Evolutionary game theory and applications

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Elements of noncoperative game theory

Our analysis is restricted to finite games in normal form

- A set of index $I = \{1, ..., n\}$ represents the set of players.
- For each $i \in I$, $S_i = \{1, ..., m_i\}$ her finite set of pure strategies.
- A vector of pure strategies $s = (s_1, ..., s_n),\ s_i \in S_i$, is a pure strategy profile.
- The set of the pure strategy profiles is $S = \times_i S_i$.
- The payoff function $\pi_i : S \to R$ is defined by the list $\{\pi_i(s) :\ s \in S\}\,,\ \forall\, i \in I$.
- For any strategy profile $s \in S, \pi_i(s) \in R$ is the associate payoff to player i.
- The payoff profile $\pi : S \to R^n$ is given by the vectorial field $\pi(s) = (\pi_1(s), ..., \pi_n(s))$.

The normal form game is summarized a triplet $G = (I, S, \pi)$. In the special case of when there are only two players:

$$S_i = \{s_{i1}, ..., s_{ik_i}\}, \; i = 1, 2$$

One may write the payoffs π_1 and π_2 in a tabular form as two $m_1 \times m_2$ matrices $A = (a_{hk})$ and $B = (b_{hk})$)

Where $\pi_1(s_{1h}, s_{2k}) = a_{hk}$ and $\pi_2(s_{1h}, s_{2k}) = b_{hk}$.

- Each row in both matrices corresponds to a pure strategy for player 1, and each column to a pure strategy of player 2.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k_2} \\ \vdots & \vdots & \vdots \\ a_{k_1 1} & \cdots & a_{k_1 k_2} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1k_2} \\ \vdots & \vdots & \vdots \\ b_{k_1 1} & \cdots & b_{k_1 k_2} \end{pmatrix}. \tag{1}$$

Example 1 *The Prisoner's Dilemma Game*

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}. \tag{2}$$

- *Strategies:* $S_i = \{don't\ confess,\ \ confess\},\ i = 1, 2.$
- *Payoffs: In years of prison.*
- *Rows the strategies of 1, Columns those of 2, payoffs for each at the intersection of the respective matrices To confess is better for both players but....*

Alternatively:

	C_1	C_2
F_1	$2, 2$	$5, 1$
F_2	$1, 5$	$4, 4$

Where $S_1 = \{F_1, F_2\}$ and $S_2 = \{C_1, C_2\}$

$$a_{hk} = \pi_1(F_h, C_k),\ h, k \in \{1, 2\}$$

$$b_{hk} = \pi_2(F_h, C_k),\ h, k \in \{1, 2\}$$

$$s^* = (F_2, C_2) = (\text{confess, confess}).$$

$$\pi(s^*) = \pi_1(F_2, C_2) = \pi_2(F_2, C_2) = (4, 4).$$

Mixed strategies

A mixed strategy for player $i \in I$ is a probability distribution over the set of pure strategies, $S_i = (s_{i1}, ..., s_{in_i})$. It can be represented by a vector

$$x_i = (x_{i1}, ..., x_{in_i}) \in \Delta_i,$$

where

$$\Delta_i = \left\{ x_i \in \Re^{n_i} : 0 \leq x_{ih} \leq 1 \ \forall \ h = 1, ..., n_i, \text{ and } \sum_{i=1}^{h} x_{ih} = 1 \right\}.$$

The set $C(x_i) = \{h \in S_i : x_{ih} > 0\}$ is called the support of x_i.

A strategic profile ia a vector of mixed strategies one for each player $x = (x_1, ..., x_n) \in \Delta = \times_{i=1}^{n} \Delta_i$

Notation The vertices or corner of this simplex are

$$e_i^1 = (1, 0, ...0), ..., e_j = (0, ..., 1, ...0), ..., e_i^n = (0, 0, ..., 1).$$

Hence, a pure strategy is a concentred distribution of probabilities.

- Every mixed strategy $x_i \in \Delta_i$ is a convex combination of this unit vectors:
$$x_i = \sum_{h=1}^{n_i} x_{ih} e_i^h.$$
- A mixed strategies profile is a vector $x = (x_1, ..., x_n)$ where $x_i \in \Delta_i$. The mixed strategies space is $\Delta = \times_{i \in I} \Delta_i$.

Notation

- Let $y = (y_1, ..y_{i-1}, y_i, y_{i+1}, .., y_n) \in \Re^n$. We represent by $y_{-i} = (y_1, ..y_{i-1}, y_{i+1}, .., y_n) \in \Re^{n-1}$ the vector of $\Re^{n-1}$ whose coordinates, with the exception of the $i - th$ that was eliminated, are equal to those of y.
- Let $y = (y_1, ..., y_i, ..., y_n) \in \Delta$ be a mixed strategies profile. We write (x_i, y_{-i}) for the strategy profile in which player $i \in I$ plays strategy $x_i \in \Delta_i$ and all others according to the profile $y \in \Delta$.
- We say that a strategy x_i is completely mixed if $C(x_i) = S_i$.

In normal form games players make their choices independiently of each other.

- Let $I = \{1, 2, ..., n\}$ be the set of players. and let $S_i = (s_{i1}, ..., s_{in_i})$ the set of pure strategies, and suppose that players are playing $x = (x_1, ..., x_n) \in \Delta$, then
- the probability that $s = (s_{1k_1}, ...s_{nk_n}) \in S = \times_{i=1}^{n} S_i$ and $k_j \in \{1, ..., n_j\}, j = 1, ..., n$ occurs if $x = (x_1, ...x_n) \in \Delta = \times_{i=1}^{n} \Delta_i$ is played is given by $x : S \to \Re$ given by

$$x(s) = x_1(s_{1k_1})x_2(s_{2k_2})....x_n(s_{nk_n}) = \Pi_{i=1}^{n} x_i(s_{ik_i}).$$

- *Where* $x(s) = \Pi_{i=1}^{n} x_i(s_{k_i})$ *is the product of probabilities assigned by each player's mixed strategy* $x_i \in \Delta_i$ *to play his pure strategy* $s_i \in S$.

Payoff functions

The payoff for the player $i \in I$ is a function $u_i : \Delta \to \Re$ deifined by *the expected value*

$$u_i(x) = \sum_{s \in S} x(s)\pi_i(s)$$

Where $s \in S$ and $x(s) = \Pi_{i=1}^{n} x_i(s_{k_i})$

The payoff $u_i(x)$ is a linear function of each player mixed strategy: to see this suppose that player j is playing his pure strategy $s_j = k$ this is equivalent to play the mixed strategy e_j^k and the payoff that player i obtain when j uses her $k-th$ pure strategy is: $u_i(e_j^k, x_{-j})$. Hence for any $x = (x_i, x_{-i}) \in \Delta_i \times \Delta_{-i}$

$$u_i(x_i, x_{-i}) = \sum_{k=1}^{n_i} x_{ik} u_i(e_i^k, x_{-i}).$$

The combined function $u : \Delta \to R^n$ defined by

$$u(x) = (u_1(x), ..., u_n(x))$$

is the combined payoff function of the game.

The mixed extension of a game $G = ((I, S, \pi)$ is given by $\Gamma = (I, \Delta, u)$.

Consider a two players game with the payoff matrices A and B. For a mixed strategy $x_1 \in \Delta_1$ and $x_2 \in \Delta_2$ we have:

$$u_1(x_1, x_2) = \sum_{h=1}^{m_1} \sum_{k=1}^{m_2} x_{1h} a_{hk} x_{2k} = x_1 A x_2$$

and (3)

$$u_2(x_1, x_2) = \sum_{h=1}^{m_1} \sum_{k=1}^{m_2} x_{1h} b_{hk} x_{2k} = x_1 B x_2 = x_2 B^T x_1.$$

Example 2 *Consider de Prisoner's dilemma again.*

- *The combined mixed strategy payoff;* $u : \Delta \to R^2$
- *The mixed strategies are* $x_1 = (x_{11}, x_{12}) \in \Delta_1; x_2 = (x_{21}, x_{22}) \in \Delta_2$.
- *The profile of pure strategies* $(c, c) = (e_1^2, e_2^2)$
- $u_1(x) = x_1 A x_2 = (2x_{21} + 5x_{22})x_{11} + (1x_{21} + 4x_{22})x_{12}$
- $u_2(x) = x_1 B x_2 = (2x_{11} + 5x_{12})x_{21} + (1x_{11} + 4x_{12})x_{22}$.
- *Note that* $u_1(e_1^1, x_2) = (2x_{21} + 5x_{22}),\ u_1(e_1^2, x_2) = (1x_{21} + 4x_{22})$
- *and that* $u_2(x_1, e_2^1) = (2x_{11} + 5x_{12}),\ u_2(x_1, e_2^2) = (1x_{11} + 4x_{12})$.
- *So,* $u_1(x_1, x_2) = u_1(e_1^1, x_2)x_{11} + u_1(e_2^2, x_2)x_{12}$
- *and* $u_2(x_1, x_2) = u_2(x_1, e_2^1)x_{21} + u_2(x_1, e_2^2)x_{22}$

Dominance relations

Definition 1 *We say that the strategy* $y_i \in \Delta_i$ *weakly dominates* $x_i \in \Delta_i$ *if* $u_i(y_i, z_{-i}) \geq u_i(x_i, z_{-i})$ *for all* $z_{-i} \in \Delta_{-i}$ with strict inequality for some $z_{-i} \in \Delta_{-i}$.
A strategy x_i *is undominated if no such strategy* y_i *exists.*

Definition 2 *We say that the strategy* $y_i \in \Delta_1$ *strictly dominates* $x_i \in \Delta_i$ *if* $u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$ *for all* $z_{-i} \in \Delta_{-i}$.

Definition 3 *A strategy* $x_i \in \Delta_i$ *is dominated if for all* $z \in \Delta$ *there exists* $y_i \in \Delta_i$ *such that* $u_i(y_i, z_{-i}) \geq u_i(x_i, z_{-i})$*, and there exists* $w_i \in \Delta_i : u_i(w_i, \alpha_{-i}) > u_i(x_i, \alpha_{-i})$ *for some* $\alpha_{-i} \in \Delta_{-i}$.

Definition 4 *A strategy* $x_i \in \Delta_i$ *is strictly dominated if for all* $z \in \Delta$ *there exists* $y_i \in \Delta_i$ *such that* $u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$.

A strategy x is weakly dominated if some other strategy exists that weakly dominates x.

The following example illustrates the possibility that a pure strategy is strictly dominated by a mixed strategy without being dominated by any pure strategy:

Example 3 *Consider the two-players game with payoff matrix:*

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \\ 1 & 1 \end{pmatrix}$$

Player I has three pure strategies.

- Her third pure strategy e_1^3 is not weakly dominated by any of the other two pure stratey.
- However, she always obtain a higher payoff by randomizing uniformly over othe pure strategies.
- Formally, let $y_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ we have $1 = u_1(e_1^3, z_2) < u_1(y_1, z_2) = \frac{3}{2}$ for all $z_2 \in \Delta_2$ So y_1 strictly dominates $x - 1$.

A rational player does not play strictly dominated strategy.

- Pure strategies strictly dominated can be deleted from a game without affecting the outcome.
- The process of to delete strictly dominate pure strategies can be iteratively repeated.

Best replies or best responses

Definition 5 *A pure best reply for player i to a strategy profile $y \in \Delta$ is a pure strategy $s_i \in S_i$ such that there is no other pure strategy available to the player given her a higher payoff against y*

$$\beta_i(y) = \left\{ h \in S_i : u_i(e_i^h, y_{-i}) \geq u_i(e_i^k, y_{-i}) \; \forall k \in S_i \right\}. \qquad (4)$$

Then β_i is a correspondence: $\beta_i : \Delta \to S_i$.

Since every mixed strategy is a convex combination of pure strategy it follows that for each $x_i = (x_{i1}, ..., x_{in_i}) \in \Delta_i$:

$$u_i(e_i^h, y_{-i}) \geq \sum_{k=1}^{n_i} u_i(e_i^k, y_{-i}) x_{ik} = u_i(x_i, y_{-i}).$$

Hence,

$$\beta_i(y) = \left\{ h \in S_i : u_i(e_i^h, y_{-i}) \geq u_i(x_i, y_{-i}) \; \forall x_i \in \Delta_i \right\}. \qquad (5)$$

Definition 6 *A mixed best reply for a player i against $y \in \Theta$ is a mixed strategy $x_i \in \Delta_i$ such that there is no other mixed strategy available to the player given her a higher payoff against y*

$$\bar{\beta}_i(y) = \{ x_i \in \Delta_i : u_i(x_i, y_{-i}) \geq u_i(z_i, y_{-i}) \; \forall z_i \in \Delta_i \}. \qquad (6)$$

Hence, $\bar{\beta}_i$ is a correspondence $\bar{\beta}_i : \Delta \to \Delta_i$.

- The subset $\bar{\beta}_i(y)$ *is always a non empty, closed and convex set of* Δ_i.
- Note that if x_i is a best reply for i against y then

$$u_i(e_i^h, y_{-i}) = u_i(e_i^k, y_{-i}) \; \forall h, k \in C(x_i).$$

The combined pure best reply is a correspondence $\beta : \Delta \to S$ defined by $\beta(y) = \times_{i=1}^n \beta_i(y)$.

The combined mixed best reply is a correspondence $\bar{\beta} : \Delta \to \Delta$ defined by $\bar{\beta}(y) = \times_{i=1}^n \bar{\beta}_i(y)$.

Proposition 1 *A pure strategy of a player in a finite strategic game is never a best response if and only if, it is strictly dominate.*

Proof: Osborne-Rubinstein.

Nash equilibrium

In terms o best reply: a strategy profile $x \in \Delta$ is a Nash equilibrium if it is a best reply against to itself, or equivalently if it is a fixed point of the correspondence $\bar{\beta}$:

Definition 7 *$x \in \Delta$ is Nash equilibrium if $x \in \bar{\beta}(x)$.*

It follows than every pure strategy s_i in the support of each component x_i of a Nash equilibrium, is a best reply against x : i.e. *if x is a Nash equilibrium and $s_i \in C(x_i)$ then $s_i \in \beta_i(x)$.*

Definition 8 *A Nash equilibrium $x \in \Delta$ is called strict if $\bar{\beta}(x) = \{x\}$.*

While a Nash equilibrium requires that no unilateral deviation should be profitable, strict NE, requires that all such deviations be costly.

In the prisoner's dilemma, $(to\ confess, to\ confess)$ is a NE, this can be represented by the mixed strategy: $((0,1),(0,1))$

Example 4 Matching pennies

	H	T
H	$+1,-1$	$-1,+1$
T	$-1,+1$	$+1,-1$

This game has not a NE in pure strategies.

- How to find a NE in mixed strategies?
- We solve the equations system:

$$\pi_1(F_1, \sigma_2^*) = \pi_1(F_2, \sigma_2^*)$$
$$\pi_2(\sigma_1^*, C_1) = \pi_2(\sigma_1^*, C_2)$$

From the first equation we obtain σ_2^* and from the first one σ_1^*.
In this case

$$\pi_1(F_1, \sigma_2^*) = 1\sigma_{21}^* - 1\sigma_{22}^* \text{ and } \pi_1(F_2, \sigma_2^*) = -1\sigma_{21}^* + 1\sigma_{22}^*.$$

$$\pi_1(F_1, \sigma_2^*) = \pi_1(F_2, \sigma_2^*)$$

it follows, $\sigma_2^* = (\frac{1}{2}, \frac{1}{2})$. Analogously for player $2 : \sigma_1^* = (\frac{1}{2}, \frac{1}{2})$

This means that each player uses in equilibrium a mixed strategy σ_i if and only if the expected value

$$E_i[s_h/\sigma_{-i}] = ... = E_i[s_k/\sigma_{-i}]$$

for each pure strategy $s_h, ..., s_k \in C(\sigma_i)$.

Note that:

- A Nash equilibrium is never a strictly dominated strategy, however it can be weakly dominated.

- Every strict Nash equilibrium is a pure strategy profile.
  Since if a NE involve a randomization, then there exist at least to strategy with the same maximal payoff.
  For instance: Let $x^* = (x_1^*, x_2^*)$ be a NE, where
  $x_1^* = (x_{11}^*, x_{12}^*) >> 0$ then

  $\{(s_1, x_2^*), (s_2, x_2^*), (x_1^*, x_2^*)\} \in B_1(x^*) \times B_2(x^*) = B(x^*)$.

- *The existence of Nash equilibrium was first established by Nash in 1950.*

Theorem 1 (Nash's theorem) *For any finite game the set of Nash equilibria* $\Delta^{NE} \neq \emptyset$.

Proof: The set Δ is nonempty, convex and compact, and so the subset $\beta(y) \in \Delta$.

By standard arguments (continuity and concavity of u_i) it can be verified that β is convex and upper hemi-continuous correspondence.

By Kakutani's fixed point theorem, there exist some $y \in \beta y$.

Refinements of Nash Equilbrium

Since 1970 there has been a flurry of refinements of the Nash Equilibrium concept. Each refinement being motivated by the desire to get rid of certain implausible or fragile Nash equilibrium.

The most well-known noncooperative refinement is that of *trembling hand perfection*

Let the game $G = \{I, S, \mu\}$ where μ is an *error function* that to each player i an pure strategy $h \in S_i$ assign a number $\mu_{ih} \in (0, 1)$ the probability that the strategy will played *by mistake* where $\sum : h\mu_{ih} < 1$. Such funtion μ defines for each player $i \in I$ the subset

$$\Delta_i = \{x \in \Delta_i : x_{ih} \geq \mu_i h\} \subset int(\Delta_i)$$

of mixed strategies that the player can implement, given the error probabilities.

We define the perturbed game

$$G(\mu) = \{I, \Delta(\mu), \mu\}\,.$$

By standard arguments we can prove that every perturbed game $G(\mu)$ has a nonempty set $\Delta^{NE}(\mu)$ of Nash equilibria.

When $\mu \to 0$ $G(\mu) \to G$ the original game.

Definition 9 *$x \in \Delta^{NE}$ perfect if, for some sequence $\{G(\mu^t)\}_{\mu^t \to 0}$ of perturbed games there exist profiles $x^t \in \Delta(\mu^t)$ such that $x^t \to x$.*

In particular, every interior Nash equilbrium is perfect.

Proposition 2 *For any finite game* $\Delta^{PE} \neq \emptyset$.

Proof: For any sequence $\{G(\mu^t)\}_{\mu^t \to 0}$ let $x^t \in \Delta^{NE}(\mu^t)$ for each t. Since x^t is a sequence in Δ a compact set it has a convergente subsequece $\{y^s\}_s$ with limit in $x^* \in \Delta$ For each $s, G(\mu^s)$ is accompanying perturbed game. By standar continuity arguments $x^* \in \Delta^{NE}$ and is perfect, because $y^s \to x^*$.

Symmetric two players games

The subclass of symmetric two-players games, provides the basic setting for much of the evolutionary game theory.

Definition 10 *A game $G = (I, S, \pi)$ is a symmetric two players game if: $I = \{1, 2\}$, $S_1 = S_2$, $\pi_1(s_1, s_2) = \pi_2(s_2, s_1)$ for all $(s_1, s_2) \in S$.*

- This is equivalent with de requirement $B = A^T$
- It follows that $\Delta_1 = \Delta_2 = \Delta$
- NOTATION; $\Delta \times \Delta = \Theta$.

  Example The prisoner dilemma is a symmetric two players game.

- If player 1 plays x and player 2 plays y player 1 gets the same result as player 2 when player 2 is the one who plays x and player 1 plays y. That is, $u_1(x, y) = u_2(y, x)$ because
- since $A = B^T \Leftrightarrow a_{ij} = b_{ji}$ for al $x \in \Delta_1$ and $y \in \Delta_2$ we have that

$$u_1(x, y) = xAy = xB^Ty = yBx = u_2(y, x)$$

- Let $(x, y) \in \Theta$ then we have that

  $x \in B_1(y)$ if and only if $x \in B_2(y)$.

  Because,

  $u_1(x, y) \geq u_1(z, y) \; \forall z \; \in \Delta_1 \Leftrightarrow u_2(y, x) \geq u_2(y, z) \; \forall z \; \in \Delta_2.$

Symmetric Nash Equilibrium

In the context of symmetric games we introduce the notation $\Theta = \Delta \times \Delta$.

- A pair $(x, y) \in \Theta$ is a NE, if and only if $x \in \beta(y)$ and $y \in \beta(x)$
- A NE (x, y) is symmetric if and only if $x = y$

We symbolize the set of symmetric NE, by

$$\Theta^{NE} = \{x \in \Delta : (x, x) \text{ is a NE}\}$$

Theorem 2 *For any symmetric two-player game:* $\Theta^{NE} \neq \emptyset$.

	H	D
H	$(v-c)/2,\ (v-c)/2$	$v,\ 0$
D	$0,\ v$	$v/2,\ v/2$

Nash equilibria:

- If $v < c$ i.e: the cost of fight exceeds the value of a victory, there are two Nash equilibrium in pure strategies

$$NE_1 = (H, D) = (e_1^1, e_2^2), \quad NE_2 = (D, H) = (e_1^2, e_2^1)$$

  and the $SNE = (x, x)$ where $x = (v/c, 1 - v/c)$

- If $v > c$ the game has a unique Nash equilibrium

$$NE = (H, H) = (e_1^1, e_2^1).$$

A classification of symmetric 2×2 games

Each symmetric 2×2 game is represented by the matrix of payoff:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The following linear transformations in the payoff of a game, do not change the characteristic of the Nash equilibria set.

Substracting a_{21} from column 1 and a_{12} from column 2, we obtain the equivalent (from the point of view of the NE) matrix

$$A' = \begin{pmatrix} a_{11} - a_{21} & 0 \\ 0 & a_{22} - a_{12} \end{pmatrix}.$$

This new matrix is a symmetric matrix, a it will be represented by:

$$A' = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Where $a_1 = a_{11} - a_{21}$ and $a_2 = a_{22} - a_{12}$.

It follows that any symmetric 2×2 game can, after normalization identified with a point $a = (a_1, a_2) \in \Re^2$ in the plain.

From the point of view of the characteristics of the Nash equilibrium, there exists three different categories depending on the signals of a_1 and a_2.

- Category I $a_1 > 0; a_2 < 0$ *or* $a_1 < 0; a_2 > 0$. There is only one Nash equilibria, in pure strategy. The prototype is the prisoner dilemma's game.

- Category II $a_1 > 0, a_2 > 0$. There are two Nash equilibrium in pure strategies (e_1^1, e_2^1) and (e_1^2, e_2^2) and the mixed symmetric equilibrium (x, x) where $x = (\frac{a_2}{a_1+a_2}, \frac{a_1}{a_1+a_2})$. The prototype is the coordination game.

- Category III $a_1 < 0, a_2 < 0$. There are two Nash equilibrium in pure strategies (e_1^1, e_2^2) and (e_1^2, e_2^1) and the mixed symmetric equilibrium (x, x) where $x = (\frac{a_2}{a_1+a_2}, \frac{a_1}{a_1+a_2})$. The prototype is the howk-dove game.

Evolutionary game theory

So far we have considered two players (populations) games in the framework of classical game theory.

The solution for this type of game (the Nash equilibrium) was based in:

- Each player is rational in the sense that:
- uses a best response to the strategy chosen by the other so,
- neither would change what they are doing.

Now, we give an alternative interpretation for the Nash equilibria (x^*, x^*), by placing the game in a population context.

Evolutionary game theory considers a population of decision makers

An strategy $x = (x_1, .., x_k)$ represents:

- the percentage of individuals adopting one of the possible k strategies
- or the percentage of times that the typical individual uses each possible strategy.

In biology: this behaviour is genetically programmed, and the payoff is identified with the expected number of offspring.

In economics: individuals that play a game many times can consciously switch strategies (by learning or imitation).

Definition 11

- *Consider an infinite population of individuals that can use some set of pure strategies S. A mixed strategy is a vector x that gives the probability $x(s)$ with which strategy $s \in S$ is played in the population.*
- *A strategic profile is a vector of distributions of probabilities over the set of available behavior or pure strategies, one for each population involved in the game.*

Example 5 *Consider a population of individuals that can use two set of pure strategies S_1 and S_2.*

- A mixed strategy $x_i = (x_{i1}, ..., x_{ik_i})$ followed by the population $i = 1, 2$ corresponds to a probability distribution over the set of possible behaviors (pure strategies) for the individuals of the $i - th$ population. Equivalently:
  - The probability of finding in the population i an individual following the behavior or the pure strategy s_{ij} (i.e,: a $j-$strategist) is equal to x_{ij}.
  - The typical individual in the population i follows the mixed strategy x_i.
- A strategic profile is a set of distributions one for each population. $x = (x_1, x_2)$.

From a modelling view point we distinguish between two types of population games:

- Games against the field (or nature). These games have the following characteristics:
  - A population of identical individuals must choose their behavior in a gives set of strategies to face a no specific *opponent*, the nature or the field.
  - The returns that each individual in the population receives, after playing the game, depends on their own strategic choice. (or behavior)
  - Generally, the behavior of the player (individually) has no influence on the field Then the field plays the same game against all participants. This also guarantees that all the individuals are playing against the field separately.

- Pairwise contest game. Describes a situation in which two individuals are chosen randomly from two different antagonistic populations whose members choose their behavior in respective sets of well-defined strategies.
- The payoff of each individual in the pair chosen depends on the strategy followed by each one.

We will start considering a pairwise contest game.

A pairwise contest game

Let G a two-player normal form game.

- Let $S = (s_1, ..., s_{n_1})$ an $S' = (s'_1, ..., s'_{n_2})$ denote a finite set of pure strategies for a game G
- Let ΔS and $\Delta S'$ denote the set of probability distributions on S and S' respectively.
- Payoffs are specified by the function $\pi : \Delta S \times \Delta S' \to R^2$.

Definition 12 *A pairwise contest game, describes a situation in which, a given individual $i \in P_1$ playing a mixed strategy $\sigma_i \in \Delta S$ against an opponent $j \in P_2$ that is randomly select (by Nature) according with the probability $\sigma'_j \in \Delta S'$. The payoff depends just on what both individuals do.*

$$u_i(\sigma, \sigma') = \sum_{s \in S} \sum_{s' \in S'} \sigma(s)\sigma'(s')u_i(s, s'); \;\; i \in \{1, 2\}.$$

The probability of obtaining $(u_i(s, s'))$ is $p(s, s') = \sigma(s)\sigma'(s')$.

So the payoffs $u(\sigma, \sigma') = (u_1(\sigma, \sigma'), u_2(\sigma, \sigma'))$ associate with the profile of mixed strategies (σ, σ') are the expected values over the set of possible matches.

Two different ways of thinking about this model:

1. $\sigma = (\sigma_1, ..., \sigma_{n_i})$, $i = 1, 2$ is a mixed strategy followed by a typical individual from the population, i.e; a distribution of probabilities over the set of available behavior for individuals of the $i - th$ population. Or equivalentley

2. $\sigma = (\sigma_1, ..., \sigma_{n_i})$, $i = 1, 2$ is a probability distribution of the $i - th$ population over the set of possible disease behaviors, that is, the percentage of individuals of each type who follow a certain disease behavior. And therefore the probability of randomly choosing a s_j strategist in the population.

Evolutionarily games theory in a classical form

The most common setting in which to discuss evolutionary games is in the framework of
a two-player (one-population) symmetric game:

- Let $S = (s_1, ..., s_n)$ denotes a finite set of pure strategies for a game G or possible behaviors of individuals in the population.
- Two players are playing G, each one must make a choice from S
- Let $\Delta = \left\{\sigma \in \Re^n_+ : \sum_{j=1}^n \sigma_j\right\}$ where σ_j denotes the probability that a player in the population follows the strategy s_j. Or equivalently,the probability to match with an individual following the $j-$th behavior.
- The payoff associated with each $s \in S$ is specified by the function $u : (S \times \Delta S) \to \Re$ defined by $E(s/\sigma \in \Delta S) = u(s, \sigma)$.

Evolutionarily stable strategies in symmetric games is one of the main concepts in evolutionary games theory. It represents a population playing against itself. Payoffs are the numbers of offprints according with the strategy followed by each individual and the strategy followed by the opponent.

Suppose that a small group of mutant appears in a large population of individuals all of whom are programmed to paly the same (mixed or pure) incumbent strategy $x \in \Delta$.

- The mutants all are programmed to play $y \in \Delta$.
- The share of mutants is $\epsilon \in (0, 1)$.
- The distribution of the postentry population is $w = \epsilon y + (1 - \epsilon)x$.
- Pairs of individuals in this bimorphic population are repeatedly drawn at randon to play the game. Every individual is drawn with the same probability.

- The probability that the opponent plays y is ϵ and $(1-\epsilon)$ is the probability that the opponent plays x.
- The payoff of the incumbent strategy is $u(x, \epsilon y + (1-\epsilon)x)$ and the payoff of the mutant strategist is $u(y, \epsilon y + (1-\epsilon)x)$.
- The distribution of the postentry population is $w = \epsilon y + (1-\epsilon)x$.

Biological intuition suggest that evolutionary forces select against the mutant strategy if and only in

$$u(x, \epsilon y + (1-\epsilon)x)) > u(y, \epsilon y + (1-\epsilon)x).$$

A strategy $x \in \Delta$ is said to be evolutionarily stable if the above inequality holds for any mutant strategy $y \neq x$ granted the population share of mutant is sufficiently small (Maynar Smith and Price 1973).

Definition 13 *$x \in \Delta$ is an evolutionary stable strategy (ESS) if for every $y \neq x$ there exists some $\bar{\epsilon}_y$ such that the inequality*

$$u(x, \epsilon_y y + (1-\epsilon_y)x) > u(y, \epsilon_y y + (1-\epsilon_y)x)$$

holds for all $\epsilon_y \in (0, \bar{\epsilon}_y)$.

An alternative framework

Equivalently, in a population made up of individuals who act according to a single predetermined plan (a mixed strategy) a percentage of individuals start acting following a new plan

The new mixed strategy is

$$w_\epsilon = (1-\epsilon)x + \epsilon y.$$

Example 6 *Consider a game with* $S = (s_1, s_2)$ *and* $x = (\frac{1}{2}, \frac{1}{2})$. *Suppose that a mutant strategy is* $y = (\frac{3}{4}, \frac{1}{4})$. *Then:*

$w_\epsilon = (1-\epsilon)x + \epsilon y = (1-\epsilon)(\frac{1}{2}, \frac{1}{2}) + \epsilon(\frac{3}{4}, \frac{1}{4}) = (\frac{1}{2} + \frac{\epsilon}{4}, \frac{1}{2} - \frac{\epsilon}{4})$.

An alternative definition of ESS for a one population game

Definition 14 *A strategy σ^* is evolutionarily stable strategy if the following two conditions are verified:*

$$u(\sigma^*, \sigma^*) \geq u(\sigma, \sigma^*) \;\; \forall\, \sigma \neq \sigma^*$$

$$u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*) \Rightarrow u(\sigma^*, \sigma) > u(\sigma, \sigma).$$

This is the original definition of M. Smith and Price (1973)

This definition has the advantage of making clear that ESS is a refinement of the N.E. concept:

- The first requirement is the condition of N.E. for (σ^*, σ^*)
- The second is a requirement of stability. Ensures that σ^* have a better performance than a mutant behavior, in a population composed by σ^* strategist and a few σ.

We will show that both definitions are equivalent:

Theorem 3 *An strategy x is ESS in a pairwise contest game, if and only if $\forall y \neq x$ either:*

1. $u(x,x) > u(y,x)$ *for all* $y \neq x$, *or*
2. *if there exist some* $y \in \Delta : \; u(x,x) = u(y,x)$ *then* $u(x,y) > u(y,y)$.

Corollary 1 *In a pairwise contest population game*

$$\Delta^{ESS} = \left\{ x \in \Delta^{NE} : u(y,y) < u(x,y) \; \forall y \in \beta(x), y \neq x \right\}.$$

This means that if x is ESS, and $y \neq x$ even when $y \in \beta(x)$ we have that $y \notin \beta(y)$.

Proof of the theorem: Suppose that x is an ESS, then

$$u(x, w_\epsilon) > u(y, w_\epsilon) \text{ where } w_\epsilon = (1-\epsilon)x + \epsilon y). \tag{7}$$

In the pairwise contest game, using the bilinearity of u this condition can rewritten as:

$$(1-\epsilon)u(x,x) + \epsilon u(x,y) > (1-\epsilon)u(y,x) + \epsilon u(y,y),$$

for all $0 \leq \epsilon < \bar{\epsilon}_y$. The inequality follows if and only if:

- $u(x,x) > u(y,x)$ and $\epsilon \leq \bar{\epsilon}_y$ or
- $u(x,x) = u(y,x)$ and $u(x,y) > u(y,y)$.•

An equivalent and useful definition of ESS is as follows: Let $f : [0, 1] \times \Delta \to \Re$ be defined by

$$f(\epsilon, y) = u(x - y, x) + \epsilon u(x - y, y - x).$$

Then $x \in ESS$ *if and only if for any* $y \neq x$ *there exists* $\bar{\epsilon}_y : 0 < f(\epsilon, y)$ *for all* $0 < \epsilon < \bar{\epsilon}_y$.

Uniform Invasion Barriers

Recall that if $x \in ESS$, then for each mutation $y \neq x$ there exist an $\epsilon_y > 0$ such that x resists the "infection" by y if it comes in smaller dose that $\epsilon_y > 0$. That is, if x is an ESS it continues being the strategy that shows the best performance even after the mutation. For example, the population distribution x ensures a greater number of descendants than if it mutates to $z = x(1 - \epsilon_y) + y\epsilon_y$.

- We are interested in knowing if there is a uniform barrier, that is, if there is some $\epsilon > 0$ that is a barrage for any mutation

Definition 15 *$x \in \Delta$ has a uniform invasion barrier if there is $\bar{\epsilon} \in (0,1)$ such that*

$$u(x, \epsilon y + (1-\epsilon)x) > u(y, \epsilon y + (1-\epsilon)x)$$

is verified for all $y \neq x$ and for all $\epsilon \in (0, \bar{\epsilon})$.

So, x has uniform invasion barrier if and only if for alll $y \neq x$

$$f(\epsilon, y) = u(x-y, x) + \epsilon u(x-y, y-x) > 0$$

for a sufficiently small ϵ.

Theorem 4 *x is ESS if and only if x has a uniform invasion barrier.*

The if part follows immediately from the definition of ESS choosing $\bar{\epsilon}_y = \bar{\epsilon}$.

For the only if part: Consider the affine function $f(\cdot, y) : [0,1] \times \Delta \to \Re$ defined by

$$f(\epsilon, y) = u(x - y, x) + \epsilon u(x - y, y - x).$$

So, if $x \in \Delta^{ESS}$ for each $y \neq x \in \Delta$ there is $0 < \epsilon_y$ such that $0 < f(\epsilon, y)$ for all $0 < \epsilon \leq \epsilon_y$.

Consider now the function: $b : Z_x \to [0,1]$ defined by

$$b(y) = \sup\{\delta \in [0,1] : f(\epsilon, y) > 0, \forall\, \epsilon \in (0,\delta)\}.$$

Where

$$Z_x = \{z \in \Delta : z_i = 0, \text{ for some } i \in C(x)\}$$

Note that if $z \in Z_x$ then $\|z - x\| \geq d = \min_{z \in Z_x} d(z,x) > 0$. d is reached because, Z_x is a compact set and $x \notin Z_x$.

- Fix $y \in Z_x$ and consider $f(\cdot, y) : [[0,1] \to \Re$ since $x \in ESS$ then there is at most one $\epsilon : f(\epsilon, y) = 0$, because
  - Since $x \in ESS$ then the vertical intercept of the affine function $u(x-y, x) \geq 0$.
  - if $u(x-y, y-x)$, the slope of the affine function, is
    - positive then, $f(\epsilon, y) > 0$ for all $\epsilon > 0$.
    - negative then, there exists at most only one $0 \leq \epsilon = \epsilon_0$ such that $f(\epsilon_0, y) = 0$.
- If $\epsilon_0 \in (0,1)$ then $u(x-y, y-x) \neq 0$ and $b(y) = \epsilon_0 = u(x-y, x)/u(x-y, x-y)$.
- Otherwise $b(y) = 1$.
- if $u(x-y, x) = 0$ then the slope $u(x-y, y-x) > 0$ and $f(\epsilon, y) > 0$ for all $y \in \Delta$.

- It is straightforward to see that $b(y)$ is a continuous function.
- Since $b(y) > 0$ for all $z \in Z_x$ and Z_x is compact subset of Δ and there is no $\{y_n\}_{n \in N} \in Z_x$ such that $y_n \to x$ then $\min_{y \in Z_x} b(y) > 0$.
- Now, for all $y \in \Delta,\ y \neq x$ there exist some $z \in Z_x$ and $\lambda \in (0, 1]$ such that $y = \lambda z + (1 - \lambda)x$.
- Note that $f(\epsilon, y) =$

$$
\begin{aligned}
&= u(x-y, x) + \epsilon u(x-y, y-x) = u(x-y, x+\epsilon(y-x)) = \\
&= u(x-y, x(1-\epsilon) + \epsilon(\lambda z + (1-\lambda)x) = \\
&= u(x-y, x(1-\lambda\epsilon) + \epsilon\lambda z) = \\
&= \lambda u(x-z, x) + \lambda^2 \epsilon u(x-z, z-x) = \ \lambda f(\lambda\epsilon, z)
\end{aligned}
$$

- It follows that $b(y) = \min\{b(z), 1\} > 0.\bullet$

It is important to note that this characterization does no imply that ESS is necessarily resistent against simultaneous multiple mutations.

Definition 16 *(Local superiority)* $x \in \Delta$ *is locally superior if it has a neighborhood* U *such that* $u(x,y) > u(y,y)$ *for all* $y \neq x$ *in* U.

Proposition 3 $x \in ESS$ *if and only if is locally superior.*

- For the part if Let $U_x \subset R^k$ such that $u(x,y) > u(y,y)$ for all $y \neq x \in U_x \cap \Delta$.
  - For any $z \neq x \in \Delta$ there exists $\bar{\epsilon}_z \in (0,1)$ such that for all $\epsilon \in (0, \bar{\epsilon}_z)$, $w = \epsilon z + (1-\epsilon)x \in U_x$.
  - By hypothesis $u(x,w) > u(w,w)$. Bilinearity of u gives:

$$u(w,w) = \epsilon u(z,w) + (1-\epsilon)u(x,w).$$

  - It follows that $u(x,w) > u(w,w) \leftrightarrow u(x,w) - u(z,w) > 0$. Hence $x \in ESS$.

- Part only if: Since $x \in \Delta^{ESS}$, then for the previous theorem, there is an uniform invasion barrier $0 < \bar{\epsilon}$.
  - For some $z \in Z_x$, and $\epsilon \in (0, \bar{\epsilon}]$ let

$$V = \{y \in \Delta : y = \epsilon z + (1 - \epsilon)x\}.$$

  - Since Z_x is closed and not containing x, there exist $U_x \subseteq R^k$ such that $U \cap \Delta \subset V$.
  - Let $y \neq x \in U \cap \Delta$. Then $y \in V$ is the post entry population and since $x \in ESS$ we have that: $u(x, y) > u(z, y)$.
  - Multiplying both terms by $\epsilon > 0$ and adding $(1 - \epsilon)u(x, y)$ to both terms of the inequality it follows that $u(y, y) < u(x, y)$.•

Structure of the Δ^{ESS}

The concept of ESS is a refinement of the concept of NE. because if $x \in \Delta^{ESS}$ then:

1. $x \in \Delta^{NE}$ (the first order condition) and
2. if $u(y,x) = u(x,x)$ then $u(y,y) < u(x,y)$, (the stability condition).

The following corollary holds:

Corollary 2 *If (x^*, x^*) is a strict NE then (x^*, x^*) is ESS.*

(x^*, x^*) is an strict NE if and only of x^* is the only best response against x^*

Note that the support of one ESS, cannot contain the support of any other symmetric NE.

Suppose that $x \in \Delta^{ESS}$ *and* $C(y) \subset C(x)$ *for some* $y \neq x$

- Then $u(y, x) = u(x, x)$.
- Since $x \in \Delta^{ESS}$ then, condition (2) implies $u(y, y) < u(x, y)$.
- Hence $y \notin \Delta^{NE}$.

The following proposition holds:

Proposition 4 *If* $x \in \Delta^{ESS}$ *and* $C(y) \subset C(x)$ *for some strategy* $y \neq x$ *the* $y \notin \Delta^{NE}$.

In particular if an ESS is interior then is the unique ESS.

Proposition 5 *If* $x \in \Delta$ *is weakly dominated, the* $x \notin \Delta^{ESS}$.

Proof: Suppose that $x \in \Delta^{NE}$ is weakly dominated by $y \in \Delta$.

- This means that $y \in \beta(x)$, then $u(y, x) = u(x, x)$.
- By the weak dominance $u(y, y) \geq u(x, y)$; x fails the second order condition.

Symmetric 2×2 games. After normalization the payoffs matrices of such games have the forms:

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

1. Category I and IV (A prisoner's dilemma variety) a_1 and a_2 are of opposite sign. There is only one N.E., symmetric and strict, so is an ESS:
   - $\Delta^{ESS} = \Delta^{NE} = \{e_2\},\ if\ a_1 < 0$ and
   - $\Delta^{ESS} = \Delta^{NE} = \{e_1\},\ if\ a_2 < 0.$

2. Category II (A coordination game variety) $a_1 > 0, a_2 > 0$. There are three Nash equilibria $\Delta^{NE} = \{e_1, e_2, x\}$ where $x = \lambda e_1 + (1-\lambda)e_2$ and $0 < \lambda = \frac{a_2}{a_1+a_2} < 1$.
   - e_1 and e_2 are strict NE then they are ESS.
   - x is not ESS, because for instance, $u(x,x) = u(e_1,x)$ but $u(e_1,e_1) = a_1 > \lambda a_1 = u(x,e_1)$.

3. Category III (A howk-dove variety) $a_1 < 0, a_2 < 0$.
   - Two strict asymmetric NE. and
   - one symmetric NE $x = \lambda e_1 + (1-\lambda)e_2$ for $\lambda = \frac{a_2}{a_1+a_2}$.
   - Is an ESS, because $u(x,y) = \lambda a_1 y_1 + (1-\lambda)a_2 y_2 = \frac{a_1 a_1}{a_1+a_2} > u(y,y) = a_1 y_1^2 + a_2 y_2^2$.

Unfortunatelly, not all game have an ESS.

Example 7 *Rock-Scissors-Paper The children simultaneously make shape of on of the items withe their hand:*

- *Rock (R) beat Scissors (S).*
- *Scissors (S) beat Paper (P).*
- *Paper (P) beat Scissors (S).*
- *if both players choose the same item, then is a drawn.*

	R	S	P
R	$0, 0$	$1, -1$	$-1, 1$
S	$-1, 1$	$0, 0$	$1, -1$
P	$1, -1$	$-1, 1$	$0, 0$

- This two players game has a unique Nash E. (σ^*, σ^*) with $\sigma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
- But this is not ESS because:
- $u(\sigma^*, \sigma^*) = u(\sigma^*, R) = 0$
- and: $u(\sigma^*, R) = u(R, R)$.

However, we have the following theorem:

Theorem 5 *All generic, ($a \neq c$ and $b \neq d$) symmetric pairwise, contest game have an ESS.*

Proof: By reduction of each game to the respective canonical form.

The Replicator dynamics

We consider a large but finite population of individuals, called replicators, who use a pre-programmed strategy, and passes this behaviour to their descendents in a symmetric two-player game.

- Let $S = (s_1, ..., s_k)$ be a finite set of different strategies.
- Let n_i be the number of individuals using s_i then the total population size is: $N = \sum_{i=1}^{k} n_i$.
- The proportion of individual using s_i is $x_i = \frac{n_i}{N}$.
- The population state is $x = (x_1, ..., x_k)$.
- Let β and δ be backgrounds per capita, birth and death rates. These are independent of the game.
- The rate of change $\beta - \delta$ is modified by the payoff for using s_i, so the rate of change of the number of individuals using s_i is:

$$\dot{n}_i = (\beta - \delta + \pi(s_i, x))n_i.$$

The rate of change of the total population size is:

$$\dot{N} = \sum_{i=1}^{k} \dot{n}_i = \sum_{i=1}^{k} [(\beta - \delta) + \pi(s_i, x)] n_i =$$
$$= (\beta - \delta) N + N \sum_{i=1}^{k} \pi(s_i, x) x_i = (\beta - \delta + \pi(x, x)) N,$$

where $\pi(x, x) = \sum_{i=1}^{k} \pi(s_i, x) x_i$.

- Thus the population grows or decline exponentially. This may be non realistic, but we can improve the model by letting, β and δ depend on N.

- Note that $\pi(s_i, x)$ depends only on the proportions x_i and not in the actual number n_i.

- From a game-theoretic point of view we are more interested in how the proportion of each type change over time.

Now $\dot{n}_i = N\dot{x}_i + x_i\dot{N}$ so:

$$N\dot{x}_i = \dot{n}_i - x_i\dot{N} =$$
$$= (\beta - \delta + \pi(e^i, x))Nx_i - x_i((\beta - \delta + \pi(x, x))N.$$

Canceling and dividing by N, we have:

$$\dot{x}_i = (\pi(s_i, x) - \pi(x, x))x_i. \quad (8)$$

Recall that

- $\pi(e^i, x) = e^i Ax$ is the expected value of an $i - th$ strategist who faces a population that is distributed according to x
- So, $\pi(x, x) = xAx$ is the expected return of a population following the distribution (or mixed strategy) x.
- We have that $\pi(e^i, x) - \pi(x, x) = \pi(e^i - x, x)$

In other words: *The proportion of individuals using strategy s_i increase (decreases) if its payoff is bigger (smaller) than the average payoff of the population.*

Once the initial conditions for the system (8, there is a unique solution for this system, which we will denote by $\xi(t, t_0, x_0)$ for all $t \in \Re_+$ and such that $\xi(t_0, t_0, x_0) = x_0$.

The equation

$$\frac{d}{dt}\left[\frac{x_i}{x_j}\right] = [\pi(e^i, x) - \pi(e^j, x)]\frac{x_i}{x_j} = \pi(e^i - e^j, x)\frac{x_i}{x_j}$$

measures the growth rate of one type of behavior over another over time.

Definition 17 *A fixed point or stationary state, of the replicator dynamics is a state satisfying* $\dot{x}_i = 0\ \forall i$.

A fixed point, describe a population that are no longer evolving. That is a solution of the dynamical system $x(t) = x^* \forall t$.

The following proposition holds:

Proposition 6 *If* x^* *is an interior fixed point, ie.,* $x_i^* > 0\ \forall$ *(all strategy is present in the population) of the replicator dynamics then* $x^* \in \Delta^{NE}$.

Proof: Note that if $x^* >> 0$ then, $\dot{x} = 0$ if and only if $\pi(e^i, x^*) = \pi(x^*, x^*)$. Hence all pure strategy must earn the same payoff therefore (x^*, x^*) is a NE.●

Example 8 *Consider a game with* $S = \{E, F\}$ *and payoffs:*

$$\pi(E,E) = 1, \;\; \pi(E,F) = 1, \;\; \pi(F,E) = 2, \;\; \pi(F,F) = 0.$$

So $\pi(E,x) = x_1 + x_2$; $\pi(F,x) = 2x_i$ *so, it follows*

$$\pi(x,x) = x_1(x_1 + x_2) + x_2(2x_1) = x_1^2 + 3x_1x_2$$

The replicator dynamics is:

$$\dot{x}_1 = x_1[x_1 + x_2 - (x_1^2 + 3x_1x_2)]$$
$$\dot{x}_2 = x_2[2x_1 - (x_1^2 + 3x_1x_2)].$$

So, fixed points are: $(0,1)$; $(1,0)$; $(\frac{1}{2}, \frac{1}{2})$.

Definition 18 *Let* $\xi(\cdot, t_0) : \Re \rightarrow \Re^n$ *be the solution of the differential equations system* $\dot{x}_i = f(x_1, x_2, .., x_n),\ i = 1, ..., n.$ *Then, an invariant set (or manifold) for the differential equations system is a connected subset* $M \subset R^n$ *: if* $\xi(t_0, t_0) = x_0 \in M$ *then* $x(t, t_0) \in M$ *for all* $t \geq t_0$*.*

Claim: The simplex S^n is an invariant set for the replicator dynamics,

$$S^n = \left\{ (x_1, ..., x_n) \in R^n : 0 \leq x_i \leq 1 \ \ \forall i, \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}.$$

Proof of the claim: Let us define $A(t) = \sum_{i=1}^{n} x_i(t)$ and t_0 such that $A(t_0) = \sum_{i=1}^{n} x_i(t_0) = 1$. Then

$$\dot{A}(t) = \overline{\sum_{i=1}^{n} x_i(t)}^{\,\cdot} = \sum_{i=1}^{n} \dot{x}_i(t) = \sum_{i=1}^{n} x_i(\pi(e_i, x) - \pi(x, x)) = 0.$$

(Note that $\pi(x, x) = \sum_{i=1}^{n} x_i \pi(e_i, x)$ and that $\sum_{i=1}^{n} x_i = 1$) and never turn x_i negative, because $x_i = 0 \Rightarrow \dot{x}_i = 0$.

- A population state is formally identical with a mixed strategy and fixed an initial state x_0 de evolution of the population is given by the solution $x(t, t_0)$ of the replicator dynamical system.

Two strategy pairwise contests.

Consider a pairwise contest game that only have two pure strategies: $S = \{s_1, s_2\}$, let $x = (x_1, x_2)$, and $x_2 = 1 - x_1$, then $\dot{x}_2 = -\dot{x}_1$.

So, writing $x = (x_1, 1 - x_1)$ it is enough to consider a single differential equation:

$$\dot{x}_1 = (\pi(e^1, x) - \pi(x, x))x_1 \;\; 0 \leq x_1 \leq 1.$$

Taking into account that $\pi(x,x) = x_1\pi(e^1,x) + (1-x_1)\pi(e_1^2,x)$. from the fact that the simplex is an invariant manifold in the replicator's dynamics, it follows the equivalent differential equation:

$$\dot{x_1} = x_1(1-x_1)(\pi(e^1,x) - \pi(e^2,x)).$$

Theorem 6 *Let* $S = \{s_1, s_2\}$, *and let* (x^*, x^*) *be a symmetric Nash equilibrium, then* $x^* = (x_1^*, x_2^*)$ *is a fixed point of the replicator dynamics,* $\dot{x}_1 = x_1(1 - x_1)(\pi(e^1, x) - \pi(e^2, x))$.

Proof:

1. If $\sigma^* = (x_1^*, (1 - x_1^*))$ is a pure strategy then $x_1^* = 0$ or $x_1^* = 1$, in either case $\dot{x}_1 = 0$.
2. If σ^* is a mixed strategy then $\pi(e^1, \sigma^*) = \pi(e^2, \sigma^*)$ consequently $\dot{x}_1 = 0. \bullet$

Note: *This theorem is also valid for games with a finite set of strategies, because if* $x^* \in \Delta^{NE}$, then for all $i = 1m, , , .n$ we have that $\pi(e^i, x) = \pi(x, x) \; \forall s_i \in C(x^*)$.

In a symmetric game, a strategy $i \in K$ is strictly dominate by $y \in \Delta$ if for all $x \in \Delta$ $u(e_i, x) < u_(y, x)$.

Theorem 7 *Granted that all pure strategies are initially present then, if a pure strategy i is strictly dominated, then $\xi(t, x_0)_{t \to \infty} \to 0$ for any $x^0 \in int(\Delta)$.*

Proof: Suppose that $i \in K$ is srictly dominated by $y \in \Delta$ and let $\epsilon = \min_{x \in \Delta} u(y - e_i, x)$. By continuity of u and compactness of Δ $\epsilon > 0$.

Define the function $v_i : int(\Delta) \to \Re$ by

$$v_i(x) = \log x_i - \sum_{j=1}^{K} y_j \log x_j .$$

The derivative along any interior solution path $\xi(t, t_0 x_0)$

$$\dot{v}(\xi(t, t_0, x_0)) = \frac{\dot{x}_i}{x_i} - \sum_{j=1}^{K} \frac{y_j \dot{x}_j}{x_j} = u(e_i - y, x) \leq -\epsilon < 0.$$

Hence $v(\xi(t, t_0, x_0))$ decreases toward $-\infty$ as $t \to \infty$. By definition of v_i tis implies that $\xi_i(t, t_0, x_0) \to 0$.

This theorem may rephrase as saying that evolution selects against irrational behaviour.

Proposition 7 *If a pure strategy i is iteratively strictly dominate then* $\xi_i(t, t_0, x_0)_{t\to\infty} \to 0$ *for all* $x_0 \in int(\Delta)$.

Definition 19 *A fixed point of a dynamical system, is said to be asymptotically stable if any small deviations from this state are eliminated by the dynamics when* $t \to \infty$.

Example 9 *Prisoner's dilemma The pure strategies are* $\{C, D\}$ *(to cooperate or to defect) the payoffs are:*

$$\pi(e^1, e^1) = \pi(C, C) = 3, \pi(e^1, e^2) = \pi(C, D) = 0,$$
$$\pi(e^2, e^1) = \pi(D, C) = 5, \pi(e^2, e^2) = \pi(D, D) = 1,$$

then

$$\pi(C, x) = 3x_1 + 0(1 - x_1) = 3x_1$$
$$\pi(D, x) = 5x_1 + 1(1 - x_1) = 1 + 4x_1.$$

$\dot{x}_1 = x_1(1 - x_1)(\pi(C, x) - \pi(D, x)) =$

$$= x_1(1 - x_1)(3x_1 - (1 + 4x_1)) = -x_1(1 - x_1)(1 + x_1).$$

The fixed points are $x_1 = 1$ or $x_1 = 0$.

The unique Nash equilibrium is to defect (to play D) i.e., $x_1^* = 0$. From the sign of $\dot{x}_1$ it follows that any population that is not at a fixed point of the dynamics, will evolve to the fixed point that correspond to a Nash equilibrium. Then, it is asymptotically stable.

Symmetric 2×2 games Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the payoff matrix of a symmetric 2×2 game.

Note that the replicator dynamic is invariant under the transformation

$$\bar{A} = A - \begin{pmatrix} c & b \\ c & b \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Consider $a_1 a_2 \neq 0$. So, after this transformation it follows that the replicator dynamics becomes:

$$\dot{x}_1 = [a_1 x_1 - a_2 x_2] x_1 x_2.$$

1. Category I and IV These are the cases where: $a_1 a_2 < 0$.
   - If $a_1 < 0$ and $a_2 > 0$ $x_1 \to 0$ from any initial value x_0.
   - If $a_1 > 0$ and $a_2 < 0$ $x_1 \to 1$ from any initial value x_0.
2. Category II and III. These are the cases where: $a_1 a_2 > 0$. The growth rate change sign when $a_1 x_1 = a_2 x_2$. This occurs when $x_1 = \lambda = \frac{a_2}{a_1 + a_2}$, $0 < \lambda < 1$.
   - If $a_1 > 0$ and $a_2 > 0$. Then

     x_1 decrease toward 0, from any initial value $x_{01} < \lambda$ and

     x_1 increase toward 1 from any initial value $x_{01} > \lambda$.
   - If $a_1 < 0$ and $a_2 < 0$. Then

     x_1 increases $x_1 \to \lambda$, from any initial value $x_{01} < \lambda$ and

     x_1 decreases $x_1 \to \lambda$, from any initial value $x_{01} > \lambda$.

Linearization and asymptotic stability

A fixed point x^* of a dynamical system $\dot{x} = \phi(x)$, is called hyperbolic if the jacobian of the vector field system evaluated at x^* has no eigenvalues with zero real part.

The following theorem justifies the use of the linearization approach, to discovering the properties of fixed point in a dynamical system in most cases.

Theorem 8 (Hartman-Grobman) *If a fixed point x^* is hyperbolic the the topology of the fixed point in the full nonlinear system is the same as the topology of the fixed point in the linearized system.*

$$\dot{x} = \phi'(x^*)(x - x^*).$$

Consider a pairwise contest game with two pure strategies A and B and the following payoffs:

$$\pi(A, A) = 3, \pi(B, B) = 1, \pi(A, B) = \pi(B, A) = 0.$$

- This is a coordination game.
- It have two pure strategy strict Nash equilibria (A, A) and (B, B).
- The mixed strategy $\sigma = (\frac{1}{4}, \frac{3}{4})$ is a NE but it is not an ESS.
- Let x_1 be the proportion of individuals using A and taking account that $x_2 = (1 - x_1)$ the replicator dynamics is given by:

$$\dot{x}_1 = x_1(1 - x_1)(\pi(A, x) - \pi(B, x)) = -x_1(1 - x_1)(1 - 4x_1).$$

- The fixed points are: $x_1^* = 0$, $x_1^* = 1$ and $x_1^* = \frac{1}{4}$.

Consider a population near $x_1^* = 0$. Let $x(0) = x_0 = (x_{10}, x_{20})$ be a perturbed state where $x_{10} = \epsilon_0,\ \epsilon_0 > 0$.

- Then, we have to solve:

$$\dot{x}_1 = -x_1(1 - x_1)(1 - 4x_1)$$

  with the initial condition $x_{10} = \epsilon_0$.

- The derivative of the vectorial field in this case is $\phi_1'(x) = -1 + 10x - 12x^2$.

- Then $x^* = 0 \Rightarrow \phi(x^*) = -1$. So we have to analyze the stability of the equilibrium of the linear equation:

$$\dot{x}_1 = -x_1 \Rightarrow \xi_1(t, x_0) = (x_1^* + \epsilon_0)e^{-t}.$$

- Then $\xi_1(t, x_0) \to 0$ and the fixed point $\xi(x^*, t) = x^* = (0, 1)$ is asymptotically stable

Consider a initial population state near $x_1^* = 1$. Let $x_{10} = 1 - \epsilon_0,\ \epsilon_0 > 0$ be the initial state at $t = 0$.

- Following the linearization procedure we find that:

$$\dot{x}_1 = -3(x_1 - 1) \text{ with the initial condition } x_1(0) = x_{10}.$$

- Which has the solution: $\xi_1(t, x_0) = (1 - \epsilon_0)e^{-3t} + 1$.
- Then $\xi_1(t, x_0) \to 1$ and the solution $\xi(t, x^*) = (1, 0)\ \forall t > 0$ is asymptotically stable.

Finally: Consider a population near $x^* = (\frac{1}{4}, \frac{3}{4})$. Let $x_{10} = \frac{1}{4} + \epsilon_0,\ \epsilon_0 > 0$ be the initial state.

- Then we have: $\dot{x}_1 = \frac{1}{16}(x_1 - \frac{1}{4})$ the solution of the linear approximation is $\xi_1(t, x_{10}) = x_{10}e^{\frac{1}{16}t} + \frac{1}{4} \to \infty$.
- So the solution $\xi(t, x^*) = (\frac{1}{4}, \frac{3}{4})\ \forall t > 0$ is not an asymptotically stable fixed point.

Note that the stability is given by the sign of the derivative in the fixed point x^*.

Theorem 9 *For any two strategy pairwise contest game, a strategy is an ESS if and only if the corresponding fixed point in the replicator dynamics is asymptotically stable.*

Proof: We uses the canonical form for the replicator dynamic:

$$\dot{x}_1 = x_1(1 - x_1)(\pi(a_1, x) - \pi(a_2, x)).$$

- We know that every ESS is asymptotically stable.
- *Reciprocally:* The fixed point for the canonical form are:

  $$x_1^* = 0, x_1^* = 1, x_1^* = \frac{a_2}{a_1 + a_2}$$

  they correspond to the following fixed points for the system:

  $(0, 1), (1, 0)$ and $(\frac{a_2}{a_1 + a_2}, \frac{a_1}{a_1 + a_2}),\ 0 < \frac{a_i}{a_1 + a_2} < 1,\ i = 1, 2.$

The associated vector field is

$$\phi(x_1) = [a_1 x_1 - a_2(1 - x_1)]x_1(1 - x_1).$$

hSo, the linear approximation for the canonical form is

$$\dot{x}_1 = \phi'(x_1^*)(x_1 - x_1^*)$$

where

$$\phi'(x_1) = (a_1 + a_2)x_1(1 - x_1) + [(a_1 + a_2)x_1 - a_2](1 - 2x_1).$$

The corresponding linear approximations are:

1. If $x_1^* = 0 \Rightarrow \dot{x}_1 = -a_2 x_1$.
2. If $x_1^* = 1 \Rightarrow \dot{x}_1 = -a_1(x_1 - 1)$.
3. If $x_1^* = \frac{a_2}{a_1+a_2} \Rightarrow \dot{x}_1 = \frac{a_1 a_2}{a_1+a_2}(x_1 - \frac{a_1}{a_1+a_2})$.

It follows that:

1. $(0,1)$ is asymptotically stable if and only if $a_2 > 0$.

   In this case $(0,1)$ *is an ESS corresponding to category I if* $a_1 < 0$ *and* $a_2 > 0$ *or category III* $a_1 > 0,$ *and* $a_2 > 0$.

2. $(1,0)$ is asymptotically stable if and only if $a_1 > 0$.

   In this case $(1,0)$ *is an ESS corresponding to category I if* $a_1 > 0,$ *and* $a_2 < 0$ *or category III* $a_1 > 0,$ *and* $a_2 > 0$.

3. $(\frac{a_2}{a_1+a_2}, \frac{a_1}{a_1+a_2})$ is asymptotically stable if and only if $a_1 a_2 > 0$ *and* $a_1 + a_2 < 0$.

   (It is easy to see that the case where $a_1 + a_2 > 0$ and $a_1 a_2 > 0$ does not need to be considered).

   In this case $(\frac{a_2}{a_1+a_2}, \frac{a_1}{a_1+a_2})$ *is an ESS corresponding to category II where* $a_1 < 0, a_2 < 0$.•

Conclusions

These conclusions are general for symmetric games, but until now they have symmetrical two strategies games.

- Let $\mathcal{F}$ be the set of fixed points for the replicator dynamics.
- Let $\mathcal{A}$ be the set of asymptotically stable fixed points.
- Let $\mathcal{N}$ be the set of symmetric Nash equilibrium.
- Let $\mathcal{E}$ be the set ESS.

Then for any two strategies, pairwise contest games the following relationship between a strategy σ^* an the corresponding population state x^* hold:

1. $\sigma^* \in \mathcal{E} \Leftrightarrow x^* \in \mathcal{A}$
2. $x^* \in \mathcal{A} \Rightarrow \sigma^* \in \mathcal{N}$, (this relation follows from the first equivalence, because $x^* \in \mathcal{E} \Rightarrow \sigma^* \in \mathcal{N}$.)
3. $\sigma^* \in \mathcal{N} \Rightarrow x^* \in \mathcal{F}$

Games with more than two strategies

If the number of strategies is n we have n differential equations:

$$\dot{x}_i = f_i(x),\ i = 1, ..., n$$

where $x \in \Delta S$.

Given that that $\sum_{i=1}^{n} x_i = 1$ the number of equation is $n - 1$

$$\dot{x}_i = f_i(x_1, ..., x_{n-1}),\ i = 1, ..., (n-1).$$

We assume the hypothesis under which, once fixed the initial distribution $x_0 \in \Delta S$ the system has one and only one solution.

We will now analyze the relationship between the stability of the dynamical equilibria of replicator dynamics, the Nash equilibria, and the evolutionarily stable equilibria in symmetric games with more than two strategies. We start with an example.

Example 10 *Consider a symmetric game characterized by the following payoffs table:*

	A	B	C
A	$0,0$	$3,3$	$1,1$
B	$3,3$	$0,0$	$1,1$
C	$1,1$	$1,1$	$1,1$

The replicator dynamics is:

$$\dot{x}_i = x_i[\pi(e_i, x) - \pi(x, x)] \quad i = 1, 2, 3.$$

$$\dot{x}_1 = x_1(3x_2 + x_3 - \pi(x, x))$$
$$\dot{x}_2 = x_2(3x_1 + x_3 - \pi(x, x))$$
$$\dot{x}_3 = x_3(1 - \pi(x, x))$$

- $\pi(x,x) = 6x_1x_2 + x_1x_3 + x_2x_3 + x_3$.
- Writing $x_1 = x_1, \;\; x_2 = x_2, \;\;$ and $x_3 = 1 - x_1 - x_2$ we reduce this system to the following two variables system:

$$\dot{x}_1 = x_1(1 - x_1 + 2x_2 - \pi(x_1, x_2))$$
$$\dot{x}_2 = x_2(1 + 2x_1 - x_2 - \pi(x_1, x_2))$$

With $\pi(x,x) = \pi(x_1, x_2) = 1 + 4x_1x_2 - x_1^2 - x_2^2$.

- The fixed points (x_1^*, x_2^*) for the system are:

$$(0,0); \;\; (0,1), \;\; (1,0); \;\; (\frac{1}{2}, \frac{1}{2}).$$

These points correspond to $x = (x_1^*, x_2^*, 1 - x_1^* - x_2^*)$ in the simplex.

- We will see that only one of them is Nash equilibrium. Precisely $x^* = (\frac{1}{2}, \frac{1}{2})$ corresponds to the Nash equilibrium $(\frac{1}{2}, \frac{1}{2}, 0)$.

Let us now study the stability of the stationary points of the system considering the behavior of its solutions on the invariant manifolds:

Example 11 *For the system*

$$\dot{x}_1 = x_1(1 - x_1 + 2x_2 - \pi(x_1, x_2))$$
$$\dot{x}_2 = x_2(1 + 2x_1 - x_2 - \pi(x_1, x_2))$$

with $\pi(x_1, x_2) = 1 + 4x_1x_2 - x_1^2 - x_2^2$.

The following manifold are invariant under this dynamic:

- The fixed points, $(0, 0), (\frac{1}{2}, \frac{1}{2}); (1, 0)$ and $(0, 1)$
- The boundaries of the simplex.
- The line $x_1 = x_2$ because $\dot{\overline{x_1 - x_2}} = 0$.
- The line $x_1 + x_2 = 1$ because $\dot{\overline{x_1 + x_2}} = 0$.

- On $y = 0$ we have: $\dot{x} = x^2(x-1)$ so $\dot{x} < 0$ for $0 < x < 1$.
- On $x = 0$ we have $\dot{y} < 0$ for $0 < y < 1$.
- On the line $x = y$ we have $\dot{x} = x^2(1-2x)$ so x and y are both increasing for $0 < x, y < \frac{1}{2}$.
- On the line $x + y - 1 = 0$ we have $\dot{x} = x(3 - 9x + 6x^2)$ hence
  - x is increasing (y is decreasing for $0 < x, y < \frac{1}{2}$,
  - and x is decreasing (y is increasing for $\frac{1}{2} < x, y < 1$.

From these considerations we can conclude that the stationary point $x^* = (1/2, 1/2)$ is asymptotically stable, and as we will see corresponds to a symmetric Nash equilibrium $\sigma^* = ((1/2, 1/2, 0)$.

The following subsets are invariant manifolds for the replicator dynamics:

- The fixed points of the dynamical system.
- The boundaries of the simplex, because $x_i = 0$ imply $\dot{x}_i = 0$.
- The simplex.

Note also that:

The ratio between any two population share $x_i > 0$ and $x_j > 0$ increases (decreases) over time if strategy i earns a higher (lower) payoff than strategy j :

$$\frac{d}{dt}\left[\frac{x_i}{x_j}\right] = [(\pi(e_i, x) - \pi(e_j, x)]\frac{x_i}{x_j}.$$

The stability from the first approximation

The qualitative picture of the dynamical system $\dot{x} = f(x)$

To obtain a qualitative picture of the solutions we consider the Taylor expansion of f in a neighborhood of each fixed points x^* it is to say that, in each point of X such that $f(x^*) = 0$, and we analyze behaviour of the linear approximation, $\dot{x} = Jf(x^*)(x - x^*)$ siendo, $Jf(x^*)$ el jacobiano de f ecaluado en x^*. Equivalently:

$$\dot{x}_i = \sum_{i=1}^{n} (x_j - x_j^*) \frac{\partial f_i}{\partial x_j}(x^*).$$

Defining $\xi_i = x_i - x_i^*$ we have: $\dot{\xi}_i = \sum_{i=1}^{n} \xi_j \frac{\partial f_i}{\partial x_j}(x^*)$.

Provided that the fixed point is not hyperbolic (*i.e.: all eigenvalues have non-zero real part*) then in a neighborhood of each fixed point the behavior of a non-linear system is the same than the linear one. (This is the Hartman-Grobman Theorem)

Combining this information with the behavior of the solutions on the invariant manifold is usually sufficient to determine a complete qualitative picture to the dynamical system.

- The linear system can be written as $\dot{\xi} = L(x^*)\xi$.

  Where the matrix L is the Jacobian Matrix of f whose components are $L_{ij} = \frac{\partial f_i}{\partial x_j}(x^*)$.

- Form the Hartman-Grobman Theorem, we conclude that the eigenvalues of the Jacobian matrix of the vector field of the system evaluated at each hyperbolic fixed point characterizes the stability of said fixed point

Returning to our example:

$$\dot{x} = x(1 - x + 2y - \pi(x, y))$$
$$\dot{y} = y(1 + 2x - y - \pi(x, y))$$

with $\pi(x, x) = 1 + 4xy - x^2 - y^2$.

- At the fixed point $(x^*, y^*) = (\frac{1}{2}, \frac{1}{2})$ we have the following linear approximation:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

  - The eigenvalues are found from $det(L - \lambda I) = 0$
  - In this case $\lambda_1 = -\frac{1}{2}$; $\lambda_2 = -\frac{3}{2}$.
  - Because the real parts are negative, the fixed point is a stable node.

Solving the eigenvector we obtain invariant manifolds for the system:

$$\begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

- For the eigenvalue $\lambda = -\frac{3}{2}$ the eigenvector is given by $\xi = -\eta$ i.e; the subspace generated by $(1, -1)$is an invariant manifold.
- For the eigenvalue $\lambda = -\frac{1}{2}$ the eigenvector is given by $\xi = \eta$ i.e; the subspace generated by $(1, 1)$ is an invariant manifold.

Close to the fixed point $(0,0)$ the linear approximation is:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

which is not hyperbolic $\lambda_1 = \lambda_2 = 0$. The linearization tell us nothing about the stability of this fixed point.

The fixed points $(x^*, y^*) = (1,0)$, and $(x^*, y^*) = (0,1)$,

- both have eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ so both points are instable nodes

Let us now study the behavior of the system on the invariant manifolds:

ESS, NE, Fixed Points and Liapunov Stability in symmetric games with more than two pure strategies

First we show that $\mathcal{NE} \subseteq \mathcal{F}$

Theorem 10 *If* (σ^*, σ^*) *is a NE. then the population state* $x^* = \sigma^*$ *is a fixed point of the replicator.*

Proof: $\pi(e_i, \sigma^*) = \pi(\sigma^*, \sigma^*)$, for all $s_i \in C(\sigma^*)$. Then $\dot{\mathbf{x}} = 0$.

Next we consider: $\mathcal{A} \subseteq \mathcal{N}$:

Theorem 11 *If* x^* *is an asymptotically stable fixed point of the replicator dynamics, then the symmetric strategy* (σ^*, σ^*) *where* $x^* = \sigma^*$ *is a Nash equilibrium.*

Proof: Since $x^* \in A$ then x^* is a fixed point for the replicator dynamic.

- First consider de case where $x_i^* > 0 \; \forall i$.

- Since $x^* = \sigma^* >> 0$ all pure strategy must earn the same payoff, i.e, $\pi(e_i, \sigma^*) = \pi(\sigma^*, \sigma^*) \; \forall \; i \in \{1, ..., n\}$.
- Then (σ^*, σ^*) is a NE.

It remains to consider the case where some strategy is absent.

- Since x^* is a fixed point, then for all $s_i \in C(x^*)$, it follows that $\pi(e_i, \sigma^*) = \pi(\sigma^*, \sigma^*)$.

- Suppose that (σ^*, σ^*) is not a NE. Then there exist $s_j \notin C(\sigma^*)$ such that $\pi(e_j, \sigma^*) > \pi(\sigma^*, \sigma^*)$.
- We have $x_j^* = \sigma_j^* = 0$. This means that there is not actually individuals in the population using the s_j strategy.
- Suppose now that a perturbation affects x^*.
- After this perturbation of x^* a proportion $x_j = \epsilon$ of individuals following the strategy s_j appear.
- The new population state is $x_\epsilon = x^* + \epsilon e_j$.

- Substituting in $\dot{x_j} = x_j[\pi(e_j, x) - \pi(x, x)]$ we obtain:

$$\dot{\epsilon} = \epsilon[\pi(e_j, x_\epsilon) - \pi(x_\epsilon, x_\epsilon)].$$

- Let $F_j(\epsilon) = [\pi(e_j, x_\epsilon) - \pi(x_\epsilon, x_\epsilon)]$. From the Taylor formula it follows that:

$$F_j(\epsilon) = F_j(0) + \epsilon F_j'(0) + \ldots.$$

- Note that $F_j(0) = [\pi(e_j, x^*) - \pi(x^*, x^*)]$. Then

$$\dot{\epsilon} = \epsilon(\pi(e_j, x^*) - \pi(x^*, x^*)) + O(\epsilon^2).$$

- Using the first order approximation we conclude that:
- The proportion x_j of individuals using s_j increases, contradicting the assumption that x^* is asymptotically stable.•

To consider the inclusion $\mathcal{E} \subseteq \mathcal{A}$ we need to introduce the Liapunov function, what we will do later

- However for the case of two strategies the proof follows considering the canonical cases.

More concisely, allowing the abuse of notation that identifies a strategy with a corresponding population state:

In the case of two pure strategies symmetric games we have already show that: $\mathcal{E} = \mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{F}$.

As we shall see for a pairwise contest game with more than two strategies these relations becomes:

$$\mathcal{E} \subseteq \mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{F}.$$

Liapunov stability

The replicator dynamical system is an autonomous, first-order, ordinary differential equation system in the form $\dot{x} = \phi(x)$, where

- $x : T \to R^k$ is a vectorial field where
  $x(t) = (x_1(t), ..., x_k(t)) \in X \subset R^k \ \forall t \in T$
- X is the state space (in the replicator dynamics $X = S^n$).
- T is an open interval in R.
- The function $\phi : X \to R^k$, given by, $\phi(x) = (\phi_1(x), ..., \phi_k(x))$.

  This vectorial field specifies at each point $x \in X$ the direction and velocity of change of the state.
- A solution of this system describe the state of the system, in the future and in the past.

Formally: A local solution trough the point $x_0 \in X$ to a dynamical system is a function

$$\xi(\cdot, x_0) : T \to X,$$

where T is an open interval containing $t = 0$ such that $x(t_0) = \xi(t_0, x_0) = x_0$, and

$$\frac{d}{dt}\xi(t, x_0) = \phi(\xi(t, x_0)) \;\; \forall t \in T.$$

Just as a triplet $\Gamma(I, S, \pi)$ defines a normal form game, a triplet $D = (T, X, \xi)$ defines a dynamic system

- on the state-space $X \subset R^k$
- over a continuous time $t \in T \subset R$,
- with solution mapping ξ.
- The existence and uniqueness of this solution, trough x_0 is guaranteed for a vectorial field sufficient smooth.

Definition 20 *A state* $x \in X$ *is Liapunov stable if every neighborhood* B *of* x *contains a neighborhood,* B^0 *of* x *such that* $\xi(t, x_0) \in B$ *for all* $x_0 \in B^0 \cap X$ *and* $t \geq 0$.

Definition 21 *A state* $x \in X$ *is Liapunov asymptotically stable if it is Liapunov stable and there exists a neighborhood* B^* *such that*

$$\lim_{t \to \infty} \xi(t, x_0) = x, \;\; \forall x_0 \in B^* \cap X$$

Proposition 8 *If a state is Liapunov stable then it is stationary.*

Proof: Suppose that $x \in X$ is a no stationary state. Then there exists $y \neq x$ such that in a finite time t we have: $\xi(t, x) = y$. So, there exist some neighborhood B of $x : y \notin B$, that is the system leaves B in finite time, then x is no a stationary state. •

The Liapunov function

Definition 22 *Let $\dot{x} = f(x)$ be a dynamical system with a fixed x^*, (i.e., $f(x^*) = 0$) Then a real function $V(x)$ defined for allowable states of the system close to x^*, such that:*

1. $V(x^*) = 0$,
2. *x^* is a strict minimum for V,*
3. *$\frac{dV}{dt}(x) \leq 0$ along the solution of the dynamical system and*
4. *there exist an arbitrarily small neighborhood $U_{x^*}(\epsilon)$ of x^* and $\epsilon > 0$ such that $\frac{dV}{dt}(x) < 0$ for all $x \notin U_x^*(\epsilon)$, and $t \geq t_0$*

   is called a (strict) Lyapunov function.

- If such function exists, then the fixed point x^* is asymptotically stable.
- If only conditions 1 and 2 are fulfilled then the fixed point will be stable

Theorem 12 *Every ESS corresponds to an asymptotically stable fixed point in the replicator dynamics. That is, if σ^* is an ESS, then the population with $x^* = \sigma^*$ is asymptotically stable.*

Proof: If σ^* is ESS, then

$$\pi(\sigma^*, x_\epsilon) > \pi(x_\epsilon, x_\epsilon)$$

where $x_\epsilon = (1-\epsilon)\sigma + \epsilon\sigma^*$. We know that $\mathcal{E} \subseteq \mathcal{N} \subseteq \mathcal{F}$ i.e, $\sigma^* = x^*$ is a fixed point of the replicator dynamics.

- Consider the relative entropy function:

$$H_{x^*}(x) = -\sum_{i=1}^{n} x_i^* \ln\left(\frac{x_i}{x_i^*}\right).$$

- Clearly $H_{x^*}(x^*) = 0$

- From the Jensen (strict) inequality (for any (strict) convex function); $E[f(x)] \geq f(E[x])$,
- $H_{x^*}(x) = -\sum_{i=1}^{n} x_i^* \ln\left(\frac{x_i}{x_i^*}\right) \geq$

$$\geq -\ln\left(\sum_{i=1}^{n} x_i^* \frac{x_i}{x_i^*}\right) = -\ln\left(\sum_{i=1}^{n} x_i\right) = 0$$

- $\frac{d}{dt} H_{x^*}(x) = -\sum_{i=1}^{n} \frac{x_i^*}{x_i} \dot{x}_i =$

$$= -\sum_{i=1}^{n} x_i^* (\pi(e_i, x) - \pi(x, x)) = -[\pi(\sigma^*, x) - \pi(x, x)] < 0 \;\forall x \neq x^*.$$

So, $V(x)$ is a Liapunov function for x^*, then the fixed point is asymptotically stable.•

We say that a state $x \in \Delta$ is reachable if there exist some interior state from which the solution trajectory converges to x.

Proposition 9 *If $x_0 \in int(\Delta)$ and $\xi(t, x_0) \to x$ then $x \in \Delta^{NE}$.*

Proof: Suppose that $x_0 \in int(\Delta)$ and $\xi(t, x_0) \to x$ but $x \notin \Delta^{NE}$.

- Then there exists some pure strategy

$$s_i : \pi(e_i, x) - \pi(x, x) = \pi(e_i - x, x) = \epsilon > 0.$$

- Since $\xi(t, x_0) \to x$ and u is continuous, then there exists some T such that for all $t > T$

$$\pi(e_i - \xi(t, x_0), \xi(t, x_0)) > \frac{\epsilon}{2}.$$

- Then $\dot{\xi}_i(t, x_0)) > \frac{\epsilon}{2}\xi_i(t, x_0)\ \forall t > T$.
- Then $\xi_i(t, x_0) > \xi_i(T, x_0) exp(\frac{t-T}{2}\epsilon)\ \ \forall t > T$.
- So, $\xi_i(t, x_0) \to \infty$. This is a contradiction.•

Recall that an interior ESS is necessarily unique. Then one may conjecture that an interior ESS strategy is globally stable.

Proposition 10 *If $x \in int(\Delta) \cap \Delta^{ESS}$ then, $\xi(t, x_0)_{t\to\infty} \to x$, for any $x_0 \in int(\Delta)$.*

Proof: Consider $x \in int(\Delta) \cap \Delta^{ESS}$:

- Then $\pi(x - y, y) > 0$ for all $y \neq x$. (Recall that $x \in ESS \leftrightarrow x \in LS$).
- It is easy to verify that $\dot{H}_x(y) < 0$ for all $y \neq x$.
- The subset $int(\Delta)$ is positively invariant in the replicator dynamics, and $\dot{H}_x(y) < 0$ means that $\xi(t, x_0) \to x$.•

Dominate strategy and the replicator's dynamics

Consider the dynamics system

$$\dot{x}_i = \pi(e_i - x, x)x_i, \; i = 1, ..., n \tag{9}$$

an let $\xi(t, x_0)$ be a solution for this system.

Proposition 11 *If a pure strategy s_i is strictly dominated, then $\xi_i(t, x_0) \to 0$ when $t \to \infty$ for any $x_0 \in int(\Delta)$.*

Proof: Suppose that $s_i \in S$ is a strictly dominated action by $y \in \Delta$. Let

$$\epsilon = \min_{x \in \Delta} \pi(y - e_i, x) > 0.$$

Define the function $v_i : int(\Delta) \to R$ by:

$$v_i(x) = \ln(x_i) - \sum_{j=1}^{n} y_j \ln(x_j).$$

The time derivative along any interior solution of (9) is, at any point $\xi(t, x_0)$,

$$\frac{d}{dt} v_i(\xi(t, x_0)) = \sum_{j=1}^{k} \frac{dv_i}{dx_j} \dot{x}_j = \frac{\dot{x}_i}{x_i} + \sum_{j=1}^{n} y_j \frac{\dot{x}_j}{x_j} =$$

$$= \pi(e_i - x, x) - \pi(y - x, x) =$$

$$= \pi(e_i - y, x) \leq -\epsilon.$$

Then, it follows that $v_i(\xi(t, x_0))_{t \to \infty} \to -\infty$. By definition of v_i, $\xi_i(t, x_0) \to 0$.•

Hence one may rephrase this result saying that:

- The evolution selects against irrational behavior, (in the sense of being a suboptimal behavior.)

Note that the result is valid, only if all pure strategies are initially present. For instance: if some strategy i is strictly dominated, but not other pure strategy is initially present then, $\xi_i(t, x_0) = 1 \; \forall t \geq 0$.

We shall analyze formal modeling of the social evolution of behaviors in a population of strategically interacting agents.

There are two basic elements common to these models:

- The time rate $r_i(x)$ at which at which agents in the population review their their strategy choice.

  This time depend on the performance of the agent's pure strategy and other aspects of the current population state.

- The probability $p_i^j(x)$, that un a reviwing $i-$strategist will switch to some pure strategy j. The vector of this probabilities is written as: $p_i(x) = (p_i^1(x), ..., p_i^k(x))$, and it is a distribution on the set K of pure strategies. So, $p_i(x) \in \Delta$.

  This distribution may depends on the current performance of the strategies and other aspects of the population state.

In a finite population one may imagine that the reviews times of an agent are the arrival time of a Poisson process with arrival time $r_i(x)$, and that at each such time the agents selects a pure strategy according to the probability distribution $p_i(x)$ over the set K.

Recall that a Poisson process is characterized by:

1. The number of changes in non overlapping intervals are independent for all intervals.
2. The probability of exactly one change in a sufficiently small interval h is $p = \nu h$, where $\nu = r_i(x)$ is the probability of one change.
3. The probability of two or more changes in sufficiently small interval h is essentially 0.

Assuming that all agents' Poisson processes are independent

- The aggregate reviewing times in the subpopulatioon i is $x_i r_i(x)$ the aggregate process in the subpopulation of $i-$strategists is itself a Poisson Process, with arrival rate $\lambda_i = x_i r_i(x)$.
- Assuming independence in switches across agents and then the process of switches from a strategy i to strategy j is a Poisson Process with arrival rate: $\Lambda_{ij} = x_i r_i p_i^j$.
- Assuming a continuum of agents and, by the large number we model these aggregate stochastic process as a deterministic flow:
- The outflow from subpopulation i thus is:

$$\sum_{j \neq i} x_i r_i(x) p_i^j(x).$$

- The inflow to this subpopulation is:

$$\sum_{j \neq i} x_j r_j(x) p_j^i(x).$$

- Doing now $\dot{x}_i = inflow\ -\ outflow$ we obtain:

Rearranging terms we obtain:

$$\dot{x}_i = \sum_{j \in K} x_j r_j(x) p_j^i(x) - r_i(x) x_i. \tag{10}$$

To guarantee that this system of differential equations induced a well defined dynamics on the space Δ we assume that

- $r_i : X \to [0, 1]$ and $p_i : X \in \Delta$ are Lipschitz continuous functions.

Then there exists in an open set X containing Δ one and only one solution through any initial state $x_0 \in \Delta$ and such that a solution trajectory never leaves Δ.
The state space Δ is *forward* invariant in this dynamics (10).

We will analyze two models of imitation:

1. Pure imitation driven by dissatisfaction.

2. Imitation of successful agent.

Pure imitation driven by dissatisfaction

This model assume that al reviewing agent adopt the strategy of *the first man they meet in the street*. Formally, for all population states $x \in \Delta$ and pure strategies $i, j \in K$:

$$p_i^j(x) = x_j.$$

Assume that agent following less successful strategies on average review their strategy at a higher rate than agents with more successful strategies, then:

$$r_i(x) = \rho(u(e^i, x), x),$$

for some Lipschitz continuous function ρ strictly decreasing in its first argument.

Under the above two assumptions, the population dynamics become:

$$\dot{x}_i = \left(\sum_{h \in K} x_h \rho(u(e^h, x), x) - \rho(u(e^i, x), x) \right) x_i. \tag{11}$$

As a special case, consider

$$\rho(u(e^i, x), x)) = \alpha - \beta u(e^i, x),$$

for some $\alpha, \beta \in R,\ \beta > 0$ and $\alpha/\beta \geq u(e^i, x)$ for all x and i, then:

$$\dot{x}_i = \beta \left[u(e^i, x) - u(x, x) \right] x_i.$$

This is the replicator dynamics.

So all result for the replicator dynamics are valid for this special case of replication by imitation.

Imitation of successful agent model

Suppose that every reviewing agent samples another agent at random from the population, with equal probability for all agents.

- He observe with some noise the average payoff of her own, and to the sampled agent's strategy.
- That is he observes payoffs: $\bar{u}(e^i, x)$ and $\bar{u}(e^j, x)$
- $\bar{u}(e^i, x)$ and $\bar{u}(e^j, x)$ are random variables.
- The difference $\bar{u}(e^j, x) - \bar{u}(e^i, x)$ is a r.v.
- The reviewing agent switches to the sampled strategy if $\bar{D} = \bar{u}(e^j, x) - \bar{u}(e^i, x) > 0$.

- Suppose that the probability to be $\bar{D} > 0$ depends on the true value of $\bar{u}(e^j, x) - \bar{u}(e^i, x)$.
- So, the conditional probability that the agent switch to strategy j is given by a continuously differentiable function $\phi : R \to [0, 1]$, defined by $\phi(u(e^j, x) - u(e^i, x))$.
- The probability that the agent sample strategy j is x_j.
- Then the resulting conditional distribution of probability is given by:

$$p_i^j(x) = \begin{cases} x_j \phi(u(e^j - e^i, x)) & i \neq j \\ 1 - \sum_{j \neq i} x_j \phi(u(e^j - e^i, x)) & \text{otherwise} \end{cases}$$

Assuming that the reviewing rates are constantly equal to one: $r_i(x) = 1$ we obtain the following selection dynamics:

$$\dot{x}_i = \left[\sum_{j \in K} x_j \left(\phi(u(e^j - e^i, x)) - \phi(u(e^i - e^j, x)) \right) \right] x_i. \qquad (12)$$

Note that if a stationary x^* is an interior point for this dynamic, then is a NE. because if

$$\sum_{j \in K} x_j^* \left(\phi(u(e^i - e^j, x^*)) - \phi(u(e^j - e^i, x^*)) \right) = 0$$

then there exists λ such that $u(e^i, x^*) = \lambda$ for all $i \in C(x^*)$.

This means that for this dynamic the set of interior stationary states coincides wit the set of interior Nash equilibria:

$$int(\Delta^0) = int(\Delta^{NE}).$$

The linearization of this system near an stationary point x^* results in:

$$\dot{x}_i \approx \sum_{j\in K} x_j \phi'(0) \left[u(e^i - e^j, x)) - u(e^j - e^i, x)\right] x_i \Rightarrow$$

$$\dot{x}_i \approx 2\phi'(0) u(e^i - x, x) x_i.$$

Hence, in a neighborhood of an interior stationary state, the vector field of the imitation dynamics, is just a positive constant times the replicator dynamics.

A model of popularity

As a generalization of pure imitation model, we assume that the choice of probabilities $p_i^j(x)$ are proportional to $j's$ popularity x_j.

- We assume that the popularity is proportional to the current payoff of $j-$strategy.
- Let the weight factor that a reviewing $i-$strategist attaches to a pure strategy j be $w_i[u(e^j, x), x] > 0$, where w_i is a Lipschitz continuous nondecreasing function in the first argument.

$$p_i^j(x) = \frac{w_i[u(e^j, x), x]x_j}{\sum_{h \in K} w_i[u(e^h, x), x]x_h}.$$

Assuming that all review rate $r_i(x) = 1$ we obtain the dynamic:

$$\dot{x}_i = \left(\sum_{j \in K} \frac{w_i[u(e^j, x), x]x_j}{\sum_{h \in K} w_i[u(e^h, x), x]x_h} - 1 \right) x_i.$$

As a special case consider: $u(z, x) = \lambda + \mu z$ for some $\lambda \in R,\ \mu > 0$ and $\lambda + \mu u(e^i, x) > 0, \forall i, x$.

The dynamic become:

$$\dot{x}_i = \frac{\mu}{\lambda + \mu u(x, x)} \left[u(e^i, x) - u(x, x) \right] x_i.$$

This dynamic have the form $\dot{x} = g(x)x$ where $x = (x_1, ..., x_k)$ and $g(x) = (g_1(x), ..., g_k(x))$. The vectorial field g is called the vector of growth rates.

Note that every NE is a fixed point for this system and that every interior fixed point is a NE.

Using the relative entropy function

$$H_x(y) = \sum_{i \in C(x)} x_i \log \frac{x_i}{y_i}$$

as a Lyapunov function for the dynamic, it follows that, if a fixed point x^* has a neighborhood U_{x^*} such that

- if $g(y)x^* > 0 \; \forall y \neq x \in U_{x^*}$ then is asymptotically stable,

  and

- if $g(y)x^* < 0 \; \forall y \neq x \in U_{x^*}$ then is instable.

Games against the field.

We are interested in strategies that represent the best possible behavior of individuals facing a given field (or a player whose identity is unknow) and that this strategy continue being a best response, even after perturbations of the field. then the following two conditions are necessary

Let Γ be a normal game of n populations and that a fixed population, for example, the $i-$th follows a strategy $\sigma^* \in \Delta S_i$ while the rest act according to a profile strategic $x^* \in \Delta_{-i}$ named the field.

The expected payoff yhe $i-$th population is denoted by $\pi(\sigma, x)$.

$$\pi(\sigma, x) = \sum_{s \in S} \sigma(s)\pi(s, x).$$

This payoff represents the success of an $\sigma-$strategist of the $i-$th population, plying in a field, given by the distribution $x \in \Delta S_{-i}$

Let $x^* = (x_1^*, ..., x_{i-1}^*, \sigma^*, x_{i+1}^*, ...x_n^*) \in \Delta$ be a Nash equilbrium.

We will say that σ_i^* is evolutionarily stable strategy for popilation i against the field x_{-i}^*, if the following two conditions are verified:

1. σ^* maximize the payoff of the $i - th$ propulation given that the field is x^* i.e.;

$$\sigma^* \in arg \max_{\sigma \in \Delta} \pi(\sigma, x^*).$$

2. If a small mutation in the field occurs the strategy σ^* continues to be a maximizing strategy against the mutated field, i

$$\pi(\sigma^*, x_\epsilon) > \pi(\sigma, x_\epsilon) \;\; \forall \sigma \in \Delta_i$$

for all $x \in \Delta_{-i}$ such that $|x - x_\epsilon| < \epsilon$.

Example: Assume that all individual in PII in a given time t_0 is following the strategy x. And in time t_1 a small part of the population, ϵ, start to following a new strategy y.

So

$$x_\epsilon = ((1-\epsilon)x + \epsilon y$$

is called post-entrant population.

Definition 23 *A mixed strategy σ^* is an ESSF (evolutionary stable strategy, against the field) given by x if (σ, x) is a Nash equilibrium and the following condition is verified:*

- *there exists $\bar{\epsilon}$ such that:*

$$\pi(\sigma^*, x_\epsilon) > \pi(\sigma, x_\epsilon) \ \ \forall \sigma \in \Delta$$

for every $0 < \epsilon < \bar{\epsilon}$ and every $\sigma \in \Delta_i$

So, every pure strategy $s \in S$ supported by an ESS σ^* adopted by individuals playing against a field x^*, must be a best response to the field.

Then if σ^* is an equilibrium it follows that $s \in C(\sigma^*)$ if and only if:

$$\pi(s, x^*) = \pi(\sigma^*, x^*).$$

If the claim is not true, then at least one strategy $s' \in C(\sigma^*)$ give a higher payoff than $\pi(\sigma^*, x^*)$, i. e.:

$$\pi(s', x^*) \geq \pi(s, x^*) \forall s \in C(\sigma^*)$$

with strict inequality for at least one $s \in C(\sigma^*)$. So,

$$\pi(\sigma^*, x^*) = \sum_{s \in C(\sigma^*)} p(s)\pi(s, x) <$$
$$< \sum_{s \in C(\sigma^*), s \neq s'} p(s)\pi(s', x) + p(s')\pi(s', x) = \pi(s', x),$$

which contradicts the definition.

In other words σ^* is an ESS if mutant that adopt any other strategy σ leave fewer offspring in the *post-entry population*, provided that the population of mutants is sufficiently small.

Evolutionary forces select against mutations if and only if

$$\pi(\sigma^*, x_\epsilon) > \pi(\sigma, x_\epsilon)$$

post-entry payoff is lower than that of begin the incumbent strategy.

Proposition 12 *$x \in (\Delta)^n$ is evolutionarily stable if and only if x is a strict Nash equilbrium.*

Proof:

1. First, assume that $x \in ESS$ and fix $i \in I$. Let $y_i \in \beta_i(x)$ and for all $j \neq i \;\; y_j = x_j$. Let $w = \epsilon y + (1-\epsilon)x$ then $u_i(x_i, w_{-i}) = u_i(y_i, w_{-i})$ and $u_j(y_j, w_{-j}) = u_j(x_j, w_{-j})$ for all $j \neq i$. So $y = x$ by ESS. Thues $\beta_i(x) = \{x\}$.
2. Suppose that x is a strict NE, and let $y \neq x$.so $u_i(x_i, x_{-i}) > u_i(y_i, x_{-i})$ for at least one $i \in I$. BY continuity of u_i we have that $u_i(x_i, w_{-i}) > u_i(y_i, w_{-i})$ for all $0 < \epsilon < \epsilon_y < 1$ and $w = \epsilon y + (1-\epsilon)x$, showing that x is ESS.

The replicator dynamics for a multipoputation game

Suppose and $n-$ mutipopulation game and the population state $x = (x_1, x_2, ..., x_n) \in \Delta$ where each component $x_i \in \Delta S_i$. The vector x_i may thus be though of as the state player population $i in I$ at time t where $x_{ih \in [0,1}$ is the proportion of individuals in population i playing the pure strategy $h \in S_i$. The replicator dynamics is

$$\dot{x}_{ih} = \left[\pi_i(e_i^h, x_{-i}) - \pi_i(x)\right] x_{ih}, \text{ for all } h \in S_i, i \in I.$$

The vector field on the right hand is in general qudratic, hence Lipschitz continuous function in the whole euclidean space $\Re^m$ containing Δ.

So the system has a unique solution $\xi(\cdot, x^0) : \Re \to \Delta$, for every initial state $x^0 \in \Delta$.

Example 12 *Suppose that two populations of individuals are interacting in an arbitrary n-players game. Let* $S_1 = \{s_1, ..., s_{n_1}\}$ *and* $S_2 = \{s_1, ..., s_{n_2}\}$ *the set of pure strategies or available behaviors for individuals of each population. Let* $\Delta S_i, i \in 1, 2$ *the respective distribution of probabilities over the set od available pure strategies.*

Let $A, B \in \mathcal{M}_{n_1 \times n_2}$ *the respective payoff matrix. Reacall taht* $u_1(x_1, x_2) = x_1 A x_2$ *y* $u_2(x, y) = x_2 B^T x_1$

Let $x_1 \in \Delta S_1$ and $x_2 \in \Delta S_2$, the replicator dynamics will be:

$$\dot{x}_{1i} = x_{1i}(e_1^i A x_2 - x_1 A x_2) =$$
$$= \left[\sum_{j \in S_2} a_{ij} x_{2j} - \sum_{i \in S_1} \sum_{j \in S_2} x_{1i} a_{ij} x_{2j} \right] x_{1i}, i = 1, 2, ...n_1$$

$$\dot{x}_{2j} = x_{2j}(e_2^j B^T x_1 - x_2 B^T x_1) =$$
$$= \left[\sum_{i \in S_1} b_{ij} x_{1i} - \sum_{j \in S_2} \sum_{i \in S_1} x_{2j} b_{ij} x_{1i} \right] x_{2j} \; j = 1, 2, ..., n_2.$$

For the special case where

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

the replicator dyanmics take the form:

$$\dot{x}_{11} = (a_1 x_{21} - a_2 x_{22}) x_{11} x_{12}$$
$$\dot{x}_{21} = (b_1 x_{11} - b_2 x_{12}) x_{21} x_{22}$$

Note that $x_{i2} = 1 - x_{i1}$ thus $\dot{x}_{i2} = -\dot{x}_{i1}$.

It can be proved that this system, regardless of the values of the payoff matrices, has no interior points asymptotically stable.

For this we must use Liouville's formula applied to the system

$$\dot{x}_{11} = \frac{(a_1 x_{21} - a_2 x_{22})}{x_{21} x_{22}}$$
$$\dot{x}_{21} = \frac{(b_1 x_{11} - b_2 x_{12})}{x_{11} x_{12}}$$

which results from the previous one after dividing the expressions on the right side by $x_{11} x_{12} x_{21} x_{22}$.

Remember that the solutions of the system do not change if all the elements of the vector field are multiplied by the same positive function. Only its speed is modified along the orbits, without affecting them

We have the following theorem:

The Liouville's Formula

Given any set A a time $t \in \Re$ such that the solution $\xi(\cdot, x^0)$ through any point $x_0 \in A$ is defined at time $t: \ \xi(t, x^0) \in X$, let $A(t)$ be the image of A under the solution mapping ξ

$$A(t) = \left\{\xi(t, x^0) : x^0 \in A\right\}$$

Then A is measurable, and its volume is $vol\,[A(t)] = \int_{A(t)} dx$. are thero The Liouville's formula states that

$$\frac{d}{dt} vol\,[A(t)] = \int_{A(t)} div[\phi(x)]dx,$$

where $div[\phi(x)] = \sum_{i=1}^{k} \frac{\partial \phi_i}{\partial x_i}(x)$ is the divergence of a vector field ϕ over the set $A \subset X$. In particular, any divergence-free vector field (i.e. $\phi(x) = 0$ for all $x \in A$) keeps all volume constant over time. In physics it corresponds to the flow of an incompressible liquid (e.g.; water at constant temperature and pressure).

Intuitively we expect that a divergence free vector field to have no asymptotically stable state. If $x \in X$ is asymptotically stable, then there is some neighborhood $B \subset X$ of x that os contracted toward the point x over time, implying that the volume of the neighborhood shrinks to zero as time increases toward infinite. By Liouville formula thsi is not possible ina any divergence free vector. (The most we van hope is stability)

The following result is stronger, it claim that a vector field that has non-negative divergence has no compact asymptotically stable set.

Theorem 13 (de Liouville) *If $X \subseteq \Re^l$ is an open set and $\phi : X \to \Re^l$ is continuously differentiable with $div[\phi(x)] \geq 0$ so the dynamics $\dot{x} = \phi(x)$ has no compact set $A \subseteq X$ asymptotically stable.*

Example 13 *Conider the pair of ordianry differential equations:*

$$\dot{x} = -(1-x)x$$
$$\dot{y} = -(1-y)y$$

arises in the standard two populations replicator dynamics and applied to the Prisoner's Dilemma.

The relevant space is $C = (0,1]^2$. *The vector field is continuously differentiable function on* $\Re^2$ *that has negative divergence:*

$$div\,[\phi(x,y)] = 2x + 2y - 2 < 0$$

for all $(x,y) \in int(C)$. *Hence according with the Liouville's formula the volume of any measurable set* A *of initial states in* $int(C)$ *shrinks over time in this dynamics . In fact,* C *is invariant, and all solutions in* $int(C)$ *converge to the origin.*

Division of both right-hand sides in this differential equations system, by some positive and Lipschitz continuos function $\pi : (0,1) \to \Re_{++}$ does not alter the solutions orbits in $int(C)$. For instance let $\phi(x,y) = (1-x)(1-y)$. Then the new dynamics is given by:

$$\dot{x} = -\tfrac{1}{(1-y)y}$$
$$\dot{y} = -\tfrac{1}{(1-x)x}$$

The new vector field is continuously differentiable on $int(C)$ but is divergence-free. Hence in this new dynamics, volume do not shrink over time, although the solutions orbits are the same as in the first pair of differential equations.

The explanation is that although orbits are unchanged, velocities along these orbits are changed. In particular velocities along orbits near the boundary of C are increased a lot (to $+\infty$ at the boundary). Hence forward images of set look very different in the two dynamics. While the origin in not reached in finite time form any interior state in the original dynamics, this point is reached in finite time by all interior initial states in the modified dynamics. Consequently, for any given set $A \subset int(C)$ there is a finite time t at which the solution through some initial state A leaves the domain $int(C)$ of the vector field inthe new dynamics. Accordingly Liouville´s formula no longer applies. However as long the image $A(t)$ belongs to $int(C)$ its volume is indeed constant by Liouville's formula.

Example 14 **Mutipopulation example**

- The population of workers split in two types of individuals,

  workers with high skills, and *workers with low skills.*

- Firms split in

  firms that invest in R & D and *firms that no invest in R & D.*

- *Workers and firms follow the behavior assuring to himself to maximize benefits. Rational behavior*

- *We model the interrelation between firms and workers using a two population-players normal form game.*

- *This game has two rational Nash equilibria,*
  - *high investment in R & D, and high skill workers* and
  - *low characteristics, the poverty trap*

The game

- By W we represent workers and

  S represents skill workers, by NS repreesnt unskilled worker

- Firms are represented by F there are two classes:

  Innovative firms I i.e, *firms that invest in R& D*
  not innovative firms NI, i.e, *firms that do not invest in R&D.*

- *When workers are contracted by a firm, they do not know the type of contracting firm.*

  Worker does not know if the contracting firm is I or NI

- but workers must present a certificate of his skills so,

- *the employer know the type of workers that he is contracting.*

- The S -type worker get a salary s in both cases when he is contracted by a firm I or NI.
- The NS-type workers get a salary s' less than s.
- *The innovative firm I at the end of the period, distributes utilities between their workers.*
  - Skill employees receive a prize pr.
  - Unskill workers receive pr'.

    We assume $0 < pr' < pr$.
- But *a no innovative firm* NI does not give prizes.
- *The cost of being a skill worker in each time is given by CS.*
- Assume that $CS > \bar{s}$ so, there is not incentives to be a skill worker if there is no prizes.
- We assume that $s + pr - CS > pr' + s'$. So, a S-worker engaged by an innovative firm obtain a higher payoff than a NS-worker.

The matrix of payoffs of this game is:

	I	**NI**
S	$s + pr - CS,\ B_I(S) - (s + pr)$	$s - CS,\ B_{NI}(S) - s$
NS	$s' + pr',\ B_I(NS) - (s' + pr')$	$s',\ B_{NI}(NS) - s'$

- Where $B_i(j)$ is the benefit of a firm of type $i \in \{I, NI\}$ contracting a worker of type $j \in \{S, NS\}$.
- The expected payoff of a $S-$type worker is:

$$E(S) = p(I)\,[s + pr] + p(NI)(s) - CS, \tag{13}$$

- For a $NS-$type worker the expected value is:

$$E(NS) = p(I)\,[s\prime + pr\prime] + P(NI)(s\prime). \tag{14}$$

Workers would prefer to be $S-$type if and only if

$$E(S) > E(NS),$$

and reciprocally.

This happen if and only if the probability of a firm be innovative $p(I)$, is large enough, i.e.:

$$p(I) > \frac{CS + s' - s}{(pr - (pr)')} = \bar{x}_I^F. \tag{15}$$

In the conditions of the model the inequalities

$$0 < \frac{CS + s' - s}{(pr - (pr)')} < 1$$

hold.

We assume the following statements:

- The benefits of an I firm contracting a $S-$worker are higher than the benefits obtained for a NI firm contracting a $S-$worker, i.e.:

$$B_I(S) - pr > B_{NI}(S).$$

- If the firm is I, the benefits of a $S-$worker is greater than the benefits of a $NS-$worker, i.e.:

$$s + pr - CS > s' + pr'.$$

- If the firm is NI the benefits of a $NS-$worker are higher than the benefits of a $S-$worker.i.e.:

$$s - CS < s'.$$

- If worker is NS, the benefits of a $NI-$firms are higher than the benefits of a innovative one, i.e.;

$$B_I(NS) - (s' - pr') < B_{NI}(NS) - s'$$

- The benefits of a I firm contracting a $S-$worker are higher than the benefits of a no innovative firm contracting $NS-$ worker, i.e.:

$$B_I(S) - pr > B_{NI}$$

This game has two Nash equilibria one of them is Pareto dominated by the other.

The fact that to be one of them Pareto dominant is not enough to guaranteed that the economy evolve to this mutually beneficial solution.

The dynamic of the model

- Suppose that the game is repeated and at the end of every period the workers and firms can change their behavior.
- Unskilled worker need to pay CS to be a skill worker in the next period.
- If a skill worker wishes to change to be a unskilled worker then he does no pay the cost CS in the next period.
- The firms can choose also to change their own behavior from innovative to no innovative or reciprocally.

Definition 24 *We say that the distribution on the population of workers* X_W*, is an ESSF given by* Y_F*, a distribution on the set of pure strategies of the firms, if there exist* $\epsilon > 0$ *such that* X_W *continues being a best response against all distribution* Y_ϵ *in a neighborhood* V_ϵ *of radium* ϵ*, centered at* Y*.*

- Intuitively, this means that the best response X_W against Y_F remains the best response against perturbations (in the distributions of the field).
- Notice that, when $Y_F \leq \pi$, the degenerate distribution $e_{US} = (0, 1)$ (i.e. all workers are unskilled) is an ESS against the field given by Y_F.
- *Then if, given the initial conditions, a rational worker chooses to be usnkilled, then he will choose the same behavior even in the case in which the initial conditions change, as long as said change is not "too great".*

According with the replicator dynamics we can express the migratory flux in the form of a differential equations system:

$$\begin{aligned} \dot{x}_S^W &= x_i^W \phi^W (E^W(S) - E^W(NS)) \\ \dot{x}_{NS}^W &= -\dot{x}_S^W \\ \dot{x}_I^F &= x_i^F \phi^F (E^F(NI) - E^F()) \\ \dot{x}_{NI}^F &= -\dot{x}_I^F \end{aligned} \tag{16}$$

- Thus, the flux from the subpopulation of the no skilled to the skilled is positive if the percentage of innovative firms is larger than the threshold value, i.e; if $x_I^F > \bar{x}_i^F$ then $(E^W(S) > E^W(NS))$,
- Analogously for innovative firms.

Note that the system can be reduced to one of two equations, for which the only region of interest is the square $C = [0, 1] \times [0, 1]$.

- Note that for the threshold values $(\bar{x}_I^F, \bar{x}_F^W)$ he have that $E^W(S) = E^W(NS)$ and $E^F(I) = E^F(NI)$.
- If $(\bar{x}_S^W, \bar{x}_I^F) \in C = [0,1] \times [0,1]$ then the distributions $\bar{x} = ((\bar{x}_S^W, 1 - \bar{x}_S^W); (\bar{x}_I^F, 1 - \bar{x}_I^F))$ corresponds to a Nash equilibrium and to an interior steady state for the replicator dynamics.

To analyze the trajectories of the system (16) the Vinogradov's theorem is of interest for solving this type of system.

The Vinograd's Theorem

Theorem 14 *Every autonomous equation is equivalent to another whose second member is bounded. Let*

$$x' = f(x) \tag{17}$$

the given differential equation. We form another equation by writing

$$x' = \frac{f(x)}{(1 + \|f(x)\|)}$$

The latter has the second bounded member and both equations are equivalent.

The claim results from the fact that at each point x the second members are proportional, that is, they are vectors of the same direction and sense and are canceled only simultaneously. Trajectories are therefore identical.

Theorem 15 (of Vinograd) *Let the equation $x' = f(x)$ be defined in an open set $A \subseteq X$, there is another differential equation system defined in all the space X that in A is equivalent to the given one.*

Proof: We designate by B the complementary set of A. If $\rho(x, y)$ is the distance between two points x and y of the space X then the distance (minimum) between a point x and a closed set C is $\rho(x, C) = \min_y \{\rho(x, y) : y \in C\}$. Then the function

$$g(x) = \frac{\rho(x, B)}{1 + \rho(x, B)}$$

it is different from zero at all points in the open A, zero at every point in B and bounded by the unit.

The differential equation

$$x' = f(x)g(x) \tag{18}$$

satisfies the conditions of the theorem. •

Being $f(x)$ bounded $f(x)g(x)$ is bounded and tends to zero for x tending to the border of A.

All the points of B are points of equilibrium because $x' = 0$. For every point of A it is either $x' = 0$ for both equations or they have the same direction and sense since it differs by a positive scalar factor. The trajectories therefore coincide

When approaching the border of A the trajectories of the equation (18) do not reach the border but nodding to $t \to \infty$ since the velocity along the path is an infinitesimal of order greater than or equal to the distance to the boundary for all finite time.