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Ergodic Theorems with Respect to Lebesgue

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Abstract: We study, from the ergodic viewpoint, the asymptotic dynamics in the future of a full Lebesgue set of initial states. The dynamical systems under research are deterministic and evolving with discrete time $n \in \mathbb{N}$ by the forward iterations of any continuous map $f : M \mapsto M$ acting on a finite-dimensional, compact and Riemannian manifold M. First, we revisit the classic definition of physical or SRB probability measures, and its generalized notion of weak physical probabilities. Then, inspired in the statistical meaning of the ergodic attractors defined by Pugh and Schub, which support ergodic physical measures, we define the more general concept of ergodic-like attractor. We prove that any such generalized attractor is the support of weak physical probabilities and conversely. Then, we revisit the proof of existence of weak physical probabilities and conclude that any continuous dynamics exhibits at least one ergodic-like attractor.

Key-Words: Ergodic theory, physical measures, ergodic attractors, topological dynamics, theoretical measure dynamics

1 Introduction

We consider, as the phase space where a dynamical system evolves, any finite dimensional, compact and Riemannian manifold M. We investigate the dynamics, evolving on M deterministically in the future, with discrete time $n \in \mathbb{N}$. Precisely, the system is obtained by iteration $f^n := f \circ f \circ \ldots \circ f$, for all $n \geq 0$, of a continuous map $f : M \mapsto M$. In the sequel we refer to it, as *a continuous dynamical system,* and denote it in brief, with f. We will focus in the abstract general scenario of all the continuous dynamical systems, from the viewpoint of the Ergodic Theory [2, 9, 14, 20, 31]. Namely, we will search for theoretical probability measures, determined by the dynamics in regime, i.e. f-invariant, that are representative of the asymptotic behavior of the orbits, for a Lebesgue-positive set of initial states in M.

We assume that the compact space M is provided with a reference Borel probability measure m, which is given independently of the dynamics. So, m is not necessarily f-invariant, or in other words, we do not restrict the theory to the so called conservative systems. The role of the given reference measure m is to describe how the *initial states* that define the different orbits of the system, are physically chosen.

Precisely, the probability distribution that m provides, states a criteria to measure all the borelian subsets $B \subset M$ of the space, according to the larger or smaller chance in which the initial state drops in B. But notice that m does not usually describe which portions of the space are more or less visited by the future states of the dynamical system, computed on times $n \geq 1$, i.e. after the deterministic dynamics f is acting. It is just a given initial distribution of the points of the space, before the dynamical system f starts its action.

One of the major problems of the modern Ergodic Theory is the existence of the so called physical or SRB (Sinai–Ruelle–Bowen) measures. Historically, these probability measures were defined forty years ago in [5, 28, 30], to describe the asymptotical statistics of deterministic dynamical systems exhibiting uniform hyperbolicity. Nevertheless, the general problem of existence of SRB measures is still open for most deterministic chaotic dynamical systems. Its relevance, from the theoretical viewpoint as well as for its applications, is exposed in [35]. First, let us recall the relevance of this subject in the recent research of theoretical mathematics. On the one hand, the observer may focus on the non singular properties of the SRB probabilities with respect to the Lebesgue measure. In this case, the study of the SRB measures is restricted to systems that are more than C^1-regular [12]. They exist for hyperbolic non invertible endomorphisms [33], for expansive (i.e. topologically hyperbolic) C^2-diffeomorphisms [19], and also for piecewise expanding maps [16]. In most of such cases they are stochastically stable, namely, they persist under the addition of noise [7]. On the other hand, one may focus on the asymptotical and statistical behaviour (called physical properties) of Lebesgue almost all the orbits, disregarding the non singular characteristics of the probability measure under research. In this latter case, recent results have proved the existence and uniqueness of an SRB probability with such physical properties, for C^1 generic dynamics with hyperbolic attractors [25]. Also from this viewpoint, generalized SRB measures have been found for a wider class of diffeomorphisms [34], and even the existence of SRB-like, physical-like or observable probability measures for the family of all the C^0 endomorphisms on a compact manifold [6]. Second, but not less important, let us summarize the relevance of the existence of such probability measures from the viewpoint of its applications. Historically, the problem of their existence was born in the theoretical physics to study the thermodynamical properties and the mechanical statistics of dynamical systems with very large number of particles. The existence of a good macroscopic probability measure that describes the distribution on the space of the attractors, provides a strong theoretical tool to study the dynamical statistics of almost all the orbits, particularly if the system is chaotic and difficult to predict from numerical experimentation. On the one hand, the theoretical general tools that we develop along this paper are new results obtained from the abstract analysis of general continuous dynamical systems, of any finite (arbitrarily large) dimension. Thus, they are particularly applicable to chaotic complex systems. On the other hand, the mathematical abstract modeling of applied systems in many fields of science and technology, make the general results on dynamical systems potentially applicable to them. For instance, recent research have modeled as dynamical systems the complexity of Internet networks [22], the problem of avoiding obstacles in the robotics with artificial sight [29], dynamical problems of control engineering [10], the evolution of populations [21] and also dynamics of models in psychological and other human sciences [32].

To construct the SRB measures, the given reference probability m in the space M is assumed to be the Lebesgue measure, or equivalent to it in the theoretical measure sense, and after a re-scaling to make $m(M) = 1$. In the sequel, we will denote m to such re-scaled measure, and still call it Lebesgue. We agree to say that a property or conclusion about the dynamics in the future, and in particular about the attractors of the system, is *relevant or observable,* if and only if it holds for all the initial states belonging to a borelian set $B \subset M$ satisfying $m(B) > 0$, namely, for a m-positive probable set of initial conditions. Besides, we say that the property or conclusion is *full probable, or globally observable,* if and only if $m(B) = 1$. Thus, even being usually $B \neq M$ (i.e. B is properly contained in M), if $m(B) = 1$ then a dynamical property $\mathbb{P}$ satisfied by the orbits with initial state in B, is almost always observed. The property $\mathbb{P}$ is full probable in this case. On the contrary, if $m(B) = 0$ then the orbits that satisfy $\mathbb{P}$ come from initial conditions in a set of zero m-probability. In this case the property $\mathbb{P}$ is zero-probable or non observable.

The purpose of the Ergodic Theory is to study the properties of the system in relation with the f- invariant probabilities μ. All the continuous dynamical systems on a compact metric space M do exhibit invariant probabilities (see for instance [2, 12, 14, 31]), and the large majority of such systems exhibit non countably many invariant probabilities. But at the same time, most continuous dynamical systems are not conservative, i.e. the reference Lebesgue measure m according to which the initial state distribute in the space M, is not invariant by f.

Any invariant measure μ describes a spacial distribution of the states, after the system has evolved in time asymptotically to the future, i.e. taking media temporal sequences depending on time n, and then $n \to +\infty$. This latter is the main consequence of the Ergodic Decomposition Theorem (see for instance [2, 12, 14, 31]). But only a few dynamical systems, even if one restricts the analysis to the C^1 differentiable dynamical systems, possess relevant invariant measures μ, so called physical or SRB measures. (See for instance [4, 18] to find the open questions about the existence and the properties of the SRB measures.) These latter measures describe the asymptotical spacial distribution of the orbits with their initial states that belong to some m-positive portion of the space. Precisely, if an invariant measure μ is supported in an attractor A whose basin includes a m-positive probable set B, then μ is called a physical measure.

One of the major subjects of research in the modern Differentiable Ergodic Theory, is to find sufficient conditions (if possible generic conditions) of a dynamical system to allow the existence of physical invariant measures. The global conjecture for generic differentiable dynamical systems in [23], the results posed in the book [4], and the state of the art, focused from the Ergodic Theory viewpoint as stated in [35], show the relevance of the problems in this subject. Particularly, for systems that are not sufficiently differentiable, most questions about the existence of SRB or physical measures remain open.

For most C^0 systems (i.e. continuous dynamical systems), even for those that are C^1 (differentiable ones), a single f-invariant measure is not in general enough to describe probabilistically the asymptotic dynamics of some relevant (or observable) portion B of the space M (i.e. satisfying $m(B) > 0$). In fact, some continuous dynamical systems may need infinitely many of its f-invariant measures μ to describe the asymptotic behavior from initial states in a subset $B \subset M$, such that $m(B) > 0$. As a consequence, there is no hope to find physical or SRB measures, nor ergodic attractors as defined in [24], for generic C^0 dynamical systems. That is why in this paper we revisit the weaker definition of observable weak physical or SRB-like measures that was introduced in [6] (see Definition 3.8). We construct generalized attractors that support those measures, which we call *ergodic-like attractors.*

The first purpose of this paper is to prove that the ergodic-like attractors have the same properties of attraction in mean as the egodic attractors, even if physical or SRB measures do not exist. (Theorems 4.7 and 4.9). The second purpose is to prove that any continuous dynamical system exhibits ergodic-like attractors (Corollary 4.10). To prove those theorems we will revisit the result of existence of observable or weak physical probabilities (Theorem 3.9), taken from [6]. We call all those new results *Ergodic Theorems with respect to Lebesgue* because they relate the asymptotic time averages of the orbits of a full Lebesgue set of initial states, with the probability distributions in the ambient manifold M, which are spatial f-invariant measures.

The methodology of research along this paper is that of pure mathematics: the theorems are rigorously proved under the rules of the classic logic, based on the basic and advanced results of the general topology theory [8, 17, 18, 27], of the measure and probability theory [1, 3] and particularly, the use of the Riesz Representation Theorem [26] to identify the space of probability measures with the dual space of the continuous real functions in the ambient manifold.

2 Revisiting basic notions

For a seek of self completeness of this paper, in this section we revisit some known definitions and results of the Functional Analysis, the Topology and the Probability Theory, that are in the foundation of the Ergodic Theory of deterministic dynamical systems.

2.1 The space of observable functions.

Denote $C^0(M,[0,1])$ to the space of all continuous real functions $\psi : M \mapsto [0,1]$, i.e. the value $\psi(x)$ satisfies $0 \leq \psi(x) \leq 1$ for all $x \in M$. We endow $C^0(M,[0,1])$ with the strong topology of the sup norm. Precisely, we define the following distance between two functions $\psi_1, \psi_2 \in C^0(M,[0,1])$:

$$\|\psi_1 - \psi_2\| := \max_{x \in M} |\psi_1(x) - \psi_2(x)|.$$

The real functions ψ as above are called *observables.* Physically each of those functions represents the real values that the observer looks from outside of the system. For instance an electronic circuit may be a discrete dynamical system, since it may evolve with time $n \geq 1$ (if the set of observation times isdiscrete instead of continuous). The state $\mathbf{x}$ of the system at each fixed time n, is not the current observed at the output branch, nor the voltage at the outpoint point of the circuit, but the vector of all the currents and voltages in all the branches and points, at each fixed instant n. So, the space M will be the set of all possible such vectors. Nevertheless, when one observes the output, only one of the coordinates of this vector, say the current x_0 along the single branch at the output, one obtains a function $\psi_0(\mathbf{x}) = x_0$. It depends continuously on the state $\mathbf{x}$. So ψ_0 is an observable.

The following result is obtained from the separable topology of the manifold M, since it is a compact metric space:

Theorem 2.1.1 *If M is a compact metric space, then there exists a countable family $\Psi := \{\psi_n\}_{n \geq 0}$ of continuous real functions $\psi_n : M \mapsto [0,1]$ such that Ψ is dense in $C^0(M,[0,1])$.*

Proof: See for instance [1, 3].

This theorem ensures that:

$$\forall\ \psi \in C^0(M,[0,1]\ \forall \epsilon > 0\ \ \exists\ \psi_n \in \Psi\ \text{ such that}$$

$$\psi(x) - \epsilon < \psi_n(x) < \psi(x) + \epsilon\ \ \forall\ x \in M.$$

The inequalities above admit take a real function from the countable family Ψ ensuring an error no greater than ϵ and, as usual, this $\epsilon > 0$ can be arbitrarily given by the customer. So, one must design the theoretical results to hold for any possible

$\epsilon > 0$. Summarizing, if one accepts an eventual error of the observations no greater than $\epsilon > 0$, then one may consider only the countable family $\Psi \subset C^0(M,[0,1])$, instead of all the possible observable functions.

In the sequel we will fix a dense countable family Ψ of observable functions as above. This is equivalent to discretize the topology in the compact given space M of all the possible states of the system.

2.2 The space $\mathcal{M}$ of probabilities

We denote $\mathcal{M}$ to the space of all the probability Borel measures in the compact space M. Recall that a Borel probability establishes a criteria to measure all the open sets of the space, and also all the Borel sets $B \subset M$ (i.e. the subsets of the σ- algebra generated by all the open subsets of M). Recall that a probability μ is, by definition, an assignment to each Borel subset $B \subset M$ of a real number $\mu(B) \in [0,1]$ such that

$$\mu(\emptyset) = 0, \quad \mu(M) = 1, \quad \mu(\biguplus_{n=0}^{+\infty}(B_m) = \sum_{n=0}^{\infty} \mu(B_n),$$

where $\biguplus$ denotes that the union is composed by all pairwise disjoint sets.

Up to the moment we have not introduced the dynamics $f : M \mapsto M$ in the definitions. Let us now consider how f acts in the space $\mathcal{M}$:

We say that a probability measure μ is f-invariant, if $\mu(f^{-1}(B)) = \mu(B)$ for all Borel set $B \subset M$. We denote $\mathcal{M}_f \subset \mathcal{M}$ to the set of all f-invariant probability measures. In particular the Lebesgue probability measure m in the compact manifold M belongs to $\mathcal{M}$, but we are not assuming that it belongs to $\mathcal{M}_f$. The following result is well known, and starting the Ergodic Theory of deterministic Dynamical Systems:

Theorem 2.2.1

For all compact metric space M and for all continuous $f : M \mapsto M$ the set $\mathcal{M}_f$ of f-invariant probability measures is not empty: $\mathcal{M}_f \neq \emptyset$.

Proof: See for instance [20, 31].

In the space $\mathcal{M}$, we define the following operator f^*, which is called the *pull back* in the space of probabilities of the dynamical system f in M:

$f^* : \mathcal{M} \mapsto \mathcal{M} : \ \forall \mu \in \mathcal{M} \ \ f^*\mu \in \mathcal{M}$ is defined by:

$$f^*\mu(B) := \mu(f^{-1}(B)) \quad \forall \text{ Borel set } B \subset M \ . \tag{1}$$

The following assertions are immediate:

$$(1) \ \ \mu \in \mathcal{M}_f \quad \Leftrightarrow \quad f^*\mu = \mu,$$

$$(2) \ \ \exists \, \mu \in \mathcal{M} \text{ such that } f^*\mu = \mu.$$

In other words, the f-invariant measures are the fixed points of $f^* : \mathcal{M} \mapsto \mathcal{M}$. and the set of those measures is not empty.

2.3 The weak* topology in the space $\mathcal{M}$

If we fix $\mu \in \mathcal{M}$ and take $\psi_1 \in C^0(M,[0,1])$, then the expected value of the observable ψ_1 with respect to the probability μ is $\int \psi_1 \, d\mu$. If ψ_2 is other observable function ϵ-near ψ_1 (namely, $\|\psi_2 - \psi_1\| < \epsilon$), then it is easy to check that the expected values of ψ_1 and ψ_2 are also ϵ-near, i.e.

$$\left| \int \psi_2 \, d\mu - \int \psi_1 \, d\mu \right| < \epsilon.$$

Now, instead of fixing $\mu \in \mathcal{M}$ we will fix $\psi \in C^0(M,[0,1]$ and take two probabilities measures μ_1, μ_2 so we can compare the expected values of the observable function ψ with respect to the two probabilities. Precisely, we define below the real application $\rho_\psi(\mu_1,\mu_2)$. It is positive, symmetric and satisfies the triangular inequality, but it is not a distance between measures in $\mathcal{M}$ since it is not necessarily strictly positive for all pair of probabilities μ_1, μ_2 such that $\mu_1 \neq \mu_2$.

$$\rho_\psi(\mu_1,\mu_2) := \left| \int \psi \, d\mu_1 - \int \psi \, d\mu_2 \right| .$$

Nevertheless when we compare as above the expected values of all the observable functions ψ, with respect to the probability μ_1 and μ_2, we can define a distance between these two probabilities, and so a topology in the space $\mathcal{M}$ of all probability measures, which is called the *weak* topology.* Even more, since the family Ψ is dense in $C^0(M,[0,1])$, to decide when two measures $\mu_1,\mu_2 \in \mathcal{M}$ are $\epsilon-$near, we can restrict to take into account the values $\rho_\psi(\mu_1,\mu_2)$ for the observable functions $\psi \in \Psi$. As a consequence we define in the next paragraph, a metric structure inducing the weak* topology in the space $\mathcal{M}$ of all the Borel probabilities on the manifold M:

Definition 2.3.1
In the sequel, we endow $\mathcal{M}$ with the so called *weak* topology.* This is the structure of open subsets in $\mathcal{M}$ that can be defined from the following metric (i.e. from the following definition of distance between two measures $\mu,\nu \in \mathcal{M}$):

$$\operatorname{dist}(\mu,\nu) := \sum_{n=0}^{\infty} \frac{|\int_M \psi_n\,d\mu - \int_M \psi_n\,d\nu|}{2^n} \qquad (2)$$

where $\Psi := \{\psi_n\}_{n\geq 0}$ is a fixed countable family of continuous real functions $\psi_n \in C^0(M,[0,1])$ such that Ψ is dense in $C^0(M,[0,1])$.

In particular, as usual in metric spaces, we define the distance between a probability measure $\mu \in \mathcal{M}$ and any non empty subset $\mathcal{L}^* \subset \mathcal{M}$, as follows:

$$\operatorname{dist}(\mu,\mathcal{L}^*) := \inf_{\nu\in\mathcal{L}^*} \operatorname{dist}(\mu,\nu). \qquad (3)$$

Also we define the distance between two subsets $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ of $\mathcal{M}$:

$$\operatorname{dist}(\mathcal{L}_1^*,\mathcal{L}_2^*) := \inf_{\mu\in\mathcal{L}_1^*} \inf_{\nu\in\mathcal{L}_2^*} \operatorname{dist}(\mu,\nu). \qquad (4)$$

The two infima in Equalities (3) and (4) exist, because any set of real numbers $\{\operatorname{dist}(\mu,\nu) : \nu \in \mathcal{L}^*\}$ is lower bounded by 0.

It is easy to check the following characterization of the weak* topology, from the definition of limit in the space $\mathcal{M}$ of probabilities, using the distance between two measures as defined above, and applying the denseness condition of the countable family $\Psi \subset C^0(M,[0,1])$ of observable functions in M:

$$\forall \text{ sequence } \{\mu_n\}_{n\geq 0} \subset \mathcal{M} :$$
$$\lim_{n\to+\infty} \mu_n = \mu \in \mathcal{M} \quad \text{if and only if}$$
$$\lim_{n\to+\infty} \int \psi\,d\mu_n = \int \psi\,d\mu \quad \forall\ \psi \in C^0(M,[0,1]). \qquad (5)$$

The following is a classic result of the basic Probability Theory, in which the Ergodic Theory of deterministic Dynamical Systems is founded:

Theorem 2.4
The space $\mathcal{M}$ of all the probability Borel measures on M, endowed with the weak topology, is compact and sequently compact.*

Explicitly, the sequently compactness of $\mathcal{M}$ has the following meaning:

For all sequence $\{\mu_n\}_{n\geq 0}$ of probabilities, there exists a subsequence $\{\mu_{n_i}\}_{i\geq 0}$ (n_i is strictly increasing with i), *such that*

$$\exists \lim_{i\to+\infty} \mu_{n_i} = \mu \in \mathcal{M}\ ,$$

where the limit in $\mathcal{M}$ is taken with the weak topology.*

Proof: See for instance [1, 3].

The following statement is a known basic result from the Functional Analysis, which is also in the basis of the Ergodic Theory:

Theorem 2.5
If $f : M \mapsto M$ is continuous, then the pull back operator $f^* : \mathcal{M} \mapsto \mathcal{M}$ defined in Equality (1), *acting in the space $\mathcal{M}$ of probability measures of M, is continuous with the weak* topology.*

Proof: We must prove that if $\lim_{n\to+\infty} \mu_n = \mu$ in the space $\mathcal{P}$ of all the Borel probability measures on the compact manifold M, and if $f : M \mapsto M$

is continuous, then $\lim_{n\to+\infty} f^*\mu_n = f^*\mu$ in $\mathcal{P}$. After Equality (5) we have

$$\lim_{n\to+\infty}\int \psi\, d\mu_n = \int \psi\, d\mu \quad \forall\ \psi \in C^0(M,[0,1]). \tag{6}$$

Since f is continuous, we have that $\psi\circ f \in C^0(M,[0,1])$ for all $\psi \in C^0(M,[0,1]$. Then Equality (6) implies that

$$\lim_{n\to+\infty}\int \psi\circ f\, d\mu_n = \int \psi\circ f\, d\mu \tag{7}$$

$$\forall\ \psi \in C^0(M,[0,1]).$$

After the definition of the operator $f^* : \mathcal{P} \mapsto \mathcal{P}$ the following assertion is satisfied by the characteristic function χ_B of any Borel set $B \subset M$ and any probability measure $\nu \in \mathcal{P}$:

$$\int \chi_B\, d\ f^*\nu = (f^*\nu)(B) = \nu(f^{-1}(B)) =$$

$$\int \chi_{f^{-1}(B)}\, d\nu = \int \chi_B \circ f\, d\nu.$$

Therefore, after the abstract definition of the integral respect to ν, the following equality is satisfied by all $\psi \in L^1(\nu)$, in particular for all $\psi \in C^0(M,[0,1])$:

$$\int \psi\, df^*\nu = \int \psi\circ f\, d\nu \quad \forall\ \nu \in \mathcal{P}.$$

Joining the last equality with (7) we obtain:

$$\lim_{n\to+\infty}\int \psi\, d\ f^*\mu_n = \int \psi\, d\ f^*\mu$$

$$\forall\ \psi \in C^0(M,[0,1]).$$

After Equality (5), the last assertion is equivalent to $\lim_{n\to+\infty} f^*\mu_n = f^*\mu$, as wanted. $\square$

Remark 2.6 As a consequence of the continuity of the operator f^*, the subset of f-invariant probability measures, which is characterized by $\mathcal{M}_f := \{\mu \in \mathcal{M} : f^*\mu = \mu\}$, is a closed subset of $\mathcal{M}$. Since $\mathcal{M}$ is a compact metric space, we conclude that $\mathcal{M}_f$ is compact.

3 Weak physical measures

We start defining the object of research when the observer analyzes the asymptotic statistics of any deterministic dynamical system:

Definition 3.1

Empiric probabilistic distributions

Let $x \in M$ be a fixed initial state. We define the *empiric probability distribution of the orbit of x up to time* $n-1 \geq 0$ to the probability

$$\nu_n(x) := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}\ , \tag{8}$$

where δ_y is the Dirac delta probability supported in the point $y \in M$. Precisely, for all Borel set $B \subset M$:

$$\delta_y(B) = 1 \quad \text{if} \quad y \in B, \qquad \delta_y(B) = 0 \quad \text{if} \quad y \notin B.$$

In other words, the empiric probability distribution $\nu_n(x)$ is supported in the finite piece of the orbit of x from time 0 up to time $n-1$, and assigns to each singleton of this finite piece of orbit, the same probability $1/n$. Physically, $\nu_n(x)$ measures the relative number of visits to the different pieces of the space, of the future orbit with initial state x, up to time $n-1$.

Note that unless $x \in M$ is a fixed point by f, the empiric distributions $\mu_n(x)$ are not f-invariant for all $n > 1$. To define and study the ergodic-like attractors, i.e. the statistics of the asymptotic behavior of the orbits, the purpose is to study the limit in the weak* topology of the measures $\nu_n(x)$ if it exists, or at least its limit set in the space $\mathcal{M}$ of probabilities which is composed by the limits of all the convergent subsequences of $\{\nu_n(x)\}_{n\geq 0} \subset \mathcal{M}$. This family of convergent subsequences is not empty, after the Theorem of sequential compactness of $\mathcal{M}$ (see Section **??**) endowed with the weak* topology. So, for all $x \in M$, we can define the non empty limit set $\mathcal{L}^*(x)$ in the space of probabilities $\mathcal{M}$, as follows:

Definition 3.2
For any given initial state $x \in M$, we call *limit set in the space of probabilities $\mathcal{M}$ of the orbit of x,* to:

$$\mathcal{L}^*(x) := \{\mu \in \mathcal{M}: \ \exists \text{ a convergent subsequence } ...$$

$$...\{\nu_{n_i}(x)\}_{i \geq 0} \quad \text{such that} \quad \lim_{i \to +\infty} \nu_{n_i}(x) = \mu\} \tag{9}$$

where the sequence $\{\nu_n\}_{n \geq 0}$ is defined in Equality (10) and the limit is taken in the weak* topology of $\mathcal{M}$.

In particular, if $\nu_n(x)$ is convergent to a probability measure μ, then the limit set is the singleton containing μ. Namely:

$$\mathcal{L}^*(x) = \{\mu\} \qquad \text{if and only if}$$

$$\exists \lim_{n \to +\infty} \nu_n(x) = \mu \in \mathcal{M} \text{ in the weak}^* \text{ topology} .$$

The following result is standard in the Ergodic Theory. For a seek of completeness we include here its proof:

Proposition 3.3
For all $x \in M \quad \mathcal{L}^*(x) \neq \emptyset \quad$ *and* $\quad \mathcal{L}^*(x) \subset \mathcal{M}_f$.

Proof: Let us first prove that $\mathcal{L}^*(x) \neq \emptyset$. In fact, the sequence $\{\nu_n(x)\}_{n \geq 0}$ of empiric probabilities, is contained in $\mathcal{M}$, which is sequentially compact after endowed with the weak* topology (see Theorem 2.4). Therefore, it has convergent subsequences. So, after Definition 3.2, the set $\mathcal{L}^*(x)$ is non empty, because it is the set of the limits of all those convergent subsequences.

Let us prove now that $\mathcal{L}^* \subset \mathcal{M}_f$. Consider the following sums of measures in the space $\mathcal{M}$ of probabilities:

$$n \cdot \nu_n(x) = \sum_{j=0}^{n-1} \delta_{f^j(x)} = \delta_x - \delta_{f^n(x)} + \sum_{j=1}^{n} \delta_{f^j(x)}$$

$$n \cdot f^* \nu_n(x) = \sum_{j=0}^{n-1} f^* \delta_{f^j(x)} = \sum_{j=0}^{n-1} \delta_{f^{j+1}(x)} =$$

$$\sum_{k=1}^{n} \delta_{f^k(x)} \; = \; \sum_{k=1}^{n-1} \delta_{f^k(x)} \; + \; \delta_{f^n(x)}$$

Therefore, after the substraction of the first sum from the second one:

$$n \cdot (f^* \nu_n(x) - \nu_n(x)) = \delta_{f^n(x)} - \delta_x$$

Take a convergent subsequence $\{\nu_{n_i}(x)\}_{i \geq 0}$ in the weak* topology of the space $\mathcal{M}$ of probabilities:

$$\lim_{i \to +\infty} \nu_{n_i}(x) = \mu \ ,$$

and compute the following limit (also in the weak* topology):

$$\lim_{i \to +\infty} f^* \nu_{n_i}(x) - \nu_{n_i}(x) = \lim_{i \to +\infty} \frac{1}{n_i} (\delta_{f^{n_i}(x)} - \delta_x) = 0.$$

The weak*-limit above is zero since $n_i \to +\infty$ and the measures $\delta_x \ \delta_{f^{n_i}(x)}$ are probabilities, so upper bounded by 1. Thus, they define bounded operator $\psi \in C^0(M, [0,1]) \ \mapsto \ \int \psi \, d\delta_{f^{n_i}(x)} \ \in [0,1] \subset \mathbb{R}$, and so, divided by $n_i \to +\infty$, they both converge to zero in the weak* topology.

Therefore, $\lim_{i \to +\infty} f^* \nu_{n_i}(x) = \lim_{i \to +\infty} \nu_{n_i}(x)$, and using Theorem 2.5, we conclude:

$$f^* \mu = f^* (\lim_{i \to +\infty} \nu_{n_i}(x)) =$$

$$\lim_{i \to +\infty} f^* \nu_{n_i}(x) = \lim_{i \to +\infty} \nu_{n_i}(x) = \mu.$$

The equality above $f^* \mu = \mu$ is equivalent to the f-invariance of the probability μ. So, we have proved that $\mu \in \mathcal{M}_f$ for all probability measure μ that is the limit of a convergent subsequence of $\{\nu_n(x)\}$, i.e. for all $\mu \in \mathcal{L}^*(x)$. Therefore $\mathcal{L}^*(x) \subset \mathcal{M}_f$, ending the proof. $\square$

Remark 3.4
From the proposition above, the measures in the non empty set $\mathcal{L}^*(x)$ are all f-invariant. But nevertheless they are not necessarily ergodic. Even if $\mathcal{L}^*(x)$ were a singleton, the example of Bowen, restated in [11], shows a case (that is besides much more than a continuous dynamical system, since

it can be constructed of C^∞ class), for which for all x in an open set $U \subset M$, the limit set $\mathcal{L}^*(x) = \{\mu\}$, but μ is not ergodic.

Up to now, with fixed $x \in M$, we look at all the measures $\mu \in \mathcal{M}$ that are limits when time $n \to +\infty$, of the convergent subsequences of the empiric probability distributions $\nu_n(x)$ supported on finite pieces of orbit with initial state x, from time 0 and up to time $n-1$. Namely, we defined $\mathcal{L}^*(x)$ choosing any fixed $x \in M$, and searching all the adequate measures $\mu \in \mathcal{M}$ that represent the statistics of the asymptotic behavior of the orbit in M with initial state x.

Dually, we will fix now *any* $\mu \in \mathcal{M}$ and look the set $B = B(\mu)$ of all the points $x \in M$ such that μ represents the statistics of the asymptotic behavior of some of the orbits in B. If this set $B(\mu) \subset M$ is non empty, it will be called *basin of attraction* of μ in $\mathcal{M}$, as we state in the following definition:

Definition 3.5
Let $\mu \in \mathcal{M}$. Denote $B(\mu)$ to the following subset of the ambient manifold M:

$$B(\mu) := \{x \in M : \ \mathcal{L}^*(x) = \{\mu\}\} \ .$$

If $B(\mu) \neq \emptyset$ we call it *basin of attraction* of μ. After Proposition 3.3, if $B(\mu)$ is not empty then $\mu \in \mathcal{M}_f$. i.e. μ is f-invariant.

Remark 3.5.1 Note that the basin of attraction $B(\mu)$ of a probability measure $\mu \in \mathcal{M}$, lays in the ambient space M (the compact manifold where f acts) and not in $\mathcal{M}$ (the space of the Borel probability measures on M). Thus, the basin of attraction of a measure μ is not defined as the basin of any attractor of the operator f^* in the space of probabilities. In general, most measures in $\mathcal{M}$, precisely most measures in $\mathcal{M}_f \subset \mathcal{M}$, define sets $B(\mu)$ that are empty, so they are not representative of the statistic of the asymptotic behavior of any orbit in M. It is folkloric believed, but wrong, that the measures that have non empty basin of attractions are only the ergodic ones. It is true the converse statement, if μ is ergodic respect to f, then its basin $B(\mu) \subset M$ includes μ-almost all points in M, and so, it is not empty. But it is false the necessary ergodicity of μ to have a nonempty basin $B(\mu)$. In fact, the Bowen example, restated in [11] as we recalled in Remark 3.4, exhibits a measure μ whose basin of attraction $B(\mu)$ is open and not empty, and μ is not ergodic.

Let us define an ϵ-weak basin of attraction of a probability measure μ, taken from [6]:

Definition 3.6
Let $\mu \in \mathcal{M}$. For all $\epsilon > 0$ we define the set:

$$B_\epsilon(\mu) = \{x \in M : \ \ \mathrm{dist}(\mu, \ \mathcal{L}^*(x)) < \epsilon\} \ ,$$

where the distance dist in the space $\mathcal{M}$ of probabilities is defined by Formulae (2) and (3).
If $B_\epsilon(\mu) \neq \emptyset$ we call it *ϵ-weak basin of attraction of μ.*

Remark 3.6.1: Note that in the definition of weak basin of attraction $B_\epsilon(\mu)$, we are not assuming that $\mathcal{L}^*$ is a singleton. In other words, we do not impose that for all x in the weak basin of attraction of μ, the sequence of empiric probabilities is convergent. On the contrary, in Definition 3.5 of (strong) basin of attraction $B(\mu)$ of a measure, we assume the convergence to μ of the empiric sequences of probabilities for all $x \in B(\mu)$.

We also notice, from Definition 3.6, that if $x \in B_\epsilon(\mu)$, then $\mathcal{L}^*(x)$ intersects, in the space $\mathcal{M}$, the ball of center μ and radius ϵ, but *is not necessarily included in that ball.*

Let us first revisit the classic definition of physical measure, and second, let us define the weak physical measures.

Definition 3.7 Physical measures
A probability measure $\mu \in \mathcal{M}$ is called *physical or SRB* if its basin of attraction $B(\mu)$ (defined in 3.5) has positive Lebesgue measure.

Physical measures μ, if they exist for some non conservative system (i.e. when the Lebesgue measure is not invariant), are in general supported on attractors of null Lebesgue measure. Even more, in most of those cases, μ is mutually singular with Lebesgue. We recall that this is also the situation in the paradigmatic non conservative example, when μ is the ergodic SRB measure of a transitive Anosov diffeormorphism which is C^1 plus Hölder. In this well known example, μ has conditional measures along the local unstable manifolds that are absolute continuous with respect to the internal Lebesgue measures of those manifolds, but it is mutually singular respect to the Lebesgue measure m in the ambient manifold M.

From Definition 3.7, if there exists a physical measure μ, then for any finite time n large enough and for a Lebesgue-positive set of points $x \in M$, the future orbit with initial state x will exhibit a empiric probability distribution $\nu_n(x)$ approaching μ. But $\nu_n(x)$ *does not equal* the probability measure μ, except, at most, for the Lebesgue-zero set of initial states in a periodic orbit on which μ could be supported. Indeed, we will always see, for a Lebesgue-positive set of orbits, an ϵ-approach to μ, with $\epsilon \neq 0$. In other words, there exists a not null error, as small as wanted if the time of experimentation is long enough, but nonzero. This $\epsilon-$approximation of the empiric probability to the physical measure is observed in the space $\mathcal{M}$, with any metrization inducing the weak*-topology on $\mathcal{M}$.

In brief, the conclusions, when using the theory of physical measures, do not hold with error $\epsilon = 0$ if time of experimentation is finite. One could better assert that the physical measure μ is the distribution in regime. But this assertion does not hold usually for a Lebesgue-positive set of initial states. In fact, for most non conservative systems that exhibit physical or SRB measures, the attractor has zero Lebesgue measure. On the other hand, the modern ergodic theory of physical and SRB measures, is based from the very beginning, in the hypothesis or exclusion of the initial states in a zero Lebesgue-measure set. So, under this hypothesis, one will never arrive exactly to the attractor, if the map f is one-to-one and such that $f^*(m) \ll m$, where m is the Lebesgue measure, and if the attractor has zero Lebesgue measure. Therefore, the exact evolution in regime is not exactly observed.

Inspired in the argument above, one does not loose the physical empiric meaning of a measure μ, if it approximates the empiric distributions $\nu_n(x)$ by 2ϵ instead of ϵ, and if this happens for all $\epsilon > 0$ when choosing adequate stop times n that are large enough (but finite) and adequate initial states x in a positive Lebesgue-measure set. In other words, we can approximate the empiric distribution $\nu_n(x)$ up to time n, for n large enough, with measures $\mu \in \mathcal{M}$ that are not exactly the limits of the convergent subsequences of $\{\nu_n(x)\}_{n\geq 0}$, but that are ϵ-near to those limits. Namely, we can consider the ϵ-weak basins of attractions $B_\epsilon(\mu)$, as defined in 3.6 (instead of the strong basin $B(\mu)$ defined in 3.5), provided that $m(B_\epsilon(\mu)) > 0$ *for all* $\epsilon > 0$.

With the weak concept of attraction described above, we are not loosing the desired physical empirical sense of the searched probability measures μ. In fact, from the considerations above, that weaker definition (which we formulate precisely in Definition 3.8) will be still strong enough to describe, like physical measures do, the asymptotic statistics of a Lebesgue positive set of orbits.

We revisit now the definitions in [6], which are generalizations of the observability notion for measures introduced in [15], and of the physical statistical properties of the SRB measures. We call this generalized notion as *weak physical.* It is indeed an ϵ-weak physical property of the measures μ, for all $\epsilon > 0$:

Definition 3.8 Weak physical measures
We say that a probability measure $\mu \in \mathcal{M}$ is

weak physical or observable if its ϵ-weak basin of attraction $B_\epsilon(\mu)$ (see Definition 3.6) has positive Lebesgue measure for all $\epsilon > 0$. We denote $\mathcal{W}_f \subset \mathcal{M}$ to the set of all weak physical measures for f.

We have the following properties of weak physical measures, first stated and proved in [6]. For a seek of completeness of this paper, we reformulate here their proofs.

Theorem 3.9

Existence of weak physical measures

For all continuous map $f : M \mapsto M$ on a compact manifold M:
(i) $\emptyset \neq \mathcal{W}_f \subset \mathcal{M}_f$
(ii) $\mathcal{W}_f$ *is a compact set in the weak** *topology of the space* $\mathcal{M}$ *of all probability measures.*

Proof: Let us first prove that $\mathcal{W}_f \neq \emptyset$. Assume by contradiction that for all $\mu \in \mathcal{M}$ there exists $\epsilon_\mu > 0$ such that $m(B_{\epsilon_\mu}(\mu)) = 0$. Fix any $\mu \in \mathcal{M}$. After Definition 3.6 of the weak basin $B_\epsilon(\nu)$ of a probability measure ν, we conclude that $m(B_{\epsilon_\mu/2}(\nu)) = 0 \quad \forall \nu \in \mathcal{M}$ such that $\text{dist}(\mu, \nu) < \epsilon_\mu/2$. So, each fixed $\mu \in \mathcal{M}$ has a neighborhood of radius $\epsilon_\mu/2$ such that all the other measures ν in that neighborhood also have an $(\epsilon_\mu/2)$-weak basin of attraction with zero Lebesgue measure (if non empty). Since $\mathcal{M}$ is compact, we can cover the whole space of probabilities with a finite number of such neighborhoods, and choose the smallest radius of them, say $\epsilon_1 > 0$. We conclude that

$$m(B_{\epsilon_1}(\nu)) = 0 \ \ \forall \, \nu \in \mathcal{M} \quad \Rightarrow$$

$$\text{dist}(\mathcal{L}^*(x), \mathcal{M}) \geq \epsilon_1 > 0 \text{ for } m-\text{a.e. } x \in M \tag{10}$$

where m is the Lebesgue measure (see Definition 3.6 of the weak basin $B_\epsilon(\mu)$). But the non empty set $\mathcal{L}^*(x)$ is contained in $\mathcal{M}_f \subset \mathcal{M}$ for all $x \in M$ (see Proposition 3.3), and thus $\text{dist}(\mathcal{L}^*(x), \mathcal{M}) = 0$ for all $x \in M$, contradicting (10). We have proved that $\mathcal{W}_f \neq \emptyset$.

Now let us end the proof of statement (i). We must prove that $\mathcal{W}_f \subset \mathcal{M}_f$. Fix $\mu \in \mathcal{W}_f$. Since $m(B_\epsilon(\mu)) > 0$ for all $\epsilon > 0$ (see Definition 3.7), there exists a sequence of points $x_n \in B_{1/n}(\mu)$ for all $n \geq 1$. After Definition 3.6 of weak basin of attraction, there exists a sequence of measures $\nu_n \in \mathcal{L}^*(x_n)$ such that $\text{dist}(\nu_n, \mu) < 1/n$ for all $n \geq 0$. Therefore $\lim_{n \to +\infty} \nu_n = \mu$. From Proposition 3.3 and since $\nu_n \in \mathcal{L}^*(x_n)$, we deduce that $\nu_n \in \mathcal{M}_f$ for all $n \geq 0$. Finally, as $\mathcal{M}_f$ is a closed subset of $\mathcal{M}$ (see the Remark after Theorem 2.5), and since $\mathcal{M}_f \ni \nu_n \to \mu$, we conclude that $\mu \in \mathcal{M}_f$, as wanted.

Finally, let us prove the statement (ii). Since $\mathcal{W}_f \subset \mathcal{M}$, and $\mathcal{M}$ is a compact metric space in the weak* topology (see Section 2), it is enough to prove that $\mathcal{W}_f$ is closed. Take $\mu_n \to \mu \in \mathcal{M}$, such that $\mu_n \in \mathcal{W}_f$ for all $n \geq 0$. We must prove that $\mu \in \mathcal{W}_f$, or in other words, we shall prove that $m(B_\epsilon(\mu)) > 0$ for all $\epsilon > 0$ (see Definition 3.8). Fix $\epsilon > 0$. Since $\mu_n \to \mu$, then $\text{dist}(\mu_n, \mu) < \epsilon/2$ for all n large enough. Fix one of such values of n. As $\mu_n \in \mathcal{W}_f$, we have $m(B_{\epsilon/2}(\mu_n)) > 0$. But from the triangular property, for all $x \in B_{\epsilon/2}(\mu_n)$ we have $\text{dist}(\mathcal{L}^*(x), \mu) \leq \text{dist}(\mathcal{L}^*(x), \mu_n) + \text{dist}(\mu_n, \mu) < (\epsilon/2) + (\epsilon/2) = \epsilon$. We conclude that $B_{\epsilon/2}(\mu_n) \subset B_\epsilon(\mu)$ and therefore $m(B_\epsilon(\mu) \geq m(B_{\epsilon/2}(\mu_n) > 0$. So we have proved that the ϵ-weak basin of attraction $B_\epsilon(\mu)$ of μ, has positive m-measure (Lebesgue measure) for all $\epsilon > 0$, as wanted. $\square$

Let us consider a restriction of f to a forward invariant portion of the space that has positive Lebesgue measure.

Definition 3.10

Restricted weak physical measures

Let $B \subset M$ a Borel subset such that $m(B) > 0$ and $f(B) \subset B$. Then $f|_B : B \mapsto B$ defines a dynamical subsystem. We say that a Borel probability measure μ (not necessarily supported on B) is *observable or weak physical for f restricted to B* if $m(B_\epsilon(\mu) \bigcap B) > 0$ for all $\epsilon > 0$, where $B_\epsilon(\mu)$ is defined in 3.8. We denote with $\mathcal{W}_{f|B}$ to the set of all weak physical measures restricted to B.

Corollary 3.11
For all continuous $f : M \mapsto M$ on a compact manifold M and for all forward invariant Borel set B with positive Lebesgue-measure:
(1) $\emptyset \neq \mathcal{W}_{f|B} \subset \mathcal{M}_f$ *and*
(2) $\mathcal{W}_{f|B}$ *is weak* compact.*

Proof: Apply the proof of Theorem 3.9 to $f|_B : B \mapsto B$ in the role of $f : M \mapsto M$. $\square$

After Theorem 3.9 and its Corollary 3.11, weak physical measures do always exist for any continuous dynamics, including in particular C^1 diffeomorphisms. This is the large difference with the observable measures defined in [15], and also with the classical definition of physical measures (see Definition 3.7). One of the major problems of the differentiable Ergodic Theory is to find probability measures that have good ergodic properties for Lebesgue almost all orbits. That is why physical or SRB measures raised in the literature of the modern Ergodic Theory. But on the other hand, it is well known that the ergodic theory of SRB or physical measures, at least from the viewpoint in which it was developed up to now, does not work for C^1 systems that are not C^1 plus Hölder. The major obstruction resides in the frequent non existence of such mild measures. On the other hand, he have proved here that weak physical measures as defined in 3.8 do exist for all continuous systems, so in particular for all C^1 systems, and as we will see in the next section, they describe a class of weak attractors from the ergodic viewpoint, for Lebesgue almost all orbits. So they can substitute the physical or SRB measure if these last probabilities do not exist. And they coincide with the physical or SRB measures in the case when they last exist. As the great advantage, the weak physical measures do exist, and describe the ergodic-like attractors, for all the continuous dynamics.

4 Ergodic-like attractors

In the sequel we will agree to say that a nonempty set $K \subset M$ is *f-invariant* if $f^{-1}(K) = K$. We recall that the continuous map $f : M \mapsto M$ is not necessarily invertible. We will agree to say that a set B is f-forward invariant if $f(B) \subset B$, or equivalently $B \subset f^{-1}(B)$.

Definition 4.1
If $K \subset M$ is not empty, compact and f-invariant, we define the following subset $V_{\epsilon,n}(K) \subset M$ for any fixed $\epsilon > 0$ and for any fixed $n \geq 0$:

$$V_{\epsilon,n}(K) = f^{-n}(V_\epsilon(K)) =$$
$$\{y \in M : \text{dist}(f^n(y), K) < \epsilon\}$$

where $V_\epsilon(K) := \{x \in M : \text{dist}(x, K) < \epsilon\}$ is called *the open ϵ-neighborhood of K.*

Note that $V_{\epsilon,0}(K) = V_\epsilon(K)$. It is immediate from the $f-$invariance of K that $K \subset V_{\epsilon,n}(K)$ for all $n \geq 0$ and for all $\epsilon > 0$.

For a fixed point $y \in M$, and for a fixed natural number $N \geq 1$ we denote $\omega_{\epsilon,N}(y, K)$ to the frequency with which the finite piece of orbit with initial state y, from time 0 and up to time N, visits the ϵ-neighborhood of K. Precisely, if $\#A$ denotes the cardinality (i.e. the number of elements) of a finite set A, we have:

$$\omega_{\epsilon,N}(y, K) := \frac{\#\{0 \leq n < N : \ y \in V_{\epsilon,n}(K)\}}{N} =$$
$$\frac{\#\{0 \leq n < N : \ \text{dist}(f^n(y), K) < \epsilon\}}{N} \qquad (11)$$

From the definition above $0 \leq \omega_{\epsilon,N}(y, K) \leq 1$ for all $y \in M$, for all $N \geq 1$ and for all $\epsilon > 0$. Note, from the f-invariance of K, that $\omega_{\epsilon,N}(y, K) = 1$ for all $y \in K$, for all $N \geq 1$ and for all $\epsilon > 0$.

For a given non empty, compact and f-invariant set K we construct the set

$$B_\epsilon(K) = \{y \in M : \liminf_{N \to +\infty} \omega_{\epsilon,N}(y, K) > 1 - \epsilon\} \tag{12}$$

where the frequency $\omega_{\epsilon,N}(y, K)$ is defined by Equality (11). We call $\omega_{\epsilon,N}(y, K)$ the *frequency of visits of the orbit of y, from time 0 up to time N, to the ϵ-neighborhood of K.* So, if $y \in B_\epsilon(K)$ and $\epsilon > 0$ is small enough, we say that the frequency of visits of the orbit of y to the ϵ-neighborhood of K is asymptotically near 1. If a point y belongs to $B_\epsilon(K)$ for all $\epsilon > 0$, we say that it is *probabilistically attracted by the set K.* In fact, the frequency of its visits to the open sets at arbitrarily small distance from K is asymptotically 1. So, with increasing probability converging to 1, in the mean times, we will find the future orbit of all the points $y \in \bigcup_{\epsilon>0} B_\epsilon(K)$ (for time large enough) as near K as wanted.

Remark 4.1.1
Note that $K \subset B_\epsilon(K)$, so for all $\epsilon > 0$ this latter set is not empty and $K \subset \bigcup_{\epsilon>0} B_\epsilon(K)$.

From the definition above $B_\epsilon(K)$ is the set of all the points $y \in M$ such that are, in the temporal averages, ϵ-attracted to K. The approximation $\epsilon > 0$ has a doubling meaning:

First, it is an spacial approximation, since the future orbit with initial state y drops in $V_\epsilon(K)$ (the ϵ-neighborhood of K) for infinitely many iterates.

Second, it is a temporal probabilistic approximation but not a topological approximation. In fact, we are not assuming that after some time N large enough all the iterates of the orbit with initial state y drop in $V_\epsilon(K)$. Nevertheless, we are assuming that a proportion near 100% of these iterates (if $\epsilon > 0$ is small enough) do drop in $V_\epsilon(K)$. In other words, the frequency according to which they approach K is larger than $1-\epsilon$. So, the orbit of y, after an iterate N large enough, has a relatively very small freedom, if ϵ is small enough, to "take a short vacation tour" far from K.

The same ϵ approximation in time-mean appears when the classical ergodic attractors, (defined as the support of physical measures) are considered. For instance the example of Bowen restated in [11], the example of Hu-Young in [13], and the C^1-generic example of Campbell and Quas, restated in [6], exhibit this weak attraction in temporal mean to the support to a physical measure, and the attraction is not topological. In those three examples there exist physical measures attracting Lebesgue almost all the points of an open set, and supported on hyperbolic fixed point of saddle type. Therefore, the attractors are not topological, but just probabilistic. Precisely, the attraction is observed in time-mean, with a frequency of visits arbitrarily near the attractor that is not exactly 1, but near 1 (namely, converging to 1 when the number N of observed iterates goes to $+\infty$). Inspired in those examples, we introduce the following Definitions 4.2 and 4.4:

Definition 4.2
For a non empty compact f-invariant set $K \subset M$ the set $B_\epsilon(K)$ constructed in Equality (12), if nonempty, is called basin of ϵ-weak ergodic attraction to K, or simply *basin of ϵ-attraction* to the set K.

Remark 4.3
We notice, from Equality (12), that the basin $B_\epsilon(K)$ is f-invariant, i.e. $f^{-1}(B_\epsilon(K) = B_\epsilon(K)$ for all $\epsilon > 0$. We remark that it is not necessarily an open set.

Definition 4.4 Ergodic-like attractor
Let $K \subset M$ be not empty, compact and f-invariant. Let $B \subset M$ be a Borel set with Lebesgue measure $m(B) = \alpha > 0$. We say that K is an *B-observable ergodic-like attractor,* if:
(1) For all $\epsilon > 0$ the basin of ϵ-attraction $B_\epsilon(K)$ of K contains Lebesgue almost all points of B. (Therefore $m(B_\epsilon(K)) \geq \alpha \; \forall \; \epsilon > 0$).
(2) K does not contain proper, compact and

nonempty subsets $K' \subset K$ that satisfy (1).

We say that a nonempty compact set $K \subset M$ is *an ergodic-like attractor* if it is B-observable ergodic-like attractor for some Borel set $B \subset M$ with positive Lebesgue measure.

We notice that the condition

$$B_\epsilon(K) \supset B \ \forall \ \epsilon > 0$$

is trivially satisfied taking $K = M$ for any dynamical system and any subset B. So, no information would be obtained about the statistics of the asymptotic dynamics if only condition (1) were assumed. To be an interesting definition, one adds the condition (2) stating the minimality of K attracting the observable set B.

In Theorem 4.9 and its Corollary 4.10 we will prove that, for all forward invariant set B with positive Lebesgue measure, there exists a B-observable ergodic-like attractor.

Now we define the exact basin of attraction of a B-observable ergodic-like attractor, and prove that it always contains B, and so it has Lebesgue measure which is not smaller than $\alpha = m(B) > 0$.

Definition 4.5

Let $K \subset M$ be a B-observable ergodic-like attractor according to Definition 4.4. Its *exact basin of attraction*, or in brief, *the basin of attraction of* K, is the following set $B(K)$:

$$B(K) := \{x \in M : \ \lim_{\epsilon \to 0^+} \liminf_{N \to +\infty} \ \omega_{\epsilon,\,N}(x, K) \ = 1\},$$

where the frequency $\omega_{\epsilon,N}(x, K)$ is defined in Equality (11).

After Remark 4.3, note that the exact basin of attraction does not necessarily contain a neighborhood of K. Nevertheless, it intersects arbitrarily small neighborhoods of K (see Definition 4.1). Besides, it has non zero Lebesgue measure, as stated in the following theorem:

Theorem 4.6
For all Borel set $B \subset M$ such that $m(B) = \alpha > 0$, and for all B-observable ergodic-like attractor $K \subset M$ (as defined in 4.4), *its exact basin of attraction $B(K) \supset K$* (as defined in 4.5) *contains m-a.e. point of B and therefore, it has Lebesgue measure larger or equal than α.*

Besides

$$B(K) = \bigcap_{\epsilon > 0} B_\epsilon(K) \quad \textit{and} \quad f^{-1}(B(K)) = B(K),$$

where $B_\epsilon(K)$ is the basin of ϵ-attraction to K (as defined in 4.2).

Proof: From Definition 4.2:

$$0 < \epsilon' < \epsilon \ \ \Rightarrow \ \ B_{\epsilon'}(K) \subset B_\epsilon(K).$$

Therefore, applying Definition 4.4, and denoting m to the Lebesgue measure:

$$m\left(\bigcap_{\epsilon>0} B_\epsilon(K)\right) = m\left(\bigcap_{n=1}^{+\infty} B_{1/n}(K)\right) =$$

$$\lim_{n \to +\infty} m(B_{1/n}(K)) \geq m(B) = \alpha > 0.$$

We assert that it is enough to prove that the exact basin of attraction $B(K)$ coincides with the following set S:

$$S = \bigcap_{n=1}^{+\infty} B_{1/n}(K) = \bigcap_{\epsilon>0} B_\epsilon(K).$$

In fact, if $B(K) = S$ then

$$m(B \setminus B(K)) = m(\bigcup_{n=1}^{+\infty} (B \setminus B_{1/n}(K)) \leq$$

$$\sum_{n=1}^{+\infty} m(B \setminus B_{1/n}(K)) = 0.$$

The sum at right is zero, because by Definition 4.4, for all $\epsilon > 0$ the basin of ϵ-attraction $B_\epsilon(K)$ contains m-almost all points of B. Then $m(B \setminus B(K)) = 0$ or, in other words, $B(K)$ also contains m-almost all points of B. Besides, if we prove that $B(K) = S$ then

$$f^{-1}(B(K)) = f^{-1}(S) = f^{-1}(\bigcap_{n=1}^{+\infty} B_{1/n}(K)\,) =$$

$$\bigcap_{n=1}^{+\infty} f^{-1}(B_{1/n}(K)) = \bigcap_{n=1}^{+\infty} B_{1/n}(K) = S = B(K).$$

So, let us prove that $B(K) = S$. Let us first show that $S \subset B(K)$. Fix a point $x \in S$. After Definition 4.2 the point x satisfies the following equalities:

$$1 \geq \liminf_{N\to+\infty}\ \omega_{\epsilon,N}\ (x) > 1-\epsilon \quad \forall\ \epsilon > 0.$$

Taking $\epsilon \to 0^+$, we deduce

$$\lim_{\epsilon\to 0^+}\ \liminf_{N\to+\infty}\ \omega_{\epsilon,N}(x) = 1.$$

Finally, recalling Definition 3.5, the equality above implies that $x \in B(K)$. This latter assertion was proved for a arbitrary point $x \in S$. Thus, we deduce $S \subset B(K)$ as wanted. The opposite inclusion $B(K) \subset S$ is immediate after Definition 4.5 and Equality (12). $\square$

One of the reasons for searching the physical probability measures, if they exist, is that their supports are ergodic-like attractors. Precisely, we state the following result:

Theorem 4.7
If K is the compact support of a physical probability measure $\mu \in \mathcal{M}_f$ (i.e. K is the minimum compact set in M such that $\mu(K) = 1$), *and if $B(\mu)$ denotes the basin of attraction of μ* (as defined in 3.7), *then K is a $B(\mu)-$observable ergodic-like attractor* (according with Definition 4.4). *Besides*

$$B(K) = B(\mu).$$

Proof: Since μ is physical, its basin of attraction $B(\mu) \subset M$ has positive Lebesgue measure, say $\alpha > 0$. Denote $K \subset M$ to the compact support of μ. Fix $\epsilon > 0$ and choose $\psi \in C^0(M, [0, 1])$ such that $\psi|_K = 1$, and $\psi(y) = 0$ if and only if $\text{dist}(y, K) \geq \epsilon$. Therefore

$$\int \psi\, d\mu = \int_K \psi\, d\mu = \mu(K) = 1.$$

Take $x \in B(\mu)$ and compute the following limit

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \psi(f^n(x)) =$$

$$\lim_{N\to+\infty} \int \psi\, d\left(\frac{1}{N}\sum_{n=0}^{\infty} \delta_{f^n(x)}\right) = \int \psi\ d\mu = 1.$$

In the equalities above we have use the characterization of the weak* limit of probabilities, Definition 3.7 of physical measure and Definition 3.5 of $B(\mu)$ which state that $\mathcal{L}^*(x) = \{\mu\}$ for all $x \in B(\mu)$. Thus, for the same previously fixed value of $\epsilon > 0$, and for all $x \in B(\mu)$ there exists $N_0(x)$ such that

$$\frac{1}{N}\sum_{n=0}^{N-1} \psi(f^n(x)) \ > \ 1-\epsilon \qquad \forall\ N \geq N_0(x),$$

and therefore

$$\frac{1}{N}\#\{0 \leq n \leq N-1:\ \psi(f^n(x)) > 0\} \ >$$

$$> \ 1-\epsilon \quad \forall\ N \geq N_0(x).$$

Then, for all $x \in B(\mu)$ and for all $N \geq 1$ the frequency $\omega_{\epsilon,N}\ (x)$, as defined in 4.1, satisfies:

$$\omega_{\epsilon,N}(x, K) =$$

$$\frac{1}{N}\ \#\{0 \leq n \leq N-1:\ \text{dist}(f^n(x), K) < \epsilon\} =$$

$$\frac{1}{N}\ \#\{0 \leq n \leq N-1:\ \psi(f^n(x)) > 0\} > 1-\epsilon.$$

After Definition 3.6, the inequality above implies that

$$B(\mu) \subset B_\epsilon(K).$$

Since $\epsilon > 0$ was arbitrary, we deduce that

$$B(\mu) \subset \bigcap_{\epsilon > 0} B_\epsilon(K) = B(K).$$

It is left to prove that K is minimal satisfying the condition above and that $B(K) = B(\mu)$. Let us first prove the minimality condition. Take a nonempty compact subset $K' \subset K$ such that $B(K') \supset B(\mu)$. We must prove that $K' = K$. It is enough to show that $\mu(K') = 1$, because by hypothesis K is the minimal compact set that satisfies that condition. Take a sequence of continuous functions $\psi_i : M \mapsto [0,1]$ such that $\psi_i|_{K'} = 1$ and $\lim_{i \to +\infty} \int \psi_i(x) = \chi_{K'}(x)$ for all $x \in M$, where $\chi_{K'}(x)$ is 1 if $x \in K'$ and is 0 if $x \notin K'$. Then, by the dominated convergence theorem:

$$\lim_{i \to +\infty} \int \psi_i \, d\mu = \int \chi_{K'}(x) \, d\mu = \mu(K'). \qquad (13)$$

Fix $x \in B(\mu) \subset B(K')$. After the definition of $B(\mu)$ we haFix $0 < \epsilon < 1$ and $i \in \mathbb{N}$. As ψ_i is continuous and $\psi_i|_{K'} = 1$, there exist $0 < \delta < \epsilon$ such that $\psi_i(y) > 1 - \epsilon$ if $\text{dist}(y, K') < \delta$. ve $\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu$ in the weak* topology of $\mathcal{M}$. After Theorem 4.6 and the definition of $B_\delta(K')$ we have $x \in B_\delta(K')$ and so, $\liminf_{n \to +\infty} \omega_{\delta,n}(x, K') > 1 - \delta$ for all n large enough. Then:

$$\int \psi_i \, d\mu = \lim_{n \to +\infty} \int \psi_i \, d \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} =$$

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_i(f^j(x)) \geq$$

$$\liminf_{n \to +\infty} (1-\epsilon) \omega_{\delta,n}(x, K') > (1-\epsilon)(1-\delta) \geq (1-\epsilon)^2.$$

The inequality above holds for all i. Joining with Inequality (13), we deduce that $\mu(K') > (1-\epsilon)^2$. Since ϵ is arbitrary, and μ is a probability measure, we conclude that $\mu(K') = 1$ as wanted. Therefore $K' = K$ and so K is a $B(\mu)$-observable ergodic-like attractor according with Definition 4.4.

Finally, let us prove that $B(K) \subset B(\mu)$. Take $x \in B(K)$ and consider its sequence of empiric probabilities $\nu_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{f^n(x)}$. We must prove that it is convergent in the weak* topology, to μ. Assume by contradiction that there exists a subsequence $\{\nu_{N_j}(x)\}_{j \geq 1}$ such that

$$\lim_{j \to +\infty} \nu_{N_j}(x) = \nu \neq \mu,$$

where the limit is taken in the weak* topology of the space $\mathcal{M}$ of probabilities. Then, there exists a continuous function $\psi \in C^0(M, [0,1])$ such that

$$\int \psi \, d\nu = 0, \qquad \int \psi \, d\mu = 1.$$

As μ is a positive measure, and ψ is a continuous real function with sup value 1, the equality $\int \psi \, d\mu = 1$ implies that $\psi(x) = 1$ for μ-a.e. $x \in M$. Therefore $\psi|_K = 1$, recalling that K is the minimum compact set such that $\mu(K) = 1$.

After the uniform continuity of ψ in the compact manifold M, there exists $0 < \epsilon \leq 1/2$ such that $\psi(y) > 1/2$ if $\text{dist}(y, K) < \epsilon$. From the assumption $x \in \bigcap_{\epsilon > 0} B_{\epsilon(K)}$, there exists $N_0(x)$ such that:

$$\frac{1}{N} \#\{0 \leq n \leq N-1 : \quad \text{dist}(f^n(x), K) < \epsilon\} >$$

$$> \; 1 - \epsilon \quad \forall \, N \geq N_0(x).$$

Therefore

$$\frac{1}{N} \#\{0 \leq n \leq N-1 : \quad \psi(f^n(x)) > \frac{1}{2}\} >$$

$$> \; 1 - \epsilon \quad \forall \, N \geq N_0(x),$$

and thus

$$\liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \psi(f^n(x)) > (1-\epsilon)/2 \geq 1/4.$$

We conclude that

$$\frac{1}{4} < \liminf_{N \to +\infty} \int \psi \, d \left(\frac{1}{N} \sum_{n=0}^{N-1} \delta_{f^n(x)} \right)$$

$$\leq \lim_{j \to +\infty} \int \psi \, d\left(\nu_{N_j}(x)\right) = \int \psi \, d\nu,$$

contradicting the fact that $\int \psi \, d\nu = 0$. $\square$

Definition 4.8 Ergodic attractors (See [24]) If an B-observable ergodic-like attractor K is the compact support of a physical measure μ (see Theorem 4.7), and if μ is ergodic, then K is called *ergodic attractor.*

It is unknown how abundant are the differentiable systems that exhibit ergodic attractors, but there are well known examples that exhibit none, since they have not physical measures or have a unique physical measure which is not ergodic (see for instance Bowen's example, cited in [11]). Nevertheless a similar result to Theorem 4.7 can be obtained using the physical-like measures defined in 3.8, and these lasts always exist for any continuous system (see Theorem 3.9 and its Corollary 3.11) So, the following result implies that any continuous system necessarily exhibits at least one ergodic-like attractor.

Theorem 4.9

Construction of Ergodic-like Attractors

Let $B \subset M$ be a forward invariant Borel set with positive Lebesgue measure. Let $\mathcal{W}|_{f|B} \subset \mathcal{M}$ be the set of physical-like measures restricted to B, as defined in 3.10. *Then, the common compact support $K \subset M$ of all the probabilities in $\mathcal{W}_{f|B}$* (i.e. the minimal compact set K such that $\mu(K) = 1$ for all $\mu \in \mathcal{W}_{f|B}$) *is a B-observable ergodic-like attractor.*

Before proving Theorem 4.9 let us state its Corollary and a Lemma:

Corollary 4.10

Existence of ergodic-like attractors

Any continuous system exhibits ergodic-like attractors.

We prove this Corollary at the end of this section.

Lemma 4.11

If B is a forward invariant Borel set with positive Lebesgue measure and if $\mathcal{W}|_{f|B}$ is the set of the physical-like measures of f restricted to B, then for Lebesgue almost all $x \in B$ the limit set $\mathcal{L}^*(x)$ (defined in 3.2) *is contained in $\mathcal{W}|_{f|B}$.*

Proof: For all $\epsilon > 0$ denote $\mathcal{V}_\epsilon = \{\mu \in \mathcal{M} : \text{dist}(\mu, \mathcal{W}_{f|B}) < \epsilon\}$, where the distance in the space $\mathcal{M}$ of probability measures is taken according to Equality (2). After Corollary 3.11, the set $\mathcal{W}_{f|B}$ is weak* compact. So

$$\mathcal{W}_{f|B} = \bigcap_{\epsilon > 0} \mathcal{V}_\epsilon = \bigcap_{N=1}^{+\infty} \mathcal{V}_{1/N}.$$

Therefore, it is enough to prove that for all $\epsilon > 0$ the set of points $x \in B$ such that

$$\mathcal{L}^* x \ \bigcap \ (\mathcal{M} \setminus \mathcal{V}_\epsilon) \ \neq \ \emptyset$$

has m-measure zero, where m denotes the Lebesgue measure in the ambient manifold M.

In fact, for all $\epsilon > 0$ any measure $\mu \in \mathcal{M} \setminus \mathcal{V}_\epsilon$ is not in $\mathcal{W}_{f|C}$, namely μ is not physical-like for f restricted to B. Therefore, applying Definition 3.10, there exists $\delta_\mu > 0$ such that $m(B_\delta(\mu)) = 0 \ \forall \, 0 < \delta \leq \delta_\mu$. Thus, for m-a.e. point $x \in C$ the set $\mathcal{L}^* x$ does not intersect $\mathcal{B}_{\delta_\mu}(\mu)$, denoting with $\mathcal{B}_{\delta_\mu}(\mu)$ the open ball with center at μ and radius δ_μ in the metric space $\mathcal{M}$ of probabilities. Now let us cover the compact set $\mathcal{M} \setminus \mathcal{V}_\epsilon$ with a finite number of such balls. We conclude that for m-a.e. point $x \in C$ the set $\mathcal{L}^* x$ does not intersect any of such balls. This implies that it does not intersect $\mathcal{M} \setminus \mathcal{V}_\epsilon$, or, in other words $\mathcal{L}^* x \subset \mathcal{V}_\epsilon$ for Lebesgue almost all $x \in B$, ending the proof. $\square$

Proof of Theorem 4.9: To prove that K is a B-observable ergodic-like attractor, we must prove that $m(B \setminus B_\epsilon(K)) = 0$ for all $\epsilon > 0$ and that K is the minimal compact set in the manifold M

with such a property. After Corollary 3.11 the set $\mathcal{W}_{f|B}$ of physical-like measures restricted to B is not empty. By hypothesis $K \subset M$ is compact and supports μ, i.e. $\mu(K) = 1$, for all $\mu \in \mathcal{W}_{f|B}$ (So, K is not empty).

Let us first prove that $m(B \setminus B_\epsilon(K)) = 0$ for all $\epsilon > 0$.

Fix $\epsilon > 0$ and choose $\psi \in C^0(M, [0,1])$ such that $\psi|_K = 1$, and $\psi(y) > \epsilon$ if $\operatorname{dist}(y, K) \geq \delta$. Therefore

$$\int \psi \, d\mu = \int_K \psi \, d\mu = \mu(K) = 1 \qquad \forall \ \mu \in \mathcal{W}_{f|B}. \tag{14}$$

After Lemma 4.11 for Lebesgue almost all point in B: $\mathcal{L}^* x \subset \mathcal{W}_{f|B}$. So, to prove that $m(B \setminus B(K)) = 0$ it is enough to prove that all points $x \in B$ such that $\mathcal{L}^* x \subset \mathcal{W}_{f|B}$ belong to $B(K)$. To do that is it enough that they belong to $B_\epsilon(K)$ for all $\epsilon > 0$. Fix such a point x and fix $\epsilon > 0$. Choose $\psi \in C^0(M, [0,1])$ such that $\psi|_K = 1$, and find $0 < \delta < \epsilon$ such that $\psi(y) = 0$ if $\operatorname{dist}(y, K) \geq \epsilon$. Therefore

$$\int \psi \, d\mu = \int_K \psi \, d\mu = \mu(K) = 1 \qquad \forall \ \mu \in \mathcal{W}_{f|B}. \tag{15}$$

Consider an increasing sequence $n_i \to +\infty$ of natural numbers such that

$$\lim_{i \to +\infty} \omega_{\epsilon, n_i}(x, K) = \liminf_{n \to +\infty} \omega_{\epsilon, n}(x, K),$$

where $\omega_{\epsilon,n}(x, K)$ is the frequency in which the future orbit of x from time 0 to time n ϵ-approaches the compact set K, as defined in (11).

Taking a subsequence of $\{n_i\}_{i \geq 1}$ if necessary, it is not restrictive to assume that the sequence of empiric probabilities of x, defined in (10), is weak*-convergent to a probability, say μ. (In fact, recall that the space $\mathcal{M}$ of probabilities is sequentially compact endowed with the weak* topology). So,

$$\lim_{i \to +\infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \delta_{f^j(x)} = \mu \in \mathcal{L}^*(x) \subset \mathcal{W}_{f|B}.$$

Therefore, applying the characterization of the weak* limit given in Equality 5:

$$1 = \int \psi \, d\mu = \lim_{i \to +\infty} \int \psi \, d \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \delta_{f^j(x_N)} =$$

$$\lim_{i \to +\infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \psi(f^j(x_N)) \leq$$

$$\lim_{i \to +\infty} \frac{1}{n_i} \# \{0 \leq j \leq n_i - 1 : \ \operatorname{dist}(f^j(x), K) < \epsilon\}$$

$$= \lim_{i \to +\infty} \omega_{\epsilon, n_i}(x, K) \leq 1.$$

To obtain the inequality above recall that

$$\psi(y) = 0 \quad \text{if} \quad \operatorname{dist}(y, K) \geq \epsilon$$

and $\psi(y) \leq 1$ otherwise. So:

$$1 - \epsilon < 1 = \lim_{i \to +\infty} \omega_{\epsilon, n_i}(x, K) = \liminf_{n \to +\infty} \omega_{\epsilon, n}(x, K).$$

Therefore we conclude that $x \in B_\epsilon(K)$, for all $x \in B$ such that $\mathcal{L}^* x \subset \mathcal{W}_{f|B}$. As we proved this assertion for all $\epsilon > 0$, we joint it with Theorem 4.6 and Lemma 4.11, to conclude that Lebesgue almost all points of B belong to $B(K)$, as wanted.

Now, let us prove that $K \subset M$ is minimal compact satisfying the condition

$$m(B \setminus B(K)) = 0.$$

Take a nonempty compact subset $K' \subset K$ such that $m(B \setminus B(K')) = 0$. We must prove that $K' = K$. Assume by contradiction that $K' \neq K$. By hypothesis K is the minimal compact set such that $\mu(K) = 1$ for all $\mu \in \mathcal{W}_{f|B}$. Then, there exists $\nu \in \mathcal{W}_{f|B}$ such that $\nu(K') < 1$.

Fix a neighborhood V of K' such that

$$\nu(V) < 1.$$

Fix $\epsilon > 0$ such that $\epsilon < 1 - \nu(V)$ and choose a continuous function $\psi : M \mapsto [0,1]$ such that $\psi|_{K'} = 1$ and $\psi(y) = 0$ if $y \notin V$. Choose $0 < \delta < \epsilon/4$ such that $\psi(y) > (1 - \epsilon/4)$ if

$\text{dist}(y, K') < \delta$. We are assuming that $B(K')$ contains Lebesgue almost all $x \in B$. Fix any such a point x and consider:

$$\left(1 - \frac{\epsilon}{4}\right) \omega_{\delta,n}(x, K') \leq$$

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(x)) \leq \omega_{\epsilon,n}(x, K') \leq 1.$$

Recall that

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(x)) = \int \psi \, d\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}.$$

So, for any sequence $n_i \to +\infty$ such that the empiric probabilities $\nu_{n_i}(x)$ of x converge to a measure $\mu \in \mathcal{L}^*(x)$, we have

$$\left(1 - \frac{\epsilon}{4}\right) \liminf_{n \to +\infty} \omega_{\delta,n}(x, K') \leq \int \psi d\mu^* \leq 1.$$

Since $x \in B(K') \subset B_\delta(K')$, we deduce that the liminf above is greater or equal than $(1-\delta)$ (recall Equality (12)). So, we conclude that

$$\int \psi \, d\mu^* \geq (1 - \epsilon/4)(1 - \delta) >$$

$$(1 - \epsilon/4)^2 > 1 - \epsilon/2 > \nu(V) + \epsilon/2 >$$

$$\int \psi \, d\nu + \epsilon/2$$

$$\forall \ \mu \in \mathcal{L}^*(x) \ \text{ for } m- \text{ a.e. } x \in B.$$

From Equality (2) defining the distance in the space of probabilities and from the denseness of the family $\Psi = \{\psi_i\}_{i \geq 1} \subset C^0(M, [0, 1])$, we conclude that there exists a continuous real function $\psi_{i_0} \in \Psi$ such that $\|\psi_{i_0} - \psi\|_{sup} \leq \epsilon/8$. Therefore, for all $\mu \in \mathcal{L}^*(x)$, for m-a.e. $x \in M$:

$$\frac{\epsilon}{2} \leq \left| \int \psi \, d\mu - \int \psi \, d\nu \right| \Rightarrow$$

$$\Rightarrow \quad \frac{\epsilon}{2} \leq \left| \int \psi \, d\mu - \int \psi_{i_0} \, d\mu \right| +$$

$$+ \left| \int \psi_{i_0} \, d\mu - \int \psi_{i_0} \, d\nu \right| + \left| \int \psi_{i_0} \, d\nu - \int \psi \, d\nu \right|$$

$$\Rightarrow \quad \frac{\epsilon}{2} \leq \left| \int \psi_{i_0} \, d\mu - \int \psi_{i_0} \, d\nu \right| + \frac{2\epsilon}{8}.$$

Then:

$$\left| \int \psi_{i_0} \, d\mu - \int \psi_{i_0} \, d\nu \right| \geq \frac{\epsilon}{2} - \frac{\epsilon}{4} = \frac{\epsilon}{4}.$$

So

$$\text{dist}(\mu, \ \nu) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \left| \int \psi_i \, d\mu - \int \psi_i \, d\nu \right| \geq$$

$$\geq \frac{1}{2^{i_0}} \cdot \frac{\epsilon}{4} = \rho > 0 \, \forall \mu \in \mathcal{L}^*(x) \quad \text{for } m- \text{ a.e. } x \in B.$$

The last inequality can be restated as follows: $m(B_\rho(\nu)) = 0$, where $B_\rho(\nu)$ is the basin of ρ-weak attraction of ν, as defined in 3.6. Then, taking into account Definition 3.10, we conclude that ν is not weak physical for $f|_B$, namely $\nu \notin \mathcal{W}_{f|B}$, contradicting its construction at the beginning. This ends the proof that K is minimal compact with the property $m(B \setminus B(K)) = 0$. Thus K is a B-observable ergodic-like attractor, and the proof of Theorem 4.9 is ended. $\square$

Proof of Corollary 4.10: Take any forward invariant Borel set $B \subset M$ with positive Lebesgue measure. Such sets always exist, since one can take for instance $B = M$. After Theorem 3.9 and its Corollary 3.11, there exists a nonempty weak*-compact set $\mathcal{W}|_{f|B}$ of physical-like probabilities of f restricted to B. Since $\mu(M) = 1$ for all $\mu \in \mathcal{W}_{f|B}$, the family Γ of all the compact sets $K \subset M$ that support all the measures $\mu \in \mathcal{W}_{f|B}$ is not empty. Define in Γ the partial order $K_1 \leq K_2$ if and only if $K_1 \subset K_2$. Any chain $\{K_n\}_{n \geq 0}$ (i.e. $K_{n+1} \leq K_n$ for all $n \geq 0$), defines a minimal element (respect to the chain) $K_\infty = \bigcap_{n \geq 0} K_n \in \Gamma$. In fact K_∞ is compact and $\mu(K_\infty) \lim_{n \to +\infty} \mu(K_n) = 1$ for all $\mu \in \mathcal{W}_{f|B}$. So K_∞ is not empty and supports all the measures of $\mathcal{W}_{f|B}$. Thus $K_\infty \in \Gamma$. Zorn Lemma states that in any partially ordered set Γ, if all chains define a minimal element (respect to the chain) in Γ, then there exist minimal elements of the whole set Γ, namely some element $K \in \Gamma$ such that all $K' \leq K$ in Γ must coincide with K. Therefore, after Zorn Lemma, there exists at least one nonempty compact set $K \in \Gamma$ such that $\mu(K) = 1$ for all $\mu \in \mathcal{W}_{f|B}$ and such that K has no proper nonempty compact subsets with that property. (Besides K is unique, as minimal element of Γ, because if there existed two of them $K_1 \neq K_2$, then $K_3 = K_1 \bigcap K_2$ would satisfy $K_3 \subset K_1, \quad K_3 \neq K_1$ and $K_3 \in \Gamma$, contradicting the minimality of K_1).

We have proved that the minimal common compact support K of all the probabilities $\mu \in \mathcal{W}_{f|B}$ exists (and is nonempty). After Theorem 4.9 this compact set K is a B-observable ergodic-like attractor. $\Box$

5 Conclusions

We have defined the weak physical probability measures for all continuous dynamical system. We have proved that any such system possesses weakly physical measures (Theorem 3.9). This result is significant because, on one hand strong physical measures mostly do not exist, and on the other hand the statistical description that physical provide, when they exist, is preserved by weakly physical probabilities. Precisely, the set $\mathcal{W}|_f$ of all weakly physical measures is a set of f-invariant probabilities that describes completely the asymptotic statistics of Lebesgue almost all orbits attracted to an ergodic-like attractor K (Theorem 4.9). Besides this attractor K, joint the the weak physical measures supported on K, have the following properties:

(1) After Lemma 4.11, the set $\mathcal{W}|_f$ of invariant probabilities contains all the limit measures of all the convergent subsequences of the empiric distributions $\nu_n(x) := (1/n)\sum_{j=0}^{n-1} \delta_{f^j(x)}$ for Lebesgue almost all x in the basin of attraction of K. In brief, the set $\mathcal{W}|_f$ gives a complete statistical description of the asymptotic time mean of the orbits of a full Lebesgue measure set of initial states. We have proved this ergodic-like result in a so general scenario that includes also all those continuous systems that do not preserve the Lebesgue measure.

(2) The description is spacial, since the measures in $\mathcal{W}_f$ are probability distributions in the ambient manifold M where the dynamics evolves, but is also temporal, since they describe the limits of the means in time.

(3) The attractor K that supports all the probability measures of the set $\mathcal{W}_f$ is minimal among the compact sets of the ambient manifold that attract weakly all the orbits of its basin. (Condition (2) of Definition 4.4). In fact, in Theorem 4.9 we have proved that the compact support K satisfies both conditions of Definition 4.4. This minimality property of the ergodic-like attractor ensures an optimality condition of our results: the Lebesgue-full attraction property of the orbits can not be obtained trying to reduce the attractor to a proper subset of K.

Summarizing, the results of this paper hold as an application of Measure Theory, to the abstract theory of continuous Dynamical Systems and Ergodic Theories. They hold for all C^0 mappings on compact manifolds. So, they include the known results about the ergodic attractors (Definition 4.8) as supports of physical measures, and the known Ergodic Theory of Differentiable Dynamics. But, on the other hands the results in this paper wide this theory, since they are not restricted to the condition of differentiability.

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References

[1] L.Ambrosio; N.Giglio; G.Savaré: *Gradient Flows in Metric Spaces and in the Space of Probabilities,* Brikhäuser–Verlag, Zürich, 2005.

[2] D.V.Anosov: Ergodic theory; in M. Hazewinkel (editor) *Encyclopaedia of Mathematics* , Springer–Verlag, Berlin–Heidelberg–New York–Tokyo, 2001

[3] P.Billingsley: *Convergence of Probability Measures.* Wiley Series in Probability and Statistics. Wiley-Interscience, New York–Toronto, 1999

[4] C.Bonatti; L.Diaz; M.Viana: *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective.* Springer–Verlag, Berlin–Heidelberg–New York–Tokyo, 2005

[5] R.Bowen; D.Ruelle: The ergodic theory of Axiom A flows. *Invent. Math.* 29, 1975, pp.281–202.

[6] E.Catsigeras; H.Enrich: SBR-like measures for C^0 dynamics. *Bull. Polish Acad. of Scienc. Math.* 59, 2011, pp. 151–164.

[7] W. Cowieson; L.S. Young: SRB measures as zero-noise limits. *Ergod. Th. Dynam. Sys.* 25, 2005, pp. 1115–1138.

[8] H.Eschrig: *Topology and geometry for physics* Lecture Notes in Physics, 822. Springer, Heidelberg, 2011

[9] M.Foreman; D.Rudolph; B.Weiss: The conjugacy problem in ergodic theory, *Ann. of Math.* 173, 2011 , pp 1529-1586

[10] M.I.García-Planas; S.Tarragona: Output observability of generalized linear systems *Int. Journ. of Pure and Applied Math.* 66, 2011, pp. 199-208

[11] T.Golenishcheva-Kutuzova; V.Kleptsyn: Convergence of the Krylov-Bogolyubov procedure in Bowan's example (Russian) *Mat. Zametki* 82, 2007, pp. 678-689; *Translation in Math. Notes* 82, 2007, pp. 608-618.

[12] J. Hatamoto: Diffeomorphisms admitting SRB measures and their regularity. *Kodai Math. Journ.* 29, 2006, pp.211-226

[13] H.Hu; L.S.Young: Nonexistence of SRB measures for some diffeomorphisms that are almost Anosov. *Ergod. Th. and Dyn. Sys.* 15, 1995, pp.67–76.

[14] S.Kalikow; R.McCutcheon: *An outline of ergodic theory. Cambridge Studies in Advanced Mathematics, 122*, Cambridge University Press, Cambridge, 2010

[15] G.Keller: *Equilibrium States in Ergodic Theory* London Math. Soc. Student Texts, Vol. 42, Cambridge University Press, Cambridge, 1998.

[16] G.Keller; C.Liverani: Uniqueness of the SRB measure for piecewise expanding weakly coupled map lattices in any dimension. *Comm. Math. Phys.* 262, 2006, pp. 33–50

[17] S.G.Krantz: *Essentials of topology with applications* Textbooks in Mathematics, CRC Press, Boca Raton Florida, 2010.

[18] G.Laures; M.Szymik: *Grundkurs Topologie* Spektrum Akademischer, Verlag GmbH, Heidelberg, 2009

[19] R.Leplaideur: Existence of SRB measures for some topologically hyperbolic diffeomorphisms. *Ergodic Theory and Dynam. Systems* 24, 2004, pp 1199–1225.

[20] R.Mañé: *Ergodic theory and differentiable dynamics.* Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Springer –Verlag, Berlin–Heidelberg–New York–Tokyo, 1987.

[21] N. Mastorakis; O.V. Avramenko: Fuzzy models of the dynamic systems for evolution of populations. *WSEAS Trans. Math.* 6 2007, pp. 667–680

[22] G.Mircea; M.Neamtu; A.Ciurdariu; D.Opris: Numerical simulations for dynamic stochastic and hybrid models of Internet networks. *WSEAS Trans. Math.* 8, 2009, pp. 679–688

[23] J.Palis: A global view of Dynamics and a conjecture on the denseness of finitude of attractors. *Astérisque* 261, 1999, pp.339–351.

[24] C.Pugh; M.Shub: Ergodic Attractors. *Trans. Amer. Math. Soc.* 312, 1989, pp. 1–54.

[25] H.Qiu: Existence and uniqueness of SRB measure on C^1 generic hyperbolic attractors. *Comm. Math. Phys.* 302, 2011, pp. 345–357.

[26] B.S.Reddy: The Riesz Theorem in random n-normed spaces *Int. Journ. of Pure and Applied Math.* 68, 2011, pp. 1-12

[27] M.Reid; B.Szendrou: *Geometry and Topology* Cambridge University Press, Cambridge, 2006

[28] D.Ruelle: A measure associated with axiom A attractors. *Amer. Journ. of Math.,* 98, 1976, pp.619–654.

[29] O.I.Sandru: Mathematical models that coordinate the movement through obstacles of the dynamic systems endowed with artificial sight. *WSEAS Trans. Math.* 7, 2008, pp. 482–491

[30] Ya.Sinai: Gibbs measure in ergodic theory. *Russ. Math. Surveys* 27, 1972 pp. 21–69.

[31] Ya.Sinai: *Ergodic Theory and Dynamical Systems. Selecta. Volume I.* Springer, New York, 2010

[32] A.G.Sumedrea; L.Sangeorzan: A mathematical theory of psychological dynamics. *WSEAS Trans. Math.* 8, 2009, pp. 604–613

[33] M.Urbanski; C.Wolf: SRB measures for Axiom A endomorphisms. *Math. Res. Lett.* 11, 2004, pp. 785–797

[34] C.Wolf: Generalized physical and SRB measures for hyperbolic diffeomorphisms. *Journ. Stat. Phys.* 122, 2006, pp.2111–1138.

[35] L.S.Young: What are SRB measures, and which dynamical systems have them? *Journ. Stat. Phys.* 108, 2002, pp.733–754.