.*

Proof. Suppose Z^A is a TFT in the sense of Atiyah, then for M an oriented n-dimensional manifold, the next isomorphism gives the correspondence

$$\begin{array}{cccc} \Psi & \mathrm{Z}^A(\Sigma_1)^* \otimes \mathrm{Z}^A(\Sigma_2) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{Z}^A(\Sigma_1), \mathrm{Z}^A(\Sigma_2)) \\ & \mathrm{Z}^A(M) & \longmapsto & \mathrm{Z}^C(M) \end{array} \tag{3.1}$$

where $\partial M = \Sigma_1^* \sqcup \Sigma_2$. Set $\mathrm{Z}^C(M) := \mathrm{Z}^A(M)$, if we identify the image of the idempotent element $\mathrm{Z}^A(\Sigma \times I)$ with the identity $1_{\mathrm{Z}^A(\Sigma)}$, then we get a functor Z^C : nCob $\to$ Vect$_{\Bbbk}$. This functor is well defined by the *functorial* and *multiplicative* axioms. Moreover, the monoidal structure is given by $\sqcup \to \otimes$ and it is symmetrical since $\mathrm{Z}^C(T_{\Sigma,\Sigma'}) = \sigma_{\mathrm{Z}^C(\Sigma),\mathrm{Z}^C(\Sigma')}$.

Conversely, given a symmetrical monoidal functor Z^C : nCob $\to$ Vect$_{\Bbbk}$, if Σ is a closed $(n-1)$-dimensional smooth manifold, set $\mathrm{Z}^A(\Sigma) := \mathrm{Z}^C(\Sigma)$. For M a n-dimensional oriented smooth manifold we take

$$\mathrm{Z}^A(M) = \mathrm{Z}^C(M')(1) \in \mathrm{Z}^C(\Sigma_{In})^* \otimes \mathrm{Z}^C(\Sigma_{Out}),$$

where M' is M reversing the orientation to the in-boundary. By hypothesis, we have $\mathrm{Z}^C(\emptyset) = \Bbbk$. Moreover, the functor Z^C is multiplicative and it is independent of the cut by the correspondence 9.1. As consequence, the axioms (iii) and (iv) are satisfied. Clearly $Z^A(\emptyset) = \widehat{1} \otimes 1$. The axiom (v) is an implication of $\Psi(Z^A(\emptyset)) = \Psi(\widehat{1} \otimes 1) = \Bbbk$. The axiom (i) is satisfied because Z^C factors through differential homotopy classes. The axioms (ii) is the proposition 3.2.5.

♣

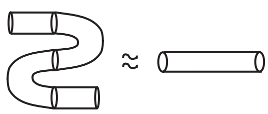
Corollary 3.2.4. *For a topological field theory* Z *of any dimension and* Σ *an object in* nCob*, the image of* Σ *under* Z *is a finite dimensional vector space.*

Proof. Let
$$\langle\, ,\, \rangle_\Sigma : Z(\Sigma) \otimes Z(\Sigma^*) \longrightarrow \Bbbk$$
and
$$\theta_\Sigma : \Bbbk \longrightarrow Z(\Sigma^*) \otimes Z(\Sigma)$$

the maps associated to  and  respectively. Since Z is a TFT, then the next diagram

$$\begin{tikzcd}
Z(\Sigma) \arrow[d, "\simeq"'] & (Z(\Sigma) \otimes Z(\Sigma)^*) \otimes Z(\Sigma) \arrow[r, "\langle,\rangle_\Sigma \otimes id_{Z(\Sigma)}"] & \Bbbk \otimes Z(\Sigma) \arrow[d, "\simeq"] \\
Z(\Sigma) \otimes \Bbbk \arrow[r, "1_{Z(\Sigma)} \otimes \theta_\Sigma"] & Z(\Sigma) \otimes (Z(\Sigma^*) \otimes Z(\Sigma)) \arrow[u, "\simeq"'] & Z(\Sigma)
\end{tikzcd}$$

is the identity map. Graphically



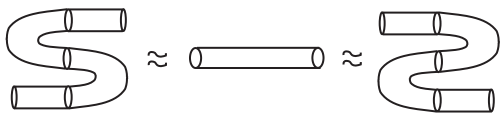
then we have $(\langle\, ,\, \rangle_\Sigma \otimes 1_{Z(\Sigma)}) \circ (1_{Z(\Sigma)} \otimes \theta_\Sigma) = 1_{Z(\Sigma)}$. For $\theta_\Sigma(1) = \sum v_j \otimes w_j$ and $a \in Z(\Sigma)$ the next implications follows

$$\begin{aligned}
a \xrightarrow{\sim} a \otimes 1 = (\langle\, ,\, \rangle_\Sigma \otimes 1_{Z(\Sigma)}) \circ (1_{Z(\Sigma)} \otimes \theta_\Sigma)(a \otimes 1) \\
= (\langle\, ,\, \rangle_\Sigma \otimes 1_{Z(\Sigma)})(\sum a \otimes v_j \otimes w_j) \\
= \sum \langle a, v_j \rangle_\Sigma \otimes w_j \xrightarrow{\sim} \sum \langle a, v_j \rangle_\Sigma w_j.
\end{aligned}$$

Then $a = \sum \langle a, v_j \rangle_\Sigma w_j$, and consequently $\{w_j\}$ generates $Z(\Sigma)$, but since $\Bbbk$ is at least a division ring, we can extract a basis from the generating set. Now since every division ring has the property of invariance of dimension then $Z(\Sigma)$ is finitely generated with $n = rank(A) \leq |\,\{w_j\}\,|$.

♣

When we study this type of structures, it is remarkable how much information they encode. For example the fact that the theory only depends on the topology implies that the next cobordisms



have associated the same linear transformation, which is the identity. In the literature this equivalences are called the *zig-zag* identities. This simple fact has as a result that for any n-dimensional **TQFT** the vector space associated to every object of $nCob$ is finite dimensional. The next proposition proves that there exists a nondegenerate pairing, which consequently entail the construction of the product and the unit for the state space.

Proposition 3.2.5. *Let* Z *be an* n*-dimensional TFT, and* Σ *an* n*-dimensional oriented closed smooth manifold, then* $\mathrm{Z}(\Sigma)$ *is equipped with a nondegenerate pairing and* $\mathrm{Z}(\Sigma^*) \simeq \mathrm{Z}(\Sigma)^*$.

Proof. Similarly to 3.2.4 we have that the next diagrams

$$\begin{tikzcd}
\mathrm{Z}(\Sigma) \arrow[d, "\simeq"'] & (\mathrm{Z}(\Sigma)\otimes \mathrm{Z}(\Sigma^*))\otimes \mathrm{Z}(\Sigma) \arrow[r, "\langle,\rangle_\Sigma \otimes 1_{\mathrm{Z}(\Sigma)}"] & \Bbbk \otimes \mathrm{Z}(\Sigma) \arrow[d, "\simeq"] \\
\mathrm{Z}(\Sigma)\otimes \Bbbk \arrow[r, "1_{\mathrm{Z}(\Sigma)}\otimes\theta_\Sigma"] & \mathrm{Z}(\Sigma)\otimes(\mathrm{Z}(\Sigma^*)\otimes \mathrm{Z}(\Sigma)) \arrow[u, "\simeq"'] & \mathrm{Z}(\Sigma)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Bbbk \otimes \mathrm{Z}(\Sigma^*) \arrow[r, "\theta_\Sigma\otimes 1_{\mathrm{Z}(\Sigma^*)}"] & (\mathrm{Z}(\Sigma^*)\otimes \mathrm{Z}(\Sigma))\otimes \mathrm{Z}(\Sigma^*) \arrow[d, "\simeq"] & \mathrm{Z}(\Sigma^*) \\
\mathrm{Z}(\Sigma^*) \arrow[u, "\simeq"] & \mathrm{Z}(\Sigma^*)\otimes(\mathrm{Z}(\Sigma)\otimes \mathrm{Z}(\Sigma^*)) \arrow[r, "1_{\mathrm{Z}(\Sigma^*)}\otimes\langle,\rangle_\Sigma"] & \mathrm{Z}(\Sigma^*)\otimes\Bbbk \arrow[u, "\simeq"']
\end{tikzcd}$$

are the identity maps of $\mathrm{Z}(\Sigma)$ and $\mathrm{Z}(\Sigma^*)$ respectively, i.e.

$$1_{\mathrm{Z}(\Sigma)} = (\langle\, ,\, \rangle_\Sigma \otimes 1_{\mathrm{Z}(\Sigma)}) \circ (1_{\mathrm{Z}(\Sigma)} \otimes \theta_\Sigma)$$

and

$$1_{\mathrm{Z}(\Sigma)} = (1_{\mathrm{Z}(\Sigma^*)} \otimes \langle\, ,\, \rangle_\Sigma) \circ (\theta_\Sigma \otimes 1_{\mathrm{Z}(\Sigma^*)})$$

An easy algebraic exercise is to prove that $\langle\ ,\ \rangle_\Sigma$ is a nondegenerate pairing and that the map

$$\begin{array}{rccc} \lambda_{\text{left}}: & \mathrm{Z}(\Sigma^*) & \longrightarrow & \mathrm{Z}(\Sigma)^* \\ & y & \longmapsto & \langle x, y\rangle_\Sigma \end{array}$$

is an isomorphism (hint:use that $\mathrm{Z}(\Sigma)$ and $\mathrm{Z}(\Sigma^*)$ are finitely generated).

♣

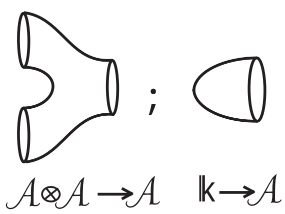
3.3 Relationship between $\mathrm{cFA}_{\Bbbk}$ and 2D-TFT

Theorem 3.3.1. (Folklore) *There is a canonical equivalence of categories*

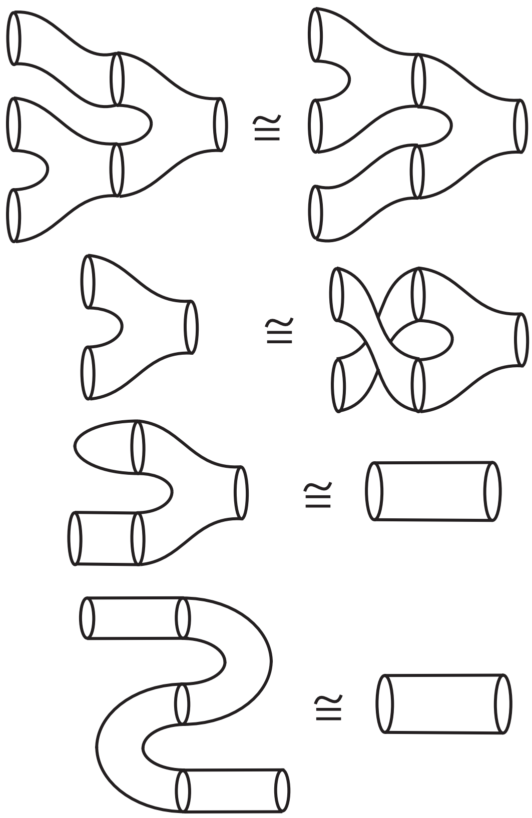
$$\text{2D-TFT}_{\Bbbk} \simeq \mathrm{cFA}_{\Bbbk}$$

where $\mathrm{cFA}_{\Bbbk}$ *is the category of commutative Frobenius algebras.*

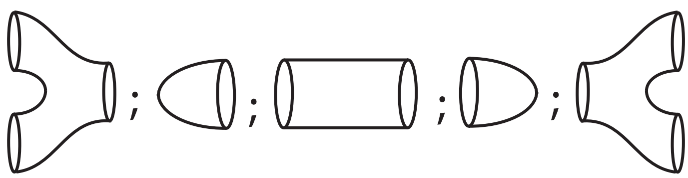
Proof. It is easy to see that a 2-TFT determines a Frobenius algebra. This is the vector space $\mathcal{A}$ associated to the circle. The next cobordisms induce a product $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a unid $u : \Bbbk \to \mathcal{A}$.



The next cobordisms implies respectively the properties of associativity, commutativity, unit and non-degenerate.



We need to prove that when we have a commutative Frobenius algebra we can assign a well defined functor from $2\,\mathrm{Cob}$ to $\mathrm{Vect}_{\Bbbk}$, for this first we note that the category is generated under composition and disjoint unions by the next five elementary cobordisms,



Moreover every decomposition in elementary cobordisms is done by a Morse function and every decomposition of a cobordism depends of all its decomposition in elementary cobordisms, where we mean by a Morse function the one which every critical point is of Morse type and all its critical values are different. The construction of a well defined functor is done if there is some way we can join any pair of Morse functions of a specific cobordism. Two Morse functions can always be connected by a good path in which every element is a Morse function except for a finite set which belongs to one of the two following cases:

1. The function has one degenerate critical point where in local coordinates (x, y) it has the form $\pm x^2 + y^3$.

2. Only two critical values of Morse type coincide.

It is understood that in any of the two cases the other critical values are different (for the case 1, they are even different to the degenerate critical point) and of Morse type. The first case is solved by the unit and counit axioms, for the second we used the identity for the Euler number

$$\chi = \sum (-1)^\lambda c_\lambda$$

with c_λ the number of critical points of index λ of its Morse function. Since every elementary cobordism has at most a critical point of index 0, 1 or 2; then for the case $\chi = 2$ the cobordism corresponding to the two critical values has Euler number -2, 0 or 2. When $\chi = 0$ or 2 the only relevant possibilities are the cylinder and the sphere while for $\chi = -2$ it is just a torus with two holes or the sphere with four holes. In the case $(1, 1, 1)$ (one entry, genus one and one exit) there is nothing to check, because, though a torus with two holes can be cut into two pair of pants by many different isotopy classes of cuts, there is only one possible composite cobordism, and we have only one possible composite map

$$\mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

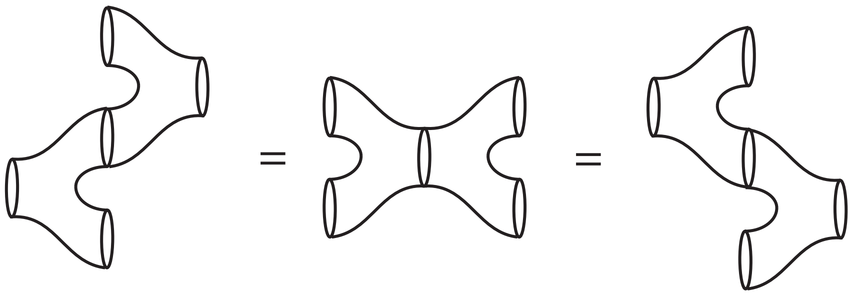
Note that the coproduct is just

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\Delta"] \arrow[d, "\lambda"'] & \mathcal{A} \otimes \mathcal{A} \\
\mathcal{A}^* \arrow[r, "m^*"'] & \mathcal{A}^* \otimes \mathcal{A}^* \arrow[u, "\lambda^{-1} \otimes \lambda^{-1}"']
\end{tikzcd}$$

where λ is the corresponding Frobenius isomorphism between $\mathcal{A}$ and its dual and for a commutative algebra is easy to prove that

$$\Delta(a) = \sum a e_i \otimes e_i^{\#} = \sum e_i \otimes e_i^{\#} a$$

with $\{e_i\}$ a basis for $\mathcal{A}$ and $\#$ denotes the dual. For the sphere with four holes when we have $(3, 0, 1)$ and $(1, 0, 3)$ these cases are covered by the associativity of the product and coassociative of the coproduct respectively. Finally for $(2, 0, 2)$ it is enough to prove that is well defined for all the possible pants decomposition; it is known that for a compact surface (m, g, n) every pants decomposition has $3g - 3 + m + n$ simple closed curves which cut the surface in $2g - 2 + m + n$ pairs of pants, hence for this case we have only a curve dividing in two pair of pants and then the only possibilities are



but this is clearly the Abrams theorem 2.1.5.

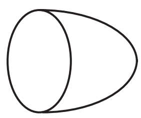
♣

3.4 TFT with positive boundary

Definition 3.4.1. A *Topological field theory with positive boundary* (TFT_+) is defined at the same at TFT but with the difference that we can write the maps of the form

$$\Psi_\Sigma : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes m}$$

only if $m > 0$. In other words we can write the linear map Ψ_Σ only if Σ has at least one outgoing boundary component. In particular, there is not linear form associated to the following surface: namely, we no longer have a trace.



$\theta\colon \mathcal{A} \longrightarrow \Bbbk$

Theorem 3.4.2. *The category* $\mathrm{nFA}_\Bbbk$ *of commutative nearly Frobenius algebras is equivalent to the category* 2D-TFT_+ *of Topological Field Theories with positive boundary.*

We have proved together with Lupercio, Segovia and Uribe this theorem in [GLSU]. Since the results of this thesis do not depend on this fact we will omit the proof here, we will only mention that it is a subtle modification of the argument given above for the usual Folk theorem.

3.4.1 String Topology

String topology is the study of the (differential and algebraic) topological properties of the spaces of smooth paths and of smooth loops on a manifold, which are themselves infinite-dimensional manifolds. The development of string topology is strongly driven by analogies with string theory in physics, which is a theory of quantum gravitation, where vibrating strings play the role of particles. As we will see, string topology provides us with a family of TFTs, one for for each manifold M.

Let M be a smooth, orientable manifold of dimension n. The *space of free loop space* is

$$\mathrm{LM} = \{\alpha : S^1 \to M\}$$

where every loop is assumed piecewise smooth.
Chas and Sullivan in [CS99] proved the next result.

Theorem 3.4.3 (Chas and Sullivan, 1999)**.** *Let M be a compact, closed, smooth, orientable manifold of dimension d. There is a commutative and associative product*

$$\mathrm{H}_p(\mathrm{LM}) \otimes \mathrm{H}_q(\mathrm{LM}) \to \mathrm{H}_{p+q-d}(\mathrm{LM})$$

- *making $\mathbb{H}_*(\mathrm{LM}) := \mathrm{H}_{*+d}(\mathrm{LM})$ an associtive, commutative algebra and*
- *is compatible with the intersection product on $\mathrm{H}_*(M)$, i.e., the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{H}_p(\mathrm{LM}) \otimes \mathrm{H}_q(\mathrm{LM}) \arrow[r] \arrow[d, "\mathrm{ev}_* \otimes \mathrm{ev}_*"'] & \mathrm{H}_{p+q-d}(\mathrm{LM}) \arrow[d, "\mathrm{ev}_*"] \\
\mathrm{H}_p\, M \otimes \mathrm{H}_q\, M \arrow[r] & \mathrm{H}_{p+q-d}\, M
\end{tikzcd}$$

In this section we present a generalization of this result when M is not necessarily compact. Moreover, we will prove that $\mathrm{H}_*(\mathrm{LM})$ is a nearly Frobenius algebra. In particular, using the Folklore Theorem we have an example of a 2D-TFT with positive boundary. In the next chapter, we will give an extension of the string theory that permits us to give a new example of 2D Open-Closed TFT with positive boundary.

Algebraic Structure

The Loop product: The Chas-Sullivan "loop product" in the homology (over a field $\Bbbk$ of zero characteristic) of the free loop space of a closed oriented d-manifold,

$$\mu : \mathrm{H_p(LM)} \otimes \mathrm{H_q(LM)} \to \mathrm{H_{p+q\text{-}d}(LM)}$$

is defined as follows.

Let $\mathrm{Map}(8, M)$ be the mapping space from the figure 8 (i.e the wedge of two circles) to the manifold M. Chose a basis point in the circle, notice that $\mathrm{Map}(8, M)$ can be viewed as the subspace of $\mathrm{LM} \times \mathrm{LM}$ consisting of those pair of loops that agree at the basepoint. In other words, there is a pullback square

$$\begin{tikzcd} \mathrm{Map}(8, M) \arrow[r, "\mathrm{e}"] \arrow[d, "\mathrm{ev}"'] & \mathrm{LM} \times \mathrm{LM} \arrow[d, "\mathrm{ev} \times \mathrm{ev}"] \\ M \arrow[r, "\Delta"'] & M \times M, \end{tikzcd} \tag{3.2}$$

where $\mathrm{ev} : \mathrm{LM} \to M$ is the fibration given by evaluating a loop at the basepoint. The map $\mathrm{ev} : \mathrm{Map}(8, M) \to M$ evaluates the map at the crossing point on the figure 8. Since $\mathrm{ev} \times \mathrm{ev}$ is a fibre bundle, $\mathrm{e} : \mathrm{Map}(8, M) \hookrightarrow \mathrm{LM} \times \mathrm{LM}$ can be viewed as a codimension d embedding, with normal bundle $\mathrm{ev}^*(\nu_\Delta) \cong \mathrm{ev}^*(TM)$.

The existence of this pullback diagram, of fiber bundles, means that there is a natural tubular neighborhood of the embedding $\mathrm{e} : \mathrm{Map}(8, M) \to \mathrm{LM} \times \mathrm{LM}$. Namely, the inverse image of a tubular neighborhood of the diagonal embedding $\Delta : M \to M \times M$. That is, $\eta_\mathrm{e} = (\mathrm{ev} \times \mathrm{ev})^{-1}(\eta_\Delta)$. Because ev is a locally trivial fibration, the tubular neighborhood η_e is homeomorphic to the total space of the normal bundle $\mathrm{ev}^*(TM)$. This induces a homeomorphism of the quotient space to the Thom space,

$$(\mathrm{LM} \times \mathrm{LM})/((\mathrm{LM} \times \mathrm{LM}) - \eta_\mathrm{e}) \cong (\mathrm{Map}(8, M))^{\mathrm{ev}^*(TM)}.$$

Combining this homeomorphism with the projection onto this quotient space, defines a Thom-collapse map

$$\tau_\mathrm{e} : \mathrm{LM} \times \mathrm{LM} \to (\mathrm{Map}(8, M))^{\mathrm{ev}^*(TM)}.$$

For notation, we refer the Thom space of the pullback bundle $\mathrm{ev}^*(TM) \to \mathrm{Map}(8, M)$ as $\mathrm{Map}(8, M)^{TM}$.

There is a functorial construction in homology which goes in the wrong direction. This is called the *Gysin map* or *Umkher map*, see [CK09]. We define an umkehr map,

$$\mathrm{e}_! : H_*(\mathrm{LM} \times \mathrm{LM}) \xrightarrow{\tau_\mathrm{e}} H_*(\mathrm{Map}(8, M)^{TM}) \xrightarrow{\cap u} H_{*-d}(\mathrm{Map}(8, M))$$

where $u \in H^d(\mathrm{Map}(8, M)^{TM})$ is the Thom class.

Chas and Sullivan also observed that given a map from the figure 8 to M then one obtains a loop in M by starting at the intersection point, traversing the top loop of the 8, and then traversing the bottom loop, this defines a map

$$\rho : \mathrm{Map}(8, M) \to \mathrm{LM}\,.$$

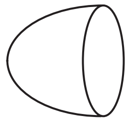


Figure 3.1: `The disc` D

Definition 3.4.4. We consider the next diagram

$$\begin{tikzcd} & \mathrm{Map}(8,M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & \\ \mathrm{LM}\times\mathrm{LM} & & \mathrm{LM} \end{tikzcd}$$

where e is defined in the diagram 3.2. The loop product in the homology of the loop space is the composition

$$\eta : H_*(\mathrm{LM})\otimes H_*(\mathrm{LM}) \to H_*(\mathrm{LM}\times\mathrm{LM}) \xrightarrow{\mathrm{e}_!} H_{*-d}(\mathrm{Map}(8,M)) \xrightarrow{\rho_*} H_{*-d}(\mathrm{LM})$$

The Loop coproduct: Notice that $\mathrm{Map}(8,M)$ can be viewed as the subspace of LM consisting in a loop that agree at the basepoint. In other words, there is a pullback square

$$\begin{tikzcd} \mathrm{Map}(8,M) \arrow[r, "\rho"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{LM} \arrow[d, "\mathrm{ev}_0\times\mathrm{ev}_{\frac{1}{2}}"] \\ M \arrow[r, "\Delta"'] & M\times M \end{tikzcd}$$

where $\mathrm{ev}_0\times\mathrm{ev}_{\frac{1}{2}} : \mathrm{LM}\to M\times M$ is the map given by evaluating a loop at 0 and $\frac{1}{2}$. Then we can define the umkehr map

$$\rho_! : \mathrm{H}_*(\mathrm{LM}) \xrightarrow{\tau_\rho} \mathrm{H}_*(\mathrm{Map}(8,M)^{TM}) \xrightarrow{\cap u} \mathrm{H}_{*-d}(\mathrm{Map}(8,M)).$$

Definition 3.4.5. The loop coproduct for the homology of the loop space is the composition

$$\Delta : \mathrm{H}_*(\mathrm{LM}) \xrightarrow{\rho_!} \mathrm{H}_{*-d}(\mathrm{Map}(8,M)) \xrightarrow{\mathrm{e}_*} \mathrm{H}_*(\mathrm{LM}\times\mathrm{LM}) = \mathrm{H}_*(\mathrm{LM})\otimes\mathrm{H}_*(\mathrm{LM}).$$

The unit and counit: Consider de disk D as a cobordism with zero incoming boundary component and one outgoing boundary component (see Figure 3.1). The restriction map to the zero incoming boundary is the map

$$\rho_{in} : \mathrm{Map}(D,M) \to \mathrm{Map}(\emptyset,M) = \text{point}.$$

Notice that the disc D is homotopy equivalent to a point, then the smooth mapping space $\mathrm{Map}(D, M)$ is homotopy equivalent to the manifold M. The umkehr map in this setting is

$$(\rho_{in})_! : \mathrm{H}_*(\mathrm{point}) \to \mathrm{H}_{*+\mathrm{d}}(\mathrm{M}),$$

which is defined by sending the generator to $[M] \in \mathrm{H}_\mathrm{d}(\mathrm{M})$. The restriction to the outgoing boundary component is the map

$$\rho_{out} : M \simeq \mathrm{Map}(D, M) \to \mathrm{LM},$$

which is given by $\iota : M \hookrightarrow \mathrm{LM}$. Thus the unit is given by

$$u : (\rho_{out})_* \circ (\rho_{in})_! = \iota_* \circ (\rho_{in})_! : \mathrm{H}_*(\mathrm{point}) \to \mathrm{H}_{*+\mathrm{d}}(\mathrm{M}) \to \mathrm{H}_{*+\mathrm{d}}(\mathrm{LM}),$$

which sends the generator to the image of the fundamental class.

The reason of the nonexistence of a counit in the Frobenius structure is formally the same to the existence of a unit. Namely, for this operation one must consider D as a cobordism with one incoming boundary, and zero outgoing boundary components. In this setting the role of the restriction maps ρ_{in} and ρ_{out} are reversed, and one obtains the diagram

$$\begin{tikzcd}
\mathrm{Map}(\emptyset, \mathrm{M}) \arrow[d, "\parallel"'] & \mathrm{Map}(\mathrm{D}, \mathrm{M}) \arrow[l, "\rho_{out}"'] \arrow[r, "\rho_{in}"] \arrow[d, "\parallel"] & \mathrm{LM} \arrow[d, "\parallel"] \\
\mathrm{point} & M \arrow[l, "\epsilon"] \arrow[r, "\iota"'] & \mathrm{LM}
\end{tikzcd}$$

where $\epsilon : M \to \mathrm{point}$ is the constant map. Now notice that in this case, the embedding $\mathrm{Map}(\mathrm{D}, \mathrm{M}) \hookrightarrow \mathrm{LM}$ is of infinite codimension for our knowledge we do not know how to define the umkher map. Ando and Morava, in [AM99], argument that if one has a theory where this umkehr map exists, one would need that the Euler class of the normal bundle $e(\nu(\iota)) \in \mathrm{H}^*(M)$ is invertible.

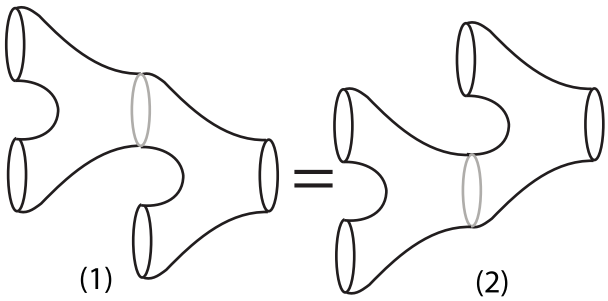
Verification of the axioms of nearly Frobenius algebra

That $\mathrm{H}_*(\mathrm{LM})$ is a nearly Frobenius algebra was first proved by Cohen and Godin in [CG04]. The proof we propose here is sufficiently different to be of independent interest.

We will use the lema 9.2.3 to prove of the next theorem. This lema appears in Appendix 2 below, and is based in a result of Quillen's that appears in [Qui71], (Proposition 3.3).

Theorem 3.4.6. $\mathrm{H}_*(\mathrm{LM})$ *is a nearly Frobenius algebra.*

Proof. 1. **Associativity of the loop product**



The loop product is defined by the next diagram.

$$\begin{tikzcd}
 & \mathrm{Map}(8,\mathrm{M}) \arrow[dl, "e"'] \arrow[dr, "\rho"] & \\
\mathrm{LM}\times\mathrm{LM} & & \mathrm{LM}
\end{tikzcd}$$

The associativity of the product is represented by the next two diagrams

(1)

$$\begin{tikzcd}
 & & \mathrm{Map}(\clubsuit,\mathrm{M}) \arrow[dl, "j"'] \arrow[dr, "i"] \arrow[ddll, bend right, "\varphi"'] \arrow[ddrr, bend left, "\psi"] & & \\
 & \mathrm{Map}(8,M)\times LM \arrow[dl, "e\times 1"'] \arrow[dr, "\rho\times 1"] & & \mathrm{Map}(8,M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & \\
\mathrm{LM}\times\mathrm{LM}\times\mathrm{LM} & & \mathrm{LM}\times\mathrm{LM} & & \mathrm{LM}
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathrm{Map}(\clubsuit,\mathrm{M}) \arrow[dl, "j'"'] \arrow[dr, "i"] \arrow[ddll, bend right, "\varphi"'] \arrow[ddrr, bend left, "\psi"] & & \\
 & \mathrm{LM}\times\mathrm{Map}(8,M) \arrow[dl, "1\times e"'] \arrow[dr, "1\times\rho"] & & \mathrm{Map}(8,M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & \\
\mathrm{LM}\times\mathrm{LM}\times\mathrm{LM} & & \mathrm{LM}\times\mathrm{LM} & & \mathrm{LM}
\end{tikzcd}$$

We will use the Quillen result to prove this property.

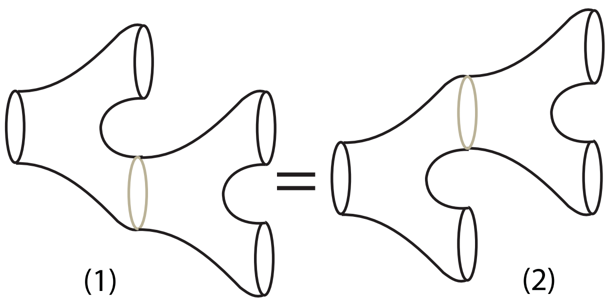
$$\begin{tikzcd}
\varphi^*(TM) = \kappa^* \mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[d, "i"] & \\
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(8, M) \arrow[r, "e"] \arrow[d, "\mathrm{ev}"'] & \mathrm{LM} \times \mathrm{LM} \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

where $\varphi = \mathrm{ev} \circ \kappa$ and $\mathrm{ev}^*(TM)$ is the normal bundle of i.

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[r, "j"] \arrow[d, "\mathrm{ev}"] & \mathrm{Map}(8, M) \times \mathrm{LM} \arrow[d, "\mathrm{ev} \times \mathrm{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

(1) We have $0 \to \mathrm{ev}^*(TM) \to \varphi^*(TM) \to F_1 \to 0$ is an exact sequence. Note that $\varphi = \mathrm{ev}$, then $F_1 = 0$. Similarly, for (2) we have $F_2 = 0$, then $e(F_1) = e(F_2)$.

2. **Coassociativity of the coproduct**



(1) (2)

(1)

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{Map}(\clubsuit, \mathrm{M}) \arrow[dll, bend right, "\psi"'] \arrow[dl, "i"'] \arrow[dr, "j'"] \arrow[drr, bend left, "\varphi"] & & \\
 & \mathrm{Map}(8, M) \arrow[dl, "\rho"] \arrow[dr, "e"] & & \mathrm{LM} \times \mathrm{Map}(8, M) \arrow[dl, "1\times\rho"'] \arrow[dr, "1\times e"] & \\
\mathrm{LM} & & \mathrm{LM} \times \mathrm{LM} & & \mathrm{LM} \times \mathrm{LM} \times \mathrm{LM}
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & \mathrm{Map}(\clubsuit,\mathrm{M}) \arrow[dl, "i"'] \arrow[dr, "j"] \arrow[dll, bend right, "\psi"'] \arrow[drr, bend left, "\varphi"] & & \\
\mathrm{Map}(8,M) \arrow[d, "\rho"'] \arrow[dr, "e"] & & \mathrm{Map}(8,M)\times LM \arrow[dl, "\rho\times 1"'] \arrow[dr, "e\times 1"] & \\
\mathrm{LM} & \mathrm{LM}\times\mathrm{LM} & & \mathrm{LM}\times\mathrm{LM}\times\mathrm{LM}
\end{tikzcd}$$

(1) In the first case we have:

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit,M) \arrow[r, "i"] \arrow[d, "\mathrm{ev}"] & \mathrm{Map}(8,M) \arrow[d, "\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
j'^*(\mathrm{ev}\times\mathrm{ev})^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit,M) \arrow[d, "j'"] & \\
 & \mathrm{LM}\times\mathrm{Map}(8,M) \arrow[r, "1\times e"] \arrow[d, "\mathrm{ev}\times\mathrm{ev}"'] & \mathrm{LM}\times\mathrm{LM} \arrow[d, "\mathrm{ev}\times\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}}"] \\
TM \arrow[r, dashed] & M\times M \arrow[r, "1\times\Delta"] & M\times M\times M
\end{tikzcd}$$

Then, we have the next exact sequence $0\to\mathrm{ev}^*(TM)\to r_2^*(\mathrm{ev}\times\mathrm{ev})^*(TM)\to F_1\to 0$. We conclude $F_1=0$ since $\mathrm{ev}^*(TM)=r_2^*(\mathrm{ev}\times\mathrm{ev})^*(TM)$.

(2) In the other case there are the diagrams

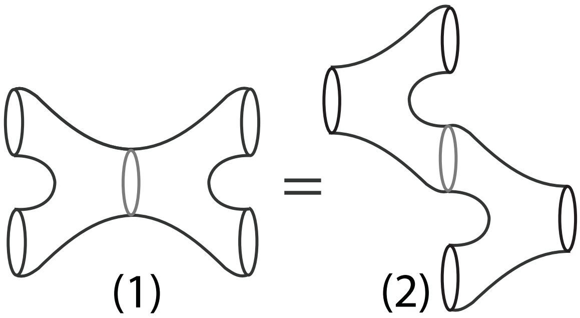
$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit,M) \arrow[r, "i"] \arrow[d, "\mathrm{ev}"] & \mathrm{Map}(8,M) \arrow[d, "\mathrm{ev}_0\times\mathrm{ev}_{\frac{1}{2}}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
j^*(\mathrm{ev}\times\mathrm{ev})^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit,M) \arrow[d, "j"] & \\
 & \mathrm{Map}(8,M)\times\mathrm{LM} \arrow[r, "e\times 1"] \arrow[d, "\mathrm{ev}\times\mathrm{ev}"'] & \mathrm{LM}\times\mathrm{LM} \arrow[d, "\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"] & M\times M\times M
\end{tikzcd}$$

Then we have the exact sequence $0 \to \mathrm{ev}^*(TM) \to j_2^*(\mathrm{ev}\times\mathrm{ev})^*(TM) \to F_2 \to 0$. Since $\mathrm{ev}^*(TM) = j_2^*(\mathrm{ev}\times\mathrm{ev})^*(TM)$ then $F_2 = 0$.

3. **Abrams condition**



(1)

$$\begin{tikzcd}
 & & \mathrm{Map}(\clubsuit, \mathrm{M}) \arrow[dl, "i'"'] \arrow[dr, "i"] \arrow[ddll, bend right, "\psi"'] \arrow[ddrr, bend left, "\varphi"] & & \\
 & \mathrm{Map}(8, M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & & \mathrm{Map}(8, M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & \\
\mathrm{LM}\times\mathrm{LM} & & \mathrm{LM} & & \mathrm{LM}\times\mathrm{LM}
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathrm{Map}(\clubsuit, \mathrm{M}) \arrow[dl, "j'"'] \arrow[dr, "j"] \arrow[ddll, bend right, "\psi"'] \arrow[ddrr, bend left, "\varphi"] & & \\
 & \mathrm{LM}\times\mathrm{Map}(8, M) \arrow[dl, "1\times\rho"'] \arrow[dr, "1\times e"] & & \mathrm{Map}(8, M)\times\mathrm{LM} \arrow[dl, "e\times 1"'] \arrow[dr, "\rho\times 1"] & \\
\mathrm{LM}\times\mathrm{LM} & & \mathrm{LM}\times\mathrm{LM}\times\mathrm{LM} & & \mathrm{LM}\times\mathrm{LM}
\end{tikzcd}$$

In the first diagram we have

$$\begin{tikzcd}
i^*\,\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[d, "i"] & \\
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(8, M) \arrow[r, "e"] \arrow[d, "\mathrm{ev}"'] & \mathrm{LM} \arrow[d, "\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[r, "i'"] \arrow[d, "\mathrm{ev}"] & \mathrm{Map}(8, M) \arrow[d, "\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

Then we have the exact sequence $0 \to \mathrm{ev}^*(TM) \to \kappa'^* \mathrm{ev}^*(TM) \to F_1 \to 0$. Since $\mathrm{ev} \circ \kappa' = \mathrm{ev}$ then $F_1 = 0$.

For the second diagram

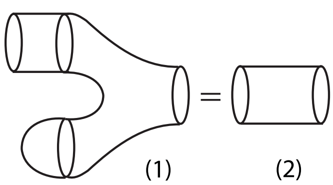
$$\begin{tikzcd}
j^*(\mathrm{ev}\times\mathrm{ev})^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[d, "j"] & \\
(\mathrm{ev}\times\mathrm{ev})^*(TM) \arrow[r, dashed] & \mathrm{Map}(8, M)\times \mathrm{LM} \arrow[r, "e\times 1"] \arrow[d, "\mathrm{ev}\times\mathrm{ev}"'] & \mathrm{LM}\times\mathrm{LM}\times\mathrm{LM} \arrow[d, "\mathrm{ev}\times\mathrm{ev}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"] & M\times M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{Map}(\clubsuit, M) \arrow[r, "j'"] \arrow[d, "\mathrm{ev}"] & \mathrm{LM}\times\mathrm{Map}(8, M) \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

Therefore we have the exact sequence $0 \to \mathrm{ev}^*(TM) \to j^*(\mathrm{ev}\times\mathrm{ev})^*(TM) \to F_2 \to 0$. Note that $\mathrm{ev}^*(TM) = j^*(\mathrm{ev}\times\mathrm{ev})^*(TM)$, then $F_2 = 0$.

4. **Unit axiom**



(1)

$$\begin{tikzcd}
& & \mathrm{LM} \arrow[dl, "\epsilon_0\times\mathrm{id}"'] \arrow[dr, "k"] \arrow[ddll, bend right, "\psi"'] \arrow[ddrr, bend left, "\varphi"] & & \\
& M\times\mathrm{LM} \arrow[dl, "\rho_{in}\times 1"] \arrow[dr, "r\times 1"] & & \mathrm{Map}(8, M) \arrow[dl, "\rho_{in}"'] \arrow[dr, "\rho_{out}"] & \\
pt\times\mathrm{LM} & & \mathrm{LM}\times\mathrm{LM} & & \mathrm{LM}
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathrm{LM} \arrow[dl, "id"'] \arrow[dr, "id"] \arrow[ddll, bend right, "\psi"'] \arrow[ddrr, bend left, "\varphi'"] & & \\
 & \mathrm{LM} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & & \mathrm{LM} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
\mathrm{LM} & & \mathrm{LM} & & \mathrm{LM}
\end{tikzcd}$$

First, we note that φ and φ' are homotopic maps, then $\varphi_* = \varphi'_*$.
In (1) we have

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{LM} \arrow[r, "\epsilon_0 \times \mathrm{id}"] \arrow[d, "\mathrm{ev}"'] & M \times \mathrm{LM} \arrow[d, "id \times \mathrm{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{ev}^*(TM) \arrow[r, dashed] & \mathrm{LM} \arrow[d, "k"] & \\
 & \mathrm{Map}(8, M) \arrow[r, "\rho_{in}"] \arrow[d, "\mathrm{ev}"'] & \mathrm{LM} \times \mathrm{LM} \arrow[d, "\mathrm{ev} \times \mathrm{ev}"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

Then $F_1 = 0$. In the second diagram is trivial to prove that $F_2 = 0$.

♣

Chapter 4

2D Open-Closed Topological Field Theory

A 2DO-CTFT is a generalization of a 2DTFT. Now the category of cobordism is modified in the sense the boundary objects are compact, oriented, one-manifolds, X, together with a labeling of the components of the boundary, ∂X, by objects of a $\mathbb{C}$-linear category $\mathcal{B}$, see figure 4.1. The morphisms generalize the usual notion of a cobordism between manifolds with boundary, but with the additional data of the labeling category $\mathcal{B}$. A cobordism Σ_{X_1,X_2} between two objects X_1 and X_2 is an oriented surface Σ, whose boundary is partitioned into three parts: the incoming boundary $\partial_{in}\Sigma$ which is identified with X_1, the outgoing boundary $\partial_{out}\Sigma$ which is identified with X_2, and the remaining part of the boundary is referred as the "free part" $\partial_{free}\Sigma$ whose path components are labeled by objects of $\mathcal{B}$. Note that $\partial_{free}\Sigma$ is a cobordism between ∂X_1 and ∂X_2, which preserves the labeling, see figure 4.2.
A monoidal functor from this category to the category of complex vector spaces will be called a (1+1)-dimensional *open-closed topological fiel theory.* We write $\mathcal{A}$ for the vector space associated to the standard circle S^1, and $\mathcal{O}_{ab} = \mathrm{Hom}(a, b)$ for the vector space associated to the interval $[0, 1]$, with ends labeled by $a, b \in \mathrm{Obj}(\mathcal{B})$.

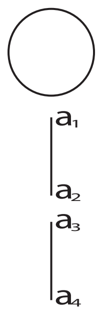


Figure 4.1: `A one manifold with labels` $a_i \in \mathrm{Obj}(\mathcal{B})$.

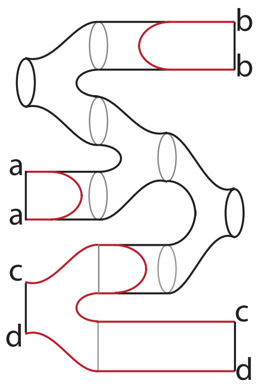


Figure 4.2: `An open-closed cobordism.`

4.1 Algebraic structure

A *Frobenius structure* consists of the following algebraic data:

1. $(\mathcal{A}, \Delta_{\mathcal{A}}, 1_{\mathcal{A}})$ is a commutative Frobenius algebra.

2. A $\mathbb{C}$-linear category $\mathcal{B}$, where $\mathscr{O}_{ab} = \mathrm{Hom}(a, b)$ for $a, b \in \mathscr{B}$.

2a. With associative linear maps η^b_{ac} and units u_a

$$\eta^b_{ac} : \mathscr{O}_{ab} \otimes \mathscr{O}_{bc} \to \mathscr{O}_{ac}, \tag{4.1}$$

$$u_a : \mathbb{C} \to \mathscr{O}_{aa}, \tag{4.2}$$

2b. The spaces $\mathscr{O}_{aa}$ have nondegenerate traces

$$\Theta_a : \mathscr{O}_{aa} \to \mathbb{C} \tag{4.3}$$

In particular, each $\mathscr{O}_{aa}$ is not necessarily a commutative Frobenius algebra.

2c. Moreover,

$$\begin{array}{ccccc} \mathscr{O}_{ab} \otimes \mathscr{O}_{ba} & \longrightarrow & \mathscr{O}_{aa} & \xrightarrow{\Theta_a} & \mathbb{C} \\ \mathscr{O}_{ba} \otimes \mathscr{O}_{ab} & \longrightarrow & \mathscr{O}_{bb} & \xrightarrow{\Theta_b} & \mathbb{C} \end{array} \tag{4.4}$$

are perfect pairings with

$$\Theta_a(\psi_1\psi_2) = \Theta_b(\psi_2\psi_1) \tag{4.5}$$

for $\psi_1 \in \mathscr{O}_{ab}$, and $\psi_2 \in \mathscr{O}_{ba}$.

3. There are linear maps:

$$\iota_a : \mathcal{A} \to \mathscr{O}_{aa}, \quad \iota^a : \mathscr{O}_{aa} \to \mathcal{A} \tag{4.6}$$

such that

3a. ι_a is an algebra homomorphism

$$\iota_a(\phi_1\phi_2) = \iota_a(\phi_1)\iota_a(\phi_2), \tag{4.7}$$

3b. the identity is preserved

$$\iota_a(1_{\mathcal{A}}) = 1_a. \tag{4.8}$$

3c. Moreover, ι_a is central in the sense that

$$\iota_a(\phi)\psi = \psi\iota_b(\phi), \tag{4.9}$$

for all $\phi \in \mathcal{A}$ and $\psi \in \mathscr{O}_{ab}$.

3d. ι_a and ι^a are adjoint

$$\Theta_{\mathcal{A}}(\iota^a(\psi)\phi) = \Theta_a(\psi\iota_a(\phi)).$$

3e. We define the map $\pi^a_b : \mathscr{O}_{aa} \to \mathscr{O}_{bb}$ as follows. Since $\mathscr{O}_{ab}$ and $\mathscr{O}_{ba}$ are in duality (using θ_a or θ_b), if we let ψ_μ be a basis for $\mathscr{O}_{ba}$ then there is a dual basis ψ^μ for $\mathscr{O}_{ab}$. Then we set

$$\pi^a_b(\psi) = \sum_\mu \psi_\mu\psi\psi^\mu, \tag{4.10}$$

and the *Cardy condition* is

$$\pi^a_b = \iota_b \circ \iota^a. \tag{4.11}$$

4.2 Pictorial representation

For the case of a closed 2D TFT the Frobenius structure is provided by the diagrams in fig. 4.3. The consistency conditions follow from fig. 4.4. In the open case, entirely analogous considerations lead to the construction of a non necessarily commutative Frobenius algebra in the open sector. The basic data are summarized in fig. 4.5. The fact that the traces are dual pairings follows from fig 4.6. The new ingredients in the open-closed theory are the open to closed and closed to open transitions. in 2D TFT these are the maps ι_a, ι^a. they are represented by fig. 4.7. There are five new consistency conditions associated with the open-closed transitions. They are illustrated in fig. 4.8 to fig 4.13.

Theorem 4.2.1. *A 2-dimensional Open-Closed Topological Fiel Theory defines and is defined by a Frobenius structure..*

The proof of this theorem is a little more complicate that the Folklore theorem, because we have to study more possibilities. You can see the proof in [MS06].

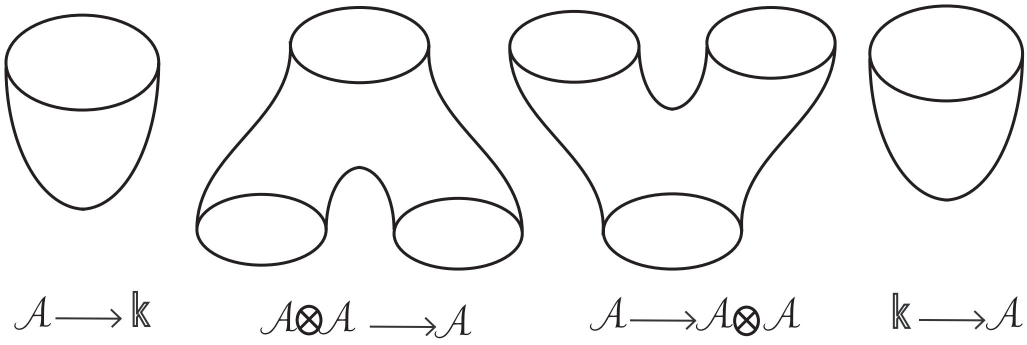


Figure 4.3: `Four diagrams defining the Frobenius structure.`

4.3 Example: Representations of a finite group G

A simple example of an open-closed TFT is the associated to a finite group G. Where the category $\mathscr{B}$ is the category $\mathscr{R}ep(G)$. If $E \in \mathrm{Obj}(\mathscr{R}ep(G))$ the trace $\theta_E : \mathscr{O}_{EE} \to \mathbb{C}$ takes $\psi : E \to E$ to $\frac{1}{|G|}\,\mathrm{tr}(\psi)$.
The algebra $\mathscr{A}$ is the center of the group algebra $\mathbb{C}[G]$ such that

$$
\begin{aligned}
\iota_E : Z(\mathbb{C}[G]) &\to \mathscr{O}_{EE}, \\
\sum_g \alpha_g g &\mapsto \sum_g \alpha_g \rho_g
\end{aligned}
$$

$$
\begin{aligned}
\iota^E : \mathscr{O}_{EE} &\to Z(\mathbb{C}[G]), \\
\psi : E \to E &\mapsto \sum_g \mathrm{tr}(\psi \rho_g|_E) g
\end{aligned}
$$

and the trace

$$
\begin{aligned}
\theta_{Z(\mathbb{C}[G])} : Z(\mathbb{C}[G]) &\to \mathbb{C} \\
\sum_g \alpha_g g &\mapsto \frac{\alpha_1}{|G|}.
\end{aligned}
$$

The next computation proves all the axioms.

1. $(Z(\mathbb{C}[G]), \theta_{Z(\mathbb{C}[G])}, 1_{Z(\mathbb{C}[G])})$ is a Frobenius algebra.
   Let $I \subset \ker(\theta_{Z(\mathbb{C}[G])})$ be an ideal of $(Z(\mathbb{C}[G])$, and $\sum_g \alpha_g g \in I$. Then $\theta_{Z(\mathbb{C}[G])}(\sum_g \alpha_g g) = \frac{\alpha_1}{|G|} = 0$, hence $\alpha_1 = 0$. If $h \in G$ we have $\sum_g \alpha_g g h^{-1} \in I$,thus $\theta_{Z(\mathbb{C}[G])}(\sum_g \alpha_g g h^{-1}) = \frac{\alpha_h}{|G|} = 0$. For this reason $\alpha_h = 0$ for any $h \in G$. Then $I = \{0\}$.

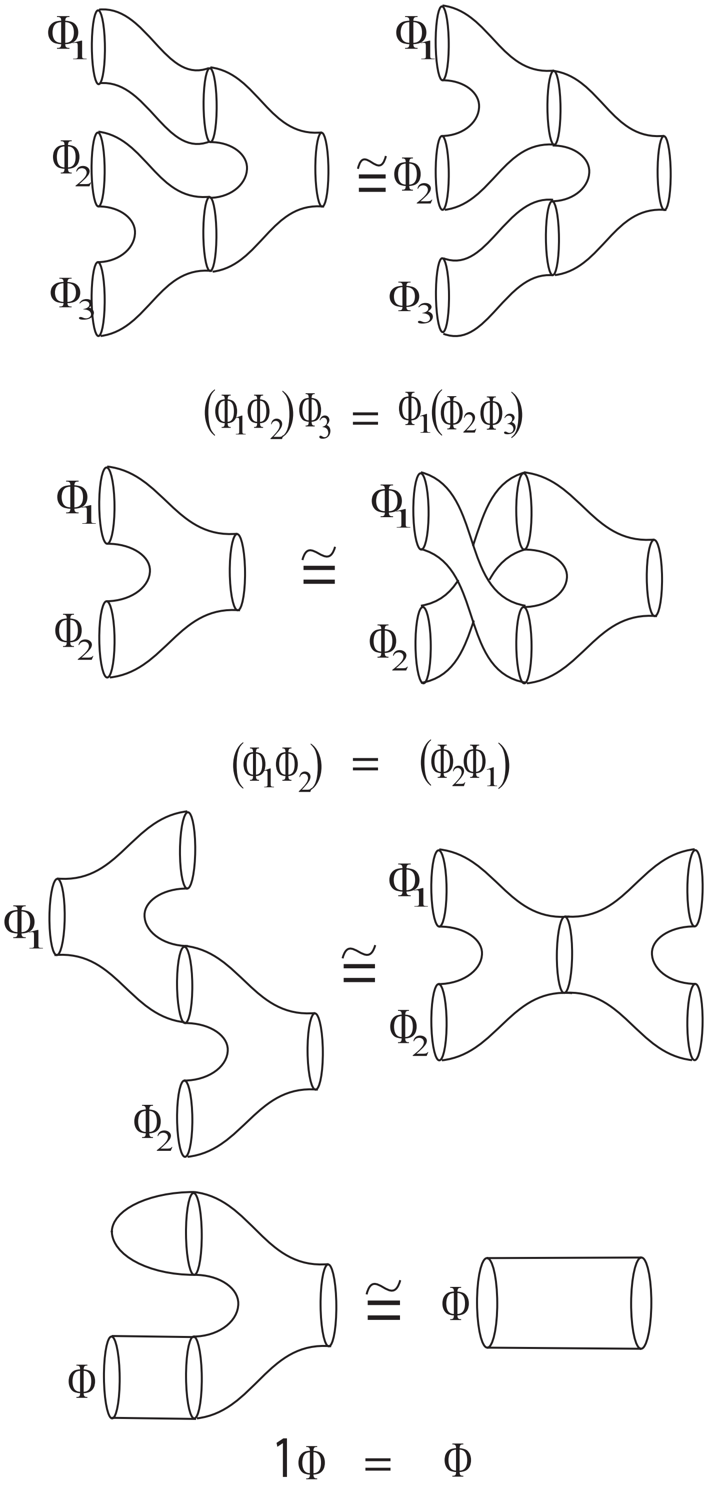


Figure 4.4: Associativity, commutativity, Abrams condition and unit constraints in the closed case.

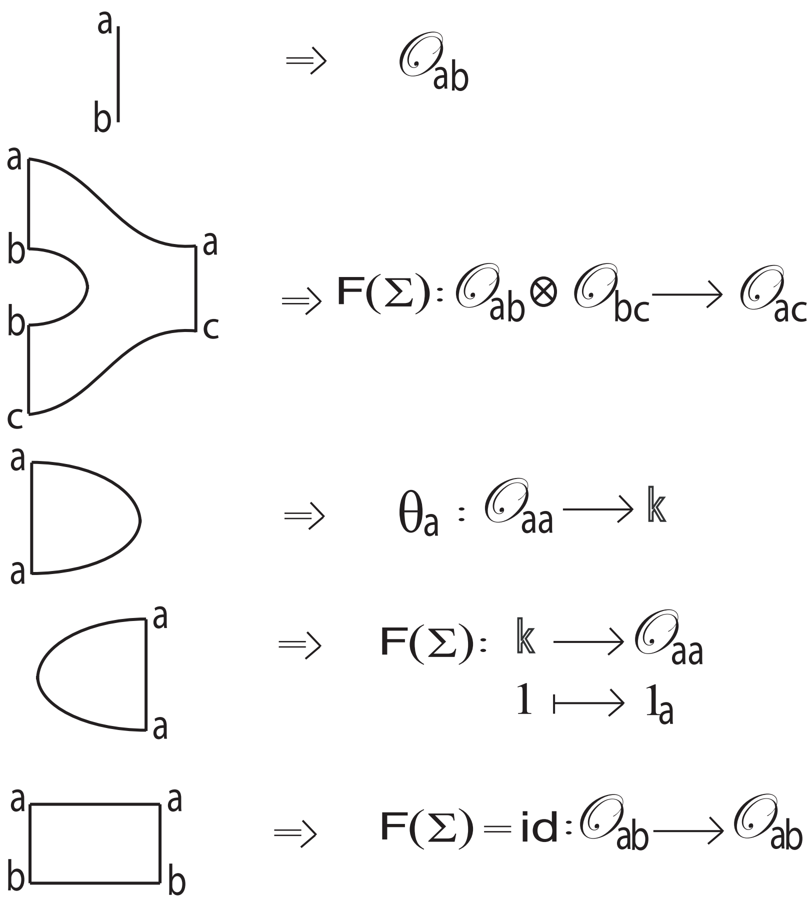


Figure 4.5: `Basic data for the open theory.`

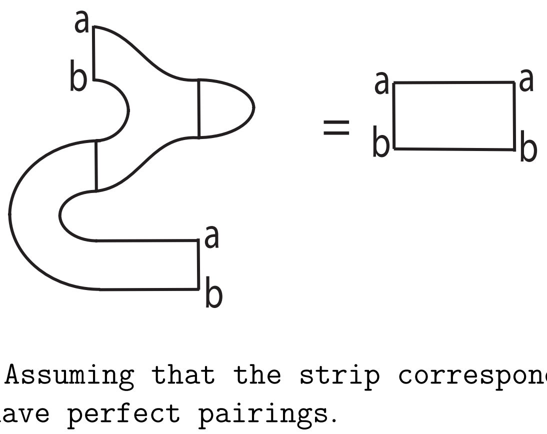


Figure 4.6: `Assuming that the strip corresponds to the identity morphism we must have perfect pairings.`

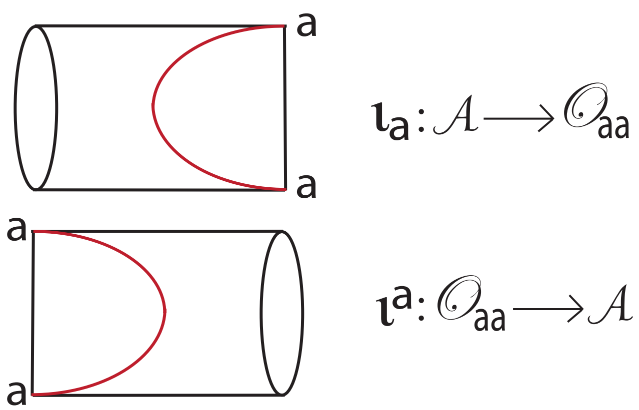


Figure 4.7: Two ways of representing open to closed and closed to open transitions.

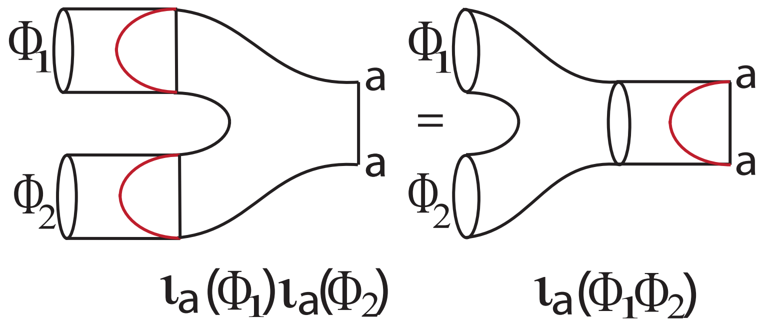


Figure 4.8: ι_a is a homomorphism.

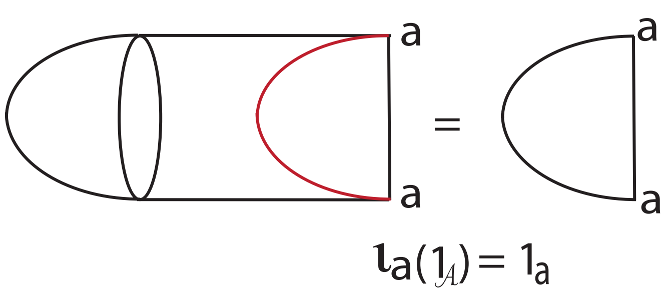


Figure 4.9: ι_a preserves the identity.

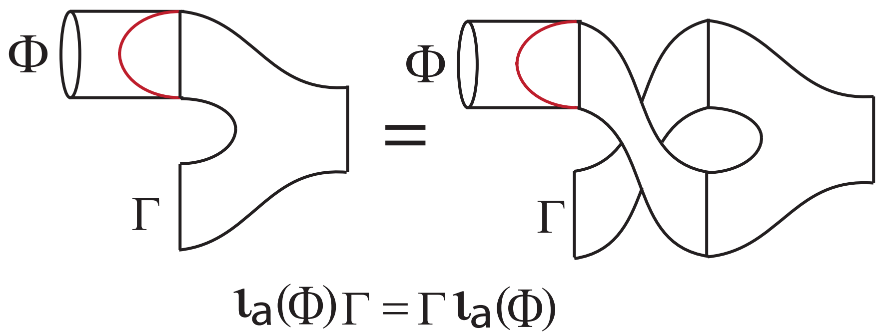


Figure 4.10: ι_a maps into the center of $\mathscr{O}_{aa}$.

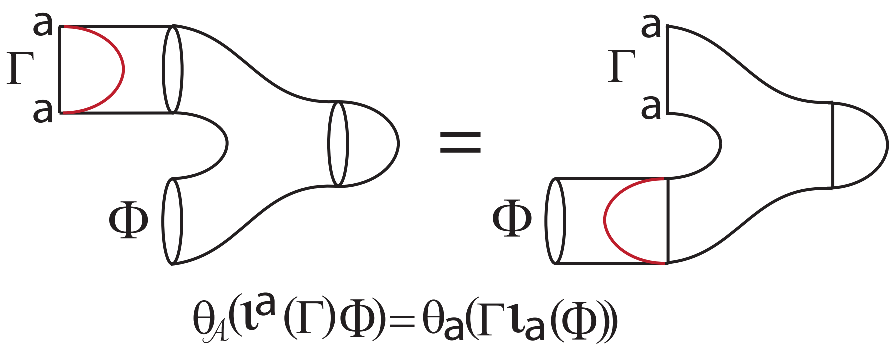


Figure 4.11: ι^a is the adjoint of ι_a.

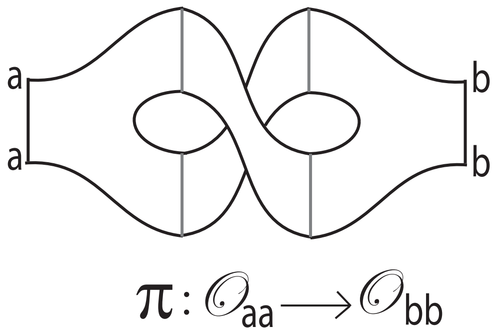


Figure 4.12: The double-twist diagram defines the map $\pi^a_b : \mathscr{O}_{aa} \to \mathscr{O}_{bb}$.

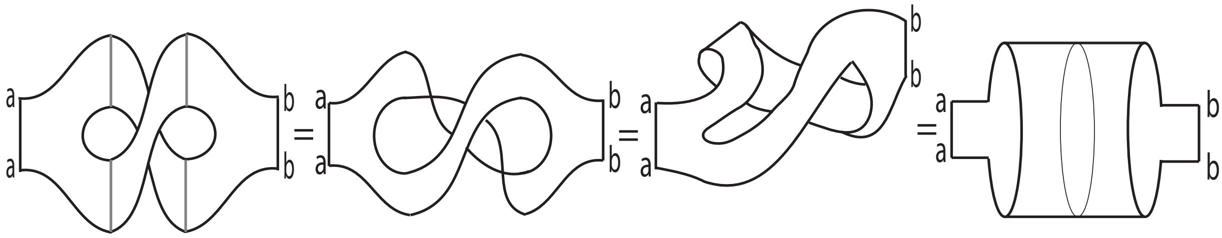


Figure 4.13: The Cardy-condition is expressing the factorization of the double-twist diagram in the closed string channel.

2a. Notation $\mathscr{O}_{ij} = \mathrm{Hom}(E_i, E_j) = \begin{cases} \mathbb{C}\,\mathrm{Id}_{E_i} & \text{if } i = j, \\ 0 & \text{in other case.} \end{cases}$
Then $\mathscr{O}_{ij} \otimes \mathscr{O}_{jk} \to \mathscr{O}_{ik}$ is zero except for $i = j = k$. In this case

$$\mathscr{O}_{ii} \otimes \mathscr{O}_{ii} \to \mathscr{O}_{ii}$$
$$\lambda\,\mathrm{Id} \otimes \mu\,\mathrm{Id} \mapsto \lambda\mu\,\mathrm{Id}$$

2b. The trace $\theta_i : \mathscr{O}_{ii} \to \mathbb{C}$ is nondegenerated. Note that if $\psi \in \mathscr{O}_{ii}$ then there exist $\lambda \in \mathbb{C}$ such that $\psi = \lambda\,\mathrm{Id}_i$, hence $\ker(\theta_i) = \{0\}$.

2c. First, suppose that $i \neq j$ then

$$\mathscr{O}_{ij} \otimes \mathscr{O}_{ji} \to \mathscr{O}_{ii} \xrightarrow{\theta_i} \mathbb{C},$$
$$\mathscr{O}_{ji} \otimes \mathscr{O}_{ij} \to \mathscr{O}_{jj} \xrightarrow{\theta_j} \mathbb{C}$$

we have $\theta_i(\psi\phi) = 0 = \theta_j(\phi\psi)$.
If $i = j$ then $\mathscr{O}_{ii} \otimes \mathscr{O}_{ii} \to \mathscr{O}_{ii} \xrightarrow{\theta_i} \mathbb{C}$. In this case $\psi = \lambda\,\mathrm{Id}$ and $\phi = \mu\,\mathrm{Id}$, hence $\psi\phi = \phi\psi$, and as a consequence $\theta_i(\psi\phi) = \theta_i(\phi\psi)$.

3a. ι_E is an algebra homomorphism.

$$\iota_E((\sum_g \alpha_g g)(\sum_h \beta_h h)) = \iota_E(\sum \alpha_g \beta_h gh) = \sum \alpha_g \beta_h \rho_{gh}$$
$$\iota_E(\sum_g \alpha_g g)\iota_E(\sum_h \beta_h h) = \sum_g \alpha_g \rho_g \sum_h \alpha_h \rho_h = \sum \alpha_g \beta_h \rho_g \rho_h$$

This expressions are the same because ρ is a group homomorphism.

3b. The identity is preserved by definition ($\iota_E(e) = \mathrm{Id}_E$).

3c. The linear map ι_E is central i.e. $\iota_E(\sum_g \alpha_g g)\psi = \psi\iota_F(\sum_g \alpha_g g)$ with $\psi \in \mathscr{O}_{EF}$.
If $\psi \in \mathscr{O}_{ij}$, then $\psi = 0$ for $i \neq j$ or $\psi = \lambda\,\mathrm{Id}_i$ for $i = j$.
If $i \neq j$ is true the statement. Now we see the case $i = j$, but since we have $\psi = \lambda\,\mathrm{Id}$ then it follows.

3d. The linear maps ι_E and ι^E are adjoint, i.e. $\theta_{Z(\mathbb{C}[G])}(\iota^E(\psi)\phi) = \theta_E(\psi\iota_E(\phi))$.

$$\begin{aligned}
\theta_E(\psi\iota_E(\phi)) &= \theta_E(\psi\sum_g \alpha_g\rho_g) = \theta_E(\sum_g \alpha_g\psi\rho_g)\\
&= \frac{1}{|G|}\operatorname{tr}(\sum_g \alpha_g\psi\rho_g) = \frac{1}{|G|}\sum_g \alpha_g \operatorname{tr}(\psi\rho_g)\\
\theta_{Z(\mathbb{C}[G])}(\iota^E(\psi)\phi) &= \theta_{Z(\mathbb{C}[G])}(\sum_g \alpha_g\iota^E(\psi)g) = \theta_{Z(\mathbb{C}[G])}(\sum_g \alpha_g \operatorname{tr}(\psi\rho_g))\\
&= \frac{1}{|G|}\sum_g \alpha_g \operatorname{tr}(\psi\rho_g)
\end{aligned}$$

3e. First

$$\begin{tikzcd}
\mathscr{O}_{ii} \arrow[r] \arrow[rrr, bend left, "\pi^i_j"] & \mathscr{O}_{ij}\otimes\mathscr{O}_{ji} \arrow[r, "\tau"] & \mathscr{O}_{ji}\otimes\mathscr{O}_{ij} \arrow[r] & \mathscr{O}_{jj}
\end{tikzcd}$$

If $i \neq j$ then $\pi^i_j = 0$. If $i = j$ we have

$$\mathscr{O}_{ii} \to \mathscr{O}_{ii}\otimes\mathscr{O}_{ii} \to \mathscr{O}_{ii}\otimes\mathscr{O}_{ii} \to \mathscr{O}_{ii}$$

$$\lambda\operatorname{Id} \mapsto \lambda\operatorname{Id}\otimes\frac{|G|}{n_i}\operatorname{Id} \mapsto \frac{|G|}{n_i}\operatorname{Id}\otimes\lambda\operatorname{Id} \mapsto \frac{|G|}{n_i}\lambda\operatorname{Id}$$

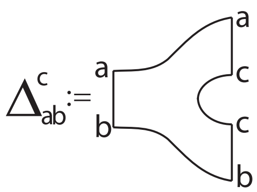
Then $\pi^i_i(\lambda\operatorname{Id}) = \frac{|G|}{n_i}\lambda$, where $n_i = \dim E_i$.
Now we need to study $\iota_i\iota^j$.
The map $\iota^i : \mathscr{O}_{ii} \to Z(\mathbb{C}[G])$ takes $\lambda\operatorname{Id}$ to $\sum_g \operatorname{tr}(\lambda\rho_g)g = \lambda\sum_g \chi_i(g)g$ and $\iota_j : Z(\mathbb{C}[G]) \to \mathscr{O}_{jj}$ takes $\sum_g \alpha_g g$ to $\sum_g \alpha_g\rho_g$. Consequently $\iota_j\iota^i(\lambda\operatorname{Id}) = \lambda\sum_g \chi_i(g)\rho_g : E_j \to E_j$.
For the map $\rho_g : E_j \to E_j$, with E_j is an irreducible representation, there exists $\mu \in \mathbb{C}$ such that $\rho_g = \mu\operatorname{Id}_j$. Hence $\operatorname{tr}(\rho_g) = \mu\dim E_j$, so $\mu = \frac{1}{n_j}\chi_j(g)$. For this $\iota_j\iota^i(\lambda\operatorname{Id}_j) = \lambda\sum_g \chi_i(g)\frac{1}{n_j}\chi_j(g)\operatorname{Id}_j = \frac{\lambda}{n_j}\sum_g \chi_i(g)\chi_j(g)\operatorname{Id}_j = \frac{\lambda}{n_j}\delta_{i^*j}|G|\operatorname{Id}_j$.
Using that the representations are real, we have that $\chi_i(g) = \overline{\chi_i}(g)$, then $\delta_{i^*j} = \delta_{ij}$ and the maps coincide.

4.4 2D Open-Closed TFT with positive boundary

In a 2D open-closed TFT we have a family of maps $\Delta_{ab}^c : \mathscr{O}_{ab} \to \mathscr{O}_{ac} \otimes \mathscr{O}_{cb}$, which are called *coproducts*, with $a, b, c \in \mathscr{B}$. These are defined by

$$\begin{tikzcd}
\mathscr{O}_{ab} \arrow[rrr, ":=\Delta_{ab}^c"] \arrow[d, "\Phi_{ab}"'] & & & \mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \\
\mathscr{O}_{ba}^* \arrow[r, "\eta_{ba}^{c*}"'] & \mathscr{O}_{bc}^* \otimes \mathscr{O}_{ca}^* \arrow[rr, "\tau"'] & & \mathscr{O}_{ca}^* \otimes \mathscr{O}_{bc}^* \arrow[u, "\Phi_{ac}^{-1} \otimes \Phi_{cb}^{-1}"']
\end{tikzcd}$$

where $\Phi_{ab} : \mathscr{O}_{ab} \to \mathscr{O}_{bc}^*$ is $\Phi_{ab}(x)(y) = \Theta_a(xy)$, for $x \in \mathscr{O}_{ab}$ and $y \in \mathscr{O}_{ba}$.



It is clear that Δ_{ab}^c is a linear map.

Remark 4.4.1. The spaces $\mathscr{O}_{ab}$ are finite dimensional with bilinear maps

$$\eta_{ab}^c : \mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \to \mathscr{O}_{ab}.$$

In the case $a = b = c$, η_{aa}^a is an associative product.
These maps satisfy the next commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{ab} \otimes \mathscr{O}_{bc} \otimes \mathscr{O}_{cd} \arrow[r, "\eta_{ac}^b \otimes 1"] \arrow[d, "1 \otimes \eta_{bd}^c"'] & \mathscr{O}_{ac} \otimes \mathscr{O}_{cd} \arrow[d, "\eta_{ad}^c"] \\
\mathscr{O}_{ad} \otimes \mathscr{O}_{bd} \arrow[r, "\eta_{ad}^b"'] & \mathscr{O}_{ad}
\end{tikzcd}$$

Lemma 4.4.2. *The maps Δ_{ab}^c are* coassociative*, i.e. the next diagram commutes*

$$\begin{tikzcd}
\mathscr{O}_{ab} \arrow[r, "\Delta_{ab}^d"] \arrow[d, "\Delta_{ab}^c"'] & \mathscr{O}_{ad} \otimes \mathscr{O}_{db} \arrow[d, "\Delta_{ad}^c \otimes 1"] \\
\mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \arrow[r, "1 \otimes \Delta_{cb}^d"'] & \mathscr{O}_{ac} \otimes \mathscr{O}_{cd} \otimes \mathscr{O}_{db}
\end{tikzcd}$$

for all $a, b, c, d \in \mathscr{B}$.

Proof. Note that in the next diagram we need to prove that the external diagram commutes.

$$
\begin{tikzcd}
 & & & & \mathscr{O}^*_{bd}\otimes\mathscr{O}^*_{da} \arrow[dr, "\tau"] & & & & \\
 & & & \mathscr{O}^*_{ba} \arrow[ur, "\eta^{d*}_{ba}"] & & \mathscr{O}^*_{da}\otimes\mathscr{O}^*_{bd} \arrow[dr, "\Phi^{-1}_{ad}\otimes\Phi^{-1}_{db}"] \arrow[ddl, "\eta^{c*}_{da}\otimes 1^*"'] & & & \\
 & & \mathscr{O}_{ab} \arrow[ur, "\Phi_{ab}"] \arrow[dl, "\Phi_{ab}"'] & \textcircled{1} & & & \mathscr{O}_{ad}\otimes\mathscr{O}_{db} \arrow[dr, "\Phi_{ad}\otimes 1"] & & \\
 & \mathscr{O}^*_{ba} \arrow[uurr, bend right, "1"'] \arrow[dl, "\eta^{c*}_{ba}"'] & & \textcircled{2} & \mathscr{O}^*_{dc}\otimes\mathscr{O}^*_{ca}\otimes\mathscr{O}^*_{bd} \arrow[dd, "\tau\otimes 1"] & \textcircled{3} & & \mathscr{O}^*_{da}\otimes\mathscr{O}_{db} \arrow[dr, "\eta^{c*}_{da}\otimes 1"] & \\
\mathscr{O}^*_{bc}\otimes\mathscr{O}^*_{ca} \arrow[dr, "\tau"'] & & & \mathscr{O}^*_{ca}\otimes\mathscr{O}^*_{bd}\otimes\mathscr{O}^*_{dc} \arrow[dr, "1\otimes\tau"] & & & & & \mathscr{O}^*_{dc}\otimes\mathscr{O}^*_{ca}\otimes\mathscr{O}_{db} \arrow[dl, "\tau\otimes 1"] \\
 & \mathscr{O}^*_{ca}\otimes\mathscr{O}^*_{bc} \arrow[urr, "1\otimes\eta^{d*}_{bc}"] \arrow[dr, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{bc}"'] & & & \mathscr{O}^*_{ca}\otimes\mathscr{O}^*_{dc}\otimes\mathscr{O}^*_{bd} \arrow[rrr, "1\otimes 1\otimes\Phi^{-1}_{db}"] \arrow[d, "1\otimes\tau"] & & & \mathscr{O}^*_{ca}\otimes\mathscr{O}^*_{dc}\otimes\mathscr{O}_{db} \arrow[dl, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{cb}\otimes 1"] & \\
 & & \mathscr{O}_{ac}\otimes\mathscr{O}_{cb} \arrow[dr, "1\otimes\Phi_{cb}"'] & \textcircled{4} & \mathscr{O}^*_{ca}\otimes\mathscr{O}^*_{bd}\otimes\mathscr{O}^*_{dc} \arrow[dd, "\Phi^{-1}_{ac}\otimes 1\otimes 1"] & \textcircled{5} & \mathscr{O}_{ac}\otimes\mathscr{O}_{cd}\otimes\mathscr{O}_{db} & & \\
 & & & \mathscr{O}_{ac}\otimes\mathscr{O}^*_{bc} \arrow[dr, "1\otimes\eta^{d*}_{bc}"'] & & \mathscr{O}_{ac}\otimes\mathscr{O}^*_{dc}\otimes\mathscr{O}^*_{bd} \arrow[ur, "1\otimes\Phi^{-1}_{cd}\otimes\Phi^{-1}_{db}"'] & & & \\
 & & & & \mathscr{O}_{ac}\otimes\mathscr{O}^*_{bd}\otimes\mathscr{O}^*_{dc} \arrow[ur, "1\otimes\tau"'] & & & &
\end{tikzcd}
$$

Note that (1) commutes trivially. The diagram (2) can be divided into four commutative diagrams

$$\begin{tikzcd}
\mathscr{O}^*_{ba} \arrow[r, "\eta^{d*}_{ba}"] \arrow[d, "\eta^{c*}_{ba}"'] & \mathscr{O}^*_{bd} \otimes \mathscr{O}^*_{da} \arrow[r, "\tau"] \arrow[d, "1\otimes\eta^{c*}_{da}"'] & \mathscr{O}^*_{da} \otimes \mathscr{O}^*_{bd} \arrow[d, "\eta^{c*}_{da}\otimes 1"] \\
\mathscr{O}^*_{bc} \otimes \mathscr{O}^*_{ca} \arrow[r, "\eta^{d*}_{bc}\otimes 1"'] \arrow[d, "\tau"'] & \mathscr{O}^*_{bd} \otimes \mathscr{O}^*_{dc} \otimes \mathscr{O}^*_{ca} \arrow[r, "\tau"] \arrow[d, "\tau"'] & \mathscr{O}^*_{dc} \otimes \mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{bd} \arrow[d, "\tau\otimes 1"] \\
\mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{bc} \arrow[r, "1\otimes\eta^{d*}_{bc}"'] & \mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{bd} \otimes \mathscr{O}^*_{dc} \arrow[r, "1\otimes\tau"'] & \mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{dc} \otimes \mathscr{O}^*_{bd}
\end{tikzcd}$$

The diagram 3 is the following

$$\begin{tikzcd}
\mathscr{O}^*_{da} \otimes \mathscr{O}^*_{bd} \arrow[r, "\Phi^{-1}_{ad}\otimes\Phi^{-1}_{db}"] \arrow[d, "\eta^{c*}_{da}\otimes 1"'] & \mathscr{O}_{ad} \otimes \mathscr{O}_{db} \arrow[r, "\Phi_{ad}\otimes 1"] & \mathscr{O}^*_{da} \otimes \mathscr{O}_{db} \arrow[d, "\eta^{c*}_{da}\otimes 1"] \\
\mathscr{O}^*_{dc} \otimes \mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{bd} \arrow[d, "\tau\otimes 1"'] & & \mathscr{O}^*_{dc} \otimes \mathscr{O}^*_{ca} \otimes \mathscr{O}_{db} \arrow[d, "\tau\otimes 1"'] \\
\mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{dc} \otimes \mathscr{O}^*_{bd} \arrow[rr, "1\otimes\Phi^{-1}_{db}"] & & \mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{dc} \otimes \mathscr{O}_{db}
\end{tikzcd}$$

and it commutes naturally. Now we check that the diagram 4 commutes

$$\begin{tikzcd}
\mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{bc} \arrow[r, "1\otimes\eta^{d*}_{bc}"] \arrow[d, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{cb}"'] & \mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{bd} \otimes \mathscr{O}^*_{dc} \arrow[r, "1\otimes\tau"] \arrow[dr, "1"'] & \mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{dc} \otimes \mathscr{O}^*_{bd} \arrow[d, "1\otimes\tau"] \\
\mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \arrow[d, "1\otimes\Phi_{cb}"'] & & \mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{bd} \otimes \mathscr{O}^*_{dc} \arrow[dl, "\Phi^{-1}_{ac}\otimes 1"] \\
\mathscr{O}_{ac} \otimes \mathscr{O}^*_{bc} \arrow[r, "1\otimes\eta^{d*}_{bc}"'] & \mathscr{O}_{ac} \otimes \mathscr{O}^*_{bd} \otimes \mathscr{O}^*_{dc} &
\end{tikzcd}$$

It commutes naturally. Finally, it rests to prove that the diagram 5 commutes. Then the external diagram commutes. The diagram 5 can be divided into the next diagrams.

$$
\begin{tikzcd}
\mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{dc}\mathscr{O}^*_{bd} \arrow[rr, "1\otimes\Phi^{-1}_{db}"] \arrow[d, "1\otimes\tau"'] \arrow[drr, bend left, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{cd}\otimes\Phi^{-1}_{db}"] \arrow[ddr, "\Phi^{-1}_{ac}\otimes 1"] & & \mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{dc} \otimes \mathscr{O}_{db} \arrow[dd, "\Phi^{-1}_{ac}\otimes\Phi^{-1}_{cd}\otimes 1"] \\
\mathscr{O}^*_{ca} \otimes \mathscr{O}^*_{bd} \otimes \mathscr{O}^*_{dc} \arrow[d, "\Phi^{-1}_{ac}\otimes 1"'] & & \\
\mathscr{O}_{ac} \otimes \mathscr{O}^*_{bd} \otimes \mathscr{O}^*_{dc} \arrow[r, "1\otimes\tau"'] & \mathscr{O}_{ac} \otimes \mathscr{O}^*_{dc} \otimes \mathscr{O}^*_{bd} \arrow[r, "1\otimes\Phi^{-1}\otimes\Phi^{-1}"'] & \mathscr{O}_{ac} \otimes \mathscr{O}_{cd} \otimes \mathscr{O}_{db}
\end{tikzcd}
$$

It is clear that they are commutative, and the coproducts are coassociative.

♣

Lemma 4.4.3. *Given the maps $\Theta_a : \mathscr{O}_{aa} \to \Bbbk$, we have that the triangles*

$$
\begin{tikzcd}
\mathscr{O}_{ab} \arrow[r, "\Delta^b_{ab}"] \arrow[d, "\cong"'] & \mathscr{O}_{ab} \otimes \mathscr{O}_{bb} \arrow[dl, "1\otimes\Theta_b"] \\
\mathscr{O}_{ab} \otimes \Bbbk &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{O}_{ab} \arrow[r, "\Delta^a_{ab}"] \arrow[d, "\cong"'] & \mathscr{O}_{aa} \otimes \mathscr{O}_{ab} \arrow[dl, "\Theta_a\otimes 1"] \\
\Bbbk \otimes \mathscr{O}_{ab} &
\end{tikzcd}
$$

commute.

Proof. Note the identity $\Theta_a = u^*_a \circ \Phi_a$. It is clear that the next diagram commutes,

$$
\begin{tikzcd}
\mathscr{O}_{ab} \arrow[r, "\Phi_{ab}"] \arrow[dd, "\cong"'] & \mathscr{O}^*_{ba} \arrow[r, "\eta^{b*}_{ba}"] & \mathscr{O}^*_{bb} \otimes \mathscr{O}^*_{ba} \arrow[d, "\tau"] \\
 & & \mathscr{O}^*_{ba} \otimes \mathscr{O}^*_{bb} \arrow[d, "\Phi^{-1}_{ab}\otimes\Phi^{-1}_{bb}"] \arrow[dl, "\Phi^{-1}_{ab}\otimes 1"'] \arrow[dll, "\Phi^{-1}_{ab}\otimes u^*_b"'] \\
\mathscr{O}_{ab} \otimes \mathbb{C} & \mathscr{O}_{ab} \otimes \mathscr{O}^*_{bb} \arrow[l, "1\otimes u^*_b"] & \mathscr{O}_{ab} \otimes \mathscr{O}_{bb} \arrow[l, "1\otimes\Phi_b"]
\end{tikzcd}
$$

the reason is that the identity $\eta^b_{ba} \circ (u_b \otimes 1) = 1$ implies that $(u^*_b \otimes 1) \circ \eta^{b*}_{ba} = 1$ then

$$(\Phi^{-1}_{ab} \otimes u^*_b) \circ \tau \circ \eta^{b*}_{ba} = \tau \circ (1 \otimes \Phi^{-1}_{ab}) \circ (u^*_b \otimes 1) \circ \eta^{b*}_{ba} = \tau \circ (1 \otimes \Phi^{-1}_{ab})$$

This proves the lema.

♣

Consider the maps

$$\overline{\eta^c_{ab}} : \mathscr{O}_{ab} \to \mathrm{Hom}(\mathscr{O}_{ca}, \mathscr{O}_{cb}) \cong \mathscr{O}_{cb} \otimes \mathscr{O}^*_{ca},$$
$$x \mapsto \cdot x : \mathscr{O}_{ca} \to \mathscr{O}_{cb}, \text{ product by the right of } x$$
$$\overline{\xi^c_{ab}} : \mathscr{O}_{ab} \to \mathrm{Hom}(\mathscr{O}_{bc}, \mathscr{O}_{ac}) \cong \mathscr{O}_{ac} \otimes \mathscr{O}^*_{bc},$$
$$x \mapsto x\cdot : \mathscr{O}_{bc} \to \mathscr{O}_{ac}, \text{ product by the left of } x$$

It is not difficult to prove that the next diagrams commute

$$\begin{tikzcd}
\mathscr{O}_{ab} \arrow[r, "\Phi_{ab}"] \arrow[d, "\Delta^c_{ab}"'] & \mathscr{O}^*_{ba} \arrow[r, "\Phi^{-1}_{ab}"] \arrow[d, "\eta^{c*}_{ba}"] & \mathscr{O}_{ab} \arrow[d, "\overline{\eta^c_{ab}}"] \\
\mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \arrow[r, "(\Phi_{cb}\otimes\Phi_{ac})\circ\tau"'] & \mathscr{O}^*_{bc} \otimes \mathscr{O}^*_{ca} \arrow[r, "\Phi^{-1}_{cb}\otimes 1"'] & \mathscr{O}_{cb} \otimes \mathscr{O}^*_{ca}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{O}_{ab} \arrow[r, "\Phi_{ab}"] \arrow[d, "\Delta^c_{ab}"'] & \mathscr{O}^*_{ba} \arrow[d, "\eta^{c*}_{ba}"] & \mathscr{O}_{ab} \arrow[l, "\Phi_{ab}"'] \arrow[d, "\overline{\xi^c_{ab}}"] \\
\mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \arrow[r, "(\Phi_{cb}\otimes\Phi_{ac})\circ\tau"'] & \mathscr{O}^*_{bc} \otimes \mathscr{O}^*_{ca} & \mathscr{O}_{ac} \otimes \mathscr{O}^*_{bc} \arrow[l, "\tau\circ(\Phi_{ac}\otimes 1)"]
\end{tikzcd}$$

Proposition 4.4.4. *The coproduct Δ^c_{ab} is a morphism of $\mathscr{O}_{da} \times \mathscr{O}_{be}$-bimodule for all d, e, i.e. the squares*

$$\begin{tikzcd}
\mathscr{O}_{da} \otimes \mathscr{O}_{ab} \arrow[r, "\eta^a_{db}"] \arrow[d, "1\otimes\Delta^c_{ab}"'] & \mathscr{O}_{db} \arrow[d, "\Delta^c_{db}"] \\
\mathscr{O}_{da} \otimes \mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \arrow[r, "\eta^a_{dc}\otimes 1"'] & \mathscr{O}_{dc} \otimes \mathscr{O}_{cb}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{O}_{ab} \otimes \mathscr{O}_{be} \arrow[r, "\eta^b_{ae}"] \arrow[d, "\Delta^c_{ab}\otimes 1"'] & \mathscr{O}_{ab} \arrow[d, "\Delta^c_{ae}"] \\
\mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \otimes \mathscr{O}_{be} \arrow[r, "1\otimes\eta^b_{ce}"'] & \mathscr{O}_{ac} \otimes \mathscr{O}_{ce}
\end{tikzcd}$$

commute.

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathscr{O}_{da} \otimes \mathscr{O}_{ab} \arrow[r, "\eta^a_{ab}"] \arrow[dl, "1\otimes\overline{\xi^c_{ab}}"'] \arrow[d, "1\otimes\Delta^c_{ab}"] \arrow[dr, phantom, "1"] & \mathscr{O}_{db} \arrow[d, "\Delta^c_{db}"'] \arrow[dr, "\overline{\xi^c_{db}}"] \arrow[dr, phantom, "2"'] & \\
\mathscr{O}_{da} \otimes \mathscr{O}_{ac} \otimes \mathscr{O}^*_{bc} \arrow[dr, "\eta^a_{dc}\otimes 1^*"'] \arrow[ur, phantom, "5"] & \mathscr{O}_{da} \otimes \mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \arrow[l, "1\otimes 1\otimes\Phi_{cb}"'] \arrow[r, "\eta^a_{dc}\otimes 1^*"'] \arrow[dr, phantom, "4"] & \mathscr{O}_{dc} \otimes \mathscr{O}_{cb} \arrow[r, "1\otimes\Phi_{cb}"'] \arrow[d, "1\otimes\Phi_{cb}"'] \arrow[dr, phantom, "3"] & \mathscr{O}_{dc} \otimes \mathscr{O}^*_{bc} \arrow[dl, "1\otimes 1^*"] \\
 & \mathscr{O}_{dc} \otimes \mathscr{O}^*_{bc} \arrow[r, "1\otimes 1^*"'] & \mathscr{O}_{dc} \otimes \mathscr{O}^*_{bc} &
\end{tikzcd}$$

If we prove that the external diagram, and the diagrams 2, 3, 4, 5 commute then the diagram 1 commutes. Note that the diagramas 2 and 5 commute using the last statement. Clearly the diagrams 3 and 4 commute, and finally the external diagram commutes by definition of $\overline{\xi^c_{ab}}$.

We use the next diagram to prove that the other diagram commutes.

$$
\begin{tikzcd}
 & \mathscr{O}_{ab} \otimes \mathscr{O}_{be} \arrow[r, "\eta^b_{ab}"] \arrow[dl, "\tau\circ(\overline{\eta^c_{ab}}\otimes 1)"'] \arrow[d, "\Delta^c_{ab}\otimes 1"'] & \mathscr{O}_{ae} \arrow[d, "\Delta^c_{ae}"] \arrow[dr, "\tau\circ\overline{\eta^c_{ae}}"] & \\
\mathscr{O}^*_{ca} \otimes \mathscr{O}_{cb} \otimes \mathscr{O}_{be} \arrow[dr, "1^*\otimes\eta^b_{ce}"'] & \mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \otimes \mathscr{O}_{be} \arrow[l, "\Phi_{ac}\otimes 1\otimes 1"] \arrow[r, "1\otimes\eta^b_{ce}"'] & \mathscr{O}_{ac} \otimes \mathscr{O}_{ce} \arrow[r, "\Phi_{ac}\otimes 1"] \arrow[d, "\Phi_{ac}\otimes 1"'] & \mathscr{O}^*_{ca} \otimes \mathscr{O}_{ce} \arrow[dl, "1^*\otimes 1"] \\
 & \mathscr{O}^*_{ca} \otimes \mathscr{O}_{ce} \arrow[r, "1^*\otimes 1"'] & \mathscr{O}^*_{ca} \otimes \mathscr{O}_{ce} &
\end{tikzcd}
$$

♣

Applying the Proposition 4.4.4 we have that the cobordisms of the figure 4.14 coincide.

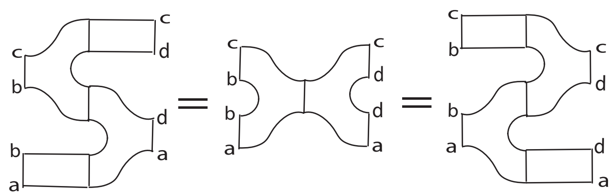
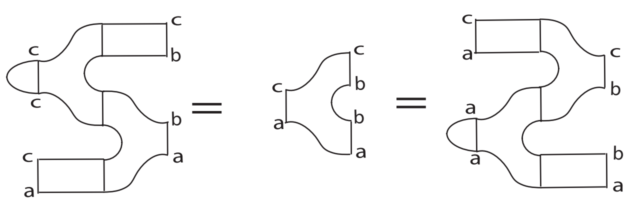
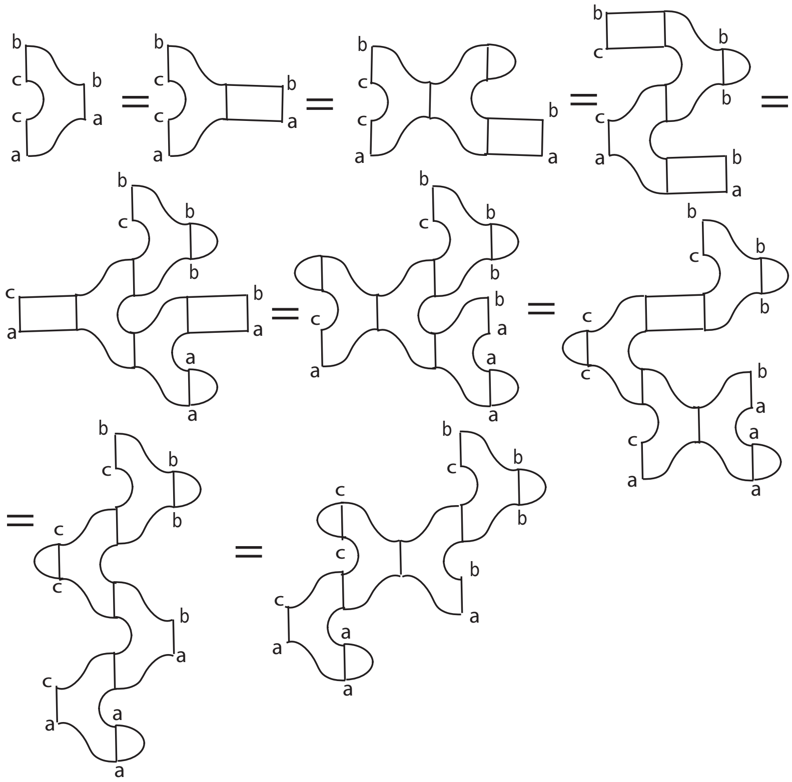


Figure 4.14: `Abrams condition`.

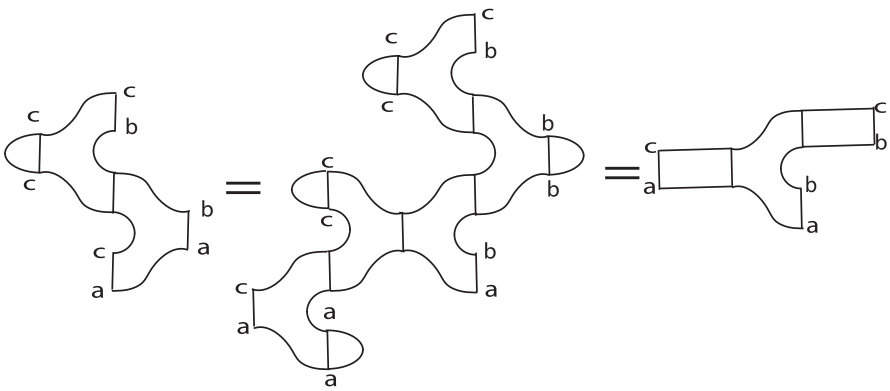
Lemma 4.4.5.



Proof.



hence



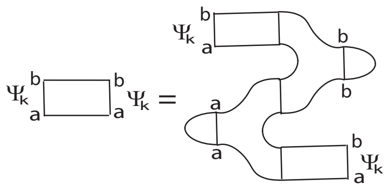
♣

Remark 4.4.6. Let $\Theta_{ab} : \mathbb{C} \to \mathscr{O}_{ab} \otimes \mathscr{O}_{ba}$ defined by

$$\Theta_{ab} = \Delta^{b}_{aa} \circ u_a,$$

where $u_a : \mathbb{C} \to \mathscr{O}_{aa}$ is the unit. Then $\Theta_{ab}(1) = \Sigma_i \Psi_i \otimes \Psi^i$, where $\{\Psi_i\}$ is a basis of $\mathscr{O}_{ab}$, and $\{\Psi^i\}$ is the dual basis of $\mathscr{O}_{ba}$, i.e. $\langle \Psi_i, \Psi^j \rangle = \delta_{ij}$.

Proof. Let be $\Theta_{ab}(1) = \Sigma_{i,j}\beta_{ij}\Psi_i \otimes \Psi^j$, where $\beta_{ij} \in \mathbb{C}$.



Then we have $(1\otimes\Theta_b)\circ(1\otimes\eta_{bb}^a)(\Sigma_{ij}\beta_{ij}\Psi_i\otimes\Psi^j\otimes\Psi_k) = (1\otimes\Theta_b)(\Sigma_{ij}\beta_{ij}\Psi_i\otimes\Psi^j\Psi_k) = \Sigma_{ij}\beta_{ij}\Theta_b(\Psi^j\Psi_k)\Psi_i = \Sigma_i\beta_{ik}\Psi_i = \Psi_k$ and hence $\beta_{ij} = \delta_{ij}$.

♣

Proposition 4.4.7. *We can modify the axioms 2 in the definition of Frobenius structure as follows: there exist a family of coassociative linear maps $\Delta_{ab}^c : \mathscr{O}_{ab} \to \mathscr{O}_{ac} \otimes \mathscr{O}_{cb}$ which are $\mathscr{O}_{aa} \times \mathscr{O}_{bb}$-bimodule morphisms and linear maps $\Theta_a : \mathscr{O}_{aa} \to \mathbb{C}$ such that*

$$\begin{tikzcd} \mathscr{O}_{ab} \arrow[r, "\Delta_{ab}^b"] \arrow[d, "\cong"'] & \mathscr{O}_{ab}\otimes\mathscr{O}_{bb} \arrow[dl, "1\otimes\Theta_b"] \\ \mathscr{O}_{ab}\otimes\mathbb{k} & \end{tikzcd} \qquad \begin{tikzcd} \mathscr{O}_{ab} \arrow[r, "\Delta_{ab}^a"] \arrow[d, "\cong"'] & \mathscr{O}_{aa}\otimes\mathscr{O}_{ab} \arrow[dl, "\Theta_a\otimes 1"] \\ \mathbb{k}\otimes\mathscr{O}_{ab} & \end{tikzcd}$$

commute.

Proof. We only need to prove that the trace $\Theta_a : \mathscr{O}_{aa} \to \mathbb{C}$ is non-degenerate. For this we consider the next commutative diagram

$$\begin{tikzcd} & & \mathscr{O}_{aa}\otimes\mathscr{O}_{aa}\otimes\mathscr{O}_{aa} \arrow[dr, "1\otimes\eta_{aa}^a"] & & \\ \mathbb{C}\otimes\mathscr{O}_{aa} \arrow[r, "u_a\otimes 1"] & \mathscr{O}_{aa}\otimes\mathscr{O}_{aa} \arrow[ur, "\Delta_{aa}^a\otimes 1"] \arrow[dr, "\eta_{aa}^a"'] & & \mathscr{O}_{aa}\otimes\mathscr{O}_{aa} \arrow[r, "1\otimes\Theta_a"] & \mathscr{O}_{aa}\otimes\mathbb{C} \\ & & \mathscr{O}_{aa} \arrow[ur, "\Delta_{aa}^a"'] & & \end{tikzcd}$$

This implies the next property

$$1\otimes x \mapsto 1_a\otimes x \mapsto (\sum_i u_i\otimes e_i)\otimes x \mapsto \sum_i u_i\otimes e_i x \mapsto \sum_i \Theta_a(e_i x)u_i = x$$

where $\{e_i\}$ is a basis of $\mathscr{O}_{aa}$. Hence $\{u_i\}$ is also a basis of $\mathscr{O}_{aa}$.
If we take $x = u_j$, then $\Theta_a(e_i u_j) = \delta_{ij}$. We suppose $y = \sum_i \alpha_i e_i$ with the property that $\Theta_a(yx) = 0$ for all $x \in \mathscr{O}_{aa}$. Therefore, if we take $x = u_j$ hence $\sum_i \alpha_i\Theta_a(e_i u_j) = \alpha_j = 0$ for all j. This prove that $y = 0$ and consequently the trace is non-degenerate.

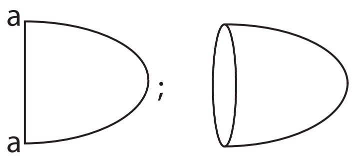


Figure 4.15: `Traces in the open theory and closed theory`.

♣

Definition 4.4.8. We will define a weaker notion of a *positive (outgoing) boundary open-closed topological field theory* (2D OC-TFT$_+$) just as we defined a 2D OC-TFT similarly with the difference that the morphisms have at least one outgoing boundary. In particular there is not linear form associated to the surfaces illustrated in the figure 4.15. Namely, we no longer have traces. Now, we describe the algebraic axioms of this theory.

A positive boundary 2D open-closed TFT is given by the following algebraic data:

1. $(\mathcal{A}, \Delta_{\mathcal{A}}, 1_{\mathcal{A}})$ is a commutative non compact Frobenius algebra.

2 $\mathscr{O}_{ab}$ is a collection of vector spaces for $a, b \in \mathscr{B}$.

2a. There is a family of associative linear maps

$$\eta^b_{ac} : \mathscr{O}_{ab} \otimes \mathscr{O}_{bc} \to \mathscr{O}_{ac} \tag{4.12}$$

2b. There is a family of co-associative linear maps

$$\Delta^c_{ab} : \mathscr{O}_{ab} \to \mathscr{O}_{ac} \otimes \mathscr{O}_{cb}.$$

2c. Moreover, Δ^c_{ab} is a morphism of $\mathscr{O}_{da} \times \mathscr{O}_{be}$-bimodule, i.e. the diagrams0

$$\begin{tikzcd}
\mathscr{O}_{da} \otimes \mathscr{O}_{ab} \arrow[r, "\eta^a_{db}"] \arrow[d, "1\otimes\Delta^c_{ab}"'] & \mathscr{O}_{db} \arrow[d, "\Delta^c_{db}"] \\
\mathscr{O}_{da} \otimes \mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \arrow[r, "\eta^a_{dc}\otimes 1"'] & \mathscr{O}_{dc} \otimes \mathscr{O}_{cb}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{O}_{ab} \otimes \mathscr{O}_{bb} \arrow[r, "\eta^b_{ae}"] \arrow[d, "\Delta^c_{ab}\otimes 1"'] & \mathscr{O}_{ae} \arrow[d, "\Delta^c_{ae}"] \\
\mathscr{O}_{ac} \otimes \mathscr{O}_{cb} \otimes \mathscr{O}_{be} \arrow[r, "1\otimes\eta^b_{ce}"'] & \mathscr{O}_{ac} \otimes \mathscr{O}_{ce}
\end{tikzcd}$$

commute.

3. There are linear maps:

$$\iota_a : \mathcal{A} \to \mathscr{O}_{aa}, \iota^a : \mathscr{O}_{aa} \to \mathcal{A} \tag{4.13}$$

such that

3a. ι_a is an algebra homomorphism

$$\iota_a(\phi_1\phi_2) = \iota_a(\phi_1)\iota_a(\phi_2) \tag{4.14}$$

3b. The identity is preserved

$$\iota_a(1_{\mathcal{A}}) = 1_a \tag{4.15}$$

3c. Moreover, ι_a is central in the sense that

$$\iota_a(\phi)\psi = \psi\iota_b(\phi) \tag{4.16}$$

for all $\phi \in \mathcal{A}$ and $\psi \in \mathscr{O}_{ab}$.

3d. We define the map

$$\pi_b^a := \eta_{bb}^a \circ \tau \circ \Delta_{aa}^b : \mathscr{O}_{aa} \to \mathscr{O}_{bb},$$

where $\tau : \mathscr{O}_{ab} \otimes \mathscr{O}_{ba} \to \mathscr{O}_{ba} \otimes \mathscr{O}_{ab}$ is the transposition map. We require the *Cardy condition*:

$$\pi_b^a = \iota_b \circ \iota^a. \tag{4.17}$$

Remark 4.4.9. This algebraic construction is equivalent to consider the categorical definition, as we do for the 2D open-closed TFT with the restriction that it do not traces in the closed and open part.

4.4.1 Open-closed String Topology

Let $\mathscr{B}$ be the category of D-branes, the objects of this category are a collection of submanifolds of M,

$$\mathrm{Obj}(\mathscr{B}) = \{D_i \subset M : \text{submanifold of } M\}.$$

Now we consider the path spaces, see figure 4.16,

$$\mathscr{P}_M(D_i, D_j) = \{\gamma : [0,1] \to M \text{ picewise smooth } : \gamma(0) \in D_i, \gamma(1) \in D_j\}$$

Then, the morphisms of the category $\mathscr{B}$ are

$$\mathrm{Hom}_{\mathscr{B}}(D_i, D_j) = \mathrm{H}_*(\mathscr{P}_M(D_i, D_j)),$$

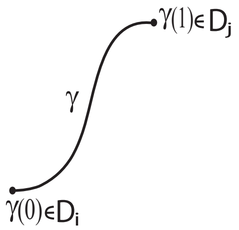
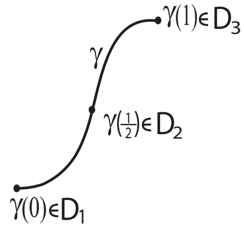


Figure 4.16: `Space` $\mathscr{P}_M(D_i, D_j)$.

for $D_i, D_j \in \mathrm{Obj}(\mathscr{B})$. In the chapter 2, we gave a nearly Frobenius structure on the homology of the free loop space, $(\mathrm{H}_*(\mathrm{LM}), \Delta, u)$. Now, we describe the other structural maps.

Consider the path space

$$\mathscr{P}_M(D_1, D_2, D_3) = \left\{ \alpha : [0,1] \to M : \alpha(0) \in D_1, \alpha\left(\frac{1}{2}\right) \in D_2, \alpha(1) \in D_3 \right\}$$



Now we consider the next diagram

$$\begin{tikzcd}
& \mathcal{P}_M(D_1, D_2, D_3) \arrow[dl, "j_{12} \times j_{23}"'] \arrow[dr, "i^2_{13}"] & \\
\mathscr{P}_M(D_1, D_2) \times \mathscr{P}_M(D_2, D_3) & & \mathscr{P}_M(D_1, D_3)
\end{tikzcd}$$

where $i^2_{13} : \mathcal{P}_M(D_1, D_2, D_3) \to \mathcal{P}_M(D_1, D_2)$ is the natural inclusion, $j_{12} : \mathcal{P}_M(D_1, D_2, D_3) \to \mathcal{P}_M(D_1, D_2)$ is defined by $j_{12}(\alpha)(t) := \alpha(\frac{t}{2})$, and $j_{23} : \mathcal{P}_M(D_1, D_2, D_3) \to \mathcal{P}_M(D_2, D_3)$ is defined by $j_{23}(\alpha)(t) := \alpha(\frac{1+t}{2})$.

The main idea to defining the product is to construct the umkehr map

$$(j_{12} \times j_{23})! : \mathrm{H}_*(\mathscr{P}_M(D_1, D_2)) \otimes \mathrm{H}_*(\mathscr{P}_M(D_2, D_3)) \to \mathrm{H}_*(\mathcal{P}_M(D_1, D_2, D_3))$$

and we define the product η^2_{13} as the composition

$$\eta^2_{13} = (i^2_{13})_* \circ (j_{12} \times j_{23})! : \mathrm{H}_*(\mathscr{P}_M(D_1, D_2)) \otimes \mathrm{H}_*(\mathscr{P}_M(D_2, D_3)) \to \mathrm{H}_*(\mathscr{P}_M(D_1, D_3)).$$

Now we observe that there is a pullback diagram of fibrations,

$$\begin{tikzcd}
\mathscr{P}_M(D_1,D_2,D_3) \arrow[r, "j_{12}\times j_{23}"] \arrow[d, "ev_{\frac{1}{2}}"'] & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3) \arrow[d, "ev_1\times ev_0"] \\
D_2 \arrow[r, "\Delta"'] & D_2\times D_2
\end{tikzcd},$$

this let us define the umkehr map $(j_{12}\times j_{23})!$.

As before we can consider the diagram

$$\begin{tikzcd}
 & \mathcal{P}_M(D_1,D_2,D_3) \arrow[dl, "i^2_{13}"'] \arrow[dr, "j_{12}\times j_{23}"] & \\
\mathscr{P}_M(D_1,D_3) & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3)
\end{tikzcd}$$

Then, we define a coproduct

$$\Delta^2_{13} : \mathrm{H}_*(\mathscr{P}_M(D_1,D_3)) \to \mathrm{H}_*(\mathscr{P}_M(D_1,D_2)) \otimes \mathrm{H}_*(\mathscr{P}_M(D_2,D_3))$$

as the composition $(j_{12}\times j_{23})_* \circ (i^2_{13})! : \mathrm{H}_*(\mathscr{P}_M(D_1,D_3)) \to \mathrm{H}_*(\mathcal{P}_M(D_1,D_2,D_3)) \to \mathrm{H}_*(\mathscr{P}_M(D_1,D_2)) \otimes \mathrm{H}_*(\mathscr{P}_M(D_2,D_3))$.
We can define the umkehr map $(i^2_{13})!$ because we have a pullback diagram of fibrations,

$$\begin{tikzcd}
\mathscr{P}_M(D_1,D_2,D_3) \arrow[r, "i^2_{13}"] \arrow[d, "ev_{\frac{1}{2}}"'] & \mathscr{P}_M(D_1,D_3) \arrow[d, "ev_{\frac{1}{2}}\times ev_{\frac{1}{2}}"] \\
D_2 \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

For the unit we consider the diagram

$$\begin{tikzcd}
 & D \arrow[dl, "r"'] \arrow[dr, "i"] & \\
\mathrm{pt} & & \mathcal{P}_M(D,D)
\end{tikzcd}$$

where $r : D \to \mathrm{pt}$ is the constant map and $i : D \to \mathcal{P}_M(D,D)$ is the inclusion. This diagram defines the unit

$$u_D : \mathrm{H}_*(\mathrm{pt}) \to \mathrm{H}_*(\mathscr{P}_M(D,D))$$

as $u_D := i_* \circ r!$, where $r! : \mathrm{H}_*(\mathrm{pt}) \to \mathrm{H}_*(D)$ sends the generator to the fundamental class $[D]$.

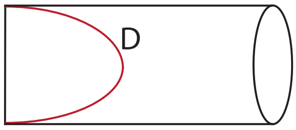


Figure 4.17: `The cobordism` I.

To finish the construction we need to define the connection maps. Consider the open-closed cobordism I between an interval, whose boundary is labeled by a D-brane D, and a circle. This cobordism is pictured in the Figure 4.17. As in the previous cases, we consider the space,

$$L_D(M) = \{\beta \in \mathrm{LM} : \beta(0) \in D\}$$

and the diagram

$$\begin{tikzcd} & \mathrm{LM} \arrow[dl, "i_D"'] \arrow[dr, "j_D"] & \\ \mathrm{LM} & & \mathscr{P}_M(D,D) \end{tikzcd}$$

We define the map ι^D by the composition,

$$\iota^D = (i_D)_* \circ (j_D)! : \mathrm{H}_*(\mathscr{P}_M(D,D)) \to \mathrm{H}_*(L_D(M)) \to \mathrm{H}_*(\mathrm{LM}).$$

For defining the umkehr map we observe that there is a pullback square

$$\begin{tikzcd} L_D(M) \arrow[r, "j_D"] \arrow[d, "ev_0"'] & \mathscr{P}_M(D,D) \arrow[d, "ev_0 \times ev_1"] \\ D \arrow[r, "\Delta"'] & D \times D \end{tikzcd}$$

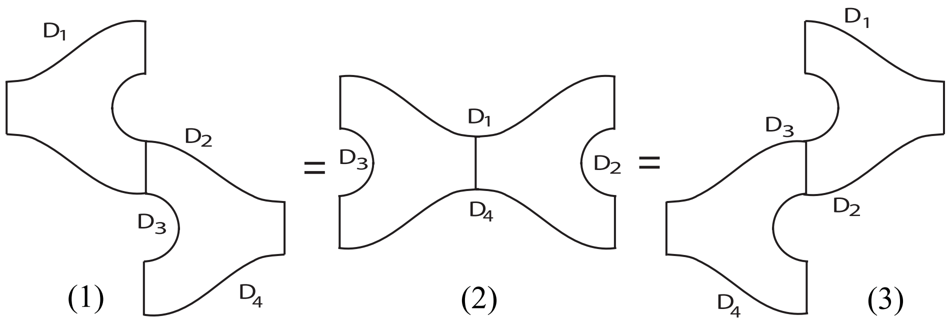
Finally we define the map $\iota_D = (j_D)_* \circ (i_D)_! : \mathrm{H}_*(\mathrm{LM}) \to \mathrm{H}_*(\mathcal{P}_M(D,D)) \to \mathrm{H}_*(\mathscr{P}_M(D,D))$, where the umkehr map $(i_D)_!$ can be defined because the existence of a pullback square,

$$\begin{tikzcd} L_D(M) \arrow[r, "j_D"] \arrow[d, "ev_0"'] & \mathrm{LM} \arrow[d, "ev_0 \times ev_0"] \\ D \arrow[r, "\Delta"'] & M \times M \end{tikzcd}$$

Theorem 4.4.10. $(\mathrm{H}_*(\mathrm{LM}), \mathscr{B})$ *is a 2D open-closed* TFT *with positive boundary.*

Proof. We only need to prove the open axioms. This because in the chapter 2 we gave the proof of the closed axioms. We will use the lema 8.2.2

1. **Abrams condition.**
   This condition is represented in the next figure.



For this we just meed to prove that the maps for (1) and (2) are the same. The same applies for the the maps for (2) and (3). The next diagrams represent these composition maps.
(1)

$$\begin{tikzcd}
& & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[dl, "i^4_{123}\times i^{12}_{34}"'] \arrow[dr, "i^{34}_{12}\times i^1_{234}"] \arrow[ddll, bend right, "\eta_1"'] \arrow[ddrr, bend left, "\xi_1"] & & \\
& \mathscr{P}_M(D_1,D_2,D_3)\times\mathscr{P}_M(D_3,D_4) \arrow[dl, "i^2_{13}\times 1"'] \arrow[dr, "j_{12}\times j_{23}\times 1"] & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3,D_4) \arrow[dl, "1\times j_{23}\times j_{34}"'] \arrow[dr, "1\times i^3_{24}"] & \\
\mathscr{P}_M(D_1,D_3)\times\mathscr{P}_M(D_3,D_4) & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3)\times\mathscr{P}_M(D_3,D_4) & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_4)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[dl, "i^2_{134}"'] \arrow[dr, "i^3_{124}"] \arrow[ddll, bend right, "\eta_2"'] \arrow[ddrr, bend left, "\xi_2"] & & \\
& \mathscr{P}_M(D_1,D_3,D_4) \arrow[dl, "j_{13}\times j_{34}"'] \arrow[dr, "i^3_{14}"] & & \mathscr{P}_M(D_1,D_2,D_4) \arrow[dl, "i^2_{14}"'] \arrow[dr, "j_{12}\times j_{24}"] & \\
\mathscr{P}_M(D_1,D_3)\times\mathscr{P}_M(D_3,D_4) & & \mathscr{P}_M(D_1,D_4) & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_4)
\end{tikzcd}$$

First, we note that $\xi_1 = \xi_2$, $\eta_1 = \eta_2$ and that the squares are pullback squares. To prove that the composition maps coincide we only need to check that the Euler class of each square coincides.

(1) In the first diagram we have the next constructions

$$\begin{tikzcd}
(i^{34}_{12} \times i^{1}_{234})^*(\mathrm{ev} \times \mathrm{ev}_{\frac{1}{2}})^*(TD_3) \arrow[r, dashed] & \mathscr{P}_M(D_1, D_2, D_3, D_4) \arrow[d, "i^{34}_{12} \times i^{1}_{234}"] & \\
 & \mathscr{P}_M(D_1, D_2) \times \mathscr{P}_M(D_2, D_3, D_4) \arrow[r, "1 \times j_{23} \times j_{34}"] \arrow[d, "\mathrm{ev} \times \mathrm{ev}_{\frac{1}{2}}"'] & \mathscr{P}_M(D_1, D_2) \times \mathscr{P}_M(D_2, D_3) \times \mathscr{P}_M(D_3, D_4) \arrow[d, "\mathrm{ev} \times \mathrm{ev} \times \mathrm{ev}"] \\
TD_3 \arrow[r, dashed] & D_2 \times D_3 \arrow[r, "1 \times \Delta"] & D_2 \times D_3 \times D_3
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{2}{3}})^*(TD_3) \arrow[r, dashed] & \mathscr{P}_M(D_1, D_2, D_3, D_4) \arrow[r, "i^{4}_{123} \times i^{12}_{34}"] \arrow[d, "\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{2}{3}}"] & \mathscr{P}_M(D_1, D_2, D_3) \times \mathscr{P}_M(D_3, D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{2}} \times \mathrm{ev} \times \mathrm{ev}"] \\
TD_3 \arrow[r, dashed] & D_2 \times D_3 \arrow[r, "1 \times \Delta"] & D_2 \times D_3 \times D_3
\end{tikzcd}$$

Note that $(\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{2}{3}})^*(TD_3) = (i^{34}_{12} \times i^{1}_{234})^*(\mathrm{ev} \times \mathrm{ev}_{\frac{1}{2}})^*(TD_3)$. Then

$$0 \to (\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{2}{3}})^*(TD_3) \to r_2^*(\mathrm{ev} \times \mathrm{ev}_{\frac{1}{2}})(TD_3) \to F_1 \to 0,$$

is exact where $F_1 = 0$.

(2) In the second case we have

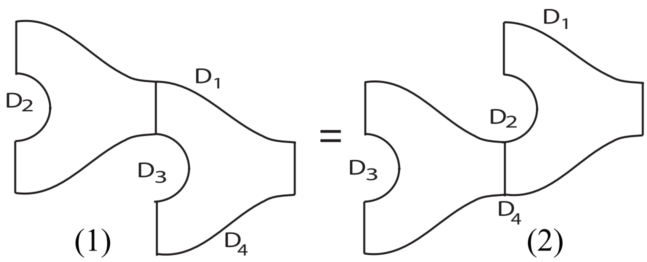
$$\begin{tikzcd}
(i^{3}_{124})^* \mathrm{ev}^*_{\frac{1}{2}}(\nu_2) \arrow[r, dashed] & \mathscr{P}_M(D_1, D_2, D_3, D_4) \arrow[d, "i^{3}_{124}"] & \\
 & \mathscr{P}_M(D_1, D_2, D_4) \arrow[r, "i^{2}_{14}"] \arrow[d, "\mathrm{ev}_{\frac{1}{2}}"'] & \mathscr{P}_M(D_1, D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{2}} \times \mathrm{ev}_{\frac{1}{2}}"] \\
\nu_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{ev}^*_{\frac{1}{2}}(\nu_2) \arrow[r, dashed] & \mathscr{P}_M(D_1, D_2, D_3, D_4) \arrow[r, "i^{2}_{134}"] \arrow[d, "\mathrm{ev}_{\frac{1}{3}}"] & \mathscr{P}_M(D_1, D_3, D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{3}} \times \mathrm{ev}_{\frac{1}{3}}"] \\
\nu_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

As $(\mathrm{ev}_{\frac{1}{3}})^*(\nu_2) = (i^{3}_{124})^*(\mathrm{ev}_{\frac{1}{2}})^*(\nu_2)$, then $F_2 = 0$.

2. **Associativity of the product.**



(1)

$$\begin{tikzcd}[column sep=tiny]
 & & & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[dll, "i^4_{123}\times i^{12}_{34}"'] \arrow[drr, "i^2_{134}"] \arrow[ddlll, bend right, "\eta"'] \arrow[ddrrr, bend left, "\xi"] & & & \\
 & \mathscr{P}_M(D_1,D_2,D_3)\times\mathscr{P}_M(D_3,D_4) \arrow[dl, "j_{12}\times j_{23}\times 1"'] \arrow[drr, "i^2_{13}\times 1"] & & & & \mathscr{P}_M(D_1,D_3,D_4) \arrow[dll, "j_{13}\times j_{34}"'] \arrow[dr, "i^3_{14}"] & \\
\mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3)\times\mathscr{P}_M(D_3,D_4) & & & \mathscr{P}_M(D_1,D_3)\times\mathscr{P}_M(D_3,D_4) & & & \mathscr{P}_M(D_1,D_4)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}[column sep=tiny]
 & & & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[dll, "i^{34}_{12}\times i^1_{234}"'] \arrow[drr, "i^3_{124}"] \arrow[ddlll, bend right, "\eta"'] \arrow[ddrrr, bend left, "\xi"] & & & \\
 & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3,D_4) \arrow[dl, "1\times j_{23}\times j_{34}"'] \arrow[drr, "1\times i^3_{24}"] & & & & \mathscr{P}_M(D_1,D_2,D_4) \arrow[dll, "j_{12}\times j_{24}"'] \arrow[dr, "i^2_{14}"] & \\
\mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3)\times\mathscr{P}_M(D_3,D_4) & & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_4) & & & \mathscr{P}_M(D_1,D_4)
\end{tikzcd}$$

First, we note that the external maps coincide.
In the diagram (1) we have

$$\begin{tikzcd}
\mathrm{ev}^*_{\frac{2}{3}}(TD_3) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[r, "i^4_{123}\times i^{12}_{34}"] \arrow[d, "\mathrm{ev}_{\frac{2}{3}}"'] & \mathscr{P}_M(D_1,D_2,D_3)\times\mathscr{P}_M(D_3,D_4) \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
TD_3 \arrow[r, dashed] & D_3 \arrow[r, "\Delta"] & D_3\times D_3
\end{tikzcd}$$

and

$$\begin{tikzcd}
(i^2_{134})^* \operatorname{ev}^*_{\frac{2}{3}}(TD_3) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[d, "i^2_{134}"'] & \\
& \mathscr{P}_M(D_1,D_3,D_4) \arrow[r, "j_{13}\times j_{34}"] \arrow[d, "\operatorname{ev}_{\frac{1}{2}}"'] & \mathscr{P}_M(D_1,D_3)\times\mathscr{P}_M(D_3,D_4) \arrow[d, "\operatorname{ev}\times\operatorname{ev}"] \\
TD_3 \arrow[r, dashed] & D_3 \arrow[r, "\Delta"] & D_3\times D_3
\end{tikzcd}$$

Note that $\operatorname{ev}_{\frac{1}{2}} \circ i^2_{134} = \operatorname{ev}_{\frac{2}{3}}$, then $\operatorname{ev}^*_{\frac{2}{3}}(TD_3) = (\operatorname{ev}_{\frac{1}{2}} \circ i^2_{134})^*(TD_3)$, and as a consequence $F_1 = 0$.

In the second diagram we have

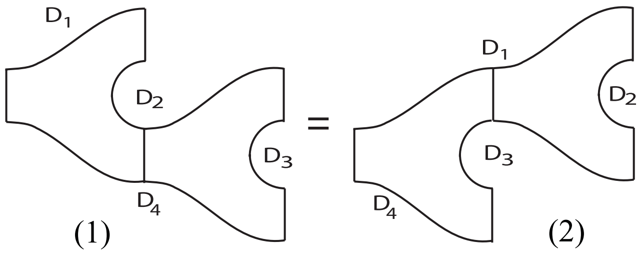
$$\begin{tikzcd}
\operatorname{ev}^*_{\frac{1}{3}}(TD_2) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[r, "i^{34}_{12}\times i^1_{234}"] \arrow[d, "\operatorname{ev}_{\frac{1}{3}}"'] & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3,D_4) \arrow[d, "\operatorname{ev}\times\operatorname{ev}"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & D_2\times D_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
(i^3_{124})^* \operatorname{ev}^*_{\frac{1}{2}}(TD_2) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[d, "i^3_{124}"'] & \\
& \mathscr{P}_M(D_1,D_2,D_4) \arrow[r] \arrow[d, "\operatorname{ev}_{\frac{1}{2}}"'] & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_4) \arrow[d, "\operatorname{ev}\times\operatorname{ev}"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & D_2\times D_2
\end{tikzcd}$$

We note that $\operatorname{ev}_{\frac{1}{3}} = \operatorname{ev}_{\frac{1}{2}} \circ i^3_{124}$. Then $\operatorname{ev}^*_{\frac{1}{3}}(TD_2) = (\operatorname{ev}_{\frac{1}{2}} \circ i^3_{124})^*(TD_2)$ and $F_2 = 0$.

3. **Coassociativity of the coproduct.**



(1)

$$\begin{tikzcd}[column sep=small]
 & & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[ddll, bend right, "\eta"'] \arrow[dl, "i^1_{234}"'] \arrow[dr, "i^3_{124}"] \arrow[ddrr, bend left, "\xi"] & & \\
 & \mathscr{P}_M(D_1,D_2,D_4) \arrow[dl, "i^2_{14}"'] \arrow[dr, "j_{12}\times j_{24}"] & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3,D_4) \arrow[dl, "1\times i^3_{24}"'] \arrow[dr, "1\times j_{23}\times j_{34}"] & \\
\mathscr{P}_M(D_1,D_4) & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_4) & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3)\times\mathscr{P}_M(D_3,D_4)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}[column sep=small]
 & & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[ddll, bend right, "\eta"'] \arrow[dl, "j_1"'] \arrow[dr, "j_2"] \arrow[ddrr, bend left, "\xi"] & & \\
 & \mathscr{P}_M(D_1,D_3,D_4) \arrow[dl, "i^3_{14}"'] \arrow[dr, "j_{13}\times j_{34}"] & & \mathscr{P}_M(D_1,D_2,D_3)\times\mathscr{P}_M(D_3,D_4) \arrow[dl, "i^2_{13}\times 1"'] \arrow[dr, "j_{12}\times j_{23}\times 1"] & \\
\mathscr{P}_M(D_1,D_4) & & \mathscr{P}_M(D_1,D_3)\times\mathscr{P}_M(D_3,D_4) & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3)\times\mathscr{P}_M(D_3,D_4)
\end{tikzcd}$$

In the first case we have

$$\begin{tikzcd}
\mathrm{ev}^*_{\frac{2}{3}}(\mu) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[r, "i^3_{124}"] \arrow[d, "\mathrm{ev}_{\frac{2}{3}}"'] & \mathscr{P}_M(D_1,D_2,D_3) \arrow[d, "\mathrm{ev}_{\frac{2}{3}}\times\mathrm{ev}_{\frac{2}{3}}"] \\
\mu \arrow[r, dashed] & D_3 \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(i^1_{234})^*(\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}})^*(\mu) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[d, "i^1_{234}"'] & \\
 & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_3,D_4) \arrow[r, "1\times i^3_{24}"] \arrow[d, "\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}}"'] & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_4) \arrow[d, "\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}_{\frac{1}{2}}"] \\
\mu \arrow[r, dashed] & D_2\times D_3 \arrow[r, "1\times\Delta"] & D_2\times M\times M
\end{tikzcd}$$

Then the sequence $0\to\mathrm{ev}^*_{\frac{2}{3}}(\mu)\to i^*_2(\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}})^*(\mu)\to F_1\to 0$ is exact, with $(i^1_{234})^*(\mathrm{ev}\times\mathrm{ev}_{\frac{1}{2}})^*(\mu)=\mathrm{ev}^*_{\frac{2}{3}}(\mu)$. And for that reason we conclude $F_1=0$.

In the second case, there is the diagram

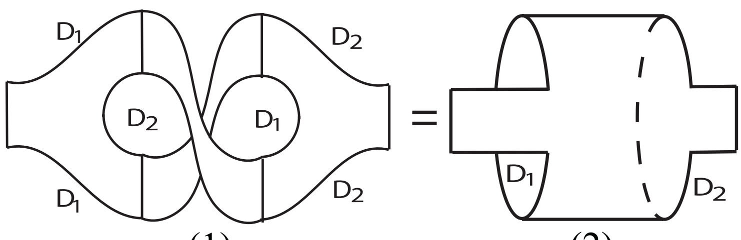
$$\begin{tikzcd}
\mathrm{ev}^*_{\frac{1}{3}}(\nu) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[r, "i^2_{134}"] \arrow[d, "\mathrm{ev}_{\frac{1}{3}}"'] & \mathscr{P}_M(D_1,D_3,D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{3}}\times\mathrm{ev}_{\frac{1}{3}}"] \\
\nu \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(i^4_{123})^*(\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev})^*(\nu) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2,D_3,D_4) \arrow[d, "i^4_{123}"'] & \\
 & \mathscr{P}_M(D_1,D_2,D_3)\times\mathscr{P}_M(D_3,D_4) \arrow[r, "i^2_{13}\times 1"] \arrow[d, "\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}"'] & \mathscr{P}_M(D_1,D_3)\times\mathscr{P}_M(D_3,D_4) \arrow[d, "\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}"] \\
\nu \arrow[r, dashed] & D_2\times D_3 \arrow[r, "1\times\Delta"] & M\times M\times D_3
\end{tikzcd}$$

As the same as before $\mathrm{ev}^*_{\frac{1}{3}}(\nu)=(i^4_{123})^*(\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev})^*(\nu)$. Consequently $F_2=0$.

4. **Cardy condition**



(1) (2)

Let be

$$L_M(D_1, D_2) = \left\{ \alpha : S^1 \to M : \ \alpha(0) \in D_1 \text{ and } \alpha\left(\frac{1}{2}\right) \in D_2 \right\}$$

(1)

$$\begin{tikzcd}
 & & L_M(D_1,D_2) \arrow[dl, "i_2"'] \arrow[dr, "i_1"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
 & \mathscr{P}_M(D_1,D_2,D_1) \arrow[dl, "i^2_{11}"'] \arrow[dr, "\tau\circ(j_{12}\times j_{21})"] & & \mathscr{P}_M(D_2,D_1,D_2) \arrow[dl, "j_{21}\times j_{12}"'] \arrow[dr, "i^1_{22}"] & \\
\mathscr{P}_M(D_1,D_1) & & \mathscr{P}_M(D_2,D_1)\times\mathscr{P}_M(D_1,D_2) & & \mathscr{P}_M(D_2,D_2)
\end{tikzcd}$$

Where τ is the transposition map.

(2)

$$\begin{tikzcd}
 & & L_M(D_1,D_2) \arrow[dl, "j_1"'] \arrow[dr, "j_2"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
 & L_{D_1}(M) \arrow[dl, "j_{D_1}"'] \arrow[dr, "i_{D_1}"] & & L_{D_2}(M) \arrow[dl, "i_{D_2}"'] \arrow[dr, "j_{D_2}"] & \\
\mathscr{P}_M(D_1,D_1) & & \mathrm{LM} & & \mathscr{P}_M(D_2,D_2)
\end{tikzcd}$$

Note that the next diagram is a pullback square

$$\begin{tikzcd}
L_M(D_1,D_2) \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathscr{P}_M(D_2,D_1,D_2) \arrow[d, "j_{21}\times j_{12}"] \\
\mathscr{P}_M(D_1,D_2,D_1) \arrow[r, "\tau\circ(j_{12}\times j_{21})"'] & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_2)
\end{tikzcd}$$

Then, for the first case

$$\begin{tikzcd}
\mathrm{ev}^*(TD_1) \arrow[r, dashed] & L_M(D_1,D_2) \arrow[r, "i_1"] \arrow[d, "\mathrm{ev}"'] & \mathscr{P}_M(D_1,D_2,D_1) \arrow[d, "\mathrm{ev}_0\times\mathrm{ev}_1"] \\
TD_1 \arrow[r, dashed] & D_1 \arrow[r, "\Delta"'] & D_1\times D_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
(i_2)^*\,\mathrm{ev}^*(TD_1) \arrow[r, dashed] & L_M(D_1,D_2) \arrow[d, "i_2"'] & \\
 & \mathscr{P}_M(D_2,D_1,D_2) \arrow[r, "\tau\circ(j_{21}\times j_{12})"] \arrow[d, "\mathrm{ev}"'] & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_1) \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
TD_1 \arrow[r, dashed] & D_1 \arrow[r, "\Delta"'] & D_1\times D_1
\end{tikzcd}$$

The next equality holds $\mathrm{ev}^*(TD_1) = (\mathrm{ev}\circ i_2)^*(TD_1)$. And we conclude $F_1 = 0$.

In the second case

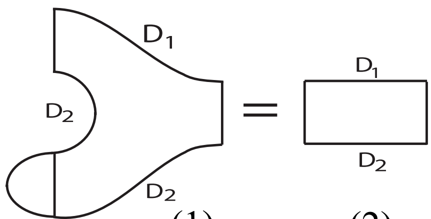
$$\begin{tikzcd}
\mathrm{ev}^*(\zeta) \arrow[r, dashed] & L_M(D_1,D_2) \arrow[r, "j_1"] \arrow[d, "\mathrm{ev}"'] & L_{D_1}(M) \arrow[d, "\mathrm{ev}_{\frac{1}{2}}\times\mathrm{ev}_{\frac{1}{2}}"] \\
\zeta \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
j_2^*\,\mathrm{ev}^*(\zeta) \arrow[r, dashed] & L_M(D_1,D_2) \arrow[d, "j_2"'] & \\
 & L_{D_2}(M) \arrow[r, "i_{D_2}"] \arrow[d, "\mathrm{ev}"'] & \mathrm{LM} \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
\zeta \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

In the same way $\mathrm{ev}^*(\zeta) = (\mathrm{ev}\circ j_2)^*(\zeta)$, then $F_2 = 0$.

5. **Unit axiom**



(1) (2)

(1)

$$\begin{tikzcd}
& & & \mathscr{P}_M(D_1,D_2) \arrow[dll, "1\times\epsilon_1"'] \arrow[drr, "i"] \arrow[ddlll, bend right, "\eta"'] \arrow[ddrrr, bend left, "\xi"] & & & \\
& \mathscr{P}_M(D_1,D_2)\times D_2 \arrow[dl, "1\times r"'] \arrow[drr, "1\times\iota"] & & & & \mathscr{P}_M(D_1,D_2,D_2) \arrow[dll, "j_{12}\times j_{22}"'] \arrow[dr, "i^2_{12}"] & \\
\mathscr{P}_M(D_1,D_2)\times \mathrm{pt} & & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_2) & & & \mathscr{P}_M(D_1,D_2)
\end{tikzcd}$$

First, we note that the next diagram is a pullback square.

$$\begin{tikzcd}
\mathscr{P}_M(D_1,D_2) \arrow[r, "i"] \arrow[d, "1\times\epsilon_1"'] & \mathscr{P}_M(D_1,D_2,D_2) \arrow[d, "j_{12}\times j_{22}"] \\
\mathscr{P}_M(D_1,D_2)\times D_2 \arrow[r, "1\times\iota"'] & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_2)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& & \mathscr{P}_M(D_1,D_2) \arrow[dl, "id"'] \arrow[dr, "id"] \arrow[ddll, bend right, "id"'] \arrow[ddrr, bend left, "id"] & & \\
& \mathscr{P}_M(D_1,D_2) \arrow[dl, "id"'] \arrow[dr, "id"] & & \mathscr{P}_M(D_1,D_2) \arrow[dl, "id"'] \arrow[dr, "id"] & \\
\mathscr{P}_M(D_1,D_2) & & \mathscr{P}_M(D_1,D_2) & & \mathscr{P}_M(D_1,D_2)
\end{tikzcd}$$

It is clear that for the second diagram we have $F_2 = 0$. Basically we have $\eta = id$ and $\xi \simeq id$, then $\xi_* = id_*$. In the first diagram the umkher map $(1\times\epsilon_1)!$ due to the next square

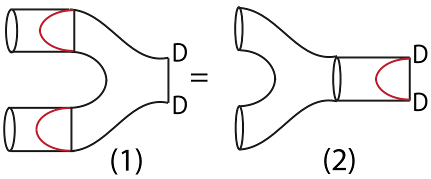
$$\begin{tikzcd}
\mathrm{ev}^*(TD_2) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2) \arrow[r, "1\times\epsilon_1"] \arrow[d, "\mathrm{ev}"'] & \mathscr{P}_M(D_1,D_2)\times D_2 \arrow[d, "\mathrm{ev}\times id"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"] & D_2\times D_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
i^*\,\mathrm{ev}^*_{\frac{1}{2}}(TD_2) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2) \arrow[d, "i"] & \\
& \mathscr{P}_M(D_1,D_2,D_2) \arrow[r, "j_{12}\times j_{22}"] \arrow[d, "\mathrm{ev}_{\frac{1}{2}}"'] & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_2) \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & D_2\times D_2
\end{tikzcd}$$

Since $(\mathrm{ev}_{\frac{1}{2}}\circ i)^*(TD_2)=\mathrm{ev}^*(TD_2)$, then $F_1=0$.

6. ι_D **is morphism of algebras**



Let be

$$\mathrm{Map}_D(8,M)=\{\alpha:8\to M:\ \alpha(0)\in D\}$$

(1)

$$\begin{tikzcd}
& & \mathrm{Map}_D(8,M) \arrow[dl, "i"'] \arrow[dr, "j"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
& \mathrm{Map}(8,M) \arrow[dl, "e"'] \arrow[dr, "\rho"] & & L_D(M) \arrow[dl, "i_D"'] \arrow[dr, "j_D"] & \\
\mathrm{LM}\times\mathrm{LM} & & \mathrm{LM} & & \mathscr{P}_M(D,D)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& & \mathrm{Map}_D(8,M) \arrow[dl, "i_1\times i_1"'] \arrow[dr, "j_1"] \arrow[ddll, bend right, "\eta"'] \arrow[ddrr, bend left, "\xi"] & & \\
& L_D(M)\times L_D(M) \arrow[dl, "i_D\times i_D"'] \arrow[dr, "j_D\times j_D"] & & \mathscr{P}_M(D,D,D) \arrow[dl, "j_{11}\times j_{11}"'] \arrow[dr, "i^1_{11}"] & \\
\mathrm{LM}\times\mathrm{LM} & & \mathscr{P}_M(D,D)\times\mathscr{P}_M(D,D) & & \mathscr{P}_M(D,D)
\end{tikzcd}$$

In the first diagram there is the square

$$
\begin{tikzcd}
\mathrm{ev}^*(\varrho) \arrow[r, dashed] & \mathrm{Map}_D(8, M) \arrow[r, "i"] \arrow[d, "\mathrm{ev}"'] & \mathrm{Map}(8, M) \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
\varrho \arrow[r, dashed] & D \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
j^*\,\mathrm{ev}^*(\varrho) \arrow[r, dashed] & \mathrm{Map}_D(8, M) \arrow[d, "j"'] & \\
 & L_D(M) \arrow[r, "i_D"] \arrow[d, "\mathrm{ev}"'] & \mathrm{LM} \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
\varrho \arrow[r, dashed] & D \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}
$$

Clearly $\mathrm{ev}^*(\varrho) = j^*\,\mathrm{ev}^*(\varrho)$. Then $F_1 = 0$.

By the other hand in (2) we have

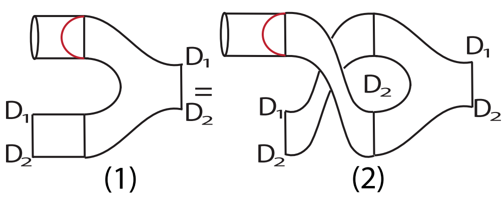
$$
\begin{tikzcd}
\mathrm{ev}^*(TD) \arrow[r, dashed] & \mathrm{Map}_D(8, M) \arrow[r] \arrow[d, "\mathrm{ev}"'] & L_D(M) \times L_D(M) \arrow[d, "\mathrm{ev}\times\mathrm{ev}"] \\
TD \arrow[r, dashed] & D \arrow[r, "\Delta"] & D \times D
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
j'^*\,\mathrm{ev}^*_{\frac{1}{2}}(TD) \arrow[r, dashed] & \mathrm{Map}_D(8, M) \arrow[d, "j_1"'] & \\
 & \mathscr{P}_M(D, D, D) \arrow[r, "j_{11}\times j_{11}"] \arrow[d, "\mathrm{ev}_{\frac{1}{2}}"'] & \mathscr{P}_M(D, D) \times \mathscr{P}_M(D, D) \arrow[d, "\mathrm{ev}_1\times\mathrm{ev}_0"] \\
TD \arrow[r, dashed] & D \arrow[r, "\Delta"] & D \times D
\end{tikzcd}
$$

As before, $j_1^*\,\mathrm{ev}^*(TD) = \mathrm{ev}^*(TD)$. Consequently $F_2 = 0$.

7. **ι is a central morphism**



(1)

$$\begin{tikzcd}
 & & & L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2) \arrow[dll, "i"'] \arrow[drr, "j"] \arrow[ddlll, bend right, "\eta"'] \arrow[ddrrr, bend left, "\xi"] & & & \\
 & L_{D_1}(M)\times\mathscr{P}_M(D_1,D_2) \arrow[dl, "i_{D_1}\times 1"'] \arrow[drr, "j_{D_1}\times 1"] & & & & \mathscr{P}_M(D_1,D_1,D_2) \arrow[dll, "j_{11}\times j_{12}"'] \arrow[dr, "i^1_{12}"] & \\
\mathrm{LM}\times\mathscr{P}_M(D_1,D_2) & & & \mathscr{P}_M(D_1,D_1)\times\mathscr{P}_M(D_1,D_2) & & & \mathscr{P}_M(D_1,D_2)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & & \mathscr{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0}L_{D_2}(M) \arrow[dll, "\tau\circ i'"'] \arrow[drr, "j'"] \arrow[ddlll, bend right, "\eta'"'] \arrow[ddrrr, bend left, "\xi'"] & & & \\
 & L_{D_2}(M)\times\mathscr{P}_M(D_1,D_2) \arrow[dl, "i_{D_2}\times 1"'] \arrow[drr, "\tau\circ(j_{D_2}\times 1)"] & & & & \mathscr{P}_M(D_1,D_2,D_2) \arrow[dll, "j_{12}\times j_{22}"'] \arrow[dr, "i^2_{12}"] & \\
\mathrm{LM}\times\mathscr{P}_M(D_1,D_2) & & & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_2) & & & \mathscr{P}_M(D_1,D_2)
\end{tikzcd}$$

Note that in the last case we have that the pullback spaces are different. For this particular case we use the corollary 9.2.4, for this, we first need to prove that $L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2)$ and $L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2)$ are homotopically equivalent spaces. For this we construct the maps.

We define the map

$$\begin{array}{rccc}\varphi: & \varphi: L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2) & \longrightarrow & \mathscr{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0}L_{D_2}(T) \\ & (\alpha,\beta) & \longmapsto & (\beta,\overline{\beta}*\alpha*\beta),\end{array}$$

and in the same way let be

$$\begin{array}{rccc}\psi: & \mathscr{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0}L_{D_2}(T) & \longrightarrow & L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2) \\ & (\gamma,\delta) & \longmapsto & (\gamma*\delta\overline{\gamma},\gamma).\end{array}$$

See this maps in the figura 4.18.

Now we check that this maps determine a homotopy equivalence.

$$\begin{aligned}\psi\circ\varphi(\alpha,\beta) &= \psi(\beta,\overline{\beta}*\alpha*\beta) = (\alpha,\overline{\alpha}*\alpha*\beta*\overline{\alpha}*\alpha)\simeq(\alpha,\beta)\\ \varphi\circ\psi(\gamma,\delta) &= \varphi(\gamma*\delta*\overline{\gamma},\gamma) \ = (\gamma,\overline{\gamma}*\gamma*\delta*\overline{\gamma}*\gamma)\simeq(\gamma,\delta).\end{aligned}$$

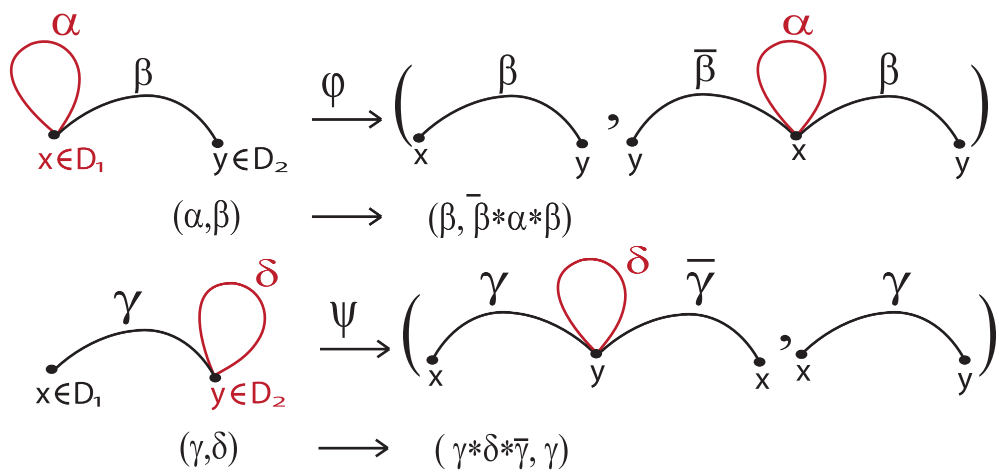


Figure 4.18: `The map` $\varphi : L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2) \to L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2)$

Finally we need to check that the external maps are homotopic.

$$\begin{aligned}
\eta'\circ\varphi(\alpha,\beta) &= \eta'(\beta,\overline{\beta}*\alpha*\beta) \quad (\overline{\beta}*\alpha*\beta,\beta)\simeq(\alpha,\beta)\\
\eta(\alpha,\beta) &= (\alpha,\beta)\\
\xi'\circ\varphi(\alpha,\beta) &= \xi'(\beta,\overline{\beta}*\alpha*\beta) = \beta*\overline{\beta}*\alpha*\beta\simeq\alpha*\beta\\
\xi(\alpha,\beta) &= (\alpha*\beta)\\
\eta\circ\psi(\gamma,\delta) &= \eta(\gamma*\delta*\overline{\gamma},\gamma) \ = (\gamma*\delta*\overline{\gamma},\gamma)\simeq(\delta,\gamma)\\
\eta'(\gamma,\delta) &= (\delta,\gamma)\\
\xi\circ\psi(\gamma,\delta) &= \xi(\gamma*\delta*\overline{\gamma},\gamma) \ = \gamma*\delta*\overline{\gamma}*\gamma\simeq\gamma*\delta\\
\xi'(\gamma,\delta) &= \gamma*\delta
\end{aligned}$$

Then, we can use the corollary. It rest to calculate the Euler class.

In the first diagram we have

$$\begin{tikzcd}
\mathrm{ev}_\infty^*(TD_1) \arrow[r, dashed] & L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2) \arrow[r, "i"] \arrow[d, "\mathrm{ev}_\infty"'] & L_{D_1}\times\mathscr{P}_M(D_1,D_2) \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TD_1 \arrow[r, dashed] & D_1 \arrow[r, "\Delta"'] & D_1\times D_1
\end{tikzcd}$$

and

$$
\begin{tikzcd}
j^* \operatorname{ev}^*_{\frac{1}{2}}(TD_1) \arrow[r, dashed] & L_{D_1}(T)_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2) \arrow[d, "j"] & \\
 & \mathscr{P}_M(D_1,D_1,D_2) \arrow[r, "j_{11}\times j_{12}"] \arrow[d, "\operatorname{ev}_{\frac{1}{2}}"] & \mathscr{P}_M(D_1,D_1)\times\mathscr{P}_M(D_1,D_2) \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TD_1 \arrow[r, dashed] & D_1 \arrow[r, "\Delta"'] & D_1\times D_1
\end{tikzcd}
$$

Note that $j^* \operatorname{ev}^*_{\frac{1}{2}}(TD_1) = \operatorname{ev}^*_{\infty}(TD_1)$. Then $F_1 = 0$.

In the second diagram there is the square

$$
\begin{tikzcd}
\operatorname{ev}^*_{\infty}(TD_2) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0}L_{D_2}(M) \arrow[r, "\tau\circ i'"] \arrow[d, "\operatorname{ev}_{\infty}"] & L_{D_2}(M)\times\mathscr{P}_M(D_1,D_2) \arrow[d, "\epsilon_0\times\epsilon_1"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & D_2\times D_2
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
j'^* \operatorname{ev}^*_{\frac{1}{2}}(TD_2) \arrow[r, dashed] & \mathscr{P}_M(D_1,D_2)_{\epsilon_1}\times_{\epsilon_0}L_{D_2}(M) \arrow[d, "j'"] & \\
 & \mathscr{P}_M(D_1,D_2,D_2) \arrow[r, "j_{12}\times j_{22}"] \arrow[d, "\operatorname{ev}_{\frac{1}{2}}"] & \mathscr{P}_M(D_1,D_2)\times\mathscr{P}_M(D_2,D_2) \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TD_2 \arrow[r, dashed] & D_2 \arrow[r, "\Delta"'] & D_2\times D_2
\end{tikzcd}
$$

Clearly $j'^* \operatorname{ev}^*_{\frac{1}{2}}(TD_2)$ and $\operatorname{ev}^*_{\infty}(TD_2)$ coincide, then $F_2 = 0$.
Finally, we need to determine that $\nu_\varphi = 0$. For this we will construct the next homotopy.

$$
\begin{array}{rccc}
H: & I\times(L_{D_1}M_{\epsilon_1}\times_{\epsilon_0}\mathscr{P}_M(D_1,D_2) & \longrightarrow & \mathrm{LM}\,{}_{\epsilon_1}\times_{\epsilon}\mathscr{P}_M(D_1,D_2)\times I \\
 & (s,(\alpha,\beta)) & \longmapsto & (\overline{\beta_s}*\alpha*\beta_s,\beta,s)
\end{array}
$$

where the map $\epsilon : I\times\mathscr{P}_M(D_1,D_2)\to M$ is given by $\epsilon(s,\beta) := \beta(s)$, and the curve $\beta_s : I\to M$ is $\beta_s(t) = \beta(st)$ for all $t,s\in I$.

Note that $H(0,(\alpha,\beta)) = (\alpha,\beta)$ and $H(1,(\alpha,\beta)) = (\overline{\beta}*\alpha*\beta,\beta) = \tau\circ\varphi(\alpha,\beta)$. Now we need to prove that these spaces of infinite dimension has a smooth structure i.e. a infinite dimensional manifold; see [KM91]. The space $W := \mathrm{LM}\,{}_{\epsilon_1}\!\times_{\epsilon}\mathscr{P}_M(D_1, D_2) \times I$ is determined by the next pullback square.

$$\begin{tikzcd}
W = \mathrm{LM}\,{}_{\epsilon_1}\!\times_{\epsilon}\mathscr{P}_M(D_1, D_2) \times I \arrow[r] \arrow[d, "\epsilon\times 1"'] & \mathrm{LM} \times \mathscr{P}_M(D_1, D_2) \times I \arrow[d, "\epsilon_0\times\epsilon\times 1"] \\
M \times I \arrow[r, "\Delta\times 1"'] & M \times M \times I
\end{tikzcd}$$

Then W is a infinite dimensional manifold. In the other hand, the next pullback square give us that the spaces $Z_s := L_{D_1}M\,{}_{\epsilon_1}\!\times_{\epsilon_s}\mathscr{P}_M(D_1, D_2)$ are submanifolds of W of codimension one.

$$\begin{tikzcd}
Z_s = L_{D_1}M\,{}_{\epsilon_1}\!\times_{\epsilon_s}\mathscr{P}_M(D_1, D_2) \times \{s\} \arrow[r] \arrow[d, "\epsilon_\infty\times s"'] & \mathrm{LM}\,{}_{\epsilon_1}\!\times_{\epsilon}\mathscr{P}_M(D_1, D_2) \times I \arrow[d, "\epsilon_\infty\times 1"] \\
M \times \{s\} \arrow[r, hook] & M \times I
\end{tikzcd}$$

In particular we have the next situation

$$\begin{tikzcd}
Z_0 = L_{D_1}M\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathscr{P}_M(D_1, D_2) \arrow[d, "\mathrm{Id}"'] & & Z_0 = L_{D_1}M\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathscr{P}_M(D_1, D_2) \arrow[d, "\varphi"] \\
Z_0 = L_{D_1}M\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathscr{P}_M(D_1, D_2) \arrow[r, phantom, "\overset{H}{\underset{\simeq}{\Longrightarrow}}"] & & Z_1 = \mathscr{P}_M(D_1, D_2)_{\epsilon_1}\!\times_{\epsilon_0} L_{D_2}M
\end{tikzcd}$$

Then $\nu_\varphi = 0$ and $e(\nu_\varphi) = 1$.

♣

Chapter 5
G-Topological Field Theory

An important construction in string theory is the orbifold construction. Abstractly, this can be carried out whenever the closed string background has a group G of automorphisms. There are two steps in defining an orbifold theory. First, one must extend the theory by introducing "external" gauge fields, which are G-bundles (with connection) on the world-sheets. Next, one must construct a new theory by summing over all possible G-bundles (and connections).

5.1 Equivariant closed theories

Let us begin with some general remarks. In n-dimensional topological field theory one begin with a category nCop whose objects are oriented $(n-1)$-manifolds and whose morphisms are oriented cobordisms. Physicists say that a theory admits a group G as a global symmetry group if G acts on the vector space associated to each $(n-1)$-manifold. The linear operator associated to each cobordism is a G-equivariant map. When we have such a "global" symmetry group G we can ask whether the symmetry can be gauged, i.e. whether elements of G can be applied independently in some sense at each point of space-time. Mathematically the process of "gauging" has a very elegant description: it amounts to extending the field theory functor from the category nCob to the category nCob_G whose objects are $(n-1)$-manifolds equipped with a principal G bundle, and whose morphisms are cobordisms with a G-bundle.

We have another interpretation of this category, this view is due to Turaev [Tur99] and it consists on working in the language of pointed homotopy theory (smooth version). For this, we set a path-connected topological space X with basis point $x \in X$. We call an X-manifold by a pair consisting of a pointed closed oriented manifold M and a characteristic map $g_M : M \to X$. For M and M' as before we can talk of a X-diffeomorphisms between them. A cobordism W from M_0 to M_1 is endowing with a map $W \to M$ sending the basis point of the boundary components into x. Both basis M_0 and M_1 are considered as X-manifolds with characteristic maps obtained by restricting the given map $W \to M$. An X-diffeomorphism of a X-cobordisms $f : (W, M_0, M_1) \to (W', M_0', M_1')$ is an orientation preserving diffeomorphism inducing a X-diffeomorphisms $M_0 \to M_0'$, $M_1 \to M_1'$ and such that $g_W = g_{W'} f$ where g_W, $g_{W'}$ are the characteristic maps of W, W' respectively.

We can glue X-cobordisms along the basiss. If (W_0, M_0, N), (W_1, N', M_1) are X-cobordisms and $f : N \to N'$ is an X-diffeomorphism then the gluing of W_0 with W_1 along f yields a new X-cobordism with basiss M_0 and M_1.

If we make a quotient by identifying diffeomorphic objects, hence any diffeomorphism becomes an identity. We get an alternative viewpoint for $nCob_G$, for this we take $X := BG$ (the classifying space of Milnor).

Definition 5.1.1. A *G-equivariant* TFT is a symmetrical monoidal functor from nCob_G to $\mathrm{Vect}_{\mathbb{C}}$.

5.2 G-Frobenius algebras

We start with the definition of the algebraic data with a proposition which is related with the Frobenius structure of the G-invariant part and with the equivariant version for the Abrams theorem. This definition was done in the seminar paper by Moore and Segal [MS06].

Definition 5.2.1. A *G-Frobenius algebra* is an algebra $\mathscr{C} = \oplus_{g \in G} \mathscr{C}_g$, where $\mathscr{C}_g$ is a vector space of finite dimension for all $g \in G$ such that

1. There is a homomorphism $\alpha : G \to \mathrm{Aut}(\mathscr{C})$, see figure 5.1, where $\mathrm{Aut}(\mathscr{C})$ is the algebra of homomorphisms of $\mathscr{C}$ such that

$$\alpha_h : \mathscr{C}_g \to \mathscr{C}_{hgh^{-1}},$$

   and for every $g \in G$ we have

$$\alpha_g|_{\mathscr{C}_g} = 1_{\mathscr{C}_g}.$$

   Note that $\alpha_e : \mathscr{C}_g \to \mathscr{C}_g$ is the identity map.

2. There is a G-invariant trace or counit $\varepsilon : \mathscr{C}_e \to \mathbb{C}$ which induce nondegenerate pairings, see figure 5.2,

$$\theta_g : \mathscr{C}_g \otimes \mathscr{C}_{g^{-1}} \to \mathbb{C}.$$

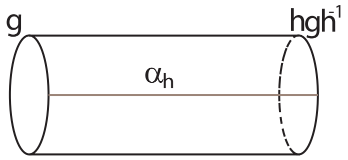


Figure 5.1: `The action` $\alpha_h : \mathscr{C}_g \to \mathscr{C}_{hgh^{-1}}$.

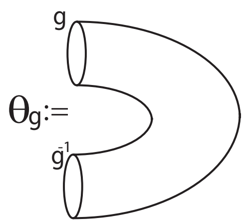


Figure 5.2: `The pairing` $\theta_g : \mathscr{C}_g \otimes \mathscr{C}_{g^{-1}} \to \mathbb{C}$.

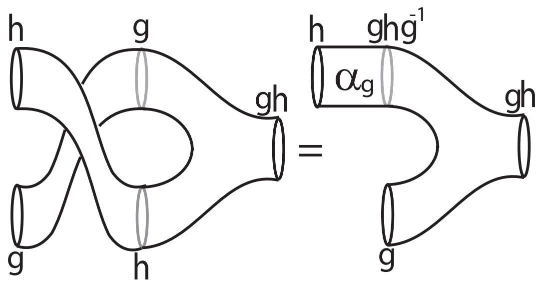


Figure 5.3: `The twisted commutativity of the product.`

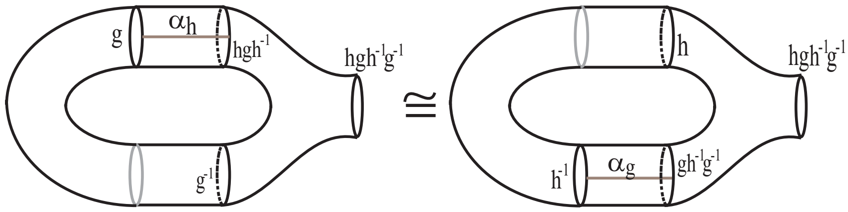


Figure 5.4: `Torus axiom`.

3. For all $x \in \mathscr{C}_g$ and $y \in \mathscr{C}_h$ we have that the product is twisted commutative (see figura 5.3), i.e.
$$xy = \alpha_g(y)x.$$

4. Let $\Delta_g = \sum_i \xi_i^g \otimes \xi_i^{g^{-1}} \in \mathscr{C}_g \otimes \mathscr{C}_{g^{-1}}$ the *Euler element*, where $\{\xi_i^g\}$ is a basis of $\mathscr{C}_g$ and $\left\{\xi_i^{g^{-1}}\right\}$ is the dual basis of $\mathscr{C}_{g^{-1}}$. For all $g, h \in G$ (see figure 5.4) the identity
$$\sum_i \alpha_h(\xi_i^g)\xi_i^{g^{-1}} = \sum_i \xi_i^h \alpha_g(\xi_i^{h^{-1}})$$

The next proposition gives us a natural consequence of this definition. This is that the G-invariant part of the G-Frobenius algebra $\mathscr{C}^G$ is a Frobenius algebra.

Proposition 5.2.2. *For $\mathscr{C}$ a G-Frobenius algebra, we have the G-invariant part of this algebra, denoted by $\mathscr{C}_{orb}$, is a Frobenius algebra.*

Proof. Let be $\mathscr{C}_{orb} := \mathscr{C}^G = (\oplus_{g\in G}\mathscr{C}_g)^G$. Note that $\mathscr{C}_{orb} \cong \oplus_{g\in T}\mathscr{C}_g^{C(g)}$ where T is a set of representatives for the conjugacy classes in G and $C(g)$ is the centralizer of $g \in G$. The maps that define this isomorphism are

$$\begin{array}{cccc} \Psi: & \bigoplus_{g\in T}\mathscr{C}_g^{C(g)} & \longrightarrow & \left(\bigoplus_{b\in G}\mathscr{C}_g\right)^G \\ & \sum_{g\in G} y_g & \longmapsto & \sum_{g\in T}\sum_{h\in[g],h=kgk^{-1}} \alpha_k(y_g) \end{array}$$

and

$$\begin{array}{cccc} \Upsilon: & \left(\bigoplus_{b\in G}\mathscr{C}_g\right)^G & \longrightarrow & \bigoplus_{g\in T}\mathscr{C}_g^{C(g)} \\ & \sum_{g\in G} x_g & \longmapsto & \sum_{g\in T} x_g. \end{array}$$

First, we prove that $\mathscr{C}_{orb}$ is an algebra. The product is simply the restriction of the product in $\mathscr{C}$, this is because for $x, y \in \mathscr{C}_{orb}$ we have that $g \cdot x = \alpha_g(x) = x$ and $g \cdot y = \alpha_g(y) = y$ for all $g \in G$, then $g \cdot xy = \alpha_g(xy) = \alpha_g(x)\alpha_g(y) = xy$.

An additional property is the commutative of the product, to check this we take $x = \sum_{g \in G} x_g$ and $y = \sum_{h \in G} y_h \in \mathscr{C}_{orb}$. The calculations are as follows:

$$xy = \sum_{g \in G} \sum_{h \in G} x_g y_h = \sum_{g,h \in G} \alpha_g(y_h) x_g = \sum_{g \in G} \alpha_g \left(\sum_{h \in G} y_h \right) x_g = \sum_{g \in G} y x_g = yx.$$

For the Frobenius structure we define the trace $\varepsilon : \mathscr{C}_{orb} \to \mathbb{C}$ as the restriction of $\varepsilon : \mathscr{C} \to \mathbb{C}$ with the value zero on $\mathscr{C}_g$ with $g \neq e$. To complete the proof we need to prove that the induced pairing is non-degenerate.
Let $x = \sum_{g \in G} x_g \in \mathscr{C}_{orb}$ and suppose $\varepsilon(xy) = 0$, for all $y \in \mathscr{C}_{orb}$. We need to prove that $x = 0$. If we show that $x_g = 0$ for all $g \in T$ we finish. This is because $x = \sum_{g \in T} \sum_{h \in [g], h = kgk^{-1}} \alpha_k(x_g)$. We can consider $y_h \in \mathscr{C}_h$, where h is the representant of $[h] \in T$, then $y := \sum_{k \in [h], k = lhl^{-1}} \alpha_l(y_h) \in \mathscr{C}_{orb}$. Now

$$\varepsilon(xy) = |[h]| \varepsilon(x_{h^{-1}}(y_h))$$

Hence $\varepsilon(x_{h^{-1}} y_h) = 0$ for all $y_h \in \mathscr{C}_h$, and then $x_{h^{-1}} = 0$ for every $h \in T$. Finally $x = 0$.

♣

Corollary 5.2.3. *The coproduct in $\mathscr{C}_{orb}$ is*

$$\Delta = (m \otimes 1) \cdot (1 \otimes \Theta)$$

where $\Theta : \mathbb{C} \to \mathscr{C}_{orb} \otimes \mathscr{C}_{orb}$ is the copairing.

Proof. We only need to construct a basis of $\mathscr{C}_{orb}$.
Let be $\{e_i^g\}$ a basis of $\mathscr{C}_g$ such that $\alpha_k(e_i^g) = e_i^{kgk^{-1}}$ is a basis of $\mathscr{C}_{kgk^{-1}}$.
For $x \in \mathscr{C}_{orb}$ there is the identity

$$x = \sum_{g \in T} \sum_{h \in [g], h = kgk^{-1}} \alpha_k(x_g),$$

where $x_g = \sum_i \lambda_i^g e_i^g \in \mathscr{C}_g$. Therefore

$$x = \sum_{g \in T} \sum_{h \in [g], h = kgk^{-1}} \sum_i \lambda_i^g \alpha_k(e_i^g) = \sum_{g \in T} \sum_i \lambda_i^g \sum_{h \in [g], h = kgk^{-1}} e_i^{kgk^{-1}} = \sum_{g \in T} \sum_i \lambda_i^g E_{i,g}$$

where $E_{i,g} = \sum_{h \in [g]} e_i^h$. This proves that $\{E_{i,g}\}$ is a generator of $\mathscr{C}_{orb}$. Now we prove that this set is linearly independent. Suppose that $\sum_{g \in T, i \in I_g} \beta_{i,g} E_{i,g} = 0$, then $\sum_{g \in T, i \in I_g} \sum_{h \in [g]} \beta_{i,g} e_i^h = \sum_{g \in G} \left(\sum_{i \in I_g} \beta_{i,g} e_i^h \right) = 0$, where $\beta_{i,g} = \beta_{i,h}$ if h and g are in the same conjugation class. As $\sum_{i\in I_g}\beta_{i,g}E_{i,g}\in\mathscr{C}_g$ hence $\sum_{i\in I_g}\beta_{i,g}E_{i,g}=0$ for all $g\in G$. We use that e_i^g is a basis of $\mathscr{C}_g$, to prove that $\beta_{i,g}=0$ for all $g\in T$, $i\in I_g$.

Note that for $E_{i,g}\in\mathscr{C}_{orb}$ and $k\in G$ we have $k\cdot E_{i,g}=\sum_{h\in[g]}\alpha_k(e_i^h)=\sum_{h\in[g]}e_i^{khk^{-1}}=\sum_{l\in[g]}e_i^l=E_{i,g}$, where $l=khk^{-1}\in[g]$.

We can construct $\left\{E_{i,g}^{\#}\right\}=\frac{1}{|[g]|}\sum_{h\in[g]}e_i^{h^{-1}}$ as the dual basis of $\mathscr{C}_{orb}$. Then

$$\Theta(1)=\sum_{g\in T,i\in I_g}E_{i,g}^{\#}\otimes E_{i,g}$$

and

$$\Delta(x)=\sum_{g\in T,i\in I_g}xE_{i,g}^{\#}\otimes E_{i,g}=\sum_{g\in T,i\in I_g}\sum_{h,k\in[g]}\frac{1}{|[g]|}xe_i^{h^{-1}}\otimes e_i^k.$$

♣

Theorem 5.2.4. *(Abrams equivariant case) Let $\mathscr{C}=\oplus_{g\in G}\mathscr{C}_g$ an algebra with an associative product $m_{g,h}:\mathscr{C}_g\otimes\mathscr{C}_h\to\mathscr{C}_{gh}$ and a unit $u:\mathbb{C}\to\mathscr{C}_e$, where every $\mathscr{C}_g$ is a finite dimension space. We have that a trace $\varepsilon:\mathscr{C}_e\to\mathbb{C}$ is non-degenerate if and only if it has a coassociative coproduct $\Delta_{g,h}:\mathscr{C}_{gh}\to\mathscr{C}_g\otimes\mathscr{C}_h$, with ε as its counit, such that for every $g,h,k\in G$ the following diagrams commute:*

$$\begin{tikzcd}
\mathscr{C}_g\otimes\mathscr{C}_{hk} \arrow[r, "m_{g,hk}"] \arrow[d, "1\otimes\Delta_{h,k}"'] & \mathscr{C}_{ghk} \arrow[d, "\Delta_{gh,k}"] \\
\mathscr{C}_g\otimes\mathscr{C}_h\otimes\mathscr{C}_k \arrow[r, "m_{g,h}\otimes 1"'] & \mathscr{C}_{gh}\otimes\mathscr{C}_k
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C}_{gh}\otimes\mathscr{C}_{k} \arrow[r, "m_{gh,k}"] \arrow[d, "\Delta_{g,h}\otimes 1"'] & \mathscr{C}_{ghk} \arrow[d, "\Delta_{g,hk}"] \\
\mathscr{C}_g\otimes\mathscr{C}_h\otimes\mathscr{C}_k \arrow[r, "1\otimes m_{h,k}"'] & \mathscr{C}_{g}\otimes\mathscr{C}_{hk}
\end{tikzcd}
\tag{5.1}$$

Proof. The necessity is the nontrivial part and for this we define the coproduct

$$\begin{tikzcd}
\mathscr{C}_{gh} \arrow[rrr, "\Delta_{g,h}"] \arrow[d, "\Phi_f"'] & & & \mathscr{C}_g\otimes\mathscr{C}_h \\
\mathscr{C}^*_{h^{-1}g^{-1}} \arrow[r, "{m_{h^{-1},g^{-1}}}^*"'] & \mathscr{C}^*_{h^{-1}}\otimes\mathscr{C}^*_{g^{-1}} \arrow[rr, "\tau"'] & & \mathscr{C}^*_{g^{-1}}\otimes\mathscr{C}^*_{h^{-1}} \arrow[u, "\Phi_g^{-1}\otimes\Phi_h^{-1}"']
\end{tikzcd}$$

where $\Phi_f(x)(y)=\varepsilon(m_{f,f^{-1}}(x\otimes y))$. This coproduct is coassociative and satisfies the two diagrams 5.1.

♣

Theorem 5.2.5. *Every 2D G-equivariant topological field theory defines and is defined by a G-Frobenius algebra, i.e. the categories associated to this structures are equivalent.*

In order to prove this result, we note that in the same way as before the only statement for checking is that the axioms for a G-Frobenius algebra are the only sewing conditions to cut a cobordism in all possible ways. A good reference for this result is [MS06].

5.3 Nearly G-Frobenius algebras

Definition 5.3.1. A *nearly G-Frobenius algebra* is an algebra $\mathscr{C} = \oplus_{g\in G}\mathscr{C}_g$, where $\mathscr{C}_g$ is a vector space for all $g \in G$ such that

1. There is a homomorphism $\alpha : G \to \mathrm{Aut}(\mathscr{C})$, where $\mathrm{Aut}(\mathscr{C})$ is the algebra of homomorphisms of $\mathscr{C}$, such that
$$\alpha_h : \mathscr{C}_g \to \mathscr{C}_{hgh^{-1}},$$
for every $g \in G$ we have
$$\alpha_g|_{\mathscr{C}_g} = \mathrm{Id}_{\mathscr{C}_g} .$$
Note that $\alpha_e : \mathscr{C}_g \to \mathscr{C}_g$ is the identity map.

2. For all $x \in \mathscr{C}_g$ and $y \in \mathscr{C}_h$ we have that the product is twisted commutative, i.e.
$$xy = \alpha_g(y)x.$$

3. There are coassociative coproducts $\Delta_{g,h} : \mathscr{C}_{gh} \to \mathscr{C}_g \otimes \mathscr{C}_h$ such that the following diagrams commute.
$$\begin{tikzcd} \mathscr{C}_g \otimes \mathscr{C}_{hf} \arrow[r, "m_{g,hf}"] \arrow[d, "1\otimes\Delta_{hf}"'] & \mathscr{C}_{ghf} \arrow[d, "\Delta_{gh,f}"] \\ \mathscr{C}_{gh} \otimes \mathscr{C}_{h^{-1}} \otimes \mathscr{C}_{hf} \arrow[r, "m_{g,h}\otimes 1"'] & \mathscr{C}_{gh} \otimes \mathscr{C}_f \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}_g \otimes \mathscr{C}_{hf} \arrow[r, "m_{g,hf}"] \arrow[d, "\Delta_{gh,h^{-1}}\otimes 1"'] & \mathscr{C}_{ghf} \arrow[d, "\Delta_{gh,f}"] \\ \mathscr{C}_{gh} \otimes \mathscr{C}_{h^{-1}} \otimes \mathscr{C}_{hf} \arrow[r, "1\otimes m_{h^{-1},hf}"'] & \mathscr{C}_{gh} \otimes \mathscr{C}_f \end{tikzcd}$$
See Figure 5.5.

4. These coproducts have the next properties: for every $g, h \in G$ the next diagram commutes
$$\begin{tikzcd} \mathbb{C} \arrow[r, "u"] \arrow[d, "u"'] & \mathscr{C}_e \arrow[r, "\Delta_h"] & \mathscr{C}_h \otimes \mathscr{C}_{h^{-1}} \arrow[r, "1\otimes\alpha_g"] & \mathscr{C}_h \otimes \mathscr{C}_{gh^{-1}g^{-1}} \arrow[d, "m_{h,gh^{-1}g^{-1}}"] \\ \mathscr{C}_e \arrow[r, "\Delta_g"'] & \mathscr{C}_g \otimes \mathscr{C}_{g^{-1}} \arrow[r, "\alpha_h\otimes 1"'] & \mathscr{C}_{hgh^{-1}} \otimes \mathscr{C}_{g^{-1}} \arrow[r, "m_{hgh^{-1},g^{-1}}"'] & \mathscr{C}_{hgh^{-1}g^{-1}} \end{tikzcd}$$

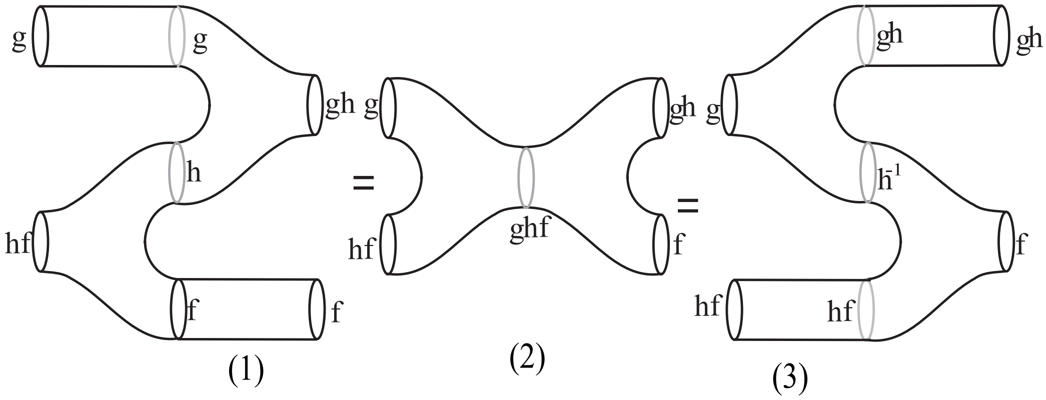


Figure 5.5: `Abrams condition`.

Remark 5.3.2. Note that the condition 3 involves the next particular case. We take the particular commutative diagrams

$$\begin{tikzcd}
\mathscr{C}_g \otimes \mathscr{C}_e \arrow[r, "m_{g,e}"] \arrow[d, "1\otimes\Delta_{h^{-1},h}"'] & \mathscr{C}_g \arrow[d, "\Delta_{gh^{-1},h}"] \\
\mathscr{C}_g \otimes \mathscr{C}_{h^{-1}} \otimes \mathscr{C}_h \arrow[r, "m_{g,h^{-1}}\otimes 1"'] & \mathscr{C}_{gh^{-1}} \otimes \mathscr{C}_h
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C}_e \otimes \mathscr{C}_g \arrow[r, "m_{e,g}"] \arrow[d, "\Delta_{gh^{-1},hg^{-1}}\otimes 1"'] & \mathscr{C}_g \arrow[d, "\Delta_{gh^{-1},h}"] \\
\mathscr{C}_{gh^{-1}} \otimes \mathscr{C}_{hg^{-1}} \otimes \mathscr{C}_g \arrow[r, "1\otimes m_{hg^{-1},g}"'] & \mathscr{C}_{gh^{-1}} \otimes \mathscr{C}_h
\end{tikzcd}$$

and $x_g \in \mathscr{C}_g$, then the next equality is satisfied

$$\sum_i x_g e_i^{h^{-1}} \otimes e_i^h = \sum_i e_i^{gh^{-1}} \otimes e_i^{hg^{-1}} x_g,$$

where $\{e_i^h\}$ is a basis of $\mathscr{C}_h$, which is a generalized condition of Lauda (see Figure 5.6).

Theorem 5.3.3. *If $\mathscr{C}$ is a nearly G-Frobenius algebra then its G-invariant part, denoted by $\mathscr{C}_{orb}$, is a nearly Frobenius algebra.*

Proof. We define the coproduct

$$\Delta : \mathscr{C}_{orb} \to \mathscr{C}_{orb} \otimes \mathscr{C}_{orb}$$

similarly as in Corollary 1.3. This is $\Delta(x) = \sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} x e_i^{h^{-1}} \otimes e_i^k$.
To prove that $(\mathscr{C}_{orb}, \Delta)$ is a nearly Frobenius algebra we only need to prove the Lauda condition, i.e.

$$\sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} x e_i^{h^{-1}} \otimes e_i^k = \sum_{g\in T, i\in I_g} \sum_{h,k\in[g]} e_i^{h^{-1}} \otimes e_i^k x.$$

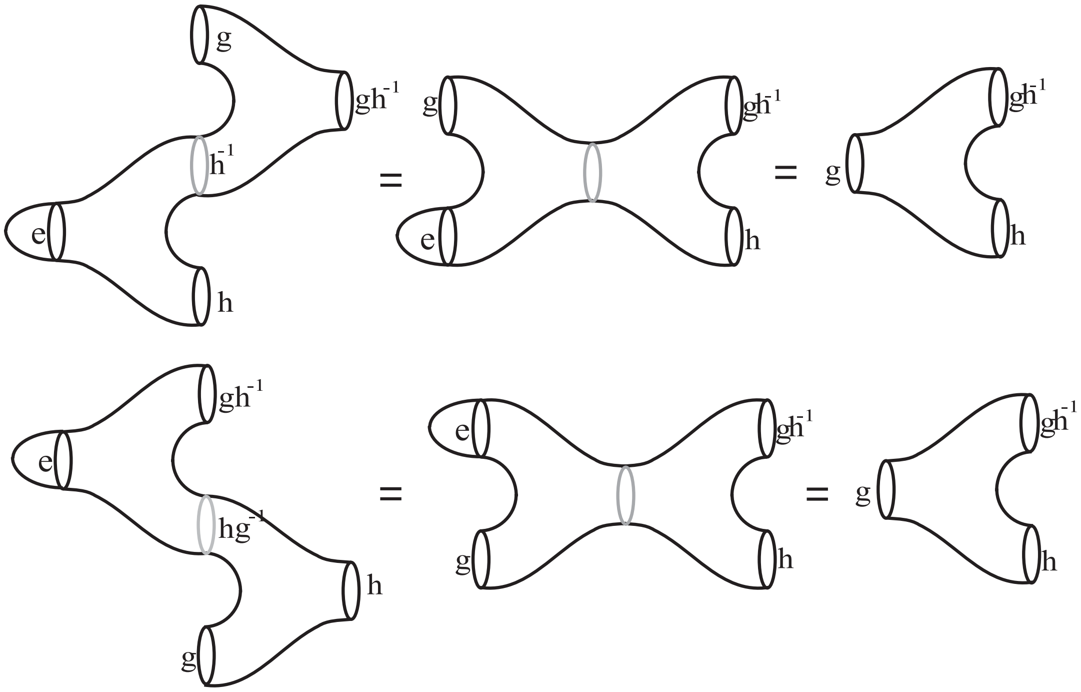


Figure 5.6: `Generalized Lauda condition`.

If $x = \sum_{l \in G} x_l$, then

$$\sum_{g \in T, i \in I_g} \sum_{h,k \in [g]} x e_i^{h^{-1}} \otimes e_i^k = \sum_{g \in T, i \in I_g} \sum_{h,k \in [g]} \sum_{l \in G} x_l e_i^{h^{-1}} \otimes e_i^k.$$

By the remark 5.3.2 we have $\sum_i x_g e_i^{h^{-1}} \otimes e_i^h = \sum_i e_i^{gh^{-1}} \otimes e_i^{hg^{-1}} x_g$. If we act the second component by $\alpha_r : \mathscr{C}_h \to \mathscr{C}_{rhr^{-1}} = \mathscr{C}_k$. Then the next identity is satisfied

$$\sum_i x_g e_i^{h^{-1}} \otimes \alpha_r(e_i^h) = \sum_i e_i^{gh^{-1}} \otimes \alpha_r(e_i^{hg^{-1}} x_g),$$

hence

$$\sum_i x_g e_i^{h^{-1}} \otimes e_i^k = \sum_i e_i^{gh^{-1}} \otimes e_i^{rhg^{-1}r^{-1}} \alpha_r(x_g).$$

Therefore

$$\sum_{g \in T, i \in I_g} \sum_{h,k \in [g]} \sum_{l \in G} x_l e_i^{h^{-1}} \otimes e_i^k = \sum_{g \in T, i \in I_g} \sum_{h,k \in [g]} \sum_{l \in G} e_i^{lh^{-1}} \otimes e_i^{rhl^{-1}r^{-1}} \alpha_r(x_l).$$

We use that lh^{-1} and $rhl^{-1}r^{-1} = krl^{-1}r^{-1}$ are in the same conjugacy class and lh^{-1}

and $rhl^{-1}r^{-1}$ vary over all G, so we can change the variables h, k for u, v. Then

$$\begin{aligned}
\Delta(x) &= \sum_{g\in T, i\in I_g} \sum_{u,v\in[g]} \sum_{l\in G} e_i^{u^{-1}} \otimes e_i^v \alpha_r(x_l) \\
&= \sum_{g\in T, i\in I_g} \sum_{u,v\in[g]} e_i^{u^{-1}} \otimes e_i^v \alpha_r\left(\sum_{l\in G} x_l\right) \\
&= \sum_{g\in T, i\in I_g} \sum_{u,v\in[g]} e_i^{u^{-1}} \otimes e_i^v \alpha_r(x) \\
&= \sum_{g\in T, i\in I_g} \sum_{u,v\in[g]} e_i^{u^{-1}} \otimes e_i^v x.
\end{aligned}$$

$\clubsuit$

5.4 Examples

5.4.1 Virtual Cohomology

Now we introduce a new structure which is defined in [LUX07]. This is a cohomology theory for orbifolds and it is other important example of a nearly G-Frobenius algebra. We will work, as the same as before, with the global quotient orbifold $[M/G]$, where M is a smooth manifold and G is a finite group acting smoothly on M. In this subsection we will describe the structure maps. For this work we only consider quotients of manifolds by a finite group.

This example generalizes two diferente families of Frobenius algebras. The first example is the *Poincaré algebra* of an oriented smooth manifold M and the second is the *Dijkgraaf-Witten model* given by a finite group G. We can relate these two structures through a smooth action

$$\begin{tikzcd}
& G \circlearrowleft M & \\
M \arrow[ur] & & G \arrow[ul]
\end{tikzcd}$$

Let G be a finite group and $M^g := \{x \in M : xg = x\}$ the set of *fixed* points of $g \in G$. If M is an oriented smooth manifold (not necessarily compact), we can define the *G-virtual cohomology*

$$\mathrm{H}^*(M, G) := \bigoplus_{g\in G} \mathrm{H}^*(M^g).$$

Definition 5.4.1. The next diagram defines the *virtual product* in $\mathrm{H}^*(M,G)$ in the following way.

$$\begin{tikzcd} & M^{g,h} \arrow[dl, hook', "\delta_{g,h}"'] \arrow[dr, hook, "i_{g,h}"] & \\ M^g \times M^h & & M^{gh} \end{tikzcd}$$

If $\alpha \in \mathrm{H}^*(M^g)$ and $\beta \in \mathrm{H}^*(M^h)$, then we can define the virtual product

$$\alpha \star \beta := i_{g,h}! \left((\nu(g,h)\delta^*_{g,h}(\alpha \times \beta) \right),$$

where $\alpha \times \beta = \pi_g^*(\alpha)\pi_h^*(\beta)$, and $\nu(g,h) = e(M; M^g, M^h)$ is the Euler class of the excess bundle $\frac{TM|_{M^{g,h}}}{TM^g|_{M^{g,h}} + TM^h|_{M^{g,h}}}$, which is called *the excess intersection class*. In the Grothendieck group of vector bundles over $M^{g,h}$ the class is

$$TM|_{M^{g,h}} + TM^{g,h} - TM^g|_{M^{g,h}} - TM^h|_{M^{g,h}}.$$

Notice that $\delta^*_{g,h}(\alpha \times \beta) = \delta^*_{g,h}(\pi_g^*(\alpha)\pi_h^*(\beta)) = (\pi_g\delta_{g,h})^*(\alpha)(\pi_h\delta_{g,h})(\beta) = i_g^*(\alpha)i_h^*(\beta)$, where $i_g : M^{g,h} \to M^g$, $i_h : M^{g,h} \to M^h$ and $\pi_g : M^g \times M^h \to M^g$, $\pi_h : M^g \times M^h \to M^h$.

This product becames graded with the degree shift

$$\dim_{virt}(\alpha) = |\alpha| + \mathrm{cod}(M^g \subseteq M).$$

We have a natural action of the group G in $\mathrm{H}^*(M;G)$ as follow

$$\alpha_g : \mathrm{H}^*(M^h) \to \mathrm{H}^*(M^{ghg^{-1}}).$$

This is induced by the map $M^{ghg^{-1}} \to M^h$, $x \mapsto xg$. Note that $\alpha_g|_{\mathrm{H}^*(M^g)} = \mathrm{id}_{\mathrm{H}^*(M^g)}$.

Now we define the *virtual coproduct* as follows

$$\begin{tikzcd} & M^{g,h} \arrow[dl, hook', "i_{g,h}"'] \arrow[dr, hook, "\delta_{g,h}"] & \\ M^{gh} & & M^g \times M^h \end{tikzcd}$$

Then for $\alpha \in \mathrm{H}^*(M^{gh})$

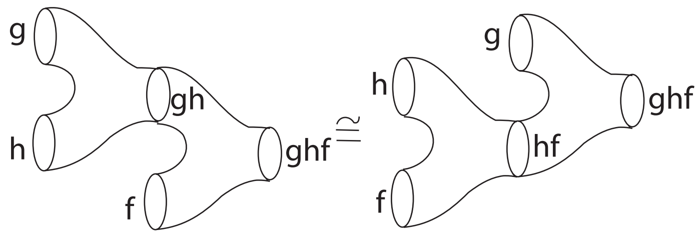
$$\Delta_{g,h}(\alpha) := \delta_{g,h}! \left(\mu(gh,g,h) i^*_{g,h}(\alpha) \right),$$

where $\mu(g,h) = e\left(\frac{TM|_{M^{g,h}}}{TM^{gh}|_{M^{g,h}}} \oplus TM^{g,h} \right)$.

Theorem 5.4.2. $\mathrm{H}^*(M;G)$ *is a nearly G-Frobenius algebra.*

Proof. We use the lema 9.1.2.

1. **Associativity of the virtual product**



(i) (ii)

(i)

$$
\begin{tikzcd}
 & & M^{g,h,f} \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & & \\
 & M^{g,h}\times M^f \arrow[dl, "\delta_{g,h}\times 1"'] \arrow[dr, "i_{g,h}\times 1"] & & M^{gh,f} \arrow[dl, "\delta_{gh,f}"'] \arrow[dr, "i_{gh,f}"] & \\
M^g\times M^h\times M^f & & M^{gh}\times M^f & & M^{ghf}
\end{tikzcd}
$$

- $e_2^*(\nu(gh,f))e(F_1)e_1^*(\nu(g,h)\times 1)$

where $e(F_1) = e\left(\frac{TM^{gh}\times M^f|_{M^{g,h,f}}}{TM^{g,h}\times M^f|_{M^{g,h,f}}+TM^{gh,f}|_{M^{g,h,f}}}\right)$, $\nu(gh,f) = e\left(\frac{TM|_{M^{gh,f}}}{TM^{gh}|_{M^{gh,f}}+TM^f|_{M^{gh,f}}}\right)$, and $\nu(g,h) = e\left(\frac{TM|_{M^{g,h}}}{TM^g|_{M^{g,h}}+TM^h|_{M^{g,h}}}\right)$.
Note that $e_1^*\left(\nu(g,h)\times 1\right) = \nu(g,h)|_{M^{g,h,f}}$. We realize the calculations in K-theory:
Set by $TM^{k_1,k_2,\cdots}|_{M^{g,h,f}} = \langle k_1,k_2,...\rangle$, then

$$
\begin{aligned}
\langle 1\rangle+\langle gh,f\rangle-\langle gh\rangle-\langle f\rangle+\langle gh\rangle+\langle f\rangle+\langle g,h,f\rangle-\langle g,h\rangle-\langle f\rangle-\langle gh,f\rangle+\langle 1\rangle+\langle g,h\rangle-\langle g\rangle-\langle h\rangle \\
= \langle 2\rangle+\langle g,h,f\rangle-\langle g\rangle-\langle h\rangle-\langle f\rangle.
\end{aligned}
$$

(ii)

$$
\begin{tikzcd}
 & & M^{g,h,f} \arrow[dl, "e_1'"'] \arrow[dr, "e_2'"] & & \\
 & M^g\times M^{h,f} \arrow[dl, "1\times\delta_{h,f}"'] \arrow[dr, "1\times i_{h,f}"] & & M^{g,hf} \arrow[dl, "\delta_{g,hf}"'] \arrow[dr, "i_{g,hf}"] & \\
M^g\times M^h\times M^f & & M^g\times M^{hf} & & M^{ghf}
\end{tikzcd}
$$

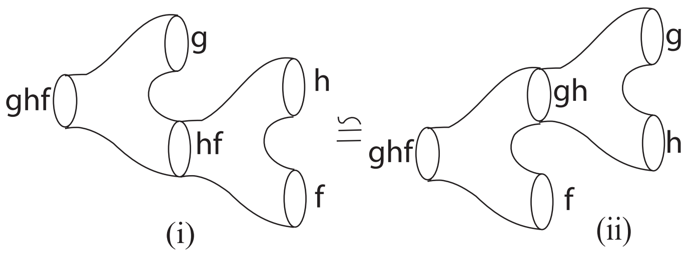
• $e_2'^*(\nu(g,hf))e(F_2)e_1'^*(1\times\nu(g,h))$
where $e(F_2)=e\left(\frac{TM^g\times M^{hf}|_{M^{g,h,f}}}{TM^g\times M^{h,f}|_{M^{g,h,f}}+TM^{g,hf}|_{M^{g,h,f}}}\right)$, $\nu(h,f)=e\left(\frac{TM|_{M^{h,f}}}{TM^h|_{M^{h,f}}+TM^f|_{M^{h,f}}}\right)$,
and $\nu(g,hf)=e\left(\frac{TM|_{M^{g,hf}}}{TM^g|_{M^{g,hf}}+TM^{hf}|_{M^{g,hf}}}\right)$.
In K-theory

$$\langle 1\rangle+\langle g,hf\rangle-\langle g\rangle-\langle hf\rangle+\langle g\rangle+\langle hf\rangle+\langle g,h,f\rangle-\langle g\rangle-\langle h,f\rangle-\langle g,hf\rangle+\langle 1\rangle+\langle h,f\rangle-\langle h\rangle-\langle f\rangle$$

$$=\langle 2\rangle+\langle g,h,f\rangle-\langle g\rangle-\langle h\rangle-\langle f\rangle.$$

Then $(\alpha\star\beta)\star\gamma=\alpha\star(\beta\star\gamma)$.

2. **Coassociativity of the virtual coproduct**



(i) (ii)

(i)

$$\begin{tikzcd}
 & & M^{g,h,f} \arrow[dl, "e_2"'] \arrow[dr, "e_1"] & & \\
 & M^{g,hf} \arrow[dl, "i_{g,hf}"'] \arrow[dr, "\delta_{g,hf}"] & & M^g\times M^{h,f} \arrow[dl, "1\times i_{h,f}"'] \arrow[dr, "1\times\delta_{h,f}"] & \\
M^{ghf} & & M^g\times M^{hf} & & M^g\times M^h\times M^f
\end{tikzcd}$$

• $e_1^*(\mu(g,hf))e(F_1)e_2^*(1\times\mu(g,h))$
where $e(F_1)=e\left(\frac{TM^g\times M^{hf}|_{M^{g,h,f}}}{TM^{g,hf}|_{M^{g,h,f}}+TM^g\times M^{h,f}|_{M^{g,h,f}}}\right)$, $\mu(g,hf)=e\left(\frac{TM|_{M^{g,hf}}}{TM^{ghf}|_{M^{g,hf}}}\,\mathrm{T}\,M^{g,hf}\right)$,
and $\mu(h,f)=e\left(\frac{TM|_{M^{h,f}}}{TM^{hf}|_{M^{h,f}}}\,\mathrm{T}\,M^{h,f}\right)$.
If we realize the calculations in K-theory, then

$$\langle 1\rangle+\langle h,f\rangle-\langle hf\rangle+\langle 1\rangle-\langle ghf\rangle+\langle g,hf\rangle+\langle g\rangle+\langle hf\rangle+\langle g,h,f\rangle-\langle g,hf\rangle-\langle g\rangle-\langle h,f\rangle$$

$$=\langle 2\rangle+\langle g,h,f\rangle-\langle ghf\rangle.$$

(ii)

$$\begin{tikzcd}
 & & M^{g,h,f} \arrow[dl, "e'_2"'] \arrow[dr, "e'_1"] & & \\
 & M^{gh,f} \arrow[dl, "i_{gh,f}"'] \arrow[dr, "\delta_{gh,f}"] & & M^{g,h} \times M^f \arrow[dl, "i_{g,h}\times 1"'] \arrow[dr, "\delta_{g,h}\times 1"] & \\
M^{ghf} & & M^{gh} \times M^f & & M^g \times M^h \times M^f
\end{tikzcd}$$

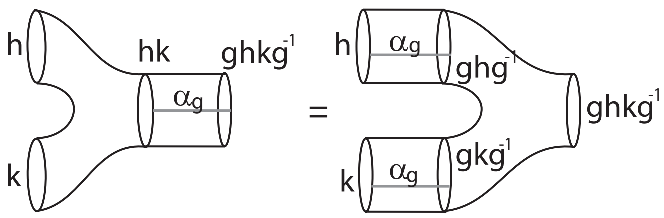
- $e_1'^*(\mu(gh,f))e(F_2)e_2'^*(\mu(g,h)\times 1)$

where $e(F_2) = e\left(\frac{TM^{gh}\times M^f|_{M^{g,h,f}}}{TM^{g,h}\times M^f|_{M^{g,h,f}}+TM^{gh,f}|_{M^{g,h,f}}}\right)$, $\mu(g,h) = e\left(\frac{TM|_{M^{g,h}}}{TM^{gh}|_{M^{g,h}}}\,\mathrm{T}\,M^{g,hf}\right)$,
and $\nu(gh,f) = e\left(\frac{TM|_{M^{gh,f}}}{TM^{ghf}|_{M^{gh,f}}}\,\mathrm{T}\,M^{gh,f}\right)$.
In K-theory

$$\langle 1\rangle+\langle g,h\rangle-\langle gh\rangle+\langle 1\rangle+\langle gh,f\rangle-\langle ghf\rangle+\langle gh\rangle+\langle f\rangle+\langle g,h,f\rangle-\langle gh,f\rangle-\langle g,h\rangle-\langle f\rangle$$

$$= \langle 2\rangle + \langle g,h,f\rangle - \langle ghf\rangle.$$

3. **The action is an algebra homomorphism**



(1) (2)

(i)

$$\begin{tikzcd}
 & & M^{ghg^{-1},gkg^{-1}} \arrow[dl, "\lambda_g"'] \arrow[dr, "i_{ghg^{-1},gkg^{-1}}"] & & \\
 & M^{h,k} \arrow[dl, "\delta_{h,k}"'] \arrow[dr, "i_{h,k}"] & & M^{ghkg^{-1}} \arrow[dl, "\alpha_g"'] \arrow[dr, "1"] & \\
M^h \times M^k & & M^{hk} & & M^{ghkg^{-1}}
\end{tikzcd}$$

- $e(F_1) = e\left(\frac{TM^{hk}|_{M^{ghg^{-1},gkg^{-1}}}}{TM^{h,k}|_{M^{ghg^{-1},gkg^{-1}}}+TM^{ghkg^{-1}}|_{MM^{ghg^{-1},gkg^{-1}}}}\right)$

and $\nu(h,k) = e\left(\frac{TM|_{M^{h,k}}}{TM^h|_{M^{h,k}}+TM^k|_{M^{h,k}}}\right)$.

Then in K-theory the calculations are

$$\langle 1\rangle + \langle h,k\rangle - \langle h\rangle - \langle k\rangle + \langle hk\rangle + \langle ghg^{-1}, gkg^{-1}\rangle - \langle h,k\rangle - \langle ghkg^{-1}\rangle$$

$$= \langle 1\rangle - \langle h\rangle - \langle k\rangle - \langle h,k\rangle.$$

(ii)

$$\begin{tikzcd}
 & & M^{ghg^{-1},gkg^{-1}} \arrow[dl, "\delta_{ghg^{-1},gkg^{-1}}"'] \arrow[dr, "1"] & & \\
 & M^{ghg^{-1},gkg^{-1}} \arrow[dl, "\alpha_g\times\alpha_g"'] \arrow[dr, "1"] & & M^{ghg^{-1},gkg^{-1}} \arrow[dl, "\delta_{ghg^{-1},gkg^{-1}}"'] \arrow[dr, "i_{ghg^{-1},gkg^{-1}}"] & \\
M^h \times M^k & & M^{hk} & & M^{ghkg^{-1}}
\end{tikzcd}$$

- $e(F_2) = e\left(\frac{TM^{ghg^{-1}}\times TM^{gkg^{-1}}|_{M^{ghg^{-1},gkg^{-1}}}}{TM^{ghg^{-1}}\times TM^{gkg^{-1}}|_{M^{ghg^{-1},gkg^{-1}}}+TM^{ghg^{-1},gkg^{-1}}|_{M M^{ghg^{-1},gkg^{-1}}}}\right)$

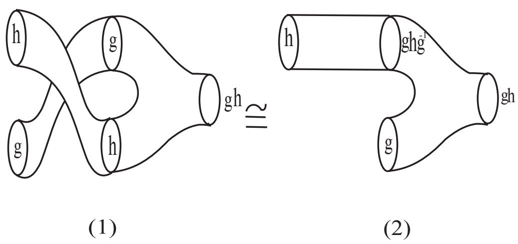
and $\nu(ghg^{-1},gkg^{-1}) = e\left(\frac{TM|_{M^{ghg^{-1},gkg^{-1}}}}{TM^{ghg^{-1}}|_{M^{h,k}}+TM^{gkg^{-1}}|_{M^{h,k}}}\right)$
Then in K-theory

$$\langle 1\rangle + \langle ghg^{-1}, gkg^{-1}\rangle - \langle ghg^{-1}\rangle - \langle gkg^{-1}\rangle + \langle ghg^{-1}, gkg^{-1}\rangle + \langle ghg^{-1}\rangle$$

$$+\langle gkg^{-1}\rangle - \langle ghg^{-1}\rangle - \langle gkg^{-1}\rangle - \langle ghg^{-1}, gkg^{-1}\rangle$$

$$= \langle 1\rangle - \langle h\rangle - \langle k\rangle - \langle h,k\rangle.$$

4. **Graded commutativity of the product**



(ii)

$$\begin{tikzcd}
 & & & M^{ghg^{-1},g} = M^{g,h} \arrow[dll, "\delta_{ghg^{-1},g}"'] \arrow[drr, "\iota"] \arrow[ddlll, bend right, "\delta_{h,g}"'] \arrow[ddrrr, bend left, "i_{g,h}"] & & & \\
 & M^{ghg^{-1}} \times M^{g} \arrow[dl, "\alpha_g \times 1"'] \arrow[drr, "\iota"] & & & & M^{ghg^{-1},g} \arrow[dll, "\delta_{ghg^{-1},g}"'] \arrow[dr, "i_{g,h}"] & \\
M^{h} \times M^{g} & & & M^{ghg^{-1}} \times M^{g} & & & M^{gh}
\end{tikzcd}$$

• $\nu(ghg^{-1}, g)e(F_2)1$
where $e(F_2) = e\left(\frac{TM^{ghg^{-1}} \times M^{g}|_{M^{g,h}}}{TM^{ghg^{-1},g}|_{M^{g,h}} + TM^{ghg^{-1}} \times M^{g}|_{M^{g,h}}}\right) = e(0) = 1.$
In K-theory

$$\langle 1 \rangle + \langle g, h \rangle - \langle ghg^{-1} \rangle - \langle g \rangle.$$

(i)

$$\begin{tikzcd}
 & & & M^{ghg^{-1},g} = M^{g,h} \arrow[dll, "\delta_{g,h}"'] \arrow[drr, "\iota"] & & & \\
 & M^{g} \times M^{h} \arrow[dl, "\tau"'] \arrow[drr, "\iota"] & & & & M^{g,h} \arrow[dll, "\delta_{g,h}"'] \arrow[dr, "i_{g,h}"] & \\
M^{h} \times M^{g} & & & M^{g} \times M^{h} & & & M^{gh}
\end{tikzcd}$$

• $\nu(g, h)e(F_1)1$
where $e(F_1) = e\left(\frac{TM^{g} \times M^{h}|_{M^{g,h}}}{TM^{g} \times M^{h}|_{M^{g,h}} + TM^{g,h}|_{M^{g,h}}}\right) = e(0) = 1.$
In K-theory

$$\langle 1 \rangle + \langle g, h \rangle - \langle g \rangle - \langle h \rangle.$$

Then $\alpha_g(\beta) \star \alpha = i_{g,h}!\left(\nu(g,h)\delta^*_{g,h}\left(\tau^*(\beta \times \alpha)\right)\right)$ if and only if

$$\langle h \rangle = \langle ghg^{-1} \rangle.$$

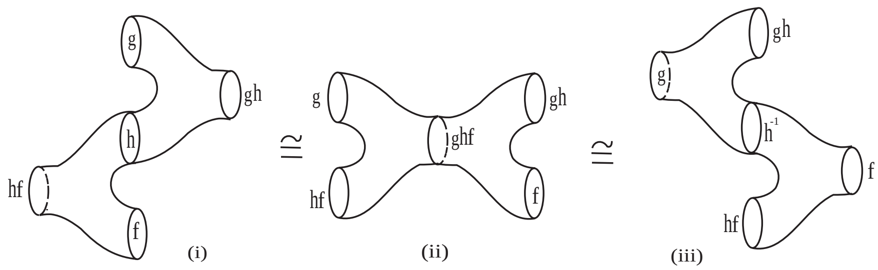
This is true because the bundles are isomorphic. Now we need to understand $\tau^*(\beta \times \alpha)$.

Let be $\tau : M^g \times M^h \to M^h \times M^g$ the transposition, and $\pi_1 : M^g \times M^h \to M^g$, $\pi_2 : M^g \times M^h \to M^h$, $\pi_1' : M^h \times M^g \to M^h$, $\pi_2' : M^h \times M^g \to M^g$. Hence

$$\begin{aligned}\tau^*(\beta \times \alpha) &= \tau^*({\pi_1'}^*(\beta))\tau^*({\pi_2'}^*(\alpha)) = (\pi_1'\tau)^*(\beta)(\pi_2'\tau)^*(\alpha)\\ &= \pi_2^*(\beta)\pi_1^*(\alpha) = (-1)^{|\alpha||\beta|}\pi_1^*(\alpha)\pi_2^*(\beta)\\ &= (-1)^{|\alpha||\beta|}\alpha \times \beta.\end{aligned}$$

Then $\alpha_g(\beta) \star \alpha = (-1)^{|\alpha||\beta|} i_{g,h}! \left(\nu(g,h)\delta^*_{g,h}(\alpha \times \beta)\right) = (-1)^{|\alpha||\beta|}\alpha \star \beta$.

5. **Abrams condition**



Remember that if $\alpha \in H^*(M^{gh})$ then

$$\Delta_{g,h}(\alpha) = \delta_{g,h}! \left(\mu(gh,g,h) i^*_{g,h}(\alpha)\right),$$

where $\mu(g,h) = e\left(\frac{TM|_{M^{g,h}}}{TM^{gh}|_{M^{g,h}}} \oplus TM^{g,h}\right)$.

(ii)

$$\begin{array}{ccccc} & & M^{g,h,f} & & \\ & \swarrow^{e_1} & & \searrow^{e_2} & \\ & M^{g,hf} & & M^{gh,f} & \\ \swarrow^{\delta_{g,hf}} & & \searrow^{i_{g,hf}} \quad \swarrow^{i_{gh,f}} & & \searrow^{\delta_{gh,f}} \\ M^g \times M^{hf} & & M^{ghf} & & M^{gh} \times M^f \end{array}$$

- $e_2^*(\mu(gh,f))e(F_1)e_1^*(\nu(g,hf))$,

where $e(F_1) = e\left(\frac{TM^{ghf}|_{M^{g,h,f}}}{TM^{g,hf}|_{M^{g,h,f}} + TM^{gh,f}|_{M^{g,h,f}}}\right)$.
Then $\langle 1\rangle + \langle gh,f\rangle - \langle ghf\rangle + \langle ghf\rangle + \langle g,h,f\rangle - \langle g,hf\rangle - \langle gh,f\rangle + \langle 1\rangle + \langle g,hf\rangle - \langle g\rangle - \langle hf\rangle = \langle 2\rangle + \langle g,h,f\rangle - \langle g\rangle - \langle hf\rangle$.

(i)

$$
\begin{tikzcd}
 & & & M^{g,h,f} \arrow[dll, "e'_1"'] \arrow[drr, "e'_2"] & & & \\
 & M^g \times M^{h,f} \arrow[dl, "1\times i_{h,f}"'] \arrow[drr, "1\times\delta_{h,f}"] & & & & M^{g,h} \times M^f \arrow[dll, "\delta_{g,h}\times 1"'] \arrow[dr, "i_{g,h}\times 1"] & \\
M^g \times M^{hf} & & & M^g \times M^h \times M^f & & & M^{gh} \times M^f
\end{tikzcd}
$$

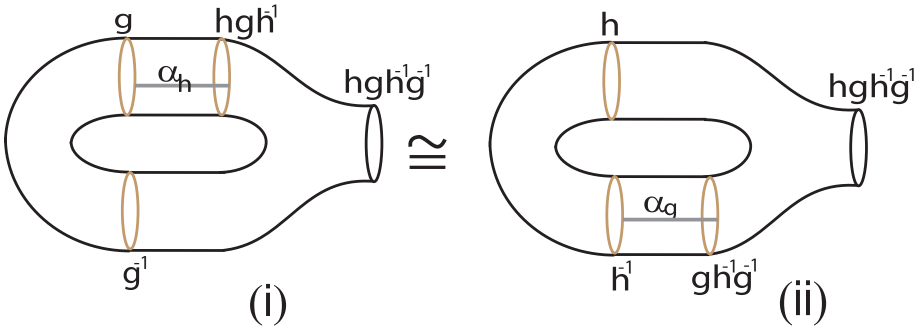
• $e_2^{'*}(\nu(g,h)\times 1)e(F_2)e_1^{'*}(1\times\mu(h,f))$,
where $e(F_2) = e\left(\frac{TM^g\times M^h\times M^f|_{M^{g,h,f}}}{TM^g\times M^{h,f}|_{M^{g,h,f}}+TM^{g,h}\times M^f|_{M^{g,h,f}}}\right)$.
Then $\langle 1\rangle+\langle g,h\rangle-\langle g\rangle-\langle h\rangle+\langle g\rangle+\langle h\rangle+\langle f\rangle+\langle g,h,f\rangle-\langle g\rangle-\langle h,f\rangle-\langle g,h\rangle-\langle f\rangle+\langle 1\rangle+\langle h,f\rangle-\langle hf\rangle=\langle 2\rangle+\langle g,h,f\rangle-\langle hf\rangle-\langle g\rangle$.

(iii)

$$
\begin{tikzcd}
 & & & M^{g,h,f} \arrow[dll, "e''_1"'] \arrow[drr, "e''_2"] & & & \\
 & M^{gh,h^{-1}} \times M^{hf} \arrow[dl, "i_{gh,h^{-1}}\times 1"'] \arrow[drr, "\delta_{gh,h^{-1}}\times 1"] & & & & M^{gh} \times M^{h^{-1},hf} \arrow[dll, "1\times\delta_{h^{-1},hf}"'] \arrow[dr, "1\times i_{h^{-1},hf}"] & \\
M^g \times M^{hf} & & & M^{gh} \times M^{h^{-1}} \times M^{hf} & & & M^{gh} \times M^f
\end{tikzcd}
$$

• $e_2^{''*}(1\times\nu(h^{-1},hf))e(F_3)e_1^{''*}(\mu(gh,h^{-1})\times 1)$,
where $e(F_3) = e\left(\frac{TM^{gh}\times M^{h^{-1}}\times M^{hf}|_{M^{g,h,f}}}{TM^{gh,h^{-1}}\times M^{hf}|_{M^{g,h,f}}+TM^{gh}\times M^{h^{-1},hf}|_{M^{g,h,f}}}\right)$.
Then $\langle 1\rangle+\langle h^{-1},hf\rangle-\langle h^{-1}\rangle-\langle hf\rangle+\langle gh\rangle+\langle h^{-1}\rangle+\langle hf\rangle+\langle g,h,f\rangle-\langle gh,h^{-1}\rangle-\langle hf\rangle-\langle gh\rangle-\langle h^{-1},hf\rangle+\langle 1\rangle+\langle gh,h^{-1}\rangle-\langle g\rangle=\langle 2\rangle+\langle g,h,f\rangle-\langle hf\rangle-\langle g\rangle$.
If we compare the three cases we have that the Abrams condition is satisfied.

6. **Torus axiom**



(i)

$$\begin{tikzcd}
 & & M^{g,h} \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & & \\
 & M^{g,g^{-1}} \arrow[dl, "i_{g,g^{-1}}"'] \arrow[dr, "\delta_{g,g^{-1}}"'] & & M^{hgh^{-1},g^{-1}} \arrow[dl, "(\alpha_h\times 1)\circ\delta_{hgh^{-1},g^{-1}}"'] \arrow[dr, "i_{hgh^{-1},g^{-1}}"] & \\
M & & M^g\times M^{g^{-1}} & & M^{hgh^{-1}g^{-1}}
\end{tikzcd}$$

- $e_2^*(\nu(hgh^{-1},g^{-1})e(F_1)e_1^*(\mu(g,g^{-1}))$,

where $e(F_1)=e\left(\frac{TM^g\times M^{g^{-1}}|_{M^{g,h}}}{TM^{g,g^{-1}}|_{M^{g,h}}+TM^{hgh^{-1},g^{-1}}|_{M^{g,h}}}\right)$ and $\mu(g,g^{-1})=e\left(\frac{TM|_{M^{g,g^{-1}}}}{TM|_{M^{g,g^{-1}}}}\oplus TM^{g,g^{-1}}\right)=e(TM^g)$. Then

$$\langle 1\rangle+\langle hgh^{-1},g^{-1}\rangle-\langle hgh^{-1}\rangle-\langle g^{-1}\rangle+\langle g\rangle+\langle g^{-1}\rangle+\langle g,h\rangle-\langle g,g^{-1}\rangle-\langle hgh^{-1},g^{-1}\rangle+\langle g,g^{-1}\rangle$$

$$=\langle 1\rangle+\langle g,h\rangle-\langle hgh^{-1}\rangle+\langle g\rangle.$$

(ii)

$$\begin{tikzcd}
 & & M^{g,h} \arrow[dl, "e_1'"'] \arrow[dr, "e_2'"] & & \\
 & M^{h,h^{-1}} \arrow[dl, "i_{h,h^{-1}}"'] \arrow[dr, "\delta_{h,h^{-1}}"'] & & M^{h,gh^{-1}g^{-1}} \arrow[dl, "\delta_{h,gh^{-1}g^{-1}}"'] \arrow[dr, "i_{h,gh^{-1}g^{-1}}"] & \\
M & & M^h\times M^{h^{-1}} & & M^{hgh^{-1}g^{-1}}
\end{tikzcd}$$

- $e_2'^{*}(\nu(h, gh^{-1}g^{-1})e(F_2)e_1'^{*}(\mu(h, h^{-1}))$,

where $e(F_2) = e\left(\frac{TM^h \times M^{h^{-1}}|_{M^{g,h}}}{TM^{h,h^{-1}}|_{M^{g,h}} + TM^{h,gh^{-1}g^{-1}}|_{M^{g,h}}}\right)$ and $\mu(h, h^{-1}) = e(TM^h)$.

Then

$$\langle 1\rangle + \langle h, gh^{-1}g^{-1}\rangle - \langle h\rangle - \langle gh^{-1}g^{-1}\rangle + \langle h\rangle + \langle h^{-1}\rangle + \langle g, h\rangle - \langle h, h^{-1}\rangle - \langle h, gh^{-1}g^{-1}\rangle + \langle h, h^{-1}\rangle$$
$$= \langle 1\rangle + \langle g, h\rangle - \langle gh^{-1}g^{-1}\rangle + \langle h\rangle.$$

Using that $\langle g\rangle = \langle hgh^{-1}\rangle$ we finish the proof.

♣

Definition 5.4.3. We define the *orbifold virtual cohomology* as the G-invariant part of $\mathrm{H}^*(M; G)$. It is denoted by $\mathrm{H}^*_{\mathrm{virt}}(M; G) = \mathrm{H}^*(M; G)^G$.

Corollary 5.4.4. *The orbifold virtual cohomology,* $\mathrm{H}^*_{\mathrm{virt}}(M; G)$*, is a nearly Frobenius algebra.*

5.4.2 Orbifold String Topology

Orbifold string topology was introduced by Lupercio, Uribe and Xicotencatl in [LUX08]. Let M be a smooth, compact, connected, oriented manifold and let G be a finite group acting on M.
We will consider the global quotient orbifold $\mathsf{X} = [M/G]$.
We define now the *loop orbifold* LX for X as follows:

Consider the space

$$\mathcal{P}_G(M) := \bigsqcup_{g\in G} \mathcal{P}_g(M) \times \{g\}$$

where

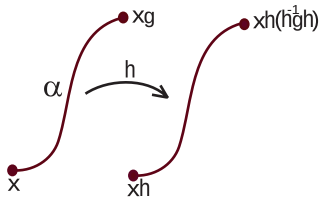
$$\mathcal{P}_g(M) = \{\gamma : [0, 1] \to Y : \gamma(0)g = \gamma(1)\},$$

together with the G-action given by

$$G \times \bigsqcup_{g\in G} \mathcal{P}_g(M) \times \{g\} \to \bigsqcup_{g\in G} \mathcal{P}_g(M) \times \{g\}$$

$$(h, (\gamma, g)) \mapsto (\gamma_h, h^{-1}gh)$$

where $\gamma_h(t) := \gamma(t)h$.



Then we define the loop orbifold as

$$\mathsf{LX} := [\mathcal{P}_G(M)/G].$$

In this section we associate a nearly G-Frobenius algebra to the loop orbifold LX. This is $\mathrm{H}_*(\mathcal{P}_G(M)) = \bigoplus_{g\in G} \mathrm{H}_*(\mathcal{P}_g(M))$, which the G-action

$$\alpha_h : \mathrm{H}_*(\mathcal{P}_g(M)) \to \mathrm{H}_*(\mathcal{P}_{hgh^{-1}}(M))$$

$$\alpha_h([\gamma]) = [\gamma_h]$$

It is important to mention that the string topology is included as $\mathrm{H}_*(\mathcal{P}_e(M))$ with $e \in G$ the identity.

We will describe the structure maps in the next section.

Algebraic structure

Orbifold string product: We will suppose that M is oriented and G acts by orientation preserving diffeomorphisms. Now we define the product for the homology of $\mathcal{P}_G(M)$. We start by defining a composition of path maps

$$\circledast : \mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) \to \mathcal{P}_{gh}(M)$$

where $\epsilon_t : \mathcal{P}_k(M) \to M$ is the evaluation map at t, given by $\gamma \mapsto \gamma(t)$ and

$$\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) = \{(\gamma_0, \gamma_1) : \gamma_0(1) = \gamma_1(0)\}.$$

The map $\circledast$ is given by

$$(\gamma_0 \circledast \gamma_1)(t) := \begin{cases} \gamma_0(2t), & 0 \le t \le \frac{1}{2} \\ \gamma_1(2t-1), & \frac{1}{2} < t \le 1 \end{cases}$$

Notice that the following diagram is a pullback square

$$\begin{tikzcd} \mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g(M) \times \mathcal{P}_h(M) \arrow[d, "\epsilon_1\times\epsilon_0"] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd} \tag{5.2}$$

where j is the inclusion, Δ is the diagonal map and $\epsilon_\infty(\gamma_0, \gamma_1) = \gamma_0(1) = \gamma_1(0)$. We observe that the pullback square 5.2 allows a Thom-Pontryagin map

$$\tau : \mathcal{P}_g(M) \times \mathcal{P}_h(M) \to (\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM},$$

where $(\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM}$ denotes the Thom space of the pullback bundle $\epsilon_\infty^*(TM)$. This is the normal bundle of the embedding j.

Set by $(\mathcal{P}_{gh}(M))^{TM}$ the Thom space of the bundle $\epsilon_{\frac{1}{2}}^*(TM)$ with $\epsilon_{\frac{1}{2}} : \mathcal{P}_{gh}(M) \to M$. The map $\circledast$ induces a map of Thom spaces

$$\tilde{\circledast} : (\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM} \to (\mathcal{P}_{gh}(M))^{TM}.$$

An immediate consequence is the next commutative diagram

$$\begin{tikzcd}
\mathcal{P}_g(M)\times\mathcal{P}_h(M) \arrow[r, "\tau"] \arrow[d, "\epsilon_1\times\epsilon_0"'] & (\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM} \arrow[r, "\tilde{\circledast}"] \arrow[d, "\epsilon_\infty"'] & (\mathcal{P}_{gh}(M))^{TM} \arrow[d, "\epsilon_{\frac{1}{2}}"] \\
M\times M \arrow[r, "\tau"] & M^{TM} \arrow[r, "="] & M^{TM}
\end{tikzcd}$$

Then, we can consider the composition

$$\begin{aligned}\eta_{g,h} : \mathrm{H}_p(\mathcal{P}_g(M))\otimes\mathrm{H}_q(\mathcal{P}_h(M)) &\xrightarrow{\times} \mathrm{H}_{p+q}(\mathcal{P}_g(M)\times\mathcal{P}_h(M)) \xrightarrow{\tau_*} \\ &\mathrm{H}_{p+q}((\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM}) \xrightarrow{\tilde{u}_*} \mathrm{H}_{p+q-d}(\mathcal{P}_{gh}(M)),\end{aligned}$$

where $\tilde{u}_*$ is the Thom isomorphism. Summing over all elements $g\in G$ we obtain the map

$$\eta : \mathrm{H}_p(\mathcal{P}_G(M))\otimes\mathrm{H}_q(\mathcal{P}_G(M)) \to \mathrm{H}_{p+q-d}(\mathcal{P}_G(M)).$$

We denote by η the *G-string product.*

Orbifold string coproduct: First, we note that the next diagram is a pullback square

$$\begin{tikzcd}
\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M) \arrow[r, "\circledast"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{gh}(M) \arrow[d, "{\epsilon_{\frac{1}{2}},\epsilon_0\cdot g}"] \\
M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

Then, we can consider the map

$$\tilde{\circledast} : \mathcal{P}_{gh}(M) \to (\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM}$$

where $(\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM}$ denotes the Thom space of the pullback bundle $\epsilon_\infty^*(TM)$, which is the normal bundle of $\circledast$.

Then, we can consider the composition

$$\begin{aligned}\Delta_{g,h} : \mathrm{H}_{p+q+d}(\mathcal{P}_{gh}(M)) &\xrightarrow{\tilde{\circledast}} \mathrm{H}_{p+q+d}(\mathcal{P}_g(M)_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h(M))^{TM} \xrightarrow{\tilde{u}_*} \mathrm{H}_{p+q}(\mathcal{P}_g(M)_{\epsilon_i}\times_{\epsilon_0}\mathcal{P}_h(M)) \xrightarrow{j_*} \\ &\mathrm{H}_{p+q}(\mathcal{P}_g(M)\times\mathcal{P}_h(M)) \longrightarrow \mathrm{H}_p(\mathcal{P}_g(M))\otimes\mathrm{H}_q(\mathcal{P}_h(M)).\end{aligned}$$

Summing over all elements $g \in G$ we obtain the map

$$\Delta : \mathrm{H}_*(\mathcal{P}_G(M)) \to \mathrm{H}_*(\mathcal{P}_G(M)) \otimes \mathrm{H}_*(\mathcal{P}_G(M))$$

We will call Δ the *G-string coproduct.*

The unit: We consider the next diagram

$$\begin{tikzcd} & M \arrow[dl, "r"'] \arrow[dr, "i_c"] & \\ \{\mathrm{pt}\} & & \mathcal{P}_e(M) \end{tikzcd}$$

where $r : M \to \{\mathrm{pt}\}$, the constant map and $i_c : M \to \mathcal{P}_e(M)$ is defined by $i_c(y) = \alpha : I \to M$ such that $\alpha(t) = y$ is the constant loop.

Then $u : \mathrm{H}_*(\{\mathrm{pt}\}) = k \xrightarrow{r!} \mathrm{H}_*(Y) \xrightarrow{i_{c*}} \mathrm{H}_*(\mathcal{P}_e(M)) \to \mathrm{H}_*(\mathcal{P}_G(M))$.

$$u : k \to \mathrm{H}_*(\mathcal{P}_G(M)).$$

Note that as the same as string topology the loop orbifold has not trace, this is because, as late, the counit is given by the diagram

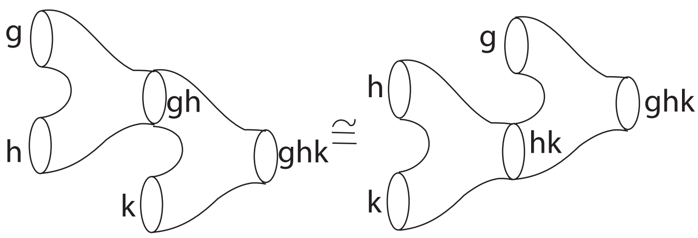
$$\begin{tikzcd} & M \arrow[dl, "r"'] \arrow[dr, hook, "i"] & \\ \mathrm{pt} & & \mathcal{P}_e(M) \end{tikzcd}$$

and the inclusion map $M \hookrightarrow \mathcal{P}_e(M)$ has infinite codimension.

Theorem 5.4.5. $\mathrm{H}_*(\mathcal{P}_G(M))$ *is a nearly G-Frobenius algebra.*

Proof. We will to check all the axioms.

1. **Associativity of the product**



(1) (2)

Remember that the product is defined from the next diagram

$$\begin{tikzcd} & \mathcal{P}_g M \, {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_h M \arrow[dl, "j"'] \arrow[dr, "*"] & \\ \mathcal{P}_g M \times \mathcal{P}_h M & & \mathcal{P}_{gh} M \end{tikzcd}$$

The associativity is encoded in the next two diagrams.

(1)

$$\begin{tikzcd}
 & & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "1\times j"'] \arrow[dr, "*\times 1"] & & \\
 & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M \times \mathcal{P}_k M \arrow[dl, "j\times 1"'] \arrow[dr, "*\times 1"] & & \mathcal{P}_{gh} M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_g M \times \mathcal{P}_h M \times \mathcal{P}_k M & & \mathcal{P}_{gh} M \times \mathcal{P}_k M & & \mathcal{P}_{ghk} M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "j\times 1"'] \arrow[dr, "1\times *"] & & \\
 & \mathcal{P}_g M \times \mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[dl, "1\times j"'] \arrow[dr, "1\times *"] & & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{hk} M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_g M \times \mathcal{P}_h M \times \mathcal{P}_k M & & \mathcal{P}_g M \times \mathcal{P}_{hk} M & & \mathcal{P}_{ghk} M
\end{tikzcd}$$

The first case involved the next constructions

$$\begin{tikzcd}
(*\times 1)^*\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[d, "*\times 1"'] & \\
\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_{gh} M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{gh} M \times \mathcal{P}_k M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\epsilon_\infty \times \epsilon_\infty)^*(TM) \arrow[r, dashed] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[r, "1\times j"] \arrow[d, "\epsilon_\infty\times\epsilon_\infty"'] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M \times \mathcal{P}_k M \arrow[d, "\epsilon_\infty\times\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \times M \arrow[r, "1\times\Delta"'] & M \times M \times M
\end{tikzcd}$$

We note that $(* \times 1)^* \epsilon_\infty^*(TM) = (\epsilon_\infty \times \epsilon_\infty)^*(TM)$. Then $F_1 = 0$.

In the second diagram we have the next constructions

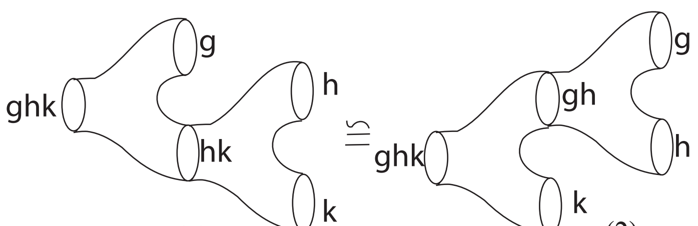
$$\begin{tikzcd}
(1\times *)^*\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[d, "1\times *"] & \\
\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{hk}M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"] & \mathcal{P}_gM \times \mathcal{P}_{hk}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\epsilon_\infty \times \epsilon_\infty)^*(TM) \arrow[r, dashed] & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[r, "j\times 1"] \arrow[d, "\epsilon_\infty\times\epsilon_\infty"] & \mathcal{P}_gM \times \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[d, "\epsilon_1\times\epsilon_0\times\epsilon_\infty"] \\
TM \arrow[r, dashed] & M \times M \arrow[r, "\Delta\times 1"'] & M \times M \times M
\end{tikzcd}$$

Similarly as before, we note that $(1 \times *)^* \epsilon_\infty^*(TM) = (\epsilon_\infty \times \epsilon_\infty)^*(TM)$. Then $F_2 = 0$. Therefore the product is associative.

2. **Coassociativity of the coproduct**



(1) (2)

In the same way as the product, the coproduct is defined from the diagram

$$\begin{tikzcd}
 & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM \arrow[dl, "*"'] \arrow[dr, "j"] & \\
\mathcal{P}_{gh}M & & \mathcal{P}_gM \times \mathcal{P}_hM
\end{tikzcd}$$

The diagrams that represent this property are

(1)

$$
\begin{tikzcd}[column sep=tiny]
 & & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "*\times 1"'] \arrow[dr, "1\times j"] & & \\
 & \mathcal{P}_{gh}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "*"'] \arrow[dr, "j"] & & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM\times\mathcal{P}_kM \arrow[dl, "*\times 1"'] \arrow[dr, "j\times 1"] & \\
\mathcal{P}_{ghk}M & & \mathcal{P}_{gh}M\times\mathcal{P}_kM & & \mathcal{P}_gM\times\mathcal{P}_hM\times\mathcal{P}_kM
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}[column sep=tiny]
 & & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "1\times *"'] \arrow[dr, "j\times 1"] & & \\
 & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{hk}M \arrow[dl, "*"'] \arrow[dr, "j"] & & \mathcal{P}_gM\times\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "1\times *"'] \arrow[dr, "1\times j"] & \\
\mathcal{P}_{ghk}M & & \mathcal{P}_gM\times\mathcal{P}_{hk}M & & \mathcal{P}_gM\times\mathcal{P}_hM\times\mathcal{P}_kM
\end{tikzcd}
$$

In the first case we have the next constructions

$$
\begin{tikzcd}
(1\times j)^*(\epsilon_\infty\times\epsilon_0)^*\eta \arrow[r, dashed] & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[d, "1\times j"'] & \\
(\epsilon_\infty\times\epsilon_0)^*\eta \arrow[r, dashed] & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM\times\mathcal{P}_kM \arrow[r, "*\times 1"] \arrow[d, "\epsilon_\infty\times\epsilon_0"'] & \mathcal{P}_{gh}M\times\mathcal{P}_kM \arrow[d, "(\epsilon_{\frac{1}{2}},\epsilon_0 g)\times\epsilon_0"] \\
\eta \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"'] & M\times M\times M
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(\epsilon_\infty\times\epsilon_0)^*\eta \arrow[r, dashed] & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[r, "*\times 1"] \arrow[d, "\epsilon_\infty\times\epsilon_0"'] & \mathcal{P}_{gh}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[d, "(\epsilon_{\frac{1}{2}},\epsilon_0 g)\times\epsilon_0"] \\
\eta \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"'] & M\times M\times M
\end{tikzcd}
$$

We note that $(1\times j)^*(\epsilon_\infty\times\epsilon_0)^*\eta = (\epsilon_\infty\times\epsilon_0)^*\eta$. Then $F_1 = 0$.

The second diagram has the next constructions

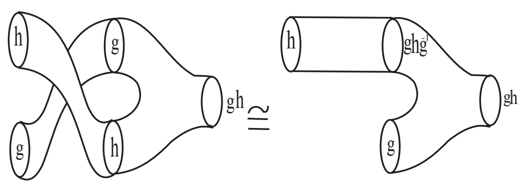
$$\begin{tikzcd}
(j\times 1)^*(\epsilon_1\times\epsilon_\infty)^*\eta \arrow[r, dashed] & \mathcal{P}_gM\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathcal{P}_hM\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathcal{P}_kM \arrow[d, "j\times 1"'] & \\
(\epsilon_1\times\epsilon_\infty)^*\eta \arrow[r, dashed] & \mathcal{P}_gM\times\mathcal{P}_hM\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathcal{P}_kM \arrow[r, "1\times *"] \arrow[d, "\epsilon_1\times\epsilon_\infty"'] & \mathcal{P}_gM\times\mathcal{P}_{hk}M \arrow[d, "\epsilon_1\times(\epsilon_{\frac{1}{2}},\epsilon_0 h)"] \\
\eta \arrow[r, dashed] & M\times M \arrow[r, "1\times\Delta"'] & M\times M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\epsilon_1\times\epsilon_\infty)^*\eta \arrow[r, dashed] & \mathcal{P}_gM\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathcal{P}_hM\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathcal{P}_kM \arrow[r, "1\times *"] \arrow[d, "\epsilon_1\times\epsilon_\infty"'] & \mathcal{P}_gM\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathcal{P}_{hk}M \arrow[d, "\epsilon_1\times(\epsilon_{\frac{1}{2}},\epsilon_0 h)"] \\
\eta \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"'] & M\times M\times M
\end{tikzcd}$$

In the same way as before, we note that $(j\times 1)^*(\epsilon_1\times\epsilon_\infty)^*\eta = (\epsilon_1\times\epsilon_\infty)^*\eta$. Then $F_2 = 0$.

3. **Graded commutativity of the product**



(1) (2)

This property is represented in the next diagrams

(1)

$$\begin{tikzcd}
 & & & \mathcal{P}_hM\,{}_{\epsilon_1}\!\times_{\epsilon_1}\mathcal{P}_gM \arrow[dll, "j\circ(1\times\alpha_{g^{-1}})\circ\tau"'] \arrow[drr, "\tau\circ(\alpha_{h^{-1}}\times 1)"] & & & \\
 & \mathcal{P}_hM\times\mathcal{P}_gM \arrow[dl, "1"'] \arrow[dr, "\tau\circ(\alpha_{h^{-1}}\times\alpha_g)"] & & & & \mathcal{P}_gM\,{}_{\epsilon_1}\!\times_{\epsilon_0}\mathcal{P}_hM \arrow[dll, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_hM\times\mathcal{P}_gM & & \mathcal{P}_gM\times\mathcal{P}_hM & & & & \mathcal{P}_{gh}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_1}\mathcal{P}_g M \arrow[dl, "i"'] \arrow[dr, "\alpha_g\times 1"] & & \\
 & \mathcal{P}_h M\times\mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "(\alpha_g\times 1)\circ\tau"] & & \mathcal{P}_{ghg^{-1}}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_h M\times\mathcal{P}_g M & & \mathcal{P}_{ghg^{-1}}M\times\mathcal{P}_g M & & \mathcal{P}_{gh}M
\end{tikzcd}$$

First, we need to check that the maps $*\circ(\alpha_g\times 1)$ and $*\circ\tau\circ(\alpha_{h^{-1}}\times 1)$ are homotopic maps and the same for $j\circ(1\times\alpha_{g^{-1}})$ and i. In each case, we will construct the homotopy. In the first case we define

$$H : I\times(\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_1}\mathcal{P}_g M)\to\mathcal{P}_{gh}M$$

by

$$H(s,(\alpha,\beta))(t) := \alpha_g(\alpha)*\beta*\alpha_{h^{-1}}(\alpha)\left(\frac{s+2t}{3}\right)$$

Note that $H(0,(\alpha,\beta))(t)=\alpha_g(\alpha)*\beta*\alpha_{h^{-1}}(\alpha)\left(\frac{2t}{3}\right)=\alpha_g(\alpha)*\beta(t)=(*\circ(\alpha_g\times 1))(\alpha,\beta)(t)$, and $H(1,(\alpha,\beta))(t)=\alpha_g(\alpha)*\beta*\alpha_{h^{-1}}(\alpha)\left(\frac{1+2t}{3}\right)=\beta*\alpha_{h^{-1}}(\alpha)(t)=(*\circ\tau(\alpha_{h^{-1}}\times 1))(\alpha,\beta)(t)$.
In the second case the next map

$$F : I\times(\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_1}\mathcal{P}_g M)\to\mathcal{P}_h M\times\mathcal{P}_g M$$

is defined by

$$F(s,(\alpha,\beta))(r,t)=\left(\alpha(r),\beta*\alpha_{g^{-1}}(\beta)\left(\frac{s+t}{2}\right)\right)$$

Note that $F(0,(\alpha,\beta))(r,t)=\left(\alpha(r),\beta*\alpha_{g^{-1}}(\beta)\left(\frac{t}{2}\right)\right)=(\alpha(r),\beta(t))=i(\alpha,\beta)(r,t)$, and $F(1,(\alpha,\beta))(r,t)=\left(\alpha(r),\beta*\alpha_{g^{-1}}(\beta)\left(\frac{1+t}{2}\right)\right)=(\alpha(r),\alpha_{g^{-1}}(\beta)(t))=j\circ(1\times\alpha_{g^{-1}})(\alpha,\beta)(r,t)$.

Now, we can determine the Euler classes. In the first case we have

$$\begin{tikzcd}
(\epsilon_\infty \circ \tau \circ (\alpha_{h^{-1}} \times 1))^*(TM) \arrow[r, dashed] & \mathcal{P}_h M {}_{\epsilon_1}\!\times_{\epsilon_1} \mathcal{P}_g M \arrow[d, "\tau\circ(\alpha_{h^{-1}}\times 1)"'] & \\
& \mathcal{P}_g M {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_h M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g M \times \mathcal{P}_h M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_1^*(TM) \arrow[r, dashed] & \mathcal{P}_h M {}_{\epsilon_1}\!\times_{\epsilon_1} \mathcal{P}_g M \arrow[r, "i"] \arrow[d, "\epsilon_1"'] & \mathcal{P}_h M \times \mathcal{P}_g M \arrow[d, "\epsilon_1\times\epsilon_1"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

We note that $\epsilon_1 = \epsilon_\infty \circ \tau \circ (\alpha_{h^{-1}} \times 1)$, then $F_1 = 0$.

For the second case

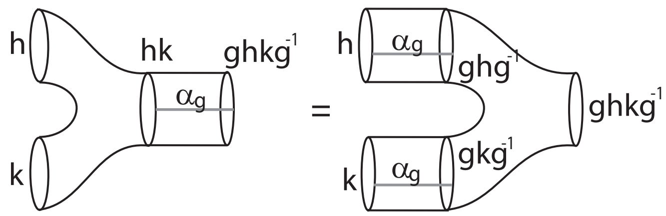
$$\begin{tikzcd}
(\epsilon_\infty \circ (\alpha_g \times 1))^*(TM) \arrow[r, dashed] & \mathcal{P}_h M {}_{\epsilon_1}\!\times_{\epsilon_1} \mathcal{P}_g M \arrow[d, "\tau\circ(\alpha_g\times 1)"'] & \\
& \mathcal{P}_{ghg^{-1}} M {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_g M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{ghg^{-1}} M \times \mathcal{P}_g M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_1^*(TM) \arrow[r, dashed] & \mathcal{P}_h M {}_{\epsilon_1}\!\times_{\epsilon_1} \mathcal{P}_g M \arrow[r, "i"] \arrow[d, "\epsilon_1"'] & \mathcal{P}_h M \times \mathcal{P}_g M \arrow[d, "\epsilon_1\times\epsilon_1"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

Similarly we note that $\epsilon_1 = \epsilon_\infty \circ (\alpha_g \times 1)$, then $F_2 = 0$.

4. **The action is an algebra homomorphism**



(1) (2)

This property is described in the next diagrams.

(1)

$$\begin{tikzcd}
 & & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "1"'] \arrow[dr, "*"] & & \\
 & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "j"'] \arrow[dr, "*"] & & \mathcal{P}_{hk}M \arrow[dl, "1"'] \arrow[dr, "\alpha_g"] & \\
\mathcal{P}_hM \times \mathcal{P}_kM & & \mathcal{P}_{hk}M & & \mathcal{P}_{ghkg^{-1}}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "j"'] \arrow[dr, "\alpha_g\times\alpha_g"] & & \\
 & \mathcal{P}_hM \times \mathcal{P}_kM \arrow[dl, "1"'] \arrow[dr, "\alpha_g\times\alpha_g"] & & \mathcal{P}_{ghg^{-1}}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{gkg^{-1}}M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_hM \times \mathcal{P}_kM & & \mathcal{P}_{ghg^{-1}}M \times \mathcal{P}_{gkg^{-1}}M & & \mathcal{P}_{ghkg^{-1}}M
\end{tikzcd}$$

In the first case is clearly that $F_1 = 0$ because the normal bundle is zero. Now we study the second case. This is

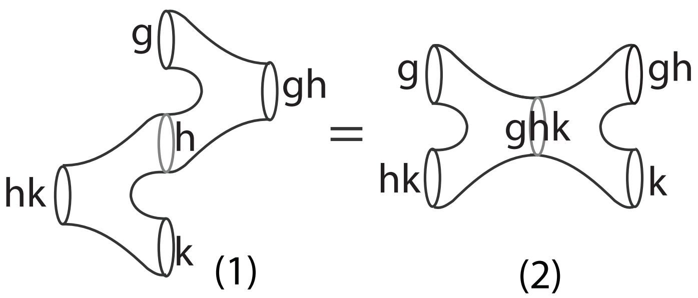
$$
\begin{tikzcd}
(\alpha_g \times \alpha_g)^* \epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_h M {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_k M \arrow[d, "\alpha_g \times \alpha_g"'] & \\
& \mathcal{P}_{ghg^{-1}} M {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_{gkg^{-1}} M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{ghg^{-1}} M \times \mathcal{P}_{gkg^{-1}} M \arrow[d, "\epsilon_1 \times \epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_h M {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_k M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_h M \times \mathcal{P}_k M \arrow[d, "\epsilon_1 \times \epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}
$$

Note that $\epsilon_\infty^*(TM) = (\alpha_g \times \alpha_g)^* \epsilon_\infty^*(TM)$, then $F_2 = 0$.

5. **Abrams condition**



This property is modeled by the next diagrams
(1)

$$
\begin{tikzcd}[column sep=tiny]
& & \mathcal{P}_g M {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_h M {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_k M \arrow[dl, "j \times 1"'] \arrow[dr, "1 \times j"] & & \\
& \mathcal{P}_g M \times \mathcal{P}_h M {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_k M \arrow[dl, "1 \times *"'] \arrow[dr, "1 \times j"] & & \mathcal{P}_g M {}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_h M \times \mathcal{P}_k M \arrow[dl, "j \times 1"'] \arrow[dr, "* \times 1"] & \\
\mathcal{P}_g M \times \mathcal{P}_{hk} M & & \mathcal{P}_g M \times \mathcal{P}_h M \times \mathcal{P}_k M & & \mathcal{P}_{gh} M \times \mathcal{P}_k M
\end{tikzcd}
$$

(2)

$$\begin{tikzcd}[column sep=tiny]
 & & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "1\times *"'] \arrow[dr, "*\times 1"] & & \\
 & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{hk}M \arrow[dl, "j"'] \arrow[dr, "*"] & & \mathcal{P}_{gh}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[dl, "*"'] \arrow[dr, "j"] & \\
\mathcal{P}_gM\times\mathcal{P}_{hk}M & & \mathcal{P}_{ghk}M & & \mathcal{P}_{gh}M\times\mathcal{P}_kM
\end{tikzcd}$$

The first case involves the following

$$\begin{tikzcd}
((\epsilon_\infty\times\epsilon_0)\circ(1\times j))^*(TM) \arrow[r, dashed] & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[d, "1\times j"] & \\
 & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM\times\mathcal{P}_kM \arrow[r, "j\times 1"] \arrow[d, "\epsilon_\infty\times\epsilon_0"] & \mathcal{P}_gM\times\mathcal{P}_hM\times\mathcal{P}_kM \arrow[d, "\epsilon_1\times\epsilon_0\times\epsilon_0"] \\
TM \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"'] & M\times M\times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\epsilon_\infty\times\epsilon_0)^*(TM) \arrow[r, dashed] & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[d, "\epsilon_\infty\times\epsilon_0"'] & \mathcal{P}_gM\times\mathcal{P}_hM\times\mathcal{P}_kM \arrow[d, "\epsilon_1\times\epsilon_0\times\epsilon_0"] \\
TM \arrow[r, dashed] & M\times M \arrow[r, "\Delta\times 1"'] & M\times M\times M
\end{tikzcd}$$

It is clear that $(\epsilon_\infty\times\epsilon_0)^*(TM) = ((\epsilon_\infty\times\epsilon_0)\circ(1\times j))^*(TM)$, then $F_1 = 0$.

In the second case we have

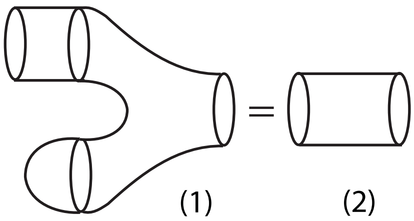
$$\begin{tikzcd}
(\epsilon_\infty\circ(*\times 1))^*(TM) \arrow[r, dashed] & \mathcal{P}_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[d, "*\times 1"'] & \\
 & \mathcal{P}_{gh}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_kM \arrow[r, "*"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{ghk}M \arrow[d, "\epsilon_{\frac{1}{2}}\times\epsilon_0 gh"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}$$

and

$$
\begin{tikzcd}
(\epsilon_1\times\epsilon_\infty)^*(TM) \arrow[r, dashed] \arrow[d, phantom] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_k M \arrow[r, "1\times *"] \arrow[d, "\epsilon_1\times\epsilon_\infty"'] & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{hk} M \arrow[d, "\epsilon_1\times\epsilon_{\frac{1}{2}}\times\epsilon_0 h"] \\
TM \arrow[r, dashed] & M\times M \arrow[r, "1\times\Delta"'] & M\times M\times M
\end{tikzcd}
$$

Finally $(\epsilon_1\times\epsilon_\infty)^*(TM) = (\epsilon_\infty\circ(*\times 1)^*(TM)$, and then $F_2 = 0$.

6. **Unit axiom**



Remember that the unit map is defined from the next diagram

$$
\begin{tikzcd}
& M \arrow[dl, "r"'] \arrow[dr, "i_c"] & \\
\mathrm{pt} & & \mathcal{P}_e M
\end{tikzcd}
$$

where $r: M\to \mathrm{pt}$ is the constant map, $\mathcal{P}_e M = \{\alpha : I\to M :\ \alpha(1)=\alpha(0)\} = \mathrm{LM}$, and $i_c : M\hookrightarrow \mathrm{LM}$ in the natural inclusion. Then $u:\mathrm{H}_*(\mathrm{pt})\to\mathrm{H}_*(\mathrm{LM}) = \mathrm{H}_*(\mathcal{P}_e M)$ is the next composition map

$$\mathrm{H}_*(\mathrm{pt}) \xrightarrow{r!} \mathrm{H}_*(M) \xrightarrow{i_{c*}} \mathrm{LM}\,.$$

The diagrams that represent the unit axiom are

(2)

$$
\begin{tikzcd}
& & \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "1"] & & \\
& \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "1"] & & \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "1"] & \\
\mathcal{P}_g M & & \mathcal{P}_g M & & \mathcal{P}_g M
\end{tikzcd}
$$

It is clear that $F_2 = 0$.
(1)

$$\begin{tikzcd}
 & & \mathcal{P}_g M \arrow[dl, "1\times\epsilon_1"'] \arrow[dr, "1\times i_c\circ\epsilon_1"] \arrow[ddll, bend right, "\varphi"'] \arrow[ddrr, bend left, "\psi"] & & \\
 & \mathcal{P}_g M \times M \arrow[dl, "1\times r"'] \arrow[dr, "1\times i_c"] & & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_e M \arrow[dl, "j"'] \arrow[dr, "*"] & \\
\mathcal{P}_g M \times \mathrm{pt} & & \mathcal{P}_g M \times \mathcal{P}_e M & & \mathcal{P}_g M
\end{tikzcd}$$

First, we note that the map ψ is homotopic to the identity $\mathrm{Id} : \mathcal{P}_g M \to \mathcal{P}_g M$, this is because

$$\psi : \alpha \mapsto (\alpha, i_c(\alpha(1))) \mapsto \alpha * i_c(\alpha(1)) \simeq \alpha.$$

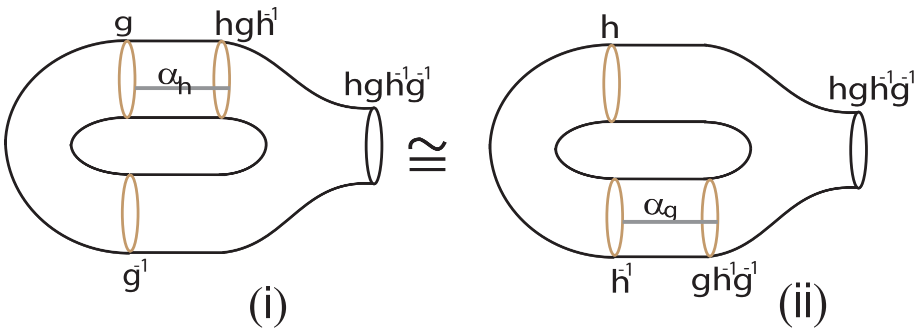
Clearly the map φ is the identity map.
Now, we determine the class of the square.

$$\begin{tikzcd}
\epsilon_1^*(TM) \arrow[r, dashed] & \mathcal{P}_g M \arrow[r, "1\times\epsilon_1"] \arrow[d, "\epsilon_1"'] & \mathcal{P}_g M \times M \arrow[d, "\epsilon_1\times 1"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

$$\begin{tikzcd}
(1\times\epsilon_1)^*\epsilon_\infty^*(TM) \arrow[r, dashed] & \mathcal{P}_g M \arrow[d, "1\times\epsilon_1"'] & \\
 & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_e M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g M \times \mathcal{P}_e M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r, dashed] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

In this case we note that $\epsilon_\infty \circ (1 \times \epsilon_1) = \epsilon_1$, this implies $\epsilon_1^*(TM) = (1 \times \epsilon_1)^*\epsilon_\infty^*(TM)$, and then $F_1 = 0$.

7. **Torus axiom**



The co-pairing map $\Theta_g : \Bbbk \to \mathrm{H}_*(\mathcal{P}_g M) \otimes \mathrm{H}_*(\mathcal{P}_{g^{-1}} M)$ is defined as the composition of the unit and the coproduct as follows,

$$\Bbbk \xrightarrow{u} \mathrm{H}_*(\mathcal{P}_e M) \xrightarrow{\Delta_{g,g^{-1}}} \mathrm{H}_*(\mathcal{P}_g M) \otimes \mathrm{H}_*(\mathcal{P}_{g^{-1}} M).$$

Now, we describe this map.

$$\begin{tikzcd}
 & & M^g \arrow[dl, "i_g"'] \arrow[dr, "f_g"] & & \\
 & M \arrow[dl, "r"'] \arrow[dr, "i_c"] & & \mathcal{P}_g M \,{}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_{g^{-1}} M \arrow[dl, "*"'] \arrow[dr, "j"] & \\
\mathrm{pt} & & \mathcal{P}_e M & & \mathcal{P}_g M \times \mathcal{P}_{g^{-1}} M
\end{tikzcd}$$

where the map $i_g : M^g \to M$ is the inclusion, and $f_g : M^g \to \mathcal{P}_g M \,{}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_{g^{-1}} M$ is given by $x \mapsto (\alpha_x, \alpha_x)$ with α_x the constant loop. The Quillen's class of this square is described as follows:

$$\begin{tikzcd}
\nu_{i_g} \arrow[r] & M^g \arrow[r, hook, "i_g"] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
f_g^* \epsilon_0^* (\nu_{(1\times\alpha_g)}) \arrow[r] & M^g \arrow[d, "f_g"'] & \\
 & \mathcal{P}_g M \,{}_{\epsilon_1}\!\times_{\epsilon_0} \mathcal{P}_{g^{-1}} M \arrow[r, "*"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_e M \arrow[d, "\epsilon_0 \times \epsilon_{\frac{1}{2}}"] \\
\nu_{(1\times\alpha_g)} \arrow[r] & M \arrow[r, "1\times\alpha_g"'] & M \times M
\end{tikzcd}$$

Note that $\epsilon_0 \circ f_g(x) = x$, this implies that $\epsilon_0 \circ f_g = i_g$ and $f_g^* \epsilon_0^*(\nu_{(1\times\alpha_g)}) = i_g^*(\nu_{(1\times\alpha_g)})$ Therefore F_g is given by the next exact sequence

$$0 \longrightarrow \nu_{i_g} \longrightarrow i_g^*(\nu_{(1\times\alpha_g)}) \longrightarrow F_g \longrightarrow 0.$$

In the next step we determine the diagram associated to the first figure.
(1)

$$\begin{tikzcd}
 & & & M^{g,h} \arrow[dl, "i^g_{g,h}"'] \arrow[drrr, "j_{g,h}"] \arrow[ddddlll, bend right, "r"'] \arrow[ddddrrrr, bend left, "i"] & & & & \\
 & & M^g \arrow[dl, "i_g"'] \arrow[dr, "f_g"] & & & & & \\
 & M \arrow[ddl, "r"'] \arrow[ddr, "i_c"] & & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{g^{-1}}M \arrow[ddl, "*"'] \arrow[dr, "j"] & & & \mathcal{P}_{hgh^{-1}} M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{g^{-1}}M \arrow[dd, "j"'] \arrow[ddr, "*"] & \\
 & & & & \mathcal{P}_g M \times \mathcal{P}_{g^{-1}}M \arrow[drr, "\alpha_h\times 1"] & & & \\
\mathrm{pt} & & \mathcal{P}_e M & & & & \mathcal{P}_{hgh^{-1}} M \times \mathcal{P}_{g^{-1}}M & \mathcal{P}_{hgh^{-1}g^{-1}}M
\end{tikzcd}$$

The class F_1 is given by

$$\nu_{i^g_{g,h}} \longrightarrow M^{g,h} \xhookrightarrow{i^g_{g,h}} M^g$$

and

$$\begin{tikzcd}
j^*_{g,h}\epsilon^*_\infty(TM) \arrow[r] & M^{g,h} \arrow[d, "j_{g,h}"] & \\
 & \mathcal{P}_{hgh^{-1}} M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{g^{-1}}M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_{hgh^{-1}} M \times \mathcal{P}_{g^{-1}}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r] & M \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

Note that $\epsilon_\infty \circ j_{g,h}(x) = \epsilon_\infty(\alpha_h(\alpha_x), \alpha_x) = x$, then $\epsilon_\infty \circ j_{g,h} = i_{g,h}$ and we have the next exact sequence

$$0 \longrightarrow \nu_{i^g_{g,h}} \longrightarrow i^*_{g,h}(TM) \longrightarrow F_1 \longrightarrow 0.$$

The second diagram is the following

(2)

$$
\begin{tikzcd}
 & & & M^{g,h} \arrow[dl, "i^h_{g,h}"'] \arrow[ddrrr, "j_{g,h}"] \arrow[ddddrrrr, bend left, "i"] \arrow[ddddlll, bend right, "r"'] & & & & \\
 & & M^h \arrow[dl, "i_h"'] \arrow[dr, "f_h"] & & & & & \\
 & M \arrow[ddl, "r"'] \arrow[ddr, "i_c"] & & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{h^{-1}}M \arrow[ddl, "*"'] \arrow[dr, "j"] & & & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{gh^{-1}g^{-1}}M \arrow[dd, "j"'] \arrow[ddr, "*"] & \\
 & & & & \mathcal{P}_hM\times\mathcal{P}_{h^{-1}}M \arrow[drr, "1\times\alpha_g"] & & & \\
\mathrm{pt} & & \mathcal{P}_eM & & & & \mathcal{P}_hM\times\mathcal{P}_{gh^{-1}g^{-1}}M & \mathcal{P}_{hgh^{-1}g^{-1}}M
\end{tikzcd}
$$

The class F_2 is associate to the next map

$$\nu_{i^h_{g,h}} \longrightarrow M^{g,h} \xhookrightarrow{i^h_{g,h}} M^h$$

in this case we have

$$
\begin{tikzcd}
j^*_{g,h}\epsilon^*_\infty(TM) \arrow[r] & M^{g,h} \arrow[d, "j_{g,h}"] & \\
 & \mathcal{P}_hM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{gh^{-1}g^{-1}}M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_hM\times\mathcal{P}_{gh^{-1}g^{-1}}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TM \arrow[r] & M \arrow[r, "\Delta"'] & M\times M
\end{tikzcd}
$$

As before there is the identity $j^*_{g,h}\epsilon^*_\infty(TM) = i^*_{g,h}(TM)$. Then

$$0 \longrightarrow \nu_{i^h_{g,h}} \longrightarrow i^*_{g,h}(TM) \longrightarrow F_2 \longrightarrow 0.$$

Applying the Quillen's we conclude

$$\circledast_* j!((\alpha_h \times 1)j)_* \circledast ! i_{c*} r!(1) = i_*(r!(1) \cap (e(i^{g*}_{g,h}(F_g)) \cup e(F_1)))$$

and

$$\circledast_* j!((1 \times \alpha_g)j)_* \circledast ! i_{c*} r!(1) = i_*(r!(1) \cap (e(i^{h*}_{g,h}(F_h)) \cup e(F_2)))$$

To prove the axiom we need to check that

$$e(i^{g*}_{g,h}(F_g)) \cup e(F_1) = e(i^{h*}_{g,h}(F_h)) \cup e(F_2),$$

or equivalently
$$i^{g*}_{g,h}(F_g) \oplus F_1 \cong i^{h*}_{g,h}(F_h) \oplus F_2.$$

The bundles are the following:

$$\begin{array}{rcccccc} E_1 & = & i^{g*}_{g,h}(F_g) \oplus F_1 & = & \frac{i^*_{g,h}(TM)}{i^{g*}_{g,h}(\nu_{i_g})} & \oplus & \frac{i^*_{g,h}(TM)}{\nu_{i^g_{g,h}}} \\ E_2 & = & i^{h*}_{g,h}(F_h) \oplus F_2 & = & \frac{i^*_{g,h}(TM)}{i^{h*}_{g,h}(\nu_{i_h})} & \oplus & \frac{i^*_{g,h}(TM)}{\nu_{i^h_{g,h}}} \end{array}$$

The information is represented in the next diagrams

$$\begin{tikzcd} & i^*_{g,h}(TM) \arrow[dr, dotted] & i^{g*}_{g,h}(\nu_{i_g}) \arrow[d, dotted] & \nu_{i_g} \arrow[d, dotted] & TM \arrow[d, dotted] \\ \nu_{i^g_{g,h}} \arrow[rr, dotted] & & M^{g,h} \arrow[r, hook, "i^g_{g,h}"'] & M^g \arrow[r, hook, "i_g"'] & M \end{tikzcd}$$

and

$$\begin{tikzcd} & i^*_{g,h}(TM) \arrow[dr, dotted] & i^{h*}_{g,h}(\nu_{i_h}) \arrow[d, dotted] & \nu_{i_h} \arrow[d, dotted] & TM \arrow[d, dotted] \\ \nu_{i^h_{g,h}} \arrow[rr, dotted] & & M^{g,h} \arrow[r, hook, "i^h_{g,h}"'] & M^h \arrow[r, hook, "i_h"'] & M \end{tikzcd}$$

Using that all the maps are inclusions we have that $i^*_{g,h}(TM) = TM|_{M^{g,h}}$ and $i^{g*}_{g,h}(\nu_{i_g}) = \nu_{i_g}|_{M^{g,h}}$. In other hand, we observe that

$$TM|_{M^{g,h}} = TM^{g,h} \oplus \nu_{i^g_{g,h}} \oplus \nu_{i_g}|_{M^{g,h}},$$

and

$$TM|_{M^{g,h}} = TM^{g,h} \oplus \nu_{i^h_{g,h}} \oplus \nu_{i_h}|_{M^{g,h}}.$$

Then

$$\nu_{i^g_{g,h}} \oplus \nu_{i_g}|_{M^{g,h}} \cong \nu_{i^h_{g,h}} \oplus \nu_{i_h}|_{M^{g,h}}$$

and in particular $E_1 \cong E_2$. This proves that $e(E_1) = e(E_2)$ and the torus axiom is satisfied.

♣

Chapter 6
G-OC-TFT

In this chapter we will consider that a group G acts on a OC-TFT and this theory is called G-equivariant open-closed theory, for this we enlarge the category $\mathscr{S}_G$ in the sence that the objects are oriented 1-manifolds with boundary, with labelled ends, equipped with principal G-bundles. The morphisms are the same cobordisms as in the non-equivariant case, but they are equipped with G-bundles.
Up to isomorphism there is only one G-bundle on the interval: it is trivial, and it admits G as an automorphism group. So an equivariant theory gives us for each pair a, b of labels a vector space $\mathscr{O}_{ab}$ with a G-action. The action of $g \in G$ on $\mathscr{O}_{ab}$ can be regarded as coming from the square cobordism with the bundle whose holonomy is g along each of its constrained edges. There is also a composition law $\mathscr{O}_{ab} \times \mathscr{O}_{bc} \to \mathscr{O}_{ac}$

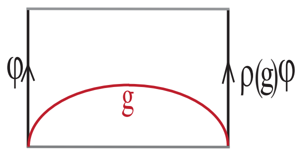


Figure 6.1: `The gray line is a constrained boundary. If there is holonomy` g `along the red path` $\mathcal{P}$ `then this morphism gives the` G`-action on` $\mathscr{O}_{ab}$.

which is G-equivariant. These maps are illustrated in Figure 6.1 and Figure 6.2.

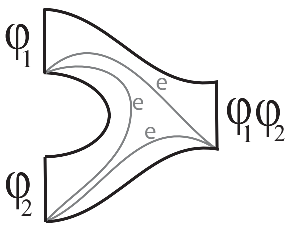


Figure 6.2: `The definition of the product in` $\mathscr{O}_{ab}$. `The holonomy on all gray path is` e, `the identity in` G.

In the open-closed case the analogous conditions are the following. We focuss first on a single label a, then the space $\mathscr{O}_{aa}$ is not necessarily a commutative Frobenius algebra together with a G-action $\rho : G \to \mathrm{Aut}(\mathscr{O})$:

$$\rho_g(\varphi_1\varphi_2) = (\rho_g\varphi_1)(\rho_g\varphi_2)$$

this action preserves the trace in the sense $\Theta_{\mathscr{O}}(\rho_g\varphi) = \Theta_{\mathscr{O}}(\varphi)$, see Figure 6.3. There

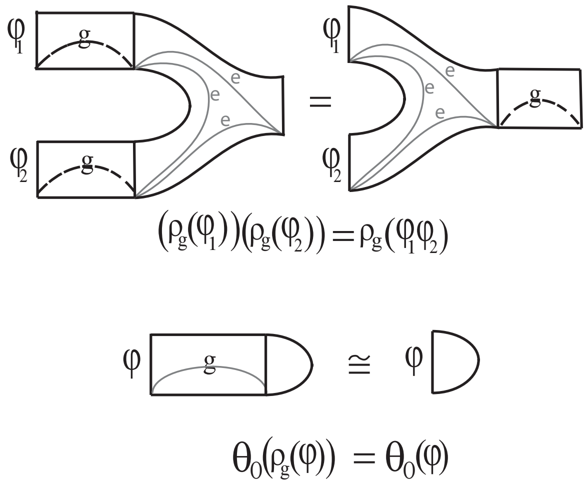


Figure 6.3: `Showing that` G `acts on` $\mathscr{O}$ `as a group of automorphisms`.

are also G-twisted open-closed transition maps

$$\iota_{g,a} = \iota_g : \mathscr{C}_g \to \mathscr{O}_{aa} = \mathscr{O},$$

$$\iota^{g,a} = \iota^g : \mathscr{O}_{aa} = \mathscr{O} \to \mathscr{C}_g,$$

which are G-equivariant, i.e. the next diagrams commute

$$\begin{tikzcd} \mathscr{C}_{g_1} \arrow[r, "\alpha_{g_2}"] \arrow[d, "\iota_{g_1}"'] & \mathscr{C}_{g_2g_1g_2^{-1}} \arrow[d, "\iota_{g_2g_1g_2^{-1}}"] \\ \mathscr{O} \arrow[r, "\rho_{g_2}"'] & \mathscr{O} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}_{g_2^{-1}g_1g_2} \arrow[r, "\alpha_{g_2}"] & \mathscr{C}_{g_1} \\ \mathscr{O} \arrow[r, "\rho_{g_2}"'] \arrow[u, "\iota^{g_2^{-1}g_1g_2}"] & \mathscr{O} \arrow[u, "\iota^{g_1}"'] \end{tikzcd}$$

These maps are illustrated in figure 6.4. The equivariant property is decried in figure 6.5.

Recall that we study in chapter 4 the definition of a G-Frobenius algebra where $\mathscr{C} = \oplus_{g\in G}\mathscr{C}_g$. The map $\iota : \mathscr{C} \to \mathscr{O}$ obtained by putting all the maps ι_g together, $\iota = \oplus_{g\in G}\iota_g$, is a ring homomorphisms (see figure 6.6), i.e.

$$\iota_{g_1}(\Phi_1)\iota_{g_2}(\Phi_2) = \iota_{g_2g_1}(\Phi_2\Phi_1),$$

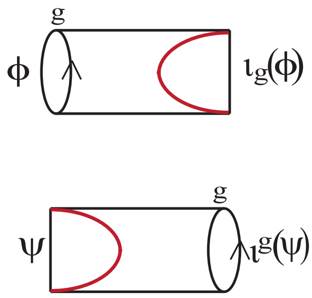


Figure 6.4: `The open-closed transitions maps` ι_g `and` ι^g.

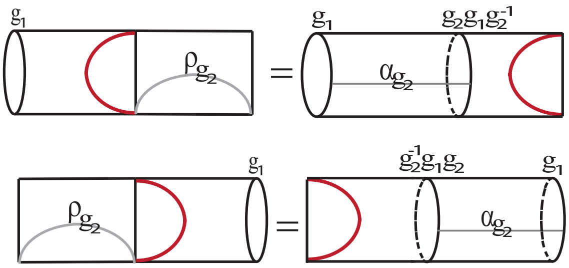


Figure 6.5: `Equivariant property of the transition maps.`

with $\Phi_1 \in \mathscr{C}_{g_1}$ and $\Phi_2 \in \mathscr{C}_{g_2}$. Moreover $\iota_e(1_{\mathscr{C}}) = 1_{\mathscr{O}}$. The G-twisted centrality condition is

$$\iota_g(\Phi)(\Psi) = (\rho_{g^{-1}}\Psi)\iota_g(\Phi),$$

with $\Phi \in \mathscr{C}_g$ y $\Psi \in \mathscr{O}$. The G-twisted adjoint condition is

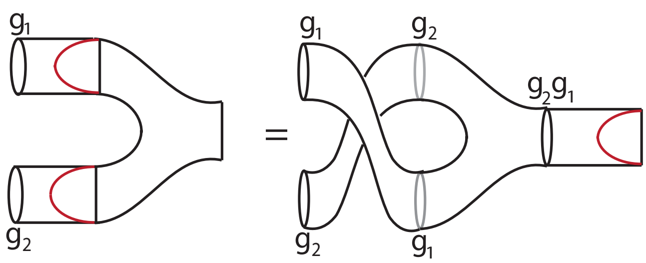


Figure 6.6: ι `is a ring homomorphism.`

$$\Theta_{\mathscr{O}}(\Psi\iota_{g^{-1}}(\Phi)) = \Theta_{\mathscr{C}}(\iota^g(\Psi)\Phi),$$

where $\Phi \in \mathscr{C}_{g^{-1}}$. Finally, the G-twisted Cardy conditions for the spaces of morphisms $\mathscr{O}_{ab}$ between the labels a and b. For each $g \in G$ we must have

$$\pi^a_{g,b} = \iota_{g,b}\iota^{g,a}.$$

Hence $\pi^a_{g,b}$ is defined by

$$\pi^a_{g,b}(\Psi) = \sum_{\mu} \psi^{\mu} \Psi(\rho_g \psi_{\mu})$$

where ψ_μ is a basis of $\mathscr{O}_{ab}$ and ψ^μ is the dual basis of $\mathscr{O}_{ba}$. See Figures 6.7 to 6.9.

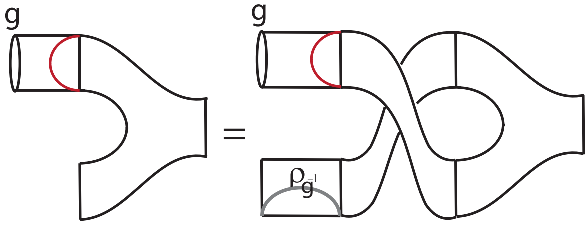


Figure 6.7: `The` G`-twisted centrality axiom`.

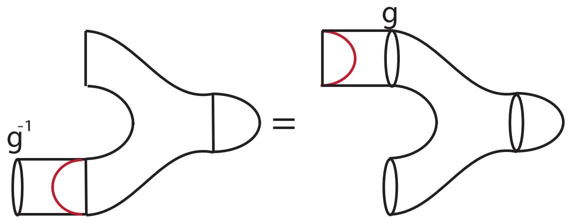


Figure 6.8: `The` G`-twisted adjoint relation`.

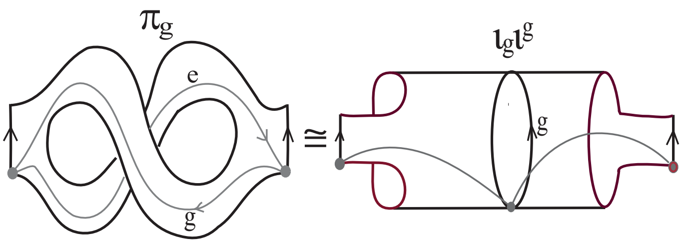


Figure 6.9: `The` G`-twisted Cardy condition`.

Theorem 6.0.6. *The G-invariant part of a G-OC-TFT is an OC-TFT.*

Proof. We apply the Proposition 6.0.6 to prove that $\mathscr{C}^G$ is a commutative Frobenius algebra.

Let be $\mathscr{O}^G_{ab}$ the G-invariant part of $\mathscr{O}_{ab}$, i.e.

$$\mathscr{O}^G_{ab} = \{\psi \in \mathscr{O}_{ab} : \rho_g(\psi) = \psi \ \forall g \in G\}$$

where $\rho : G \to \mathrm{Aut}(\mathscr{O}_{ab})$ is the action.
The next step are to define the structural maps and to check the properties associated to an OC-TFT.

1. We define the product by
$$\eta_{ac}^{b,G} : \mathscr{O}_{ab}^G \otimes \mathscr{O}_{bc}^G \to \mathscr{O}_{ac}^G$$
which is the restriction of the product $\eta_{ac}^b : \mathscr{O}_{ab} \otimes \mathscr{O}_{bc} \to \mathscr{O}_{ac}$. We note that, if $\psi \in \mathscr{O}_{ab}^G$ and $\varphi \in \mathscr{O}_{bc}^G$ then $\eta_{ac}^b(\psi \otimes \varphi) \in \mathscr{O}_{ac}^G$. This is because
$$\rho_g(\eta_{ac}^b(\psi \otimes \varphi)) = \eta_{ac}^b((\rho_g(\psi)) \otimes (\rho_g(\varphi))) = \eta_{ac}^b(\psi \otimes \varphi),$$
with $g \in G$.

2. Note that $u_a(1) \in \mathscr{O}_{aa}^G$, since $\rho_g(u_a(1)) = u_a(1)$ for all $g \in G$, so we define $u_a^G = u_a : \mathbb{C} \to \mathscr{O}_{aa}^G$.

3. The trace $\Theta_a^G : \mathscr{O}_{aa}^G \to \mathbb{C}$ is the restriction $\Theta_a^G = \Theta_a|_{\mathscr{O}_{aa}^G}$ of the trace $\Theta_a : \mathscr{O}_{aa} \to \mathbb{C}$.

4. The connection map $\iota_a^G : \mathscr{C}^G \to \mathscr{O}_{aa}^G$ is the restriction of the map $\iota_a : \mathscr{C} \to \mathscr{O}_{aa}$ to $\mathscr{C}^G$. We need to prove that $\iota_a^G(\phi) \in \mathscr{O}_{aa}^G$ for all $\phi \in \mathscr{C}^G$.
We know that $\rho_h(\iota_{g,a}(\phi)) = \iota_{hgh^{-1}}(\alpha_h(\phi))$, where $\phi \in \mathscr{C}_g$. If $\phi = \sum_{g\in G} \phi_g$, with $\phi_g \in \mathscr{C}_g$ then $\iota_a(\phi) = \sum_{g\in G} \iota_{g,a}(\phi_g)$. Finally we have
$$\begin{aligned}\rho_h(\iota_a(\phi)) &= \sum_{g\in G} \rho_h(\iota_{g,a}(\phi_g)) = \sum_{g\in G} \iota_{hgh^{-1}}(\alpha_h(\phi_g))\\ &= \sum_{g\in G} \iota_{hgh^{-1}}(\phi_{hgh^{-1}}) = \sum_{k\in G} \iota_k(\phi_k)\\ &= \iota_a(\phi).\end{aligned}$$

5. The map ι_a^G is central, i.e. $\iota_a^G(\phi)\psi = \psi\iota_a^G(\phi)$ for $\phi \in \mathscr{C}^G$ and $\psi \in \mathscr{O}_{aa}^G$. Let $\phi = \sum_{g\in G} \phi_g \in \mathscr{C}^G$, since $\iota_{g,a}(\phi_g)(\rho_g(\psi)) = \psi\iota_{g,a}(\phi_g)$ for $\phi_g \in \mathscr{C}_g$ and $\psi \in \mathscr{O}_{aa}$ the next identity holds
$$\begin{aligned}\iota_a^G(\phi)\psi &= \left(\sum_{g\in G} \iota_{g,a}(\phi_g)\right)\psi = \sum_{g\in G}(\iota_{g,a}(\phi_g)\rho_g(\psi))\\ &= \sum_{g\in G} \psi\iota_{g,a}(\phi_g) = \psi\sum_{g\in G} \iota_{g,a}(\phi_g)\\ &= \psi\iota_a(\phi).\end{aligned}$$

6. The connection map is $\iota^{a,G} : \mathscr{O}^G_{aa} \to \mathscr{C}^G$ is the restriction of the map $\iota^a : \mathscr{O}_{aa} \to \mathscr{C}$ to $\mathscr{O}^G_{aa}$. We know that $\iota^{g,a} \circ \rho_h = \alpha_h \circ \iota^{h^{-1}gh}$, and let $\psi \in \mathscr{O}^G_{aa}$ then

$$\begin{aligned}\iota^{a,G}(\psi) = \iota^{a,G}(\rho_h(\psi)) = \sum_{g\in G} \iota^{g,a}(\rho_h(\psi)) &= \sum_{g\in G} \alpha_h(\iota^{h^{-1}gh,a}(\psi)) \\ &= \alpha_h \left(\sum_{g\in G} \iota^{h^{-1}gh,a}(\psi) \right) = \alpha_h \left(\sum_{k\in G} \iota^{k,a}(\psi) \right) \\ &= \alpha_h(\iota^a(\psi)).\end{aligned}$$

7. Similarly, we have the adjoint property $\Theta_{\mathscr{C}^G}(\iota^{a,G}(\psi)\phi) = \Theta^G_a(\psi\iota^G_a(\phi))$. To prove the statement remember that $\Theta_a(\psi\iota_{g^{-1}}(\phi)) = \Theta_{\mathscr{C}}(\iota^{g,a}(\psi)\phi)$. Then

$$\begin{aligned}\Theta_{\mathscr{C}^G}(\iota^{a,G}(\psi)\phi) = \Theta_{\mathscr{C}^G}\left(\sum_{g,h\in G} \iota^{g,a}(\psi)\phi_h \right) &= \sum_{g,h\in G} \Theta_{\mathscr{C}^G}(\iota^{g,a}(\psi)\phi_h) \\ &= \sum_{g,h\in G} \Theta_a(\psi\iota_{a,g^{-1}}(\phi_h)) = \Theta_a\left(\sum_{g,h\in G} \psi\iota_{a,g^{-1}}(\phi_h) \right) \\ &= \Theta_a\left(\psi \sum_{g\in G} \iota_{a,g^{-1}}(\phi) \right) = \Theta^G_a(\psi\iota^G_a(\phi)).\end{aligned}$$

8. We need to define the coproduct $\Delta^{c,G}_{ab} : \mathscr{O}^G_{ab} \to \mathscr{O}^G_{ac} \otimes \mathscr{O}^G_{cb}$. Note that it is enough to give the map $\Theta^G : \mathbb{C} \to \mathscr{O}^G_{ac} \otimes \mathscr{O}^G_{ca}$ which is associated to figure 6.10.

   Now we consider the basis $\{\psi_\mu\}$ of $\mathscr{O}_{ac}$ and $\{\psi^\mu\}$ the dual basis in $\mathscr{O}_{ca}$. In

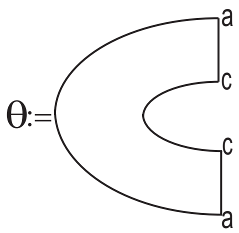


Figure 6.10: `Copairing` Θ.

   this case we define $\Theta : \mathbb{C} \to \mathscr{O}_{ac} \otimes \mathscr{O}_{ca}$ by $\Theta(1) = \sum_{\mu\in I} \psi_\mu \otimes \psi^\mu$. Now we take $\xi_\mu := \frac{1}{|G|}\sum_{g\in G} \rho_g(\psi_\mu) \in \mathscr{O}^G_{ac}$ and $\xi^\mu := \frac{1}{|G|}\sum_{g\in G} \rho_g(\psi^\mu) \in \mathscr{O}^G_{ca}$ and we define

$$\Theta^G(1) = \sum_{\mu\in I} \xi_\mu \otimes \xi^\mu \in \mathscr{O}^G_{ac} \otimes \mathscr{O}^G_{ca}.$$

Finally we need to prove that $\Delta^{c,G}_{ab}(\psi) := \sum_{\mu\in I}\psi\xi_\mu\otimes\xi^\mu = \sum_{\mu\in I}\xi_\mu\otimes\xi^\mu\psi$. We know that $\sum_{\mu\in I}\psi\psi_\mu\otimes\psi^\mu = \sum_{\mu\in I}\psi_\mu\otimes\psi^\mu\psi$. Then

$$\begin{aligned}\sum_{\mu\in I}\psi\xi_\mu\otimes\xi^\mu &= \sum_{\mu\in I}\sum_{g,h\in G}\psi\rho_g(\psi_\mu)\otimes\rho_h(\psi^\mu) = \sum_{\mu\in I}\sum_{g,h\in G}\rho_g(\psi)\rho_g(\psi_\mu)\otimes\rho_h(\psi^\mu)\\ &= \sum_{\mu\in I}\sum_{g,h\in G}\rho_g(\psi\psi_\mu)\otimes\rho_h(\psi^\mu) = \sum_{g,h\in G}(\rho_g\otimes\rho_h)\left(\sum_{\mu\in I}\psi\psi_\mu\otimes\psi^\mu\right)\\ &= \sum_{g,h\in G}(\rho_g\otimes\rho_h)\left(\sum_{\mu\in I}\psi_\mu\otimes\psi^\mu\psi\right) = \sum_{\mu\in I}\xi_\mu\otimes\xi^\mu\psi.\end{aligned}$$

9. The Cardy condition, i.e. the map $\pi^{a,G}_b := \eta^{a,G}_{bb}\circ\tau\circ\Delta^{b,G}_{aa} : \mathscr{O}^G_{aa}\to\mathscr{O}^G_{bb}$ coincides with the map $\iota^G_b\circ\iota^{a,G}$.
We know that $\pi^a_{g,b}(\psi) = \sum_{\mu\in I}\psi^\mu\psi(\rho_g\psi_\mu) = \iota_{g,b}\circ\iota^{g,a}$, for all $g\in G$. We use that $\Delta^{b,G}_{aa}(\psi) = \sum_{\mu\in I}\xi_\mu\otimes\xi^\mu\psi$ then

$$\begin{aligned}\pi^{a,G}_b(\psi) &= \sum_{\mu\in I}\xi^\mu\psi\xi_\mu = \sum_{g,h\in G}\sum_{\mu\in I}\rho_g(\psi^\mu)\psi\rho_h(\psi_\mu) = \sum_{g\in G}\sum_{\mu\in I}\left(\rho_g(\psi^\mu)\rho_g(\psi)\sum_{k\in G}\rho_{gk}(\psi_\mu)\right)\\ &= \sum_{g\in G}\rho_g\left(\sum_{\mu\in I}\sum_{k\in G}(\psi^\mu\psi\rho_k(\psi_\mu))\right) = \sum_{g,k\in G}\rho_g\left(\sum_{\mu\in I}\psi^\mu\psi\rho_g(\psi_\mu)\right)\\ &= \sum_{g,k\in G}\rho_g(\iota_{k,b}\circ\iota^{k,a}(\psi)) = \sum_{g,k\in G}\iota_{ghg^{-1}}(\alpha_g(\iota^{k,a}(\psi)))\\ &= \sum_{g,k\in G}\iota_{gkg^{-1},b}\circ\iota^{gkg^{-1},a}(\rho_g(\psi)) = \sum_{g,k\in G}\iota_{gkg^{-1},b}\circ\iota^{gkg^{-1},a}(\psi)\\ &= \iota^G_b\circ\iota^{a,G}(\psi).\end{aligned}$$

♣

In the next section we give the notion of a G-open-closed Topological Field Theory with positive boundary.

6.1 G-OC-TFT with positive boundary

As before we define the notion of a G-open-closed theory with positive boundary as a G-open-closed theory but with the restriction that the morphisms have at least one outgoing boundary.
The algebraic characterization is the following.

1. A nearly G-Frobenius algebra associated to the circle.

2. For each pair a, b of labels a vector space $\mathscr{O}_{ab}$ with a G-action

$$\rho : G \to \mathrm{Aut}(\mathscr{O}_{ab})$$

such that

$$\rho_g(\eta^c_{ab}(\varphi_1 \otimes \varphi_2)) = \eta^c_{ab}(\rho_g(\varphi_1) \otimes \rho_g(\varphi_2)),$$
$$\Delta^c_{ab}(\rho_g(\varphi)) = (\rho_g \otimes \rho_g)\Delta^c_{ab}(\varphi),$$

for $\varphi_1 \in \mathscr{O}_{ac}$, $\varphi_2 \in \mathscr{O}_{cb}$, $\varphi \in \mathscr{O}_{ab}$ and $g \in G$. This conditions are represented in the figures 6.11 and 6.12.

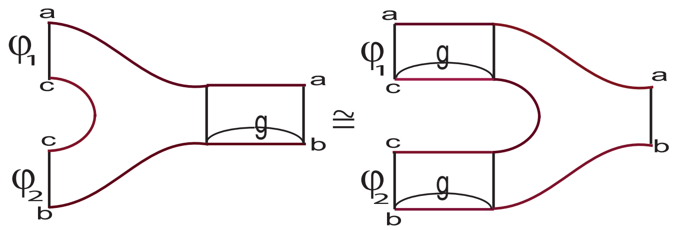


Figure 6.11: `The product is a` G`-morphism with the diagonal action.`

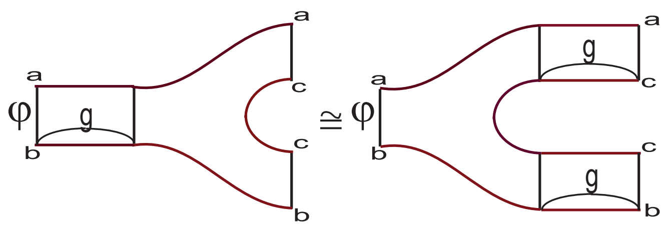


Figure 6.12: `The coproduct is a` G`-morphism with the diagonal action.`

3. For every label a the vector space $\mathscr{O}_{aa}$ is non necessarily a commutative nearly Frobenius algebra.

4. There are also G-twisted open-closed transition maps

$$\iota_{g,a} : \mathscr{C}_g \to \mathscr{O}_{aa},$$
$$\iota^{g,a} : \mathscr{O}_{aa} \to \mathscr{C}_g,$$

which are equivariant.
The map $\iota : \mathscr{C} \to \mathscr{O}$ is obtained by putting the ι_g together, i.e. $\iota = \oplus_{g \in G}\iota_g$ is a ring homomorphisms, then

$$\iota_{g_1}(\Phi_1)\iota_{g_2}(\Phi_2) = \iota_{g_2 g_1}(\Phi_2\Phi_1),$$

with $\Phi_1 \in \mathscr{C}_{g_1}$ and $\Phi_2 \in \mathscr{C}_{g_2}$. Moreover $\iota_e(1_{\mathscr{C}}) = 1_{\mathscr{O}_{aa}}$. The G-twisted centrality condition is

$$\iota_g(\Phi)(\rho_g \Psi) = \Psi \iota_g(\Phi),$$

where $\Phi \in \mathscr{C}_g$ y $\Psi \in \mathscr{O}_{aa}$.

5. The G-twisted Cardy conditions. For each $g \in G$ we must have

$$\pi^a_{g,b} = \iota_{g,b} \iota^{g,a}.$$

Hence $\pi^a_{g,b}$ is defined by

$$\pi^a_{g,b} := \eta^a_{bb} \circ \tau \circ (1 \otimes \rho_g) \circ \Delta^b_{aa} : \mathscr{O}_{aa} \to \mathscr{O}_{bb}$$

where $\tau : \mathscr{O}_{ab} \otimes \mathscr{O}_{ba} \to \mathscr{O}_{ba} \otimes \mathscr{O}_{ab}$ is the transposition map, see Figure 6.13.

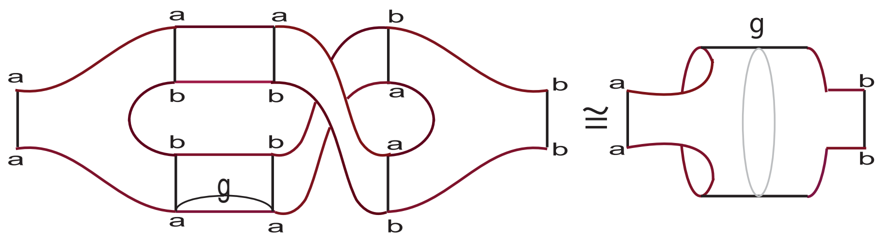


Figure 6.13: `G-twisted Cardy condition`.

Theorem 6.1.1. *The G-invariant part of a G-OC TFT with positive boundary is an OC-TFT with positive boundary.*

6.2 Examples

6.2.1 Open-closed Virtual Cohomology

As in the model for the loop orbifold we saw that the virtual cohomology has the structure of a G-topological field theory with positive boundary. Now we extend this to an open-closed theory, the open part is the following: Let be $\mathcal{B} = \{X \subset M\ G\text{-invariante}\}$ such that, if $X, Y \in \mathcal{B}$ then $TX|_{(X\cap Y)^g} \cong TY|_{(X\cap Y)^g}$ for all $g \in G$. We define $\mathrm{Hom}_{\mathcal{B}}(X,Y) = \mathrm{H}^*(X \cap Y)$, for $X, Y \in \mathcal{B}$.

Now we consider the diagram

$$\begin{tikzcd} & X \cap Y \cap Z \arrow[dl, "(i^Z_{XY} \times i^X_{YZ}) \circ \Delta"'] \arrow[dr, "i^Y_{XZ}"] & \\ (X \cap Y) \times (Y \cap Z) & & X \cap Z \end{tikzcd}$$

where $i^Z_{XY} : X \cap Y \cap Z \hookrightarrow X \cap Y$ is the inclusion map.

We define the product $\eta^Y_{XZ} : \mathrm{H}^*(X \cap Y) \otimes \mathrm{H}^*(Y \cap Z) \to \mathrm{H}^*(X \cap Z)$ by

$$\eta^Y_{XZ}(\alpha \otimes \beta) = i^Y_{XZ}!\left(E_{XYZ}((i^Z_{XY} \times i^X_{YZ}) \circ \Delta)^*(\alpha \otimes \beta)\right)$$

with

$$E_{XYZ} = e\left(\frac{TY|_{X\cap Y\cap Z}}{T(X\cap Y)|_{X\cap Y\cap Z} + T(Y\cap Z)|_{X\cap Y\cap Z}}\right).$$

In a similar way, we define the coproduct $\Delta^Y_{XZ} : \mathrm{H}^*(X \cap Z) \to \mathrm{H}^*(X \cap Y) \otimes \mathrm{H}^*(Y \cap Z)$ by

$$\Delta^Y_{XZ}(\gamma) := ((i^Z_{XY} \times i^X_{YZ}) \circ \Delta)!\left(E(X,Y,Z)i^{Y*}_{XZ}(\gamma)\right)$$

where

$$E(X,Y,Z) = e\left(\frac{TM|_{X\cap Y\cap Z}}{T(X\cap Z)|_{X\cap Y\cap Z}} \oplus T(X\cap Y\cap Z)\right).$$

The next step consists in defining the connection maps. For this we consider the next diagram

$$\begin{tikzcd} & X^g \arrow[dl, "j_g"'] \arrow[dr, "i_g"] & \\ X & & M^g \end{tikzcd}$$

Then we define $\iota_{g,X} : \mathrm{H}^*(M^g) \to \mathrm{H}^*(X)$ as follows

$$\iota_{g,X}(\alpha) := j_g!\left(e(E_g)i^*_g(\alpha)\right)$$

where $E_g = \frac{TM|_{X^g}}{TX|_{X^g}+TM^g|_{X^g}}$. In the same way, the map $\iota^{g,X} : \mathrm{H}^*(X) \to \mathrm{H}^*(M^g)$ is defined by

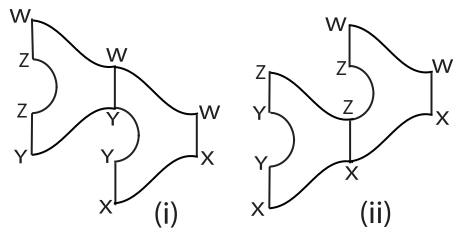
$$\iota^{g,X}(\beta) := i_g!\left(e(F_g)j^*_g(\beta)\right)$$

with $F_g = TX|_{X^g} + TX^g$.

Theorem 6.2.1. *The virtual cohomology together with the category $\mathscr{B}$ as the D-branes is a G-OC-TFT with positive boundary.*

Proof. We already know that the virtual cohomology is a G-TFT with positive boundary, then it remaind to prove the open axioms.

1. **Associativity**



The diagrams are

(i)

$$\begin{tikzcd}[column sep=tiny]
 & & & X\cap Y\cap Z\cap W \arrow[dll, "i^{ZW}_{XY}\times i^{X}_{YZW}"'] \arrow[drr, "i^{Z}_{XYW}"] & & & \\
 & (X\cap Y)\times(Y\cap Z\cap W) \arrow[dl, "1\times(i^{W}_{YZ}\times i^{Y}_{ZW})\circ\Delta"'] \arrow[drr, "1\times i^{Z}_{YW}"'] & & & & X\cap Y\cap W \arrow[dll, "(i^{W}_{XY}\times i^{X}_{YW})\circ\Delta"] \arrow[dr, "i^{Y}_{XW}"] & \\
(X\cap Y)\times(Y\cap Z)\times(Z\cap W) & & & (X\cap Y)\times(Y\cap W) & & & X\cap W
\end{tikzcd}$$

We conclude the following

$$F_1=\frac{T(X\cap Y)\times(Y\cap W)|_{X\cap Y\cap Z\cap W}}{T(X\cap Y)\times(Y\cap Z\cap W)|_{X\cap Y\cap Z\cap W}+T(X\cap Y\cap W)|_{X\cap Y\cap Z\cap W}},$$

$$E_{XYW}=\frac{TY|_{X\cap Y\cap W}}{T(X\cap Y)|_{X\cap Y\cap W}+T(Y\cap W)|_{X\cap Y\cap W}},$$

and

$$E_{YZW}=\frac{TZ|_{Y\cap Z\cap W}}{T(Y\cap Z)|_{Y\cap Z\cap W}+T(Z\cap W)|_{Y\cap Z\cap W}}$$

In K-theory the calculations are

$$\begin{aligned}
&\langle X,Y\rangle+\langle Y,W\rangle+\langle X,Y,Z,W\rangle-\langle X,Y\rangle-\langle Y,Z,W\rangle-\langle X,Y,W\rangle\\
&+\langle Y\rangle+\langle X,Y,W\rangle-\langle X,Y\rangle-\langle Y,W\rangle\\
&+\langle Z\rangle+\langle Y,Z,W\rangle-\langle Y,Z\rangle-\langle Z,W\rangle\\
=&\langle X,Y,Z,W\rangle+\langle Y\rangle+\langle Z\rangle-\langle X,Y\rangle-\langle Y,Z\rangle-\langle Z,W\rangle
\end{aligned}$$

(ii)

$$\begin{tikzcd}[column sep=tiny]
 & & & X\cap Y\cap Z\cap W \arrow[dll, "i^{W}_{XYZ}\times i^{XY}_{ZW}"'] \arrow[drr, "i^{Y}_{XZW}"] & & & \\
 & (X\cap Y\cap Z)\times(Z\cap W) \arrow[dl, "(i^{Z}_{XY}\times i^{X}_{YZ})\circ\Delta\times 1"'] \arrow[drr, "i^{Y}_{XZ}\times 1"'] & & & & X\cap Z\cap W \arrow[dll, "(i^{W}_{XZ}\times i^{X}_{ZW})\circ\Delta"] \arrow[dr, "i^{Z}_{XW}"] & \\
(X\cap Y)\times(Y\cap Z)\times(Z\cap W) & & & (X\cap Z)\times(Z\cap W) & & & X\cap W
\end{tikzcd}$$

Hence the identity

$$F_2=\frac{T(X\cap Z)\times(Z\cap W)|_{X\cap Y\cap Z\cap W}}{T(X\cap Y\cap Z)\times(Z\cap W)|_{X\cap Y\cap Z\cap W}+T(X\cap Z\cap W)|_{X\cap Y\cap Z\cap W}},$$

$$E_{XYZ}=\frac{TY|_{X\cap Y\cap Z}}{T(X\cap Y)|_{X\cap Y\cap Z}+T(Y\cap Z)|_{X\cap Y\cap Z}},$$

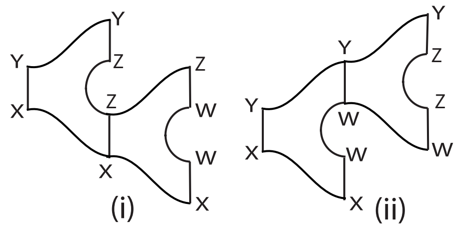
and

$$E_{XZW} = \frac{TZ|_{X\cap Z\cap W}}{T(X\cap Z)|_{X\cap Z\cap W} + T(Z\cap W)|_{X\cap Z\cap W}}$$

Then

$$\begin{aligned}
&\langle X,Z\rangle + \langle Z,W\rangle + \langle X,Y,Z,W\rangle - \langle X,Y,Z\rangle - \langle Z,W\rangle - \langle X,Z,W\rangle \\
&+ \langle Y\rangle + \langle X,Y,Z\rangle - \langle X,Y\rangle - \langle Y,Z\rangle \\
&+ \langle Z\rangle + \langle X,Z,W\rangle - \langle X,Z\rangle - \langle Z,W\rangle \\
=&\langle X,Y,Z,W\rangle + \langle Y\rangle + \langle Z\rangle - \langle X,Y\rangle - \langle Y,Z\rangle - \langle Z,W\rangle
\end{aligned}$$

2. **Coassociativity**



(i)

$$\begin{tikzcd}
 & & X\cap Y\cap Z\cap W \arrow[dl, "i^W_{XZY}"'] \arrow[drr, "(i^Y_{XZW}\times i^{XW}_{YZ})\circ\Delta"] & & & \\
 & X\cap Y\cap Z \arrow[dl, "i^Z_{XY}"'] \arrow[dr, "(i^Y_{XZ}\times i^X_{ZY})\circ\Delta"'] & & & (X\cap Z\cap W)\times(Z\cap Y) \arrow[dll, "i^W_{XZ}\times 1"] \arrow[dr, "(i^Z_{XW}\times i^X_{ZW})\circ\Delta\times 1"] & \\
(X\cap Y) & & (X\cap Z)\times(Z\cap Y) & & & (X\cap W)\times(W\cap Z)\times Z\cap Y
\end{tikzcd}$$

In this case

$$F_1 = \frac{T(X\cap Z)\times(Z\cap Y)|_{X\cap Y\cap Z\cap W}}{T(X\cap Y\cap Z)|_{X\cap Y\cap Z\cap W} + T(X\cap Z\cap W)\times(Z\cap Y)|_{X\cap Y\cap Z\cap W}},$$

$$E(X,Z,Y) = \frac{TM|_{X\cap Y\cap Z}}{T(X\cap Y)|_{X\cap Y\cap Z}} \oplus T(X\cap Y\cap Z),$$

and

$$E(X,W,Z) = \frac{TM|_{X\cap Z\cap W}}{T(X\cap Z)|_{X\cap Z\cap W}} \oplus T(X\cap Z\cap W)$$

Then in K-theory

$$\begin{aligned}
&\langle X,Z\rangle + \langle Z,Y\rangle + \langle X,Y,Z,W\rangle - \langle X,Y,Z\rangle - \langle Z,Y\rangle - \langle X,Z,W\rangle \\
&+ \langle M\rangle - \langle X,Y\rangle + \langle X,Y,Z\rangle \\
&+ \langle M\rangle - \langle X,Z\rangle + \langle X,Z,W\rangle \\
=&\langle M\rangle + \langle M\rangle - \langle X,Y\rangle
\end{aligned}$$

(ii)

$$\begin{tikzcd}[column sep=small]
 & & X\cap Y\cap Z\cap W \arrow[dl, "i^{Z}_{XYW}"'] \arrow[drr, "(i^{YZ}_{XW}\times i^{X}_{YZW})\circ\Delta"] & & & \\
 & X\cap Y\cap W \arrow[dl, "i^{W}_{XY}"'] \arrow[dr, "(i^{Y}_{XW}\times i^{X}_{WY})\circ\Delta"'] & & & (X\cap W)\times(Y\cap Z\cap W) \arrow[dll, "1\times i^{Z}_{WY}"] \arrow[dr, "1\times(i^{Y}_{WZ}\times i^{W}_{ZY})\circ\Delta"] & \\
(X\cap Y) & & (X\cap W)\times(W\cap Y) & & & (X\cap W)\times(W\cap Z)\times Z\cap Y
\end{tikzcd}$$

We conclude

$$F_2=\frac{T(X\cap W)\times(W\cap Y)|_{X\cap Y\cap Z\cap W}}{T(X\cap Y\cap W)|_{X\cap Y\cap Z\cap W}+T(X\cap W)\times(Y\cap Z\cap W)|_{X\cap Y\cap Z\cap W}},$$

$$E(X,W,Y)=\frac{TM|_{X\cap Y\cap W}}{T(X\cap Y)|_{X\cap Y\cap W}}\oplus T(X\cap Y\cap W),$$

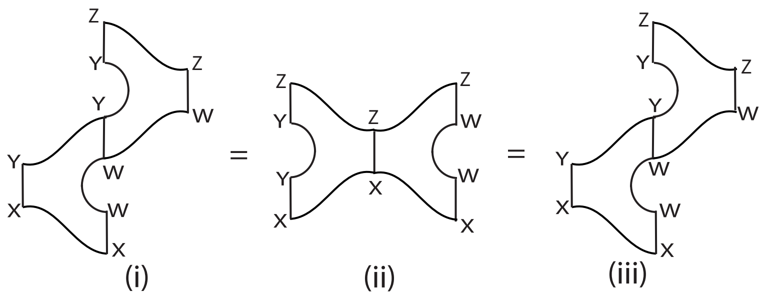
and

$$E(W,Z,Y)=\frac{TM|_{Y\cap Z\cap W}}{T(W\cap Y)|_{Y\cap Z\cap W}}\oplus T(Y\cap Z\cap W)$$

In K-theory

$$\begin{aligned}
&\langle X,W\rangle+\langle W,Y\rangle+\langle X,Y,Z,W\rangle-\langle X,Y,W\rangle-\langle X,W\rangle-\langle Y,Z,W\rangle\\
&+\langle M\rangle-\langle X,Y\rangle+\langle X,Y,W\rangle\\
&+\langle M\rangle-\langle Y,W\rangle+\langle Y,Z,W\rangle\\
=&\langle M\rangle+\langle M\rangle-\langle X,Y\rangle
\end{aligned}$$

3. **Abrams condition**



(i)

$$\begin{tikzcd}[column sep=small]
 & & X\cap Y\cap Z\cap W \arrow[dl, "(i^{Z}_{XYW}\times i^{XW}_{YZ})\circ\Delta"'] \arrow[drr, "(i^{YZ}_{XW}\times i^{X}_{YZW})\circ\Delta"] & & & \\
 & (X\cap Y\cap W)\times(Y\cap Z) \arrow[dl, "i^{W}_{XY}\times 1"'] \arrow[dr, "(i^{Y}_{XW}\times i^{X}_{WY})\circ\Delta\times 1"'] & & & (X\cap W)\times(Y\cap Z\cap W) \arrow[dll, "1\times(i^{Z}_{WY}\times i^{W}_{YZ})\circ\Delta"] \arrow[dr, "1\times i^{Y}_{WZ}"] & \\
(X\cap Y)\times(Y\cap Z) & & (X\cap W)\times(W\cap Y)\times(Y\cap Z) & & & (X\cap W)\times(W\cap Z)
\end{tikzcd}$$

Hence

$$F_1 = \frac{T(X\cap W)\times(W\cap Y)\times(Y\cap Z)|_{X\cap Y\cap Z\cap W}}{T(X\cap Y\cap W)\times(Y\cap Z)|_{X\cap Y\cap Z\cap W} + T(X\cap W)\times(Y\cap Z\cap W)|_{X\cap Y\cap Z\cap W}},$$

$$E(X,W,Y) = \frac{TM|_{X\cap Y\cap W}}{T(X\cap Y)|_{X\cap Y\cap W}} \oplus T(X\cap Y\cap W),$$

and

$$E_{W,Y,Z} = \frac{TY|_{Y\cap Z\cap W}}{T(W\cap Y)|_{Y\cap Z\cap W} + T(Y\cap Z)|_{Y\cap Z\cap W}}$$

Then in K-theory

$$\begin{aligned}
&\langle X,W\rangle + \langle W,Y\rangle + \langle Y,Z\rangle + \langle X,Y,Z,W\rangle - \langle X,Y,W\rangle - \langle Y,Z\rangle - \langle X,W\rangle - \langle Y,Z,W\rangle \\
&+ \langle M\rangle - \langle X,Y\rangle + \langle X,Y,W\rangle \\
&+ \langle Y\rangle - \langle Y,W\rangle + \langle Y,Z,W\rangle - \langle Y,Z\rangle \\
=&\langle M\rangle + \langle Y\rangle + \langle X,Y,Z,W\rangle - \langle Y,Z\rangle - \langle X,Y\rangle
\end{aligned}$$

(ii)

$$\begin{tikzcd}
& & & X\cap Y\cap Z\cap W \arrow[dll, "i^W_{XYZ}"'] \arrow[drr, "i^Y_{XZW}"] & & & \\
& X\cap Y\cap Z \arrow[dl, "(i^Z_{XY}\times i^X_{YZ})\circ\Delta"'] \arrow[drr, "i^Y_{XZ}"'] & & & & X\cap Z\cap W \arrow[dll, "i^W_{XZ}"] \arrow[dr, "(i^Z_{XW}\times i^X_{ZW})\circ\Delta"] & \\
(X\cap Y)\times(Y\cap Z) & & & (X\cap Z) & & & (X\cap W)\times(W\cap Z)
\end{tikzcd}$$

As a consequence

$$F_2 = \frac{T(X\cap Z)|_{X\cap Y\cap Z\cap W}}{T(X\cap Y\cap Z)|_{X\cap Y\cap Z\cap W} + T(X\cap Z\cap W)|_{X\cap Y\cap Z\cap W}},$$

$$E(X,W,Z) = \frac{TM|_{X\cap Z\cap W}}{T(X\cap Z)|_{X\cap Y\cap W}} \oplus T(X\cap Z\cap W),$$

and

$$E_{X,Y,Z} = \frac{TY|_{Y\cap Z\cap X}}{T(X\cap Y)|_{Y\cap Z\cap X} + T(Y\cap Z)|_{Y\cap Z\cap X}}$$

In K-theory

$$\begin{aligned}
&\langle X,Z\rangle + \langle X,Y,Z,W\rangle - \langle X,Y,Z\rangle - \langle X,Z,W\rangle \\
&+ \langle M\rangle - \langle X,Z\rangle + \langle X,Z,W\rangle \\
&+ \langle Y\rangle - \langle X,Y\rangle + \langle X,Y,Z\rangle - \langle Y,Z\rangle \\
=&\langle M\rangle + \langle Y\rangle + \langle X,Y,Z,W\rangle - \langle Y,Z\rangle - \langle X,Y\rangle
\end{aligned}$$

(iii)

$$
\begin{tikzcd}[column sep=tiny]
 & & X\cap Y\cap Z\cap W \arrow[dl, "(i^{ZW}_{XY}\times i^{X}_{YZW})\circ\Delta"'] \arrow[dr, "(i^{Z}_{XYW}\times i^{XY}_{ZW})\circ\Delta"] & & \\
 & (X\cap Y)\times(Y\cap Z\cap W) \arrow[dl, "1\times i^{W}_{YZ}"'] \arrow[dr, "1\times(i^{Z}_{YW}\times i^{Y}_{WZ})\circ\Delta"] & & (X\cap Y\cap W)\times(Z\cap W) \arrow[dl, "(i^{W}_{XY}\times i^{X}_{YW})\circ\Delta\times 1"'] \arrow[dr, "i^{Y}_{XW}\times 1"] & \\
(X\cap Y)\times(Y\cap Z) & & (X\cap Y)\times(Y\cap W)\times(W\cap Z) & & (X\cap W)\times(W\cap Z)
\end{tikzcd}
$$

Hence

$$F_3 = \frac{T(X\cap Y)\times(W\cap Y)\times(W\cap Z)|_{X\cap Y\cap Z\cap W}}{T(X\cap Y)\times(Y\cap Z\cap W)|_{X\cap Y\cap Z\cap W} + T(X\cap Y\cap W)\times(Z\cap W)|_{X\cap Y\cap Z\cap W}},$$

$$E(Y,W,Z) = \frac{TM|_{Z\cap Y\cap W}}{T(Z\cap Y)|_{Z\cap Y\cap W}} \oplus T(Z\cap Y\cap W),$$

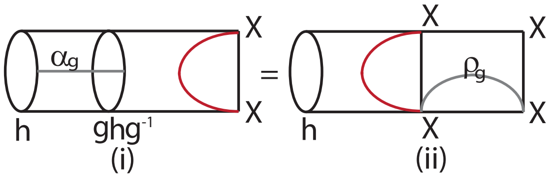
and

$$E_{X,Y,W} = \frac{TY|_{X\cap Y\cap W}}{T(X\cap Y)|_{X\cap Y\cap W} + T(Y\cap W)|_{X\cap Y\cap W}}$$

In K-theory

$$
\begin{aligned}
&\langle X,Y\rangle + \langle W,Y\rangle + \langle W,Z\rangle + \langle X,Y,Z,W\rangle - \langle Z,Y,W\rangle - \langle Y,X\rangle - \langle Z,W\rangle - \langle Y,X,W\rangle\\
&+ \langle M\rangle - \langle Z,Y\rangle + \langle Z,Y,W\rangle\\
&+ \langle Y\rangle - \langle Y,X\rangle + \langle Y,X,W\rangle - \langle Y,W\rangle\\
=&\langle M\rangle + \langle Y\rangle + \langle X,Y,Z,W\rangle - \langle Y,Z\rangle - \langle X,Y\rangle
\end{aligned}
$$

4. **The map ι_g is an equivariant map**



(i)

$$
\begin{tikzcd}
 & & X^{ghg^{-1}} \arrow[dl, "i_{ghg^{-1}}"'] \arrow[dr, "\mathrm{id}"] & & \\
 & M^{ghg^{-1}} \arrow[dl, "\alpha_g"'] \arrow[dr, "\mathrm{id}"'] & & X^{ghg^{-1}} \arrow[dl, "i_{ghg^{-1}}"] \arrow[dr, "j_{ghg^{-1}}"] & \\
M^{h} & & M^{ghg^{-1}} & & X
\end{tikzcd}
$$

$$F_1 = \frac{TM^{ghg^{-1}}|_{X^{ghg^{-1}}}}{TM^{ghg^{-1}}|_{X^{ghg^{-1}}} + TX^{ghg^{-1}}|_{X^{ghg^{-1}}}} = 0,$$

and

$$E_{ghg^{-1}} = \frac{TM|_{X^{ghg^{-1}}}}{TX|_{X^{ghg^{-1}}} + TM^{ghg^{-1}}|_{X^{ghg^{-1}}}}$$

Then in K-theory

$$\langle 1\rangle_M + \langle ghg^{-1}\rangle_X - \langle ghg^{-1}\rangle_M - \langle 1\rangle_X$$

(ii)

$$\begin{tikzcd}
 & & X^{ghg^{-1}} \arrow[dl, "\rho_g"'] \arrow[dr, "j_{ghg^{-1}}"] & & \\
 & X^h \arrow[dl, "i_h"'] \arrow[dr, "j_h"] & & X \arrow[dl, "\rho_g"'] \arrow[dr, "\mathrm{id}"] & \\
M^h & & X & & X
\end{tikzcd}$$

$$F_2 = \frac{TX|_{X^{ghg^{-1}}}}{TX|_{X^{ghg^{-1}}} + TX^h|_{X^{ghg^{-1}}}} = 0,$$

and

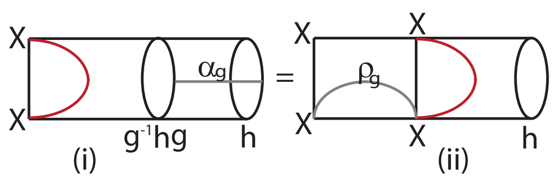
$$E_h = \frac{TM|_{X^h}}{TX|_{X^h} + TM^h|_{X^h}}$$

Then in K-theory

$$\langle 1\rangle_M + \langle h\rangle_X - \langle h\rangle_M - \langle 1\rangle_X$$

We use that $X^{ghg^{-1}} \cong X^h$, then $\langle h\rangle_X = \langle ghg^{-1}\rangle_X$ and in the same way, we have $\langle h\rangle_M = \langle ghg^{-1}\rangle_M$.

5. **The map ι^g is an equivariant map**



(i)

$$\begin{tikzcd}
 & & X^h \arrow[dl, "\rho_g"'] \arrow[dr, "i_h"] & & \\
 & X^{g^{-1}hg} \arrow[dl, "j_{g^{-1}hg}"'] \arrow[dr, "i_{g^{-1}hg}"] & & M^h \arrow[dl, "\alpha_g"'] \arrow[dr, "\mathrm{id}"] & \\
X & & M^{g^{-1}hg} & & M^h
\end{tikzcd}$$

Hence

$$F_1 = \frac{TM^{g^{-1}hg}|_{X^h}}{TM^h|_{X^h} + TX^{g^{-1}hg}|_{X^h}} = 0,$$

this is because $TM^{g^{-1}hg} \cong TM^h$, and we conclude

$$F_{g^{-1}hg} = TX|_{X^{g^{-1}hg}} + TX^{g^{-1}hg}$$

Then in K-theory

$$\langle 1\rangle_X + \langle g^{-1}hg\rangle_X \tag{6.1}$$

(ii)

$$\begin{tikzcd}
 & & X^h \arrow[dl, "j_h"'] \arrow[dr, "\mathrm{id}"] & & \\
 & X \arrow[dl, "\rho_g"'] \arrow[dr, "\mathrm{id}"] & & X^h \arrow[dl, "j_h"'] \arrow[dr, "i_h"] & \\
X & & X & & M^h
\end{tikzcd}$$

Similarly

$$F_2 = \frac{TX|_{X^h}}{TX|_{X^h} + TX^h|_{X^h}} = 0,$$

and

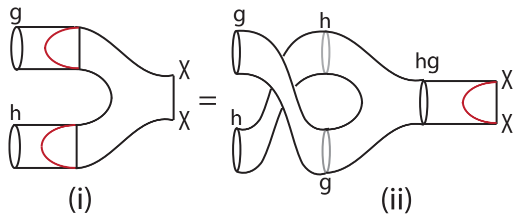
$$F_h = TX|_{X^h} + TX^h.$$

Then, in K-theory

$$\langle 1\rangle_X + \langle h\rangle_X \tag{6.2}$$

9.1 and 6.2 are the same because $X^h \cong X^{g^{-1}hg}$.

6. **The map ι_g is a ring homomorphism**



(i)

$$\begin{tikzcd}
 & & X^{g,h} \arrow[dl, "j_{g,h}"'] \arrow[dr, "i_{g,h}"] & & \\
 & X^g \times X^h \arrow[dl, "j_g \times j_h"'] \arrow[dr, "i_g \times i_h"] & & X \arrow[dl, "\Delta"'] \arrow[dr, "\mathrm{id}"] & \\
M^g \times M^h & & X \times X & & X
\end{tikzcd}$$

Hence

$$F_1 = \frac{T(X \times X)|_{X^{g,h}}}{T(X^g \times X^h)|_{X^{g,h}} + TX|_{X^{g,h}}},$$
$$E_g = \frac{TM|_{X^g}}{TX|_{X^g} + TM^g|_{X^g}},$$
$$E_h = \frac{TM|_{X^h}}{TX|_{X^h} + TM^h|_{X^h}},$$

and

$$E_{XXX} = \frac{TX|_X}{TX|_X + TX|_X} = 0$$

Then in K-theory

$$\begin{aligned}
&\langle 1\rangle_X + \langle 1\rangle_X + \langle g,h\rangle_X - \langle g\rangle_X - \langle h\rangle_X - \langle 1\rangle_X \\
&+ \langle 1\rangle_M + \langle g\rangle_X - \langle 1\rangle_X - \langle g\rangle_M \\
&+ \langle 1\rangle_M + \langle h\rangle_X - \langle 1\rangle_X - \langle h\rangle_M \\
=&\langle 1\rangle_M + \langle 1\rangle_M - \langle 1\rangle_X + \langle g,h\rangle_X - \langle g\rangle_M - \langle h\rangle_M
\end{aligned}$$

(ii)

$$\begin{tikzcd}
& & X^{g,h} \arrow[dl, "j_{h,g}"'] \arrow[dr, "k_{h,g}"] & & \\
& M^{h,g} \arrow[dl, "\tau\circ\delta_{h,g}"'] \arrow[dr, "i_{h,g}"'] & & X^{hg} \arrow[dl, "j_{hg}"] \arrow[dr, "i_{hg}"] & \\
M^g \times M^h & & M^{hg} & & X
\end{tikzcd}$$

We conclude

$$F_2 = \frac{TM^{hg}|_{X^{g,h}}}{TM^{g,h}|_{X^{g,h}} + TX^{hg}|_{X^{g,h}}},$$
$$E_{hg} = \frac{TM|_{X^{hg}}}{TX|_{X^{hg}} + TM^{hg}|_{X^{hg}}},$$

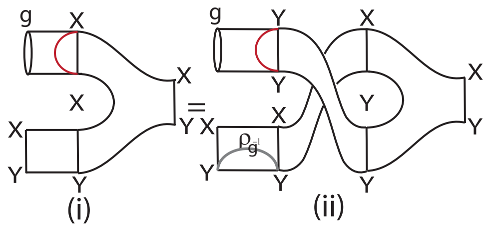
and

$$\nu(g,h) = \frac{TM|_{M^{g,h}}}{TM^g|_{M^{g,h}} + TM^h|_{M^{g,h}}},$$

Then in K-theory

$$\begin{aligned}
&\langle hg\rangle_M + \langle g,h\rangle_X - \langle g,h\rangle_M - \langle hg\rangle_X \\
&+ \langle 1\rangle_M + \langle hg\rangle_X - \langle 1\rangle_X - \langle hg\rangle_M \\
&+ \langle 1\rangle_M + \langle g,h\rangle_M - \langle g\rangle_M - \langle h\rangle_M \\
=&\langle 1\rangle_M + \langle 1\rangle_M - \langle 1\rangle_X + \langle g,h\rangle_X - \langle g\rangle_M - \langle h\rangle_M
\end{aligned}$$

7. **G-twisted centrality condition**



(i)

$$
\begin{tikzcd}[column sep=small]
 & & & (X\cap Y)^g \arrow[dll, "\Delta"'] \arrow[drr, "i_g"] & & & \\
 & X^g\times(X\cap Y) \arrow[dl, "j_g\times 1"'] \arrow[drr, "i_g\times 1"] & & & & X\cap Y \arrow[dll, "\Delta"] \arrow[dr, "\mathrm{id}"] & \\
M^g\times(X\cap Y) & & & X\times(X\cap Y) & & & X\cap Y
\end{tikzcd}
$$

Hence

$$F_1=\frac{T(X\times(X\cap Y))|_{(X\cap Y)^g}}{T(X^g\times(X\cap Y))|_{(X\cap Y)^g}+T(X\cap Y)|_{(X\cap Y)^g}},$$

$$E_g=\frac{TM|_{X^g}}{TX|_{X^g}+TM^g|_{X^g}},$$

and

$$E_{YXX}=\frac{TX|_{X\cap Y}}{TX|_{X\cap Y}+T(X\cap Y)|_{X\cap Y}}=0$$

Then in K-theory

$$
\begin{aligned}
&\langle 1\rangle_X+\langle X,Y\rangle+\langle g\rangle_{X,Y}-\langle g\rangle_X-\langle X,Y\rangle-\langle X,Y\rangle\\
&+\langle 1\rangle_M+\langle g\rangle_X-\langle 1\rangle_X-\langle g\rangle_M\\
=&\langle 1\rangle_M+\langle g\rangle_{X,Y}-\langle X,Y\rangle-\langle g\rangle_M
\end{aligned}
$$

(ii)

$$
\begin{tikzcd}[column sep=small]
 & & & (X\cap Y)^g \arrow[dll, "\Delta"'] \arrow[drr, "i_g"] & & & \\
 & Y^g\times(X\cap Y) \arrow[dl, "j_g\times\rho_{g^{-1}}"'] \arrow[drr, "i_g\times 1"] & & & & X\cap Y \arrow[dll, "\tau\circ\Delta"] \arrow[dr, "\mathrm{id}"] & \\
M^g\times(X\cap Y) & & & Y\times(X\cap Y) & & & X\cap Y
\end{tikzcd}
$$

As a consequence

$$F_2 = \frac{T(Y \times (X \cap Y))|_{(X\cap Y)^g}}{T(Y^g \times (X\cap Y))|_{(X\cap Y)^g} + T(X\cap Y)|_{(X\cap Y)^g}},$$

$$E_g = \frac{TM|_{Y^g}}{TY|_{Y^g} + TM^g|_{Y^g}},$$

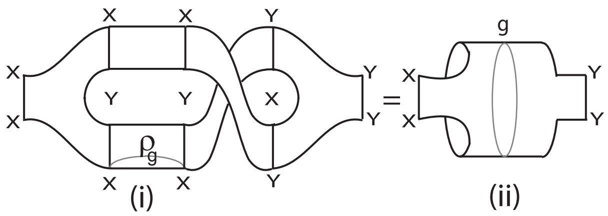
and

$$E_{YYX} = \frac{TY|_{X\cap Y}}{TY|_{X\cap Y} + T(X\cap Y)|_{X\cap Y}} = 0$$

Then in K-theory

$$\begin{aligned} &\langle 1\rangle_Y + \langle X,Y\rangle + \langle g\rangle_{X,Y} - \langle g\rangle_Y - \langle X,Y\rangle - \langle X,Y\rangle \\ &+ \langle 1\rangle_M + \langle g\rangle_Y - \langle 1\rangle_Y - \langle g\rangle_M \\ =&\langle 1\rangle_M + \langle g\rangle_{X,Y} - \langle X,Y\rangle - \langle g\rangle_M \end{aligned}$$

8. **Cardy condition**



(i)

$$\begin{tikzcd} & & (X\cap Y)^g \arrow[dl, "i_g"'] \arrow[dr, "i_g"] & & \\ & X\cap Y \arrow[dl, "i_X^Y"'] \arrow[dr, "\Delta"] & & X\cap Y \arrow[dl, "(1\times\rho_g)\circ\tau\circ\Delta"'] \arrow[dr, "i_X^Y"] & \\ X & & (X\cap Y)\times(Y\cap X) & & Y \end{tikzcd}$$

Hence

$$F_1 = \frac{T((X\cap Y)\times(X\cap Y))|_{(X\cap Y)^g}}{T(X\cap Y)|_{(X\cap Y)^g} + T(X\cap Y)|_{(X\cap Y)^g}},$$

$$E_{YXY} = \frac{TX|_{X\cap Y}}{T(Y\cap X)|_{X\cap Y} + T(X\cap Y)|_{X\cap Y}},$$

and

$$E(X,Y,X) = \frac{TM|_{X\cap Y}}{TX|_{X\cap Y}} \oplus T(X\cap Y)$$

In K-theory

$$\begin{aligned}&\langle X,Y\rangle+\langle X,Y\rangle+\langle g\rangle_{X,Y}-\langle X,Y\rangle-\langle X,Y\rangle\\&+\langle 1\rangle_X+\langle X,Y\rangle-\langle X,Y\rangle-\langle X,Y\rangle\\&+\langle 1\rangle_M-\langle 1\rangle_X+\langle X,Y\rangle\\=&\langle g\rangle_{X,Y}+\langle 1\rangle_M\end{aligned}$$

(ii)

$$\begin{tikzcd} & & (X\cap Y)^g \arrow[dl, "j"'] \arrow[dr, "i"] & & \\ & X^g \arrow[dl, "i_g"'] \arrow[dr, "j_g"] & & Y^g \arrow[dl, "j_g"'] \arrow[dr, "i_g"] & \\ X & & M^g & & Y \end{tikzcd}$$

Then

$$F_2=\frac{TM^g|_{(X\cap Y)^g}}{TX^g|_{(X\cap Y)^g}+TY^g|_{(X\cap Y)^g}},$$

$$E_g=\frac{TM|_{Y^g}}{TY|_{Y^g}+TM^g|_{Y^g}},$$

and

$$F_g=TX|_{X^g}+TX^g$$

In K-theory

$$\begin{aligned}&\langle g\rangle_M+\langle g\rangle_{X,Y}-\langle g\rangle_X-\langle g\rangle_Y\\&+\langle 1\rangle_M+\langle g\rangle_Y-\langle 1\rangle_Y-\langle g\rangle_M\\&+\langle 1\rangle_X+\langle g\rangle_X\\=&\langle g\rangle_{X,Y}+\langle 1\rangle_M\end{aligned}$$

♣

6.2.2 Open-closed Loop Orbifold

In the previous chapter we saw that the homology of the Loop Orbifold has the structure of a G-topological field theory with positive boundary. Now we describe the open part of this theory.
The category of branes is the following:

$$\mathscr{B}=\{X\subset M\ G\text{-ivariant submanifold with } X\pitchfork Y \text{ transverse for } X\neq Y\}$$

Now we consider the sets $\mathcal{P}_{X,Y}M = \{\alpha : I \to M : \ \alpha(0) \in X, \alpha(1) \in Y\}$, for $X, Y \in \mathscr{B}$. We define $\mathrm{Hom}_{\mathscr{B}}(X,Y) = \mathrm{H}_*(\mathcal{P}_{X,Y}M)$. Note that G acts in $\mathrm{H}_*(\mathcal{P}_{X,Y}M)$ as follows

$$\begin{aligned} \rho : G \to & \mathrm{Aut}(\mathrm{H}_*(\mathcal{P}_{X,Y}M)) \\ g \mapsto & \ \ \rho_g : \mathrm{H}_*(\mathcal{P}_{X,Y}M) \to \mathrm{H}_*(\mathcal{P}_{X,Y}M) \\ & \qquad\qquad\qquad \alpha \mapsto \alpha.g \end{aligned}$$

where $\alpha.g(t) = \alpha(t)g$ for $t \in I$.
The product and coproduct is the same that the product and coproduct defined in the open-closed string topology.
Now we describe the connection maps. For this we consider the next diagram

$$\begin{tikzcd} & \mathcal{P}^X_g M \arrow[dl, "j"'] \arrow[dr, "i"] & \\ \mathcal{P}_g M & & \mathcal{P}_{X,X}M \end{tikzcd}$$

where $\mathcal{P}^X_g M = \{\alpha : I \to M : \ \alpha(1) = \alpha(0)g, \ \alpha(0) \in X\}$.
First, we will prove that the map $j_! : \mathrm{H}_*(\mathcal{P}_g M) \to \mathrm{H}_*(\mathcal{P}^X_g M)$ exists. This is because the next diagram is a pullback square.

$$\begin{tikzcd} \mathcal{P}^X_g M \arrow[r, "j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g M \arrow[d, "\epsilon_0 \times \epsilon_1"] \\ X \arrow[r, hook, "(\mathrm{id},g)"'] & M \times M \end{tikzcd}$$

Clearly the map $(\mathrm{id}, g) : X \to M \times M$ is an embedding. Then, we can define the map $\iota_{g,X}$ as the conposition

$$\mathrm{H}_*(\mathcal{P}_g M) \xrightarrow{j_!} \mathrm{H}_*(\mathcal{P}^X_g M) \xrightarrow{i_*} \mathrm{H}_*(\mathcal{P}_{X,X}M).$$

For the other map we consider the same diagram

$$\begin{tikzcd} & \mathcal{P}^X_g M \arrow[dl, "j"'] \arrow[dr, "i"] & \\ \mathcal{P}_g M & & \mathcal{P}_{X,X}M \end{tikzcd}$$

and we use the next pullback square

$$\begin{tikzcd} \mathcal{P}^X_g M \arrow[r, "i"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{X,X} M \arrow[d, "\epsilon_0 \times \epsilon_1"] \\ X \arrow[r, "(\mathrm{id},g)"'] & X \times X \end{tikzcd}$$

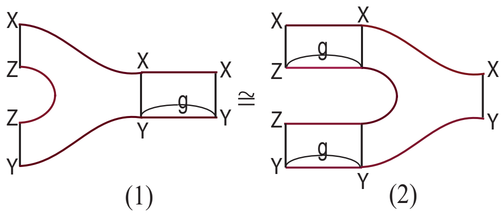
to define the map $\iota^{g,X}$ as the composition

$$\iota^{g,X} : \mathrm{H}_*(\mathcal{P}_{X,X}M) \xrightarrow{i_!} \mathrm{H}_*(\mathcal{P}_g^X M) \xrightarrow{j_*} \mathrm{H}_*(\mathcal{P}_g M).$$

Theorem 6.2.2. *The following* $(\mathrm{H}_*(\mathcal{P}_G(M)), \mathscr{B})$ *is a* G-OC-TFT *with positive boundary.*

Proof. We will check the open axioms.

1. **The action respects the product**



(1) (2)

The property is the following
(1)

$$\begin{tikzcd}
 & & \mathcal{P}_{XZY}M \arrow[dl, "1"'] \arrow[dr, "i_{XY}"] & & \\
 & \mathcal{P}_{XZY}M \arrow[dl, "j_{XZ}\times j_{ZY}"'] \arrow[dr, "i_{XY}"] & & \mathcal{P}_{XY}M \arrow[dl, "1"'] \arrow[dr, "\rho_g"] & \\
\mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M & & \mathcal{P}_{XY}M & & \mathcal{P}_{XY}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_{XZY}M \arrow[dl, "j_{XZ}\times j_{ZY}"'] \arrow[dr, "\rho_g"] & & \\
 & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M \arrow[dl, "1"'] \arrow[dr, "\rho_g\times\rho_g"] & & \mathcal{P}_{XZY}M \arrow[dl, "j_{XZ}\times j_{ZY}"'] \arrow[dr, "i_{XZ}"] & \\
\mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M & & \mathcal{P}_{XY}M
\end{tikzcd}$$

In the first diagram is clear that $F_1 = 0$, this because the normal bundles are zero. In the second diagram we have

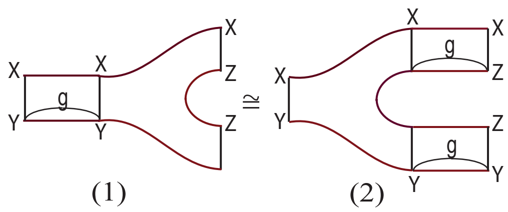
$$\begin{tikzcd}
(\rho_g \circ \epsilon_{\frac{1}{2}})^*(\eta) \arrow[r, dashed] & \mathcal{P}_{XZY}M \arrow[d, "\rho_g"'] & \\
& \mathcal{P}_{XZY}M \arrow[r, "j_{XZ}\times j_{ZY}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
\eta \arrow[r, dashed] & Z \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon^*_{\frac{1}{2}}(\eta) \arrow[r, dashed] & \mathcal{P}_{XZY}M \arrow[r, "j_{XZ}\times j_{ZY}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XZ} \times \mathcal{P}_{ZY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
\eta \arrow[r, dashed] & Z \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

We note, as before, that $(\rho_g \circ \epsilon_{\frac{1}{2}})^*(\eta) = \epsilon^*_{\frac{1}{2}}(\eta)$. Then $F_2 = 0$.

2. **The action respects the coproduct**



(1) (2)

(1)

$$\begin{tikzcd}
& & \mathcal{P}_{XZY}M \arrow[dl, "i_{XY}"'] \arrow[dr, "\rho_g"] & & \\
& \mathcal{P}_{XY}M \arrow[dl, "1"'] \arrow[dr, "\rho_g"] & & \mathcal{P}_{XZY}M \arrow[dl, "i_{XY}"'] \arrow[dr, "j_{XZ}\times j_{ZY}"] & \\
\mathcal{P}_{XY}M & & \mathcal{P}_{XY}M & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& & \mathcal{P}_{XZY}M \arrow[dl, "1"'] \arrow[dr, "j_{XZ}\times j_{ZY}"] & & \\
& \mathcal{P}_{XZY}M \arrow[dl, "i_{XY}"'] \arrow[dr, "j_{XZ}\times j_{ZY}"] & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M \arrow[dl, "1"'] \arrow[dr, "\rho_g\times\rho_g"] & \\
\mathcal{P}_{XY}M & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M & & \mathcal{P}_{XZ}M \times \mathcal{P}_{ZY}M
\end{tikzcd}$$

In the second diagram it is clear that $F_2 = 0$. In the first diagram the calculus as the following

$$\begin{tikzcd}
(\epsilon_{\frac{1}{2}} \circ \rho_g)(\vartheta) \arrow[r, dashed] & \mathcal{P}_{XZY}M \arrow[d, "\rho_g"'] & \\
 & \mathcal{P}_{XZY}M \arrow[r, "i_{XY}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XY}M \arrow[d, "\epsilon_{\frac{1}{2}} \times \epsilon_{\frac{1}{2}}"] \\
\vartheta \arrow[r, dashed] & Z \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

and

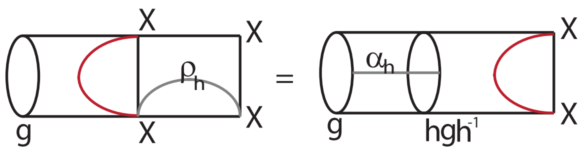
$$\begin{tikzcd}
\epsilon^*_{\frac{1}{2}}(\vartheta) \arrow[r, dashed] & \mathcal{P}_{XZY}M \arrow[r, "i_{XY}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XY}M \arrow[d, "\epsilon_{\frac{1}{2}} \times \epsilon_{\frac{1}{2}}"] \\
\vartheta \arrow[r, dashed] & Z \arrow[r, "\Delta"'] & M \times M
\end{tikzcd}$$

Since ρ_g is a isomorphism, then the next bundles are isomorphic,

$$\epsilon^*_{\frac{1}{2}}(\vartheta) \simeq (\epsilon_{\frac{1}{2}} \circ \rho_g)(\vartheta)$$

hence $F_1 = 0$.

3. **The map ι_g is an equivariant map**



(1) (2)

Remember that the connection maps are defined using the next diagram

$$\begin{tikzcd}
 & \mathcal{P}^X_g M \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathcal{P}_g M & & \mathcal{P}_{XX}M
\end{tikzcd}$$

where $\mathcal{P}^X_g M = \{\alpha : I \to M : \alpha(1) = \alpha(0)g,\, \alpha(0) \in X\}$.
We defined $\iota_{g,X}$ by the composition

$$\mathrm{H}_*(\mathcal{P}_g M) \xrightarrow{j_!} \mathrm{H}_*(\mathcal{P}^X_g M) \xrightarrow{i_*} \mathrm{H}_*(\mathcal{P}_{XX}M)$$

The diagrams that model this properties are:
(1)

$$\begin{tikzcd}
 & & \mathcal{P}_g^X M \arrow[dl, "1"'] \arrow[dr, "i"] & & \\
 & \mathcal{P}_g^X M \arrow[dl, "j"'] \arrow[dr, "i"] & & \mathcal{P}_{XX} M \arrow[dl, "1"'] \arrow[dr, "\rho_h"] & \\
\mathcal{P}_g M & & \mathcal{P}_{XX} M & & \mathcal{P}_{XX} M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_g^X M \arrow[dl, "i"'] \arrow[dr, "\alpha_h"] & & \\
 & \mathcal{P}_g M \arrow[dl, "1"'] \arrow[dr, "\alpha_h"] & & \mathcal{P}_{hgh^{-1}}^X M \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathcal{P}_g M & & \mathcal{P}_{hgh^{-1}} M & & \mathcal{P}_{XX} M
\end{tikzcd}$$

In the first case it is clear that $F_1 = 0$. This because the normal bundles are zero. For the second case we have

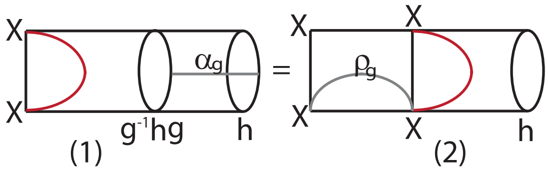
$$\begin{tikzcd}
(\epsilon_0 \circ \alpha_h)^*(\vartheta) \arrow[r, dashed] & \mathcal{P}_g^X M \arrow[d, "\alpha_h"'] & \\
 & \mathcal{P}_{hgh^{-1}}^X M \arrow[r, "j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{hgh^{-1}} M \arrow[d, "\epsilon_0"] \\
\vartheta \arrow[r, dashed] & X \arrow[r, "\iota"'] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_0^*(\vartheta) \arrow[r, dashed] & \mathcal{P}_g^X M \arrow[r, "i"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g M \arrow[d, "\epsilon_0"] \\
\vartheta \arrow[r, dashed] & X \arrow[r, "\iota"'] & M
\end{tikzcd}$$

The bundles $\epsilon_0^*(\vartheta)$ and $(\epsilon_0 \circ \alpha_h)^*(\vartheta)$ are isomorphic because the action α_h is a diffemorphism. Then, in particular is $F_2 = 0$.

4. **The map ι^h is an equivariant map**



The diagrams are the followings
(1)

$$\begin{tikzcd}[column sep=small]
 & & \mathcal{P}^X_{g^{-1}hg}M \arrow[dl, "1"'] \arrow[dr, "j"] & & \\
 & \mathcal{P}^X_{g^{-1}hg}M \arrow[dl, "i"'] \arrow[dr, "j"] & & \mathcal{P}_{g^{-1}hg}M \arrow[dl, "1"'] \arrow[dr, "\alpha_g"] & \\
\mathcal{P}_{XX}M & & \mathcal{P}_{g^{-1}hg}M & & \mathcal{P}_hM
\end{tikzcd}$$

(2)

$$\begin{tikzcd}[column sep=small]
 & & \mathcal{P}^X_{g^{-1}hg}M \arrow[dl, "i"'] \arrow[dr, "\alpha_g"] & & \\
 & \mathcal{P}_{XX}M \arrow[dl, "1"'] \arrow[dr, "\rho_g"] & & \mathcal{P}^X_hM \arrow[dl, "i"'] \arrow[dr, "j"] & \\
\mathcal{P}_{XX}M & & \mathcal{P}_{XX}M & & \mathcal{P}_hM
\end{tikzcd}$$

For the first case, it is an easy consequence that $F_1 = 0$. This because the normal bundles are zero.

The second case involves the following diagrams

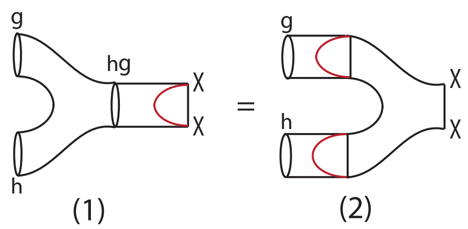
$$\begin{tikzcd}
(\epsilon_0 \circ \alpha_g)^*(\vartheta) \arrow[r, dashed] & \mathcal{P}^X_{g^{-1}hg}M \arrow[d, "\alpha_g"'] & \\
 & \mathcal{P}^X_hM \arrow[r, "i"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{XX}M \arrow[d, "\epsilon_0\times\epsilon_1"] \\
\vartheta \arrow[r, dashed] & X \arrow[r, "1\times\alpha_h"'] & X \times X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_0^*(\vartheta) \arrow[r, dashed] & \mathcal{P}^X_{g^{-1}hg}M \arrow[r, "i"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{XX}M \arrow[d, "\epsilon_0\times\epsilon_1"] \\
\vartheta \arrow[r, dashed] & X \arrow[r, "1\times\alpha_{g^{-1}hg}"'] & X\times X
\end{tikzcd}$$

Note that the bundles $\epsilon_0^*(\vartheta)\simeq(\epsilon_0\circ\alpha_g)^*(\vartheta)$ since α_g is a diffeomorphism. Then $F_2=0$.

5. **The map ι_g is a ring homomorphism**



In this case the diagrams that model this property are the following
(1)

$$\begin{tikzcd}
 & & \mathcal{P}^X_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}^X_h M \arrow[dl, "j\times j"'] \arrow[dr, "*"] & & \\
 & \mathcal{P}_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_h M \arrow[dl, "j"'] \arrow[dr, "*"] & & \mathcal{P}^X_{gh}M \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathcal{P}_g M\times\mathcal{P}_h M & & \mathcal{P}_{gh}M & & \mathcal{P}_{XX}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}^X_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}^X_h M \arrow[dl, "j\times j"'] \arrow[dr, "*"] & & \\
 & \mathcal{P}^X_g M\times\mathcal{P}^X_h M \arrow[dl, "j\times j"'] \arrow[dr, "i\times i"] & & \mathcal{P}_{XXX}M \arrow[dl, "j_{12}\times j_{23}"'] \arrow[dr, "j_{13}"] & \\
\mathcal{P}_g M\times\mathcal{P}_h M & & \mathcal{P}_{XX}M\times\mathcal{P}_{XX}M & & \mathcal{P}_{XX}M
\end{tikzcd}$$

For the first case we have

$$\begin{tikzcd}
(\epsilon_0 \circ *)^*(\eta) \arrow[r, dashed] & \mathcal{P}^X_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}^X_h M \arrow[d, "*"'] & \\
& \mathcal{P}^X_{gh} M \arrow[r, "j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{gh} M \arrow[d, "\epsilon_0\times\epsilon_1"] \\
\eta \arrow[r, dashed] & X \arrow[r, "1\times g"'] & M \times M
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_0^*(\eta) \arrow[r, dashed] & \mathcal{P}^X_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}^X_h M \arrow[r, "j\times j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g M_{\epsilon_0}\times_{\epsilon_1}\mathcal{P}_h M \arrow[d, "\epsilon_0\times\epsilon_1"] \\
\eta \arrow[r, dashed] & X \arrow[r, "1\times g"'] & M \times M
\end{tikzcd}$$

We note that $\epsilon_0 \circ * = \epsilon_0$, then $\epsilon_0^*(\eta) = (\epsilon_0 \circ *)^*(\eta)$ and $F_1 = 0$.
The second case has the following diagrams

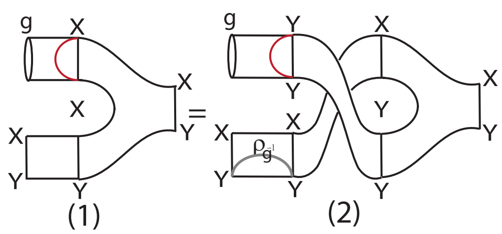
$$\begin{tikzcd}
(\epsilon_{\frac{1}{2}} \circ *)^*(TX) \arrow[r, dashed] & \mathcal{P}^X_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}^X_h M \arrow[d, "*"'] & \\
& \mathcal{P}_{XXX} M \arrow[r, "j_{12}\times j_{23}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XX} M \times \mathcal{P}_{XX} M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TX \arrow[r, dashed] & X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_\infty^*(TX) \arrow[r, dashed] & \mathcal{P}^X_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}^X_h M \arrow[r, "j"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}^X_g M \times \mathcal{P}_h M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TX \arrow[r, dashed] & X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

We note that $\epsilon_{\frac{1}{2}} \circ * = \epsilon_\infty$, hence $\epsilon_\infty^*(\eta) = (\epsilon_{\frac{1}{2}} \circ *)^*(\eta)$ and $F_2 = 0$.

6. ***G*-twisted centrality condition**



This condition is modeled by the next diagrams.
(1)

$$\begin{tikzcd}
 & & & \mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M \arrow[dll, "j'"'] \arrow[drr, "*\circ\iota"] \arrow[ddlll, bend right, "\psi_1"'] \arrow[ddrrr, bend left, "\varphi_1"] & & & \\
 & \mathcal{P}_g^X M \times \mathcal{P}_{XY}M \arrow[dl, "i\times 1"'] \arrow[drr, "j\times 1"] & & & & \mathcal{P}_{XXY}M \arrow[dll, "j_{12}\times j_{23}"'] \arrow[dr, "j_{13}"] & \\
\mathcal{P}_g M \times \mathcal{P}_{XY}M & & & \mathcal{P}_{XX}M \times \mathcal{P}_{XY}M & & & \mathcal{P}_{XY}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & & \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M \arrow[dll, "\tau\circ(1\times\rho_g)"'] \arrow[drr, "*\circ\iota'"] \arrow[ddlll, bend right, "\psi_2"'] \arrow[ddrrr, bend left, "\varphi_2"] & & & \\
 & \mathcal{P}_g^Y M \times \mathcal{P}_{XY}M \arrow[dl, "i\times 1"'] \arrow[drr, "\tau\circ(j\times\rho_{g^{-1}})"] & & & & \mathcal{P}_{XYY}M \arrow[dll, "j_{12}\times j_{23}"'] \arrow[dr, "j_{13}"] & \\
\mathcal{P}_g M \times \mathcal{P}_{XY}M & & & \mathcal{P}_{XY}M \times \mathcal{P}_{XYY}M & & & \mathcal{P}_{XY}M
\end{tikzcd}$$

We first check that the spaces $\mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M$ and $\mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M$ are homotopic. We define the maps as follow:

$$\begin{array}{rccc}
\varphi: & \mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M & \longrightarrow & \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M \\
 & (\alpha,\beta) & \longmapsto & (\beta, \overline{\beta} * \rho_{g^{-1}}(\alpha) * \rho_{g^{-1}}(\beta))
\end{array}$$

$$\begin{array}{rccc}
\psi: & \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M & \longrightarrow & \mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M \\
 & (\gamma,\delta) & \longmapsto & (\rho_g(\gamma) * \rho_g(\delta) * \overline{\gamma}, \gamma)
\end{array}$$

$\psi\circ\varphi(\alpha,\beta) = \psi(\beta,\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta)) = (\rho_g(\beta)*\rho_g(\overline{\beta})*\alpha*\beta*\overline{\beta},\beta) \simeq (\alpha,\beta)$,

$\varphi\circ\psi(\gamma,\delta) = \varphi(\rho_g(\gamma)*\rho_g(\delta)*\overline{\gamma},\gamma) = (\gamma,\overline{\gamma}*\gamma*\delta*\rho_{g^{-1}}(\overline{\gamma})*\rho_{g^{-1}}(\gamma)) \simeq (\gamma,\delta)$.

Then

$$\psi\circ\varphi \simeq \mathrm{Id} \quad \text{and} \quad \varphi\circ\psi \simeq \mathrm{Id}\,.$$

Now we check the external maps for the diagrams (1) and (2).

- $\varphi_2\circ\varphi(\alpha,\beta) = \varphi_2(\beta,\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta)) = \beta*\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta) \simeq \alpha*\beta$,
- $\varphi_1(\alpha,\beta) = \alpha*\beta$.
- $\psi_2\circ\varphi(\alpha,\beta) = \psi_2(\beta,\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta)) = (\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta),\rho_g(\beta)) \simeq (\alpha,\beta)$,
- $\psi_1(\alpha,\beta) = (\alpha,\beta)$.

- $\varphi_1 \circ \psi(\gamma, \delta) = \varphi_1(\rho_g(\gamma) * \rho_g(\delta) * \overline{\gamma}, \gamma) = \rho_g(\gamma) * \rho_g(\delta) * \overline{\gamma} * \gamma \simeq \gamma * \delta$,
- $\varphi_2(\gamma, \delta) = \gamma * \delta$.
- $\psi_1 \circ \psi(\gamma, \delta) = \psi_1(\rho_g(\gamma) * \rho_g(\delta) * \overline{\gamma}, \gamma) = (\rho_g(\gamma) * \rho_g(\delta) * \overline{\gamma}, \gamma) \simeq (\delta, \rho_g(\gamma))$,
- $\psi_2(\gamma, \delta) = (\delta, \rho_g(\gamma))$.

Finally we need to calculate the Euler class in each diagram. For the first case we have

$$\begin{tikzcd}
(\epsilon_{\frac{1}{2}} \circ * \circ \iota)^*(TX) \arrow[r, dashed] & \mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M \arrow[d, "*\circ\iota"'] & \\
 & \mathcal{P}_{XXY}M \arrow[r, "j_{12}\times j_{23}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XX}M \times \mathcal{P}_{XY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TX \arrow[r, dashed] & X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_\infty^*(TX) \arrow[r, dashed] & \mathcal{P}_g^X M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M \arrow[r, "j'"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g^X M \times \mathcal{P}_{XY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TX \arrow[r, dashed] & X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

We note that $\epsilon_\infty^*(TX) = (\epsilon_{\frac{1}{2}} \circ * \circ \iota)^*(TX)$. Then $F_1 = 0$.
The second case has associated the next diagrams

$$\begin{tikzcd}
(\epsilon_{\frac{1}{2}} \circ * \circ \iota')^*(TY) \arrow[r, dashed] & \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M \arrow[d, "*\circ\iota'"'] & \\
 & \mathcal{P}_{XYY}M \arrow[r, "j_{12}\times j_{23}"] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{XY}M \times \mathcal{P}_{YY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TY \arrow[r, dashed] & Y \arrow[r, "\Delta"'] & Y \times Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
\epsilon_\infty^*(TY) \arrow[r, dashed] & \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_g^Y M \arrow[r, "\tau\circ(1\times\rho_g)"] \arrow[d, "\epsilon_\infty"'] & \mathcal{P}_g^Y M \times \mathcal{P}_{XY}M \arrow[d, "\epsilon_0\times\epsilon_1"] \\
TY \arrow[r, dashed] & Y \arrow[r, "\Delta"'] & Y \times Y
\end{tikzcd}$$

As the same as before it holds the identity $\epsilon_\infty^*(TY) = (\epsilon_{\frac{1}{2}} \circ * \circ \iota')^*(TY)$. Then $F_2 = 0$.

To finish the proof we only need to check that $\nu_\varphi = 0$. For this, we construct the next homotopy:

$$\begin{array}{rccc} H: & I\times(\mathcal{P}^X_g M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M) & \longrightarrow & \mathcal{P}_{XY}M_{\epsilon}\times_{\epsilon_0}\mathcal{P}_gM\times I \\ & (s,(\alpha,\beta)) & \longmapsto & (\beta,\overline{\beta_s}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta),s)\end{array}$$

where $\epsilon : I \times \mathcal{P}_{XY}M \to M$ is given by $\epsilon(s,\beta) := \beta(s)$. The next pullback square proves that $W := \mathcal{P}_{XY}M_{\epsilon}\times_{\epsilon_0}\mathcal{P}_gM\times I$ is an infinite manifold.

$$\begin{tikzcd} W = \mathcal{P}_{XY}M_{\epsilon}\times_{\epsilon_0}\mathcal{P}_gM\times I \arrow[r] \arrow[d, "\epsilon_\infty\times 1"'] & \mathcal{P}_{XY}M\times\mathcal{P}_gM\times I \arrow[d, "\epsilon\times\epsilon_0\times 1"] \\ M\times I \arrow[r, "\Delta\times 1"'] & M\times M\times I \end{tikzcd}$$

Similarly, the next pullback square

$$\begin{tikzcd} Z_s := \mathcal{P}_{XY}M_{\epsilon_s}\times_{\epsilon_0}\mathcal{P}_gM\times\{s\} \arrow[r] \arrow[d, "\epsilon_s\times\{s\}"'] & \mathcal{P}_{XY}M_{\epsilon}\times_{\epsilon_0}\mathcal{P}_gM\times I \arrow[d, "\epsilon\times 1"] \\ M\times\{s\} \arrow[r, hook] & M\times I \end{tikzcd}$$

proves that Z_s is a sub-manifold of codimension one of W for all s.
Note that the homotopy H satisfies that

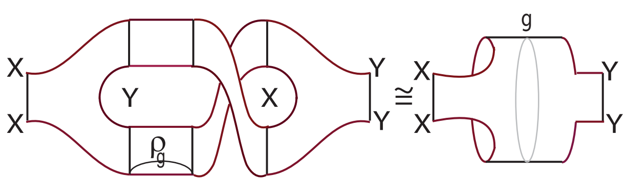
$$\begin{array}{rclcl} H(0,(\alpha,\beta)) & = & (\beta,\rho_{g^{-1}}(\alpha)) & = & (1\times\rho_{g^{-1}})\circ\tau(\alpha,\beta) \\ H(1,(\alpha,\beta)) & = & (\beta,\overline{\beta}*\rho_{g^{-1}}(\alpha)*\rho_{g^{-1}}(\beta)) & = & \varphi(\alpha,\beta)\end{array}$$

Then, in particular we have the next situation

$$\begin{tikzcd} \mathcal{P}^X_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M \arrow[d, "(1\times\rho_{g^{-1}})\circ\tau"', "\text{diffemorphism}"] \arrow[r, phantom, "\overset{H}{\underset{\simeq}{\Longrightarrow}}"] & \mathcal{P}^X_gM_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}_{XY}M \arrow[d, "\varphi"] \\ Z_0 = \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}^Y_gM & Z_1 = \mathcal{P}_{XY}M_{\epsilon_1}\times_{\epsilon_0}\mathcal{P}^Y_gM \end{tikzcd}$$

Since $\nu_{(1\times\rho_{g^{-1}})\circ\tau} = 0$ then $\nu_\varphi = 0$ and $e(\nu_\varphi) = 1$.

7. **Cardy condition**



(1) (2)

(1)

$$\begin{tikzcd}
 & & \mathcal{P}_g^{X\cap Y}M \arrow[dl, "i_X"'] \arrow[dr, "i_Y"] \arrow[ddll, bend right, "g_1"'] \arrow[ddrr, bend left, "f_1"] & & \\
 & \mathcal{P}_g^X M \arrow[dl, "i"'] \arrow[dr, "j"] & & \mathcal{P}_g^Y M \arrow[dl, "j"'] \arrow[dr, "i"] & \\
\mathcal{P}_{XX}M & & \mathcal{P}_g M & & \mathcal{P}_{YY}M
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
 & & \mathcal{P}_{g,X}^{Y}M \arrow[dl, "i"'] \arrow[dr, "*\circ(\rho_g\times 1)\circ\tau\circ(j\times j)"] \arrow[ddll, bend right, "g_2"'] \arrow[ddrr, bend left, "f_2"] & & \\
 & \mathcal{P}_{XYX}M \arrow[dl, "i_{XX}^Y"'] \arrow[dr, "\tau\circ(1\times\rho_g)\circ(j\times j)"] & & \mathcal{P}_{YXY}M \arrow[dl, "j\times j"'] \arrow[dr, "i_{YY}^X"] & \\
\mathcal{P}_{XX}M & & \mathcal{P}_{YX}M\times\mathcal{P}_{XY}M & & \mathcal{P}_{XX}M
\end{tikzcd}$$

In this particular case, the maps are illustrated in the figure 6.14, and they are homotopic to the cobordism illustrated in the figure 6.15. We will suppose

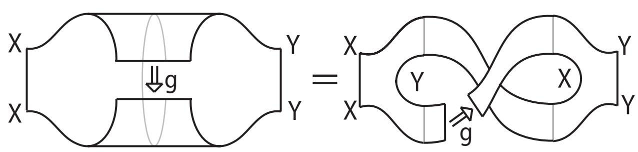


Figure 6.14: `The composition maps in the Cardy condition.`

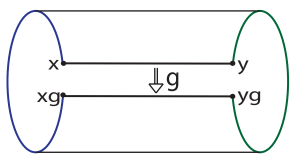


Figure 6.15: `The cobordisms associated to the compositions.`

that the intersection $X \cap Y$ is non-empty, this because if it is empty then the two composition maps are zero. In the second cobordism the composition is zero by definition, and in the first this is because for an empty intersection the composition of the umhker maps is zero since the tubular neighborhoods are disjoint.

We prove that $\mathcal{P}_g^{X\cap Y}M$ and $\mathcal{P}_{g,X}^{Y}M$ are homotopically equivalent spaces. First we describe the maps between the spaces. Suppose that $z \in X \cap Y$, and if we take M arco-connected then for $x \in M$ exists $\eta : I \to M$ such that $\eta(0) = z$ and $\eta(1) = x$.

$$\begin{array}{rccc} \varphi : & \mathcal{P}_g^{X\cap Y}M & \longrightarrow & \mathcal{P}_{g,Y}^{X}M \\ & \alpha & \longmapsto & \alpha * \overline{\alpha} * \alpha \\ \psi : & \mathcal{P}_{g,Y}^{X}M & \longrightarrow & \mathcal{P}_g^{X\cap Y}M \\ & \delta & \longmapsto & \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \end{array}$$

The composition maps are

$$\psi \circ \varphi(\alpha) = \psi(\alpha * \overline{\alpha} * \alpha) = \eta * \alpha * \overline{\alpha} * \alpha * \rho_{g^{-1}}(\overline{\eta}) \simeq \eta * \rho_{g^{-1}}(\overline{\eta}) \simeq \alpha.$$

$$\varphi \circ \psi(\delta) = \varphi(\eta * \delta * \rho_{g^{-1}}(\overline{\eta})) = \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) * \rho_{g^{-1}}(\eta) * \overline{\delta} * \overline{\eta} * \eta * \delta * \rho_{g^{-1}}(\overline{\eta})$$
$$\simeq \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \simeq \delta.$$

The composition with the external maps is the following. First we note that the maps $f_1 : \mathcal{P}_g^{X\cap Y}M \hookrightarrow \mathcal{P}_{XX}M$, $g_1 : \mathcal{P}_g^{X\cap Y}M \hookrightarrow \mathcal{P}_{YY}M$ and $g_2 : \mathcal{P}_{g,Y}^{X}M \hookrightarrow \mathcal{P}_{XX}M$ are natural inclusion maps. Finally, the map $f_2 : \mathcal{P}_{g,Y}^{X}M \to \mathcal{P}_{YY}M$ is given by $f_2(\alpha * \beta) = \rho_g(\beta) * \alpha$. Then

$$\delta \xmapsto{\psi} \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \xmapsto{g_1} \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \simeq \delta$$

$$\delta \xmapsto{g_2} \delta$$

$$\delta \xmapsto{\psi} \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) \xmapsto{f_1} \eta * \delta * \rho_{g^{-1}}(\overline{\eta}) = \eta * \alpha * \beta * \rho_{g^{-1}}(\overline{\eta}) \simeq \rho_g(\beta) * \alpha$$

$$\delta = \alpha * \beta \xmapsto{f_2} \rho_g(\beta) * \alpha$$

$$\alpha \xmapsto{g_1} \alpha$$

$$\alpha \xmapsto{\varphi} \alpha * \overline{\alpha} * \alpha \xmapsto{g_2} \alpha * \overline{\alpha} * \alpha \simeq \alpha$$

$$\alpha \xmapsto{f_1} \alpha$$

$$\alpha \xmapsto{\varphi} \alpha * \overline{\alpha} * \alpha \xmapsto{f_2} \rho_g(\overline{\alpha} * \alpha) * \alpha \simeq \alpha$$

Now we need to determine the Euler class for this case. First, we calcule that $e(\nu_\psi) = 0$. Let be the homotopy

$$\begin{array}{rccc} H : & I \times \mathcal{P}_{g,Y}^{X}M & \longrightarrow & \mathcal{P}_{g,Y,X}^{I}M \times I \\ & (s, \delta) & \longmapsto & (\eta_s * \delta * \rho_{g^{-1}}(\overline{\eta_s}), s) \end{array}$$

where $\eta_s : I \to M$ is given by $\eta_s(t) = \eta((1-s)t + s)$, then $\eta_s(0) = \eta(s)$ and $\eta_s(1) = \eta(1)$. See figure 6.16.

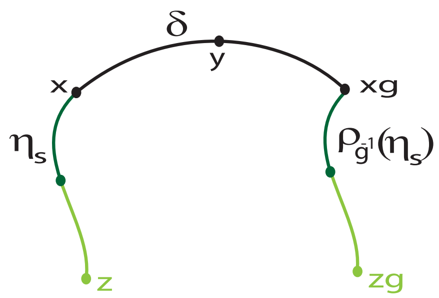


Figure 6.16: `The homotopy` H.

Note that $H(0,\delta) = (\eta_0 * \delta * \rho_{g^{-1}}(\overline{\eta_0}), 0) = (\eta * \delta * \rho_{g^{-1}}(\overline{\eta}) = \psi(\delta)$, and $H(1,\delta) = (\eta_1 * \delta * \rho_{g^{-1}}(\overline{\eta_1}), 1) = \delta = \mathrm{Id}(\delta)$. Then, we have the next situation

$$\begin{tikzcd} \mathcal{P}^X_{g,Y}M \arrow[d, "\mathrm{Id}"'] & & \mathcal{P}^X_{g,Y}M \arrow[d, "\psi"] \\ Z_0 = \mathcal{P}^X_{g,Y}M \arrow[rr, phantom, "\overset{H}{\underset{\simeq}{\Longrightarrow}}"] & & Z_1 = \mathcal{P}^{X\cap Y}_g M \end{tikzcd}$$

For the space $Z_s := \mathcal{P}^s_{g,Y,X}M \times \{s\} = \{\eta_s * \delta * \rho_{g^{-1}}(\overline{\eta_s}) : \delta \in \mathcal{P}^X_{g,Y}M\} \times \{s\} \subset W := \mathcal{P}^I_{g,Y,X}M \times I = \{\eta_s * \delta * \rho_{g^{-1}}(\overline{\eta_s}) :\ s \in I\} \times I$ we have Z_s is a submanifold of W of codimension one. This is because the next diagram is a pullback square.

$$\begin{tikzcd} Z_s := \mathcal{P}^s_{g,Y,X}M \times \{s\} \arrow[r] \arrow[d, "\epsilon_\infty \times \{s\}"'] & \mathcal{P}^I_{g,Y,X}M \times I \arrow[d, "\epsilon_\infty \times 1"] \\ X \times \{s\} \arrow[r, hook] & X \times I \end{tikzcd}$$

It is clear that W is a manifold of infinite dimension. Then $\nu_\psi = \nu_{\mathrm{Id}} = 0$ and $e(\nu_\psi) = 1$.

Finally, we need to determine the Euler class of the following two diagrams. The first diagram is

$$\begin{tikzcd} \epsilon_0^*(\nu_{i_2}) \arrow[r] & \mathcal{P}^{X\cap Y}_g M \arrow[d, "i_Y"] & \\ & \mathcal{P}^Y_g M \arrow[r, "j"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g M \arrow[d, "\epsilon_0"] \\ \nu_{i_2} \arrow[r] & Y \arrow[r, hook, "i_2"'] & M \end{tikzcd}$$

and the second

$$\begin{tikzcd}
\epsilon_0^*(\nu_{i_1}) \arrow[r] & \mathcal{P}_g^{X\cap Y}M \arrow[r, "i_X"] \arrow[d, "\epsilon_0"'] & \mathcal{P}_g^X M \arrow[d, "\epsilon_0"] \\
\nu_{i_1} \arrow[r] & X\cap Y \arrow[r, hook, "i_1"'] & X
\end{tikzcd}$$

If we suppose that $X \pitchfork Y$ then $e(\epsilon_0^*(\nu_{i_1})) = e(\epsilon_0^*(\nu_{i_2}))$, and $F_1 = 0$.
In the second case we have

$$\begin{tikzcd}
\epsilon_0^*(\nu_{(1\times\alpha_g)}) \arrow[r] & \mathcal{P}_{g,Y}^X M \arrow[r] \arrow[d, "\epsilon_0"'] & \mathcal{P}_{XYX}M \arrow[d, "\epsilon_0\times\epsilon_1"] \\
\nu_{(1\times\alpha_g)} \arrow[r] & X \arrow[r, "1\times\alpha_g"'] & X\times X
\end{tikzcd}$$

and

$$\begin{tikzcd}
f^*\epsilon_{\frac{1}{2}}^*(TX) \arrow[r] & \mathcal{P}_{g,Y}^X M \arrow[d, "f=*\circ(\rho_g\times 1)\circ\tau\circ(j\times j)"] & \\
 & \mathcal{P}_{YXY}M \arrow[r] \arrow[d, "\epsilon_{\frac{1}{2}}"'] & \mathcal{P}_{YX}M\times\mathcal{P}_{XY}M \arrow[d, "\epsilon_1\times\epsilon_0"] \\
TX \arrow[r] & X \arrow[r, "\Delta"'] & X\times X
\end{tikzcd}$$

Note that $f^*\epsilon_{\frac{1}{2}}^*(TX) \simeq \epsilon_0^*(\nu_{1\times\alpha_g})$, this is because $\nu_{(1\times\alpha_g)} \cong TX$. Then $F_2 = 0$.

♣

Chapter 7

2D OC-TFT of the derived category of a Calabi-Yau manifold

Associated to a Calabi-Yau manifold X there is a standard 2D-OC-TFT coming from string theory, the boundary conditions are supposed to be generated by complex submanifolds of X so the boundary conditions are taken to be complexes of coherent sheaves on X; the open string category is then supposed to be the derived category of coherent sheaves on X. The closed string part $\mathscr{C}$ should be $\mathrm{Hom}^*_{\mathcal{D}(X\times X)}(\mathscr{O}_\Delta, \mathscr{O}_\Delta)$, in other words, the Hochschild cohomology $\mathrm{HH}^*(X)$ of X. This example is developed in [CW07].

First we describe briefly the concept of derived category.

7.1 Derived categories

Derived and triangulated categories were introduced by Grothendieck and Verdier in the early sixties in order to establish a relative version of Serre duality for a "nice" morphism $f : X \to Y$ of schemes, [Ver96]. The Grothendieck-Verdier duality theory involves the construction of derived categories $\mathcal{D}(X)$ and $\mathcal{D}(Y)$, whose objects consist of complexes of sheaves with quasi-coherent cohomology, together with a derived push-forward functor $Rf_* : \mathcal{D}(X) \to \mathcal{D}(Y)$ and a right adjoint $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$.

Let $\mathcal{A}$ be an abelian category and denote by $\mathrm{Kom}(\mathcal{A})$ the category whose objects are complexes of objects in $\mathcal{A}$ and whose morphisms are maps of complexes. $\mathrm{Kom}(\mathcal{A})$ is again abelian.

Denote by $\mathrm{K}(\mathcal{A})$ the homotopy category of $\mathcal{A}$, which has the same objects as $\mathrm{Kom}(\mathcal{A})$, but whose morphisms are homotopy classes of maps of complexes. $\mathrm{K}(\mathcal{A})$ is not abelian in general (the component-wise kernel in $\mathrm{Kom}(\mathcal{A})$ is not well-defined up to homotopy). There is a natural functor $\mathrm{Kom}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ which is the identity in objects and sends a morphism to its homotopy class.

Theorem 7.1.1. *Given an abelian category $\mathcal{A}$, there exists a category $\mathcal{D}(\mathcal{A})$, called the* derived category *of $\mathcal{A}$, such that*

1. *There is a functor $Q : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ which sends quasi-isomorphisms to isomorphisms.*

2. *The functor Q is universal with respect to property 1.: given any category $\mathcal{B}$ and any functor $\mathscr{F} : \mathrm{Kom}(\mathcal{A}) \to \mathcal{B}$ which sends quasi-isomorphisms to isomorphisms, there is a unique functor $\mathscr{G} : \mathcal{D}(\mathcal{A}) \to \mathcal{B}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Kom}(\mathcal{A}) \arrow[rr, "Q"] \arrow[dr, "\mathscr{F}"'] & & \mathcal{D}(\mathcal{A}) \arrow[dl, "\mathscr{G}"] \\ & \mathcal{B} & \end{tikzcd}$$

A very good reference for this topic is the book [GM80].

Now we give a short description of Serre functors, this is a principal property which derives in our example in a Frobenius structure.

7.2 Serre functors

One of the most useful theorems in algebraic geometry is Serre duality, which is formalized in the following way.

Definition 7.2.1. Let $\mathscr{C}$ be a Hom-finite $\Bbbk$-linear category. A *Serre functor* is a $\Bbbk$-linear equivalence

$$\mathcal{S} : \mathscr{C} \to \mathscr{C}$$

together with isomorphisms

$$\mathrm{Hom}(F, G) \longrightarrow \mathrm{Hom}(G, \mathcal{S}F)^* \tag{7.1}$$

natural in F, $G \in \mathscr{C}$.

Letting $G = F$, the image of $1 \in \mathrm{Hom}(F, F)$ under the isomorphism 7.1 gives a canonical *trace* element

$$\mathrm{Tr}_F : \mathrm{Hom}(F, \mathcal{S}F) \to \Bbbk.$$

The composition of morphisms followed by the trace,

$$\mathrm{Hom}(G, \mathcal{S}F) \otimes \mathrm{Hom}(F, G) \longrightarrow \mathrm{Hom}(F, \mathcal{S}F) \xrightarrow{\mathrm{Tr}_F} \Bbbk,$$

is a non-degenerate pairing and realizes the duality in 7.1.

Definition 7.2.2. A triangulated category $\mathscr{T}$ with Serre functor S is called an n-*Calabi-Yau* (n-CY) if there is a natural isomorphism of functors $S \simeq [n]$.

Remark 7.2.3. When Serre functors exist, they are unique up to natural isomorphism.

To understand the concept of a triangulated category you can see [GM80] or [Cal05a] and for more detail about Serre functors you can see [Bra08].

7.2.1 The Serre functor on $\mathcal{D}(X)$

If X is a space, we can consider an associated derived category. This is the derived category of coherent sheaves on X, $\mathcal{D}^b(\mathrm{Coh}(X))$. We denote this category as $\mathcal{D}(X)$. In this case we can define the Serre functor as follows

$$\mathcal{S} : \mathcal{D}(X) \to \mathcal{D}(X); \quad \mathscr{E} \mapsto \omega_X[\dim X] \otimes \mathscr{E},$$

where ω_X is the canonical bundle of X. Serre duality gives natural, bifunctorial isomorphisms

$$\eta_{\mathscr{E},\mathscr{F}} : \mathrm{Hom}_{\mathcal{D}(X)}(\mathscr{E}, \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(X)}(\mathscr{F}, \mathcal{S}\mathscr{E})^*$$

for any objects $\mathscr{E}, \mathscr{F} \in \mathcal{D}(X)$, where $*$ denotes the dual vector space.

From this data, for any object $\mathscr{E} \in \mathcal{D}(X)$, the Serre trace is defined as follows:

$$\mathrm{Tr} : \mathrm{Hom}(\mathscr{E}, \mathcal{S}\mathscr{E}) \to \Bbbk; \quad \mathrm{Tr}(\alpha) := \eta_{\mathscr{E},\mathscr{E}}(\mathrm{Id}_{\mathscr{E}})(\alpha).$$

Note that from this trace we can recover $\eta_{\mathscr{E},\mathscr{F}}$, because

$$\eta_{\mathscr{E},\mathscr{F}}(\alpha)(\beta) = \mathrm{Tr}(\beta \circ \alpha).$$

We also have the identity

$$\mathrm{Tr}(\beta \circ \alpha) = \mathrm{Tr}(\mathcal{S}\alpha \circ \beta).$$

Another way to encode this data is as a perfect pairing, the *Serre pairing*:

$$\langle\, , \rangle_{\mathcal{S}} : \mathrm{Hom}(\mathscr{E}, \mathscr{F}) \otimes \mathrm{Hom}(\mathscr{F}, \mathcal{S}\mathscr{E}) \to \Bbbk; \quad \langle \alpha, \beta \rangle_S := \mathrm{Tr}(\beta \circ \alpha).$$

7.2.2 Serre kernel and functor on $\mathcal{D}(X \times X)$

Definition 7.2.4. For a space X, the *Serre kernel* Σ_X is defined to be $\Delta_* \omega_X[\dim X] \in \mathcal{D}(X \times X)$, where $\Delta : X \to X \times X$ is the diagonal map and $\Delta_* : \mathcal{D}(X) \to \mathcal{D}(X) \otimes \mathcal{D}(X)$ is the induced map in the derived category. Similarly the *anti-Serre kernel* Σ_X^{-1} is defined to be $\Delta_* \omega_X^{-1}[-\dim X] \in \mathcal{D}(X \times X)$.

Proposition 7.2.5. *For spaces X and Y the Serre functor $S_{X,Y} : \mathcal{D}(X \times Y) \to \mathcal{D}(X \times Y)$ can be taken to be $\Sigma_Y \circ - \circ \Sigma_X$.*

The next section gives us the Frobenius structure in the closed part of the theory.

7.3 The Hochschild structure

For a space X, we denote by Id_X and Σ_X^{-1} the objects of $\mathcal{D}(X \times X)$ given by

$$\mathrm{Id}_X := \Delta_* \mathscr{O}_X \quad \text{and} \quad \Sigma_X^{-1} := \Delta_* \omega_X^{-1}[-\dim X]$$

where $\Delta : X \to X \times X$ is the diagonal map, and ω_X^{-1} is the anti-canonical line bundle of X.
The *Hochschild structure* of the space X consists of the following data:

- the graded ring $\mathrm{HH}^*(X)$, the *Hochschild cohomology* of X, whose i-th graded piece is defined as

$$\mathrm{HH}^i(X) := \mathrm{Hom}^i_{\mathcal{D}(X \times X)}(\mathrm{Id}_X, \mathrm{Id}_X),$$

- the graded left $\mathrm{HH}^*(X)$-module $\mathrm{HH}_*(X)$, the *Hochschild homology* module over X, defined as

$$\mathrm{HH}_i(X) := \mathrm{Hom}^{-i}_{\mathcal{D}(X \times X)}(\Sigma_X^{-1}, \mathrm{Id}_X),$$

- a non-degenerate graded pairing $\langle\, , \rangle_M : \mathrm{HH}_*(X) \otimes \mathrm{HH}_*(X) \to \mathbb{C}$, the *generalized Mukai pairing.*

The definition of the Mukai pairing the Hochschild homology is quite subtle and we refer the reader directly to [CW07].
The above definitions of Hochschild homology and cohomology agree with the usual ones for quasi-projective schemes (see [Cal05b]).

7.4 Open-closed 2D TFT from a Calabi-Yau manifold

Finally we complete the structure of this example for a Calabi-Yau manifold X. For each $\mathscr{E}$ and $\mathscr{F}$, we have the pairing

$$\mathrm{Hom}^*_{\mathcal{D}(X)}(\mathscr{E}, \mathscr{F}) \otimes \mathrm{Hom}^*_{\mathcal{D}(X)}(\mathscr{F}, \mathscr{E}) \to \Bbbk[\dim X]$$

which comes from the Serre pairing as a Calabi-Yau manifold.

As X is Calabi-Yau, a trivialization of the canonical bundle induces an isomorphism between Hochschild cohomology and Hochschild homology, up to a shift. This means that the closed string space $\mathscr{C}$ has both the cohomological product and the Mukai pairing, and these make $\mathscr{C}$ into a Frobenius algebra.

We need to specify the algebraic maps $\iota_{\mathscr{E}} : \mathscr{C} \to \mathscr{O}_{\mathscr{E}\mathscr{E}}$. These are

$$\iota_{\mathscr{E}} : \mathrm{Hom}^*(\mathscr{O}_\Delta, \mathscr{O}_\Delta) \to \mathrm{Hom}^*(\mathscr{E}, \mathscr{E})$$

which can be given by interpreting $\mathscr{E}$ as a kernel pt $\to X$ and taking $\iota_{\mathscr{E}}$ to be convolution with the identity on $\mathscr{E}$.

The other map is $\iota^{\mathscr{E}} : \mathrm{Hom}^*(\mathscr{E}, \mathscr{E}) \to \mathrm{Hom}^*(\mathscr{O}_\Delta, \mathscr{O}_\Delta)$, which is given by taking the trace. This definition relies on the fact that X is Calabi-Yau.

In the case of a general manifold X we have the next result. This is a baggy Cardy condition. In the particular case that X is Calabi-Yau this property is equivalent to the Cardy Condition.

Theorem 7.4.1 (Theorem 15 of [CW07])**.** *Suppose that $\mathscr{O}$ is a Calabi-Yau category and $\mathscr{C}$ is an inner product space, such that for each $\mathcal{A} \in \mathscr{O}$ there are adjoint maps $\iota^{\mathcal{A}} : \mathscr{O}_{\mathcal{A}\mathcal{A}} \to \mathscr{C}$ and $\iota_{\mathcal{A}} : \mathscr{C} \to \mathscr{O}_{\mathcal{A}\mathcal{A}}$. Then the Cardy condition*

$$\mu^{\mathcal{A}}_{\mathcal{B}\mathcal{B}} \circ \tau \circ \Delta^{\mathcal{B}}_{\mathcal{A}\mathcal{A}} = \iota_{\mathcal{B}} \circ \iota^{\mathcal{A}}$$

is equivalent to the following equality holding for all $\mathsf{a} \in \mathscr{O}_{\mathcal{A}\mathcal{A}}$ and $\mathsf{b} \in \mathscr{O}_{\mathcal{B}\mathcal{B}}$, where the map ${}_{\mathsf{a}}m_{\mathsf{b}}$ is the map obtained by pre-composing with a and post-composing with b:

$$\langle \iota^{\mathcal{B}} -, \iota^{\mathcal{A}} - \rangle_{\mathscr{C}} = \mathrm{Tr}\, {}_{-}m_{-}.$$

The alternative condition given in the above theorem holds for the derived category and Hochschild homology of any space: in particular, the Cardy Condition holds for Calabi-Yau spaces.

Proposition 7.4.2 (Theorem 16 (The Baggy Cardy Condition) of [CW07])**.** *Let X be a space, let $\mathscr{E}$ and $\mathscr{F}$ be objects in $\mathcal{D}(X)$ and consider morphisms*

$$\mathsf{e} \in \mathrm{Hom}_{\mathcal{D}(X)}(\mathscr{E}, \mathscr{E}) \text{ and } \mathsf{f} \in \mathrm{Hom}_{\mathcal{D}(X)}(\mathscr{F}, \mathscr{F}).$$

Define the operator

$${}_{\mathsf{f}}m_{\mathsf{e}} : \mathrm{Hom}^*_{\mathcal{D}(X)}(\mathscr{E}, \mathscr{F}) \longrightarrow \mathrm{Hom}^*_{\mathcal{D}(X)}(\mathscr{E}, \mathscr{F})$$

to be post-composition by f and pre-composition by e. Then we have

$$\mathrm{Tr}\, {}_{\mathsf{f}}m_{\mathsf{e}} = \langle \iota^{\mathscr{E}}(\mathsf{e}), \iota^{\mathscr{F}}(\mathsf{f}) \rangle_M,$$

where $\iota^{\mathscr{E}}$, $\iota^{\mathscr{F}}$ are the maps defined previously, and Tr *denotes the (super)trace.*

We finish this chapter with a conjecture. We proved before that the cohomology of non-compact manifold satisfies the axioms of a nearly Frobenius algebra, which is the same as an ordinary TFT. In this chapter we have seen how to a Calabi-Yau manifold one can associate a 2D-OC-TFT using the derived category.

Question: Is it true that the derived category construction applied to a non-compact Calabi-Yau manifold (orbifold)associates to it a 2D-OC-TFT with positive boundary satisfying the definition given in this thesis?

It does not seem impossible for the answer to be affirmative. We will return to this issue elsewhere.

Chapter 8

Appendix 1
Monoidal categories

A monoidal structure in a set is known as a *monoid* (or *semigroup*). More explicitly, a monoid is an algebraic structure with an associative binary operation and an identity element. In category theory, a monoid can be regarded as a category with only one object.

The extension of this concept to an additional degree of complexity is known as a *monoidal category*. Similarly, a monoidal category can be regarded as a 2-category with only one object (or *bicategory*). Then, this concept is a bridge between the category theory and the theory of 2-categories.

Monoidal categories were introduced in the 1960s by Bénabou, Lane and others.

Definition 8.0.3. A monoidal category (or tensor category) consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, called the monoidal product(or tensor product), an object $u \in \mathrm{Ob}(\mathcal{C})$, called the unit and natural isomorphisms

- $\alpha_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z$,
- $\lambda_x : u \otimes x \longrightarrow x$,
- $\rho_x : x \otimes u \longrightarrow x$,

called *associativity*, *left unit* and *right unit* such that the following diagrams commutative:

$$\begin{tikzcd}
x \otimes (y \otimes (w \otimes z)) \arrow[r, "\alpha_{x,y,w\otimes z}"] \arrow[d, "1\otimes\alpha_{y,w,z}"'] & (x \otimes y) \otimes (w \otimes z) \arrow[r, "\alpha_{x\otimes y,w,z}"] & ((x \otimes y) \otimes w) \otimes z \\
x \otimes ((y \otimes w) \otimes z) \arrow[rr, "\alpha_{x,y\otimes w,z}"'] & & (x \otimes (y \otimes w)) \otimes z \arrow[u, "\alpha_{x,y,w}\otimes 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes (u \otimes y) \arrow[rr, "\alpha_{x,u,y}"] \arrow[dr, "1\otimes\lambda_y"'] & & (x \otimes u) \otimes y \arrow[dl, "\rho_x\otimes 1"] \\
& x \otimes y &
\end{tikzcd}$$

for $x, y, w, z \in \mathrm{Ob}(\mathcal{C})$, and also

$$\lambda_u = \rho_u : u \otimes u \longrightarrow u.$$

A monoidal category is called *strict monoidal category*, if the morphisms α, λ, ρ are the identity morphisms.

8.0.1 Monoidal Functors

Definition 8.0.4. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, u_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, u_{\mathcal{D}})$ be monoidal categories. A *monoidal functor* is a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ together with natural isomorphisms

- $\xi_{x,y} : F(x) \otimes_{\mathcal{D}} F(y) \longrightarrow F(x \otimes_{\mathcal{C}} y)$
- $\xi_0 : u_{\mathcal{D}} \longrightarrow F(u_{\mathcal{C}})$

which satisfy the following commutative diagrams:

$$\begin{tikzcd}
F(x) \otimes_{\mathcal{D}} (F(y) \otimes_{\mathcal{D}} F(z)) \arrow[r, "1\otimes_{\mathcal{D}}\xi_{y,z}"] \arrow[d, "\alpha_{F(x),F(y),F(z)}"'] & F(x) \otimes_{\mathcal{D}} F(y \otimes_{\mathcal{C}} z) \arrow[r, "\xi_{x,y\otimes_{\mathcal{C}} z}"] & F(x \otimes_{\mathcal{C}} (y \otimes_{\mathcal{C}} z)) \arrow[d, "F(\alpha_{x,y,z})"] \\
(F(x) \otimes_{\mathcal{D}} F(y)) \otimes_{\mathcal{D}} F(z) \arrow[r, "\xi_{x,y}\otimes_{\mathcal{D}} 1"'] & F((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{D}} F(z) \arrow[r, "\xi_{x\otimes_{\mathcal{C}} y,z}"'] & F((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{C}} z)
\end{tikzcd}$$

$$\begin{tikzcd}
u_{\mathcal{D}} \otimes_{\mathcal{D}} F(x) \arrow[r, "\xi_0\otimes_{\mathcal{D}} 1"] \arrow[dr, "\lambda_{F(x)}"'] & F(u_{\mathcal{C}}) \otimes_{\mathcal{D}} F(x) \arrow[r, "\xi_{u,x}"] & F(u_{\mathcal{C}} \otimes_{\mathcal{C}} x) \arrow[dl, "F(\lambda_x)"] \\
& F(x) &
\end{tikzcd}$$

$$\begin{tikzcd}
F(x) \otimes_{\mathcal{D}} u_{\mathcal{D}} \arrow[r, "1\otimes_{\mathcal{D}}\xi_0"] \arrow[dr, "\rho_{F(x)}"'] & F(x) \otimes_{\mathcal{D}} F(u_{\mathcal{C}}) \arrow[r, "\xi_{x,u}"] & F(x \otimes_{\mathcal{C}} u_{\mathcal{C}}) \arrow[dl, "F(\rho_x)"] \\
& F(x) &
\end{tikzcd}$$

The functor is called *strict monoidal functor* if ξ and ξ_0 are the identity morphisms.

Remark 8.0.5. For any monoidal functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$. Let (ξ, ξ_0) and (ξ', ξ_0') the natural isomorphisms of F and G, respectively. The natural isomorphisms (ξ'', ξ_0'') for the composition $F \circ G : \mathcal{C} \longrightarrow \mathcal{E}$ are defined by

$$\begin{tikzcd} G \circ F(x) \otimes_{\mathcal{E}} G \circ F(y) \arrow[r, "\xi'"] \arrow[rr, bend right, "\xi''"'] & G(F(x) \otimes_{\mathcal{D}} F(y)) \arrow[r, "G(\xi)"] & G \circ F(x \otimes_{\mathcal{C}} y) \end{tikzcd}$$

$$\begin{tikzcd} u_{\mathcal{E}} \arrow[r, "\xi_0'"] \arrow[rr, bend right, "\xi_0''"'] & G(u_{\mathcal{D}}) \arrow[r, "G(\xi_0)"] & G \circ F(u_{\mathcal{C}}) \end{tikzcd}$$

Example 8.0.1. The most important ones are

$(Set, \times, \{*\})$,	the category of sets with the cross product.
$(Set, \sqcup, \emptyset)$,	the category of sets with the disjoint union.
$(\mathrm{Vect}_{\Bbbk}, \otimes, \Bbbk)$,	the category of vector spaces with the tensor product over $\Bbbk$.
$(\mathrm{Top}, \times, *)$,	the category of topological spaces with the cross product.
$(\mathrm{Ab}, \otimes, \mathbb{Z})$,	the category of abelian groups with the usual tensor product over $\mathbb{Z}$.
$(\mathrm{nCob}, \sqcup, \emptyset)$,	the category of n-cobordisms whit the disjoint union.

8.0.2 Monoidal Natural Transformation

Definition 8.0.6. Let $F, F' : \mathcal{C} \longrightarrow \mathcal{D}$ monoidal functors. A natural transformation $\sigma : F \Longrightarrow F'$ between monoidal functors is called a *monoidal natural transformation* if the diagrams

$$\begin{tikzcd} F(x) \otimes_{\mathcal{D}} F(y) \arrow[r, "\xi_{x,y}"] \arrow[d, "\sigma_x \otimes_{\mathcal{D}} \sigma_y"'] & F(x \otimes_{\mathcal{C}} y) \arrow[d, "\sigma_{x \otimes_{\mathcal{C}} y}"] \\ F'(x) \otimes_{\mathcal{D}} F'(y) \arrow[r, "\xi'_{x,y}"'] & F'(x \otimes_{\mathcal{C}} y) \end{tikzcd}$$

$$\begin{tikzcd} u \arrow[r, "\xi_0"] \arrow[dr, "\xi_0'"'] & F(u) \arrow[d, "\sigma_u"] \\ & F'(u) \end{tikzcd}$$

commute.

Let $\mathcal{C}$ and $\mathcal{D}$ monoidal categories. A monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called a *monoidal equivalence* if there exists a monoidal functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ and monoidal natural isomorphisms $\varphi : G \circ F \cong 1_{\mathcal{C}}$ and $\psi : F \circ G \cong 1_{\mathcal{D}}$.

8.0.3 Braided Monoidal Categories

A *braided monoidal category* consists of a monoidal category $\mathcal{C}$ together with a *braiding*, which is defined by a family of isomorphims

$$\varsigma_{x,y} : x \otimes y \longrightarrow y \otimes x.$$

They are natural for x and y in $\mathscr{M}$, and satisfy for the unit u the commutative diagram

$$\begin{tikzcd} x \otimes u \arrow[rr, "\varsigma_{x,u}"] \arrow[dr, "\rho_x"'] & & u \otimes x \arrow[dl, "\lambda_x"] \\ & x, & \end{tikzcd}$$

Moreover the maps $\varsigma_{x,y}$, with the associativity α, make commutative the following hexagonal diagrams:

$$\begin{tikzcd} & (x \otimes y) \otimes z \arrow[r, "\varsigma"] \arrow[dl, "\alpha^{-1}"'] & z \otimes (x \otimes y) \arrow[dr, "\alpha"] & \\ x \otimes (y \otimes z) \arrow[dr, "1\otimes\varsigma"'] & & & (z \otimes x) \otimes y \arrow[dl, "\varsigma\otimes 1"] \\ & x \otimes (z \otimes y) \arrow[r, "\alpha"'] & (x \otimes z) \otimes y, & \end{tikzcd}$$

$$\begin{tikzcd} & x \otimes (y \otimes z) \arrow[r, "\varsigma"] \arrow[dl, "\alpha"'] & (y \otimes z) \otimes x \arrow[dr, "\alpha^{-1}"] & \\ (x \otimes y) \otimes z \arrow[dr, "\varsigma\otimes 1"'] & & & y \otimes (z \otimes x) \arrow[dl, "1\otimes\varsigma"] \\ & (y \otimes x) \otimes z \arrow[r, "\alpha^{-1}"'] & y \otimes (x \otimes z). & \end{tikzcd}$$

8.0.4 Symmetric Monoidal Categories

A *symmetric monoidal category* is a monoidal category with a braiding, which satisfies the identity

$$\varsigma_{y,x} \circ \varsigma_{x,y} = 1.$$

8.0.5 Symmetric monoidal functor

Let F be a monoidal functor between two symmetric categories $(\mathcal{C}, \otimes, u, \varsigma)$ to $(\mathcal{C}', \otimes', u', \varsigma')$. The functor F is a *symmetric monoidal functor* if $F(\varsigma) = \varsigma'$.

8.0.6 Symmetric monoidal natural transformation

A *symmetric monoidal natural transformation* is a monoidal natural transformation between symmetric monoidal functors.

Chapter 9

Appendix 2
Some technical lemmas

This chapter is dedicated to describe some technical lemmas that we use throughout the thesis. In what follows we will suppose that all the manifolds are almost complex manifolds. We use this only to avoid carrying signs in the calculations.

9.1 Cohomological results

Let Y, Z be closed submanifolds of X which intersect *cleanly*, that is, $W = Y \cap Z$ is a submanifold of X and at each point x of W the tangent space of W at x is the intersection of the tangent spaces of Y and Z. Let F be the *excess* bundle of the intersection, i.e., the vector bundle over W which is the quotient of the tangent bundle of X by the sum of the tangent bundles of Y and Z restricted to W. Thus $F = 0$ if and only if Y and Z intersect transversally. If the relevant inclusion maps are denoted

$$\begin{tikzcd} W \arrow[r, "j'"] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ Y \arrow[r, "j"'] & X \end{tikzcd}$$

then F fits into an exact sequence

$$0 \longrightarrow \nu_{i'} \longrightarrow j^{'*}\nu_i \longrightarrow F \longrightarrow 0.$$

We call this square *Quillen's square* [Qui71].

Proposition 9.1.1. *If $z \in \mathrm{H}^*(Z)$, then*

$$j^* i_*(z) = i'_*(e(F) \cdot j^{'*}(z))$$

in $\mathrm{H}^{*+a}(Y)$, where a is the real dimension of ν_i.

Proposition 9.1.2. *For the diagram*

$$\begin{tikzcd}
 & & Z \arrow[dl, "\varepsilon_i"'] \arrow[dr, "\xi_i"] \arrow[ddll, bend right, "\varphi"'] \arrow[ddrr, bend left, "\psi"] & & \\
 & B_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\beta_i"] & & D_i \arrow[dl, "\gamma_i"'] \arrow[dr, "\delta_i"] & \\
A & & C_i & & E
\end{tikzcd}$$

classes $e_{B_i} \in H^*(B_i)$ *and* $e_{D_i} \in H^*(D_i)$, *let be* $\varphi = \alpha_i\varepsilon_i$, $\psi = \delta_i\xi_i$ *and*

$$e_{B_i,D_i} = \xi_i^*(e_{D_i})e(F_i)\varepsilon_i^*(e_{B_i})$$

where $e(F_i)$ *is the excess intersection class of the Quillen square. If* $e_{B_{i_1},D_{i_1}} = e_{B_{i_2},D_{i_2}}$ *then for* $z \in H^*(A)$ *we have the identity*

$$\delta_{i_1}!\left(e_{D_{i_1}}\gamma_{i_1}^*\left(\beta_{i_1}!\left(e_{B_{i_1}}\alpha_{i_1}^*(z)\right)\right)\right) = \delta_{i_2}!\left(e_{D_{i_2}}\gamma_{i_2}^*\left(\beta_{i_2}!\left(e_{B_{i_2}}\alpha_{i_2}^*(z)\right)\right)\right)$$

Proof. We use the projection formula $f!(x)y = f!(xf^*(y))$. Then the Quillen formula is

$$\begin{aligned}
\delta_i!\left(e_{D_i}\gamma_i^*\left(\beta_i!\left(e_{B_i}\alpha_i^*(z)\right)\right)\right) &= \delta_i!\left(e_{D_i}\xi_i!\left(e(F_i)\varepsilon_i^*\left(e_{B_i}\alpha_i^*(z)\right)\right)\right) \\
&= \delta_i!\left(e_{D_i}\xi_i!\left(e(F_i)\varepsilon_i^*(e_{B_i})\varphi^*(z)\right)\right) \\
&= \delta_i!\xi_i!\left(\xi_i^*(e_{D_i}e(F_i)\varepsilon_i(e_{B_i})\varphi^*(z)\right)) \\
&= \psi!\left(\xi_i^*(e_{D_i})e(F_i)\varepsilon_i(e_{B_i})\varphi^*(z)\right)
\end{aligned}$$

♣

9.2 Homological results

In this section we describe the analogue result of the last section but in homology in al little more general way.

Lemma 9.2.1. *Let* $i : Z \hookrightarrow X$ *an inclusion of manifolds with* $k = \dim X - \dim Z$*.Then, for* $z \in \mathrm{H}_*(Z)$

$$i!i_*(z) = e(\nu_i) \cap z,$$

where ν_i *is the normal bundle of the inclusion* i.

Proof.

$$\begin{tikzcd}
Z \arrow[r, hook, "i"] \arrow[rr, bend right, "s"'] & X \arrow[r, "\tau"] & \frac{X}{X-\eta_i} \arrow[r, "\pi"] & Z
\end{tikzcd}$$

In homology is

$$
\begin{tikzcd}
i!: \mathrm{H}_*(Z) \arrow[r, "i_*"] \arrow[rrr, bend left, "s_*"] & \mathrm{H}_*(X) \arrow[r, "\tau_*"] & \mathrm{H}_*(X, X-\eta_i) \arrow[r, "e"] & \mathrm{H}_*(E(\nu_i), E(\nu_i)_0) \arrow[r, "\cap \mathrm{Th}"] \arrow[rr, bend left, "\phi"] & \mathrm{H}_{*-k}(E(\nu_i)) \arrow[r, "\pi_*"] & \mathrm{H}_{*-k}(Z)
\end{tikzcd}
$$

$$z \longmapsto i_*(z) \longmapsto s_*(z) \longmapsto \pi_*(\mathrm{Th} \cap s_*(z))$$

Note that we can give another expression for $\pi_*(\mathrm{Th} \cap s_*(z))$, that is $\pi_*(\mathrm{Th} \cap s_*(z)) = \pi_* s_*(s^*(\mathrm{Th}) \cap z) = (\pi \circ s)_*(e(\nu_i) \cap z) = (\mathrm{Id})_*(e(\nu_i) \cap z) = e(\nu_i) \cap z$, where $e(\nu_i) = s^*(\mathrm{Th})$ because in cohomology the umkehr map is

$$i_* : \mathrm{H}^*(Z) \xrightarrow{\Phi} \mathrm{H}^{*+k}(\nu_i, \nu_0) \xrightarrow{\tau^*} \mathrm{H}^{*+k}(X)$$

$$\alpha \longmapsto \mathrm{Th} \cup \alpha \longmapsto \tau^*(\mathrm{Th} \cup \alpha)$$

Then $i^*(\tau^*\Phi(1)) = i^* i_*(1) = e(\nu_i)$, by Quillen's result. In the other hand, $i^*(\tau^*\Phi(1)) = (\tau \circ i)^*(\Phi(1)) = s^*(\Phi(1)) = s^*(\mathrm{Th})$.
Finally we obtain that $i! i_*(z) = e(\nu_i) \cap z$, for $z \in \mathrm{H}_*(Z)$.

♣

Proposition 9.2.2. *Let Y, Z be closed submanifolds of X which intersect cleanly and $W = Y \cap Z$ is a submanifold of X such that at each point of W the tangent space of W at x is the intersection of the tangent spaces of Y and Z.*

$$
\begin{tikzcd}
W \arrow[r, "j'"] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\
Y \arrow[r, "j"'] & X
\end{tikzcd}
\tag{9.1}
$$

and $z \in \mathrm{H}_*(Z)$, then

$$j! i_*(z) = i'_*(e(F) \cap j'!(z))$$

where

$$0 \longrightarrow \nu_{i'} \longrightarrow j'^* \nu_i \longrightarrow F \longrightarrow 0$$

is an exact sequence.

Proof. We can replace X by a tubular neighborhood of W. Thus we may suppose that 9.1 is of the form

$$
\begin{tikzcd}
W \arrow[r, "j'"] \arrow[d, "i'"'] & E_1 \arrow[d, "i"] \\
E_2 \arrow[r, "j"'] & E_1 \oplus E_2 \oplus F
\end{tikzcd}
$$

where E_1 is a complex vector bundle over W with zero section j', E_2 is a complex vector bundle with zero section i', and i and j are the obvious inclusions. Let $i_\epsilon : E_\epsilon \to E_1 \oplus E_2$, $\epsilon = 1, 2$ and $k : E_1 \oplus E_2 \to E_1 \oplus E_2 \oplus F$ be the inclusion map. Hence

$$\begin{array}{rclcll} j!i_*(z) & = & i_2!k!k_*i_{1*}(z) & = & i_2!(e(\nu_k) \cap i_{1*}(z)) & \text{by the lemma 9.2.1} \\ & = & i_2!i_{1*}(i_1^*e(\nu_k) \cap z) & = & i'_*j'!(i_1^*e(\nu_k) \cap z) & \text{by affirmation 1} \\ & = & i'_*j'!(\pi^*(e(F)) \cap z) & & & \text{by affirmation 2} \\ & = & i'_*(e(F) \cap j'!(z)) & & & \text{by affirmation 3} \end{array}$$

- *Affirmation 1:* We consider the next commutative diagram

$$\begin{tikzcd} W \arrow[r, "i'"] \arrow[d, "j'"'] & E_2 \arrow[d, "i_2"'] \arrow[dr, "i"] & \\ E_1 \arrow[r, "i_1"'] \arrow[rr, bend right, "j"'] & E_1 \oplus E_2 \arrow[r, "k"'] & E_1 \oplus E_2 \oplus F \end{tikzcd}$$

Then $i_2!i_{1*} = i'_*j'!$. To prove this we check that the next diagrams commute in homology.

$$\begin{tikzcd} E_1 \arrow[r, "\tau_1"] \arrow[d, "i_1"'] & \frac{E_1}{E_1 - \eta_{j'}} \arrow[rr, "\pi_1"] \arrow[d, "l"] & & W \arrow[d, "i'"] \\ E - 1 \oplus E_2 \arrow[r, "\tau_2"'] & \frac{E_1 \oplus E_2}{E_1 \oplus E_2 - \eta_{i_2}} \cong \frac{E_1}{E_1 - \eta_{j'}} \oplus E_2 \arrow[rr, "\pi_2"'] & & E_2 \end{tikzcd}$$

The first commutes by definition of the maps, and the second commutes by the following:
Let $x \in \mathrm{H}_*\left(\frac{E_1}{E_1 - \eta_{j'}}\right)$, then $\pi_{2*}(\mathrm{Th}_2 \cap l_*(x)) = \pi_{2*}l_*(l^*(\mathrm{Th}_2) \cap x) = i'_*\pi_{1*}(l^*(\mathrm{Th}_2) \cap x) = i'_*\pi_{1*}(\mathrm{Th}_1 \cap x)$.
Finally, if $x \in \mathrm{H}_*(E_1)$: $\pi_{2*}(\mathrm{Th}_2 \cap \tau_{2*}i_{1*}(x)) = \pi_{2*}(\mathrm{Th}_2 \cap l_*\tau_{1*}(x))$
$= \pi_{2*}l_*(l^*(\mathrm{Th}_2) \cap \tau_{1*}(x)) = i'_*\pi_{1*}(\mathrm{Th}_1 \cap \tau_{1*}(x))$. Then, $i_2!i_*(x) = i'_*j'!(x)$.

- *Affirmation 2:* The bundles $i_1^*(\nu_k)$ and $\pi^*(F)$ coincide, in particular $i_1^*(e(\nu_k)) = \pi^*(e(F))$.
To prove this, we consider the pullback square

$$\begin{tikzcd} \pi^*(F) \arrow[r] \arrow[d] & F \arrow[d, "\pi_F"] \\ E_1 \arrow[r, "\pi"'] & W \end{tikzcd}$$

where $\pi^*(F) = \{(x,z) \in E_1 \times F : \pi(x) = \pi_F(z)\} = E_1 \oplus F$ bundle over E_1. Hence is enough to prove $i_1^*(\nu_k) = E_1 \oplus F$. First we note that the next diagram commute

$$\begin{tikzcd} E_1 \oplus F \arrow[r, "j"] \arrow[d, "\pi_1"'] & \nu_k \arrow[d, "\pi_k"] \\ E_1 \arrow[r, "i_1"'] & E_1 \oplus E_2 \end{tikzcd}$$

where $k : E_1 \oplus E_2 \to E_1 \oplus E_2 \oplus F$, $\pi_1 : E_1 \oplus F \to E_1$ is the projection and $j : E_1 \oplus F \to \nu_k$ is given by $j(x,y) = (x,0,y) \in \nu_k$.
This square commute by $i_1 \circ \pi_1(x,y) = i_1(x) = (x,0)$ and $\pi_k \circ j(x,y) = \pi_k(x,0,y) = (x,0)$.
To finish we need to check that $E_1 \oplus F$ is the pullback square of the maps

$$\begin{tikzcd} & \nu_k \arrow[d, "\pi_k"] \\ E_1 \arrow[r, "i_1"'] & E_1 \oplus E_2 \end{tikzcd}$$

Let Z be a manifold such that

$$\begin{tikzcd} Z \arrow[drr, "g"] \arrow[ddr, "f"'] & & \\ & E_1 \oplus F \arrow[r, "j"] \arrow[d, "\pi_1"] & \nu_k \arrow[d, "\pi_k"] \\ & E_1 \arrow[r, "i_1"'] & E_1 \oplus E_2 \end{tikzcd}$$

$\pi_k \circ g = i_1 \circ f$. We define $h : Z \to E_1 \oplus F$ by $h(z) = (f(z), \pi_3 \circ g(z))$.
Note that $\pi_k \circ g(z) = (f(z), 0)$ since $\pi_k \circ g = i_1 \circ f$. Then $j \circ h(z) = j(f(z), \pi_3 \circ g(z)) = (f(z), 0, \pi_3 \circ g(z)) = (\pi_k(g(z)z), \pi_3(g(z))) = g(z)$, and $\pi_1(h(z)) = \pi_1(f(z, \pi_3(g(z)))) = f(z)$.

- *Affirmation 3:* If $\varphi \in \mathrm{H}^*(W)$ and $z \in \mathrm{H}_*(E_1)$ is $j'!(\pi^*(\varphi) \cap z) = \varphi \cap j'!(z)$. This is an immediately consequence of the definition of the umkehr map, that is: $j' : W \to E_1$ and $\begin{tikzcd} E_1 \arrow[r, "\tau"] \arrow[rr, bend right, "\pi"'] & \frac{E_1}{E_1 - \eta_{j'}} \arrow[r, "p"] & W \end{tikzcd}$.

  Then

$$\begin{aligned} \varphi \cap j'!(z) &= \varphi \cap p_*(\mathrm{Th} \cap \tau_*(z)) &&= p_*(p^*(\varphi) \cap \mathrm{Th} \cap \tau_*(z)) \\ &= p_*(\mathrm{Th} \cap \tau_*(\tau^* p^*(\varphi) \cap z)) &&= p_*(\mathrm{Th} \cap \tau_*(\pi^*(\varphi) \cap z)) \\ &= j'!(\pi^*(\varphi) \cap z) \end{aligned}$$

♣

Proposition 9.2.3. *For the diagram*

$$\begin{tikzcd}
 & & Z \arrow[dl, "\varepsilon_i"'] \arrow[dr, "\xi_i"] \arrow[ddll, bend right, "\varphi_i"'] \arrow[ddrr, bend left, "\psi_i"] & & \\
 & B_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\beta_i"] & & D_i \arrow[dl, "\gamma_i"'] \arrow[dr, "\delta_i"] & \\
A & & C_i & & E
\end{tikzcd}$$

let be $\varphi_i = \alpha_i\varepsilon_i$, $\psi_i = \delta_i\xi_i$ such that $(\varphi_1)! = (\varphi_2)!$, $(\psi_1)_* = (\psi_2)_*$ and $e(F_1) = e(F_2)$ where $e(F_i)$ is the excess intersection class of the Quillen square. Then for $z \in \mathrm{H}_*(A)$ we have the identity

$$\delta_{1*}(\gamma_1!(\beta_{1*}(\alpha_1!(z)))) = \delta_{2*}(\gamma_2!(\beta_{2*}(\alpha_2!(z))))$$

Proof. We use the Quillen's formula, then

$$\begin{aligned}
\delta_{1*}(\gamma_1!(\beta_{1*}(\alpha_1!(z)))) &= \delta_{1*}(\xi_{1*}(e(F_1) \cap \varepsilon_1!(\alpha_1!(z)))) \\
&= \delta_{1*}\xi_{1*}((e(F_1) \cap \varepsilon_1!(\alpha_1!(z)))) \\
&= (\psi_1)_*(e(F_1) \cap \varphi_1!(z)) \\
&= (\psi_2)_*(e(F_2) \cap \varphi_2!(z)) \\
&= \delta_{2*}(\gamma_2!(\beta_{2*}(\alpha_2!(z))))
\end{aligned}$$

♣

Corollary 9.2.4. *We consider the next diagrams, $i = 0, 1$.*

$$\begin{tikzcd}
 & & Z_i \arrow[dl, "\varepsilon_i"'] \arrow[dr, "\xi_i"] \arrow[ddll, bend right, "\varphi_i"'] \arrow[ddrr, bend left, "\psi_i"] & & \\
 & B_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\beta_i"] & & D_i \arrow[dl, "\gamma_i"'] \arrow[dr, "\delta_i"] & \\
A & & C_i & & E
\end{tikzcd}$$

where the squares are Quillen's squares, i.e. the intersection of B_i and D_i is clean and the spaces Z_1 and Z_2 are homotopically equivalents,

$$\begin{tikzcd}
Z_1 \arrow[r, shift left, "f_1"] & Z_2 \arrow[l, shift left, "f_2"]
\end{tikzcd}$$

such that

$$\begin{tikzcd}
& Z_2 \arrow[ddl, bend right, "\varphi_2"'] \arrow[d, shift left, "f_2"] \arrow[ddr, bend left, "\psi_2"] & \\
& Z_1 \arrow[u, shift left, "f_1"] \arrow[dl, "\varphi_1"'] \arrow[dr, "\psi_1"] & \\
A & & E
\end{tikzcd}$$

commutes up to homotopy. Then, if $f_2^*(e(\nu_{f_1} \oplus F_1)) = e(F_2)$*, for* $z \in \mathrm{H}_*(A)$ *we have*

$$\delta_{1*} \circ \gamma_1! \circ \beta_{1*} \circ \alpha_1!(z) = \delta_{2*} \circ \gamma_2! \circ \beta_{2*} \circ \alpha_2!(z).$$

Proof.

$$\begin{aligned}
\delta_{1*} \circ \gamma_1! \circ \beta_{1*} \circ \alpha_1!(z) &= \delta_{1*}\xi_{1*}\left(e(F_1) \cap \varepsilon_1!(\alpha_1!(z))\right) && \text{property of Quillen} \\
&= \psi_{1*}\left(e(F_1) \cap \varphi_1!(z)\right) && \text{by } \delta_1\xi_1 \simeq \psi_1, \alpha_1 \circ \varepsilon_1 \simeq \varphi_1 \\
&= \psi_{2*}f_{1*}\left(e(F_1) \cap f_1!\varphi_2!(z)\right) && \text{by } \psi_1 \simeq \psi_2 \circ f_1, \varphi_1 \simeq \varphi_2 \circ f_1 \\
&= \psi_{2*}f_{1*}\left(f_1^* f_2^* e(F_1) \cap f_1!\varphi_2!(z)\right) && \text{by } f_2 \circ f_1 \simeq 1 \\
&= \psi_{2*}\left(f_2^*(e(F_1)) \cap f_{1*}f_1!(\varphi_2!(z))\right) && \text{by the projection formula}
\end{aligned}$$

Now we need to understand the map $f_* \circ f! : \mathrm{H}_*(B) \to \mathrm{H}_*(A) \to \mathrm{H}_*(B)$, where $f : A \to B$. First we consider the next Quillen's diagram.

$$\begin{tikzcd}
& A \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\
A \arrow[dr, "f"'] & & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

For the Quillen's property we have $f!f_*(z) = e(\nu_f) \cap z$, where $z \in \mathrm{H}_*(A)$ and ν_f is the normal bundle of the map $f : A \to B$.

$$\begin{aligned}
& f_1!f_{1*}f_{2*}(z) = f_1!(z) && \text{by } f_{1*}f_{2*} = \mathrm{Id}, \\
& e(\nu_{f_1}) \cap f_{2*}(z) = f_1!(z) && \text{using that } f_1!f_{1*}(z) = z \cap e(\nu_{f_1}), \\
& f_{1*}(e(\nu_{f_1}) \cap f_{2*}(z)) = f_{1*}f_1!(z) && \text{composition with } f_{1*}, \\
& f_{1*}(f_1^* f_2^*(e(\nu_{f_1})) \cap f_{2*}(z)) = f_{1*}f_1!(z) && \text{using that } f_1^* f_2^* = \mathrm{Id}, \\
& f_2^*(e(\nu_{f_1})) \cap f_{1*}f_{2*}(z) = f_{1*}f_1!(z) && \text{by the projection formula}, \\
& f_2^*(e(\nu_{f_1})) \cap z = f_{1*}f_1!(z) && \text{using that } f_{1*}f_{2*} = \mathrm{Id}\,.
\end{aligned}$$

Then $f_{1*}f_1!(z) = f_2^*(e(\nu_{f_1})) \cap z$, for all $z \in \mathrm{H}_*(B)$.

Finally, returning to the calculations, we have

$$\begin{aligned}
\psi_{2*}\left(f_2^*(e(F_1)) \cap f_{1*}f_1!(\varphi_2!(z))\right) &= \psi_{2*}\left(f_2^*(e(F_1)) \cap f_2^*(e(\nu_{f_1})) \cap \varphi_2!(z)\right) \\
&= \psi_{2*}\left((f_2^*(e(F_1)) \cup f_2^*(e(\nu_{f_1}))) \cap \varphi_2!(z)\right) \\
&= \psi_{2*}\left(f_2^*(e(F_1) \cup e(\nu_{f_1})) \cap \varphi_2!(z)\right) \\
&= \psi_{2*}\left(f_2^*(e(\nu_{f_1} \oplus F_1)) \cap \varphi_2!(z)\right)
\end{aligned}$$

Since $f_2^*(e(\nu_{f_1} \oplus F_1)) = e(F_2)$ then

$$\psi_{2*}\left(\varphi_2!(z) \cap f_2^*(e(\nu_{f_1}\, \mathrm{F}_1))\right) = \psi_{2*}\left(\varphi_2!(z) \cap e(F_2)\right) = \delta_{2*} \circ \gamma_2! \circ \beta_{2*} \circ \alpha_2!(z).$$

♣

In particular we have the next result.

Corollary 9.2.5. *In the hypothesis of the last corollary, if Z_1 and Z_2 are diffeomorphic spaces, where $f_1 : Z_1 \to Z_2$ is the diffeomorphism between them, then the identity $e(F_1) = e(F_2)$, implies*

$$\delta_{1*} \circ \gamma_1! \circ \beta_{1*} \circ \alpha_1!(z) = \delta_{2*} \circ \gamma_2! \circ \beta_{2*} \circ \alpha_2!(z).$$

Proof. This is because if f_1 is a diffeomorphism then $\nu_{f_1} = 0$.

♣

Theorem 9.2.6. *Let $f, g : A \to X$ be cofibration maps, and $H : A \times \mathrm{I} \to X$ an homotopy between them, i.e $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for $x \in A$. Then*

$$f! = g! : \mathrm{H}_*(X) \to \mathrm{H}_*(A)$$

Proof. Remember that the umkehr map $f!$ is defined as the next steps

Step 1: We consider the projection map

$$\tau_f : X \to \frac{X}{X - \eta_f(A)}$$

where η_f is the tubular neighborhood of f.

Step 2: We use the exponential function $(E(\varepsilon), E_0(\varepsilon)) \to (\eta_f, \eta_f - A) \subset (X, X - A)$ and by excision we have the next isomorphisms

$$\mathrm{H}_*(X, X - A) \cong \mathrm{H}_*(E(\varepsilon), E_0(\varepsilon)) \cong \mathrm{H}_*(E, E_0)$$

then

$$\mathrm{H}_*(X/(X - A)) \cong \mathrm{H}_*(E/E_0) \xrightarrow{Thom} \mathrm{H}_{*-k}(A)$$

Finally, the next diagram gives the umkehr map

$$\begin{tikzcd} \mathrm{H}_*(X) \arrow[r, "(\tau_f)_*"] \arrow[rrr, bend right, "f_!"'] & \mathrm{H}_*(X/(X-A)) \arrow[r, "\cong"] & \mathrm{H}_*(E/E_0) \arrow[r, "Thom"] & \mathrm{H}_{*-k}(A) \end{tikzcd}$$

Note that $(X, f(A))$ and $(X, g(A))$ are good pairs, i.e. $f(A) \hookrightarrow X$ and $g(A) \hookrightarrow X$ are cofibrations. Then the homotopy

$$H' : f(A) \times \mathrm{I} \to X$$

given by $H'(f(x), t) = H(x, t)$ extends to X such that

$$H'|_{\eta_f \times \{1\}} = \eta_g$$

and

$$H'|_{A\times\{0\}} = f, \quad H'|_{A\times\{1\}} = g$$

Set by $f' := H'(-, 0)$ and $g' := H'(-, 1)$.

Let $\alpha \in \mathrm{H}_*(X)$ with $\alpha = f'_{\#}(\beta) + \gamma$, where $\beta \in C_{\#}(\eta_f)$, $\gamma \in C_{\#}(X - A)$ and $f_{\#}$ is the map induced in the chain complexes. This is posible by the using of the barycentric subdivision $C_{\#}(\eta_f + (X - A)) \xrightarrow{\cong} C_{\#}(X)$. Since we have the Quillen diagram

$$\begin{tikzcd} & \eta_f \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\ \eta_f \arrow[dr, "f'"'] & & \eta_f \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

then $f'_! f'_*(\beta) = \beta\, \mathrm{e}(\nu_{f'})$.
Finally

$$f'_!(\alpha) = f'_!(f'_*(\beta) + \gamma) = \beta\, \mathrm{e}(\nu_{f'}) + f'_!(\gamma)$$

where we note that $f'_!(\gamma) = 0$ because $\gamma \in C_{\#}(X - A)$. Then $f'_!(\alpha) = \beta\, \mathrm{e}(\nu_{f'})$.
In other hand, using the homotopy $H' : X \times \mathrm{I} \to X$ we can find a new representant of α in $C_{\#}(\eta_g + (X - g(A)))$ of the form $g'_{\#}(\beta') + \gamma'$ with $\beta' \in C_{\#}(\eta_g)$ and $\gamma' \in C_{\#}(X - g(A))$. Then

$$g'_!(g'_*(\beta') + \gamma') = g'_! g'_*(\beta') + g'_!(\gamma') = g'_! g'_*(\beta') = \beta'\, \mathrm{e}(\nu_{g'}) = \beta\, \mathrm{e}(\nu_{f'}).$$

Therefore $f'_!(\alpha) = g'_!(\alpha)$, and in particular $f_!(\alpha) = g_!(\alpha)$.

♣

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