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Self-synchronization of networks with a strong kernel of integrate and fire excitatory neurons

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Abstract: We study the global dynamics of networks of pulsed-coupled neurons that are modeled as integrate and fire oscillators. We focus on excitatory networks with a strong kernel. We prove the synchronization of the whole network from any initial state, and find a bound from above of the transients until the full synchronization is achieved. The methodology of research is by exact mathematical definitions and statements and by deductive proofs, from standard arguments of the mathematical theory of abstract dynamical systems. We include examples of applications to diverse fields, and also a brief review of other mathematical methods of research on general networks of dynamically interacting units.

Key-Words: Pulse-coupled networks, synchronization, neural networks, integrate and fire oscillators.

1 Introduction

The large-scaled dynamics of systems that change on time and are composed by many interacting units, emerges from the free dynamics of each unit and from the rules of interactions among them. Such dynamical systems are called, in a general and abstract context, *networks*. Applications of the mathematical analysis of networks are abundant in diverse fields of Science and Technology (see Section 4).

In particular, some class of networks come from mathematical dynamical models of the living nervous system, and are called neuronal networks. Among these networks we are focusing on those composed by *integrate and fire* neurons (e.g. [4, 7]), that are assumed to be *pulsed coupled* (e.g. [19]).

The *pulse-coupling* means that the interactions are instantaneous and are not produced all along time, but at certain discrete sequence of instants that are separated by regular or irregular time intervals. An autonomous system decides by itself the instants at which the interactions among the units of the network are produced. In other words, the network self-organizes. In Section 2, we pose the exact statements of the mathematical model under study.

The emergent dynamics of pulsed-coupled networks is usually modeled by *impulsive differential or integro-differential equations* (e.g. [18, 38]). For instance, the spikes of the neurons during the synaptic activity in the living nervous system is mathematically studied by the solution of impulsive differential equa-

tions (e.g.[6]).

The global dynamics of impulsive differential or integro-differential equations is a source of many mathematically open questions, whose answers are mostly unknown, except in particular cases or in low dimensions (e.g. [46, 51]). In general, dynamical systems of coupled units (even if they are not pulsed-coupled or modeled by impulsive equations), pose new open problems to Mathematics, which are particularly difficult to solve if the interactions' parameters belong to an intermediate range, neither too strong nor too weak (e.g. [49]).

The research in this paper focusses on the *synchronization* mathematical problem of pulsed-coupled networks of integrate and fire neurons without delay.

The synchronization phenomenon of several identical (or at least similar) dynamical units appears in Physics: For instance, the global synchronization of networks with large complexity was mostly studied for mutually coupled identical oscillators (e.g. [33] and references therein). Those results were applied to study the behavior of Light Controlled Oscillators (LCO) in [35, 36]. LCO systems are used to study diverse biological systems, as for instance, the emergent synchronized dynamics of populations of the south-eastern fireflies: a large number of insects flash altogether as the result of the mutual interactions produced by the light impulsive signals among them. Such an experimental result with electronic simulated fireflies was obtained in [36], while [37] proved math-

ematically the synchronization for two electronic fireflies, which are governed by linear differential equations during the time-intervals between consecutive fires.

Other results of synchronization for arbitrarily large networks of pulsed-coupled units, are already known and rigorously proved: For instance, [32] proved the synchronization of large completely connected networks of identical units, with constant positive interactions, and assuming that the evolution among pulses is linear on time. [7] proved it for large completely connected networks of identical units, with non constant positive interactions, and non linear dependence on time. [4] prove it for completely connected networks, non constant positive interactions, and arbitrary dependence on time, provided that certain stability property holds.

In Theorem 8 of this paper, we generalize the previous results cited above: We prove the synchronization of an arbitrarily large number of dynamical units, governed by impulsive differential equations of any type, provided that the state variable of each neuron is increasing on time (during the time-intervals between consecutive interactive impulses). We assume arbitrary, positive or null interactions, and a network's graph that is not necessarily complete, but has a *strong kernel* with strictly positive weights (Definition 6).

The synchronization of neuronal networks is relevant in the nervous system, not because the whole system synchronizes (it certainly does not), but because some specialized subnetworks synchronize. These latter groups of cells, allow a living individual acquire stable biological rhythms that coexist with other non synchronized regions of the brain. The existence of stable biological rhythms are essential for life. For instance, the heart pacemaker neurons work in synchrony [27]. Stable partial synchronization is also necessary for the regulation of the information that is generated or processed by the nervous system. This information is not properly chaotic. Namely, it does not necessarily increase with a positive rate forever in the future, but acquires a formed of self-controlled structured information [47].

Periodic limit cycles that are not necessarily synchronized orbits, appear also in mathematical models of biological neuronal networks that are not all excitatory [8, 7]. Nevertheless, since the periods (and also the finite number of periodic orbits) may be arbitrarily large (e.g. [8]), the observed behaviour during a finite interval or time seems to be irregular. This phenomenon is called *virtual chaos* in [8], or *stable chaos* in [34]. The same argument shows that also when the network finally synchronizes, it may look exhibiting virtual chaos during the transitory times, if these transients last too long.

As said above, in Theorem 8 we prove synchronization of the network. Besides, by Formula (8) we bound from above the transient duration until full synchronization is achieved. Along the proofs of these results we obtain intermediate other results, such as the formation, during the transients, of different patterns of clustered cells that mutually synchronize and do not include all the cells of the network (Lemmas 11 and 12). The proofs assume that the network's dynamics is autonomous and deterministic, the impulses are excitatory (i.e. positive or null), no delay exists, and the cells are integrate and fire oscillators. Nevertheless, if those hypothesis on the model do not hold, still synchronization, or at least phase locking, can be proved. In fact, [30] proved the synchronization of coupled oscillators that are stochastically modeled. Also synchronization was proved in some pulsed-coupled networks with delayed interactions [48]. Synchronization among two chaotic dynamical units (the cells are not oscillators) was proved when the interactions are continuous on time, governed by a system of differential equations of fractional order [45]. Finally, in [43] some mutually coupled chaotic circuits were proved to exhibit the so called *inverse lag synchronization*, which is, roughly speaking, the phase lock on opposite phases.

2 The mathematical model

Along this paper, each neuron i of the network is modeled as an *integrate and fire* oscillator. This means that its instantaneous state is described by a real variable x_i which has two phases: a integration phase, which is the integral flow that solves an ordinary differential equation, and a firing phase, which is governed by a spiking rule according to which the state of the neuron resets and sends instantaneous actions to the other neurons of the network.

2.1 The integration phase

During the integration phase the variable x_i (if the neuron i were hypothetically isolated from the network) is the solution of an autonomous differential equation with initial condition $x_i(0)$, of the following type:

$$\frac{dx_i}{dt} = f(x_i) \quad \text{if } L \leq x_i < U, \quad L \leq x_i(0) < U \quad (1)$$

where $t \in \mathbb{R}$ is time, $U > 0$ and $L < 0$ are constants, and $f : [L, U] \mapsto \mathbb{R}^+$ is a positive Lipschitz-continuous function: As the domain of f is the compact interval $[L, U]$ and f is continuous, the positive values of f are bounded away from zero. Namely, there exists a constant $a > 0$ such that

$$f(x) \geq a > 0 \quad \forall x \in [L, U]. \quad (2)$$

Therefore, from the differential equation (1) we obtain that $x_i(t)$ is strictly increasing on time t , and its derivative with respect to t is larger than a positive constant a . In other words, the velocity according to which the variable $x_i(t)$ increases, is larger than $a > 0$. Since the initial condition $x_i(0)$ is lower than U , we deduce that there exists a finite time $T_i > 0$ (which depends on the initial state $x_i(0)$ of the neuron i) such that x_i reaches the upper bound U . Precisely,

$$x_i(T_i^-) := \lim_{t \rightarrow T_i^-} x_i(t) = U. \quad (3)$$

Definition 1 The constant upper bound $U > 0$ of the state x_i of each neuron, is called the threshold level. The lowest constant value $L < 0$ that x_i can hypothetically take, is called the low bound. The model states that the differential equation (1) holds while the state x_i has not arrived to its threshold level and is not smaller than the low bound.

2.2 The instantaneous firing phase

Definition 2 At any instant $T > 0$ at which the state x_i arrives to (or is larger than) the threshold level U - in particular, at the first instant $T_i > 0$ satisfying Equality (3), - we say that the neuron i spikes or fires. Such an instant T is called a spiking instant.

A firing of a neuron, by hypothesis, produces two consequences:

- First, at each instant $T > 0$ for which $x_i(T^-) = U$, the state x_i of the neuron *resets*. By changing the origin in the real axis of the variable x_i , if necessary, it is not restrictive to assume that the reset value is zero. Explicitly:

$$\forall T > 0, \text{ if } \lim_{t \rightarrow T^-} x_i(t) = U, \text{ then } x_i(T) = 0. \quad (4)$$

In particular, for the first instant $T_i > 0$ at which the neuron i fires, we have $x_i(T_i) = 0$.

- Second, at each spiking instant $T > 0$, the neuron i sends an *instantaneous action signal* $A_{i,j}$ to the other neurons $j \neq i$ of the network. This model assumes that $A_{i,j}$ is a real number that depends only on i and j , but not on time.

Definition 3 A neuron i is *excitatory* if $A_{i,j} \geq 0$ for all $j \neq i$, and it is *inhibitory* if $A_{i,j} \leq 0$ for all $j \neq i$. We say that a network \mathcal{N} is *excitatory* if all its neurons are excitatory. Along this paper we only consider excitatory networks. Note that not all the interactions must be strictly positive.

2.3 The interactions

If $T > 0$ is an instant at which only one neuron $j \neq i$ spikes, then the state x_i of the neuron i that receives the action signal $A_{j,i} \geq 0$ from the neuron j - suffers a discontinuity jump which is defined by the following rule:

$$\begin{aligned} x_i(T) &= x_i(T^-) + A_{j,i} \text{ if } < U, \\ x_i(T) &= 0 \text{ otherwise} \end{aligned}$$

If many neurons j_1, \dots, j_N (all different from i) spike simultaneously at an instant $T > 0$, then the state x_i of the neuron i suffers a discontinuity jump defined by:

$$x_i(T) = x_i(T^-) + \sum_{l=1}^N A_{j_l,i} \text{ if } < U, \quad (5)$$

$$x_i(T) = 0 \text{ otherwise} \quad (6)$$

2.4 Strong excitatory kernel

Definition 4 The network's graph is defined such that its vertices are the neurons, and their edges are directed and weighted with the value $A_{i,j}$ for each nonzero action signal $A_{i,j} \neq 0$ from the neuron (or vertex) i to the neuron (or vertex) $j \neq i$. Note that the network's graph is not necessarily symmetric, i.e. $A_{i,j}$ may differ from $A_{j,i}$. Also maybe only one of the two directed edges between i and j may exist (in such a case the action in the opposite direction is zero).

Definition 5 A kernel \mathcal{K} of an excitatory network \mathcal{N} is, if it exists, a complete subgraph, such that $A_{i,j} > 0$ for all $i \in \mathcal{K}$ and for all $j \in \mathcal{N}$, $j \neq i$. In Figure 1 we draw a simple example of a network of six neurons with a kernel of three neurons.

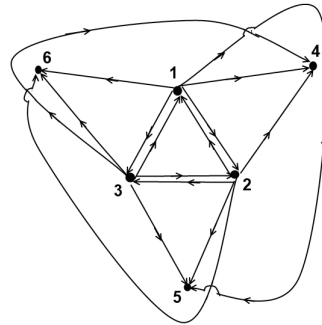


Figure 1: The network $\mathcal{N} = \{1, 2, 3, 4, 5, 6\}$ has a kernel $\mathcal{K} = \{1, 2, 3\}$.

The neurons that are not in the kernel may have null or positive interactions among them, and from them to the neurons in the kernel.

Definition 6 We say that a kernel \mathcal{K} of an excitatory network \mathcal{N} is strong if the number k of neurons in \mathcal{K} is at least 3 and if the minimum excitatory action from a neuron in \mathcal{K} to the other neurons of \mathcal{N} is strong enough to satisfy the following inequality:

$$\min\{A_{i,j}: i \in \mathcal{K}, j \in \mathcal{N}, i \neq j\} \geq \frac{U - L}{\sqrt{k}}, \quad (7)$$

where $U > 0$ is the threshold level and $L < 0$ the low bound of the states of the neurons (cf. Definition 1).

In the above definition, the interactions $A_{i,j}$ for $i \in \mathcal{K}$, $j \in \mathcal{N}$, $i \neq j$, need not to be actually strong, in an absolute sense. In fact, the number k of neurons in the kernel \mathcal{K} may be large enough, so Inequality (7) holds for a small value of $\min\{A_{i,j}: i \in \mathcal{K}, j \in \mathcal{N}, i \neq j\}$. In other words, for arbitrarily small positive interactions $A_{i,j}$, the kernel \mathcal{K} is strong, (i.e. it satisfies Inequality (7)), if it has a sufficiently large number of neurons. On the contrary, if the kernel has a small number of neurons (say for instance, $k = 3$), it will be strong only if the minimum positive interaction $A_{i,j}$ is large enough to satisfy Inequality (7).

3 Synchronization

3.1 Statement of the main result

Let \mathcal{N} be a network composed by m integrate and fire neurons. We call the m -th. vector

$$\mathbf{x}(0) = (x_1(0), \dots, x_i(0), \dots, x_m(0))$$

the initial state of the network. We denote by

$$\mathbf{x}(t) = (x_1(t), \dots, x_i(t), \dots, x_m(t)),$$

the state of network at instant t , and call $\{\mathbf{x}(t)\}_{t \geq 0}$ the orbit with initial state $\mathbf{x}(0)$.

Definition 7 We say that the orbit $\{\mathbf{x}(t)\}_{t \geq 0}$ is synchronized or that the network with initial state $\mathbf{x}(0)$ is synchronized, if

$$x_i(t) = x_j(t) \quad \forall t \geq 0 \quad \forall i \neq j.$$

We say that the orbit - or the network- with initial state \mathbf{x}_0 synchronizes after a transitory time T_0 , if $T_0 \geq 0$ is the minimal non negative real number such that

$$x_i(t) = x_j(t) \quad \forall t \geq T_0 \quad \forall i \neq j.$$

Theorem 8 Let \mathcal{N} be an excitatory network with a strong kernel \mathcal{K} . Then, from any initial state there exists a transitory time $T_0 \geq 0$ (which, in general, depends on the initial state) such that the network synchronizes for all $t \geq T_0$. Besides,

$$T_0 \leq \frac{U - L}{\min\{f(x) : L \leq x \leq U\}}, \quad (8)$$

where $U > 0$ is the threshold level, $L < 0$ is the low bound of the state x_i of each neuron $i \in \mathcal{N}$, and f is the real function in the second member of the differential equation (1).

We start the proof of Theorem 8 with Lemma 9 in the following Subsection, and end it in Subsection 3.3.

3.2 Lemmas

Assume that \mathcal{N} is an excitatory network with a strong kernel \mathcal{K} as in the hypothesis of Theorem 3.3. To simplify the notation, in the sequel we agree that $i \neq j$ when we write i, j to denote two neurons of the network \mathcal{N} .

Let N be the minimum positive natural number such that

$$N \geq \frac{U - L}{\min\{A_{j,i}: j \in \mathcal{K}, i \in \mathcal{N}\}}. \quad (9)$$

Lemma 9 In the hypothesis of Theorem 8, if at least N neurons of the kernel \mathcal{K} spike simultaneously at some instant $T_0 > 0$, then the neurons of the whole network spike altogether at instant T_0 .

Proof: Arguing by contradiction, let us assume that there exists a neuron $i \in \mathcal{N}$ that does not spike at instant T_0 . We have

$$x_i(T_0) < U. \quad (10)$$

Since $L < 0$ is the low bound of the state x_i of any neuron, we know

$$L \leq x_i(T_0^-) < U,$$

where $x_i(T_0^-)$ denotes the limit when $t \rightarrow T_0^-$ of the state $x_i(t)$ at instant t , which is governed by the differential equation (1). By hypothesis, there exists at least N neurons j_1, j_2, \dots, j_N of the kernel \mathcal{K} spiking at instant T_0 . Thus, these neurons j_l are sending actions $A_{j_l,i}$ to the neuron i . Since the whole network is excitatory, we have $A_{j,i} \geq 0$ for all $j \neq i$. In particular, for all $j \in J_{T_0}$ (where J_{T_0} denotes the subset

of all the neurons that spike at instant T_0), we have $A_{j,i} \geq 0$. Applying Equality (5), we obtain

$$x_i(T_0) = x_i(T_0^-) + \sum_{j \in J_{T_0}} A_{j,i} \geq x_i(T_0^-) + \sum_{l=1}^N A_{j_l,i}.$$

From Definition 5, $A_{j_l,i} > 0$ for all $j_l \in \mathcal{K}$, $\forall i \in \mathcal{N}$. Besides $x_i(T_0^-) \geq L$. Thus, we obtain

$$x_i(T_0) \geq L + N \cdot \min\{A_{j,i} : j \in \mathcal{K}, i \in \mathcal{N}\}.$$

Taking into account the definition of the natural number N as the minimum $N \geq 1$ that satisfies Inequality (9), we deduce

$$x_i(T_0) \geq L +$$

$$\frac{U - L}{\min\{A_{j,i} : j \in \mathcal{K}, i \in \mathcal{N}\}} \cdot \min\{A_{j,i} : j \in \mathcal{K}, i \in \mathcal{N}\} \\ = L + (U - L) = U.$$

Therefore, we deduce that $x_i(T_0) \geq U$, which contradicts to inequality (10) and ends the proof of Lemma 9. \square

Lemma 10 *There exists a strictly increasing sequence $\{t_n\}_{n \geq 1}$ of instants $t_n > 0$ such that:*

- (i) *For all $n \geq 1$ there exists at least one neuron of the kernel \mathcal{K} spikes at instant t_n .*
- (ii) *No neuron of the kernel \mathcal{K} spikes during the time intervals (t_n, t_{n+1}) for all $n \geq 1$, and also during $[0, t_1)$.*

Proof: Such a sequence $\{t_n\}_{n \geq 1}$ exists, due to Equality (3). In fact, denote $t_0 = 0$. Since any neuron i (and in particular any neuron of the kernel) has a minimal spiking instant $T_i > 0$, then, from any instant $t_n > 0$, there always exists a first next instant $t_{n+1} > t_n$ for which at least one neuron of the kernel spikes. Since t_{n+1} can be chosen minimal with such a property, no neuron spikes during the time intervals $[0, t_1)$ and (t_n, t_{n+1}) . \square

Lemma 11 *In the hypothesis of Theorem 8, for each initial state there exists a minimal instant $T_0 > 0$ such that at least N neurons of the kernel \mathcal{K} spike simultaneously at T_0 . Moreover, $T_0 = t_{n_0}$ for some $1 \leq n_0 \leq N$, where $N \geq 1$ is the minimum positive natural number that satisfies Inequality (9) and $\{t_n\}_{n \geq 1}$ is the strictly increasing sequence of Lemma 10.*

Proof To prove Lemma 11, it is enough to prove the following assertion:

Assertion (A) to be proved: There exists a minimal positive natural number $n_0 \geq 1$ such that at instant

t_{n_0} at least N neurons of the kernel \mathcal{K} spike simultaneously.

In fact, if we prove Assertion (A), then we deduce Lemma 11 by defining $T_0 := t_{n_0}$.

Consider the instant t_1 . By Lemma 10, which asserts the existence of the sequence $\{t_n\}_{n \geq 1}$, there exists at least one neuron, say $i_1 \in \mathcal{K}$, that spikes at instant t_1 . Thus, if $N = 1$, Assertion (A) holds trivially taking $n_0 = 1$. If $N \geq 2$, then either there exist at least N neurons in \mathcal{K} that spike at instant t_1 (joint with i_1), or there exists at most $N - 1$ neurons that do so. In the first case, Assertion (A) holds for $n_0 = 1$, and there is nothing more to prove. In the second case, denote by I_{t_1} the subset of neurons in \mathcal{N} that spike at instant t_1 , and denote by $\mathcal{K} \cap I_{t_1}$ the subset of neurons in the kernel \mathcal{K} that spike at instant t_1 . We are assuming that

$$\#(\mathcal{K} \cap I_{t_1}) \leq N - 1, \quad (11)$$

where $\#$ denotes “the number of elements of”.

By hypothesis \mathcal{K} is strong, so it satisfies Definition 6. Thus, $k = \#\mathcal{K} \geq 3$ satisfies Inequality (7). We have:

$$\sqrt{k} > \frac{U - L}{\min\{A_{i,j} : i \in \mathcal{K}, j \in \mathcal{N}\}}. \quad (12)$$

Thus, recalling that $N \geq 1$ is the minimum natural number that satisfies Inequality (9), we deduce

$$N + 1 < \frac{U - L}{\min\{A_{i,j} : i \in \mathcal{K}, j \in \mathcal{N}\}}.$$

Joining with Inequality (12), we obtain

$$\sqrt{k} > N + 1, \quad k > (N + 1)^2. \quad (13)$$

If we denote by $\#(\mathcal{K} \setminus I_{t_1})$ the number of cells in the kernel \mathcal{K} that do not spike at instant t_1 , we have

$$\begin{aligned} \#(\mathcal{K} \setminus I_{t_1}) &= (\#\mathcal{K}) - (\#\mathcal{K} \cap I_{t_1}) \\ &= k - (\#\mathcal{K} \cap I_{t_1}) \geq (N + 1)^2 - (\#\mathcal{K} \cap I_{t_1}) \end{aligned} \quad (14)$$

Joining Inequalities (11) and (14) we obtain:

$$\#(\mathcal{K} \setminus I_{t_1}) \geq (N + 1)^2 - (N - 1) > N^2 - (N - 1). \quad (15)$$

Since at least the neuron i_1 spikes at time t_1 , and the network is excitatory, we deduce - from Equality (5) applied to any neuron $j \in \mathcal{K} \setminus I_{t_1}$ - the following property:

$$U > x_j(t_1) = x_j(t_1^-) + \sum_{i \in I_{t_1}} A_{i,j} \geq L + A_{i_1,j}$$

$$\geq L + \min\{A_{i,j}: i \in \mathcal{K}, j \in \mathcal{N}\}.$$

Applying inequality (9), we obtain:

$$U > x_j(t_1) \geq L + \frac{U-L}{N} \quad \forall j \in \mathcal{K} \setminus I_{t_1}. \quad (16)$$

Now, we will generalize Inequalities (15) and (16) for other instants t_h with $h \geq 1$, as follows:

Assertion (B): If at the instants $t_1, \dots, t_l, \dots, t_h$ (for some $h \geq 1$), at most $N-1$ cells of the kernel \mathcal{K} spike, then (denoting by I_{t_l} the subset of all the cells that spike at instant t_l) the following inequalities hold:

$$\#(\mathcal{K} \setminus (\bigcup_{l=1}^h I_{t_l})) > N^2 - h(N-1). \quad (17)$$

$$U > x_j(t_h) \geq L + \frac{(U-L)h}{N} \quad \forall j \in \mathcal{K} \setminus \left(\bigcup_{l=1}^h I_{t_l}\right). \quad (18)$$

Let us prove Assertion (B) by induction on the natural number $h \geq 1$. In (15) and (16), we have proved (B) if $h = 1$. Assume that (17) and (18) hold for some $h \geq 1$ and let us prove them for $h+1$. Since at most $N-1$ cells of the kernel \mathcal{K} spike at the instant t_l for all $1 \leq l \leq h+1$, then in particular for $l = h+1$ we obtain:

$$\#(\mathcal{K} \cap I_{t_{h+1}}) \leq N-1.$$

Therefore, using Inequality (17) for h , we obtain:

$$\begin{aligned} \#(\mathcal{K} \setminus (\bigcup_{l=1}^{h+1} I_{t_l})) &= \\ \#(\mathcal{K} \setminus (\bigcup_{l=1}^h I_{t_l})) - (\#(\mathcal{K} \cap I_{t_{h+1}})) &> \\ N^2 - h(N-1) - (\#(\mathcal{K} \cap I_{t_{h+1}})) &\geq \\ N^2 - h(N-1) - (N-1) &= N^2 - (h+1)(N-1). \end{aligned}$$

Thus, we have proved Inequality (17) for $h+1$. Now, let us prove Inequality (18) for $h+1$, assuming that it holds for h . Let us fix any neuron

$$j \in \mathcal{K} \setminus \left(\bigcup_{l=1}^{h+1} I_{t_l}\right) \subset \mathcal{K} \setminus \left(\bigcup_{l=1}^h I_{t_l}\right).$$

Since the neuron j had not spiked in instants t_1, \dots, t_h, t_{h+1} , its variable $x_j(t)$ had increased with t , since the instant t_1 up to instant t_{h+1} , and is smaller than the threshold level U . Therefore, applying Inequality (18) for h , we get:

$$x_j(t_{h+1}^-) \geq x_j(t_h) \geq L + \frac{(U-L)h}{N}. \quad (19)$$

By Lemma 10, at instant t_{h+1} at least one cell, say $i_{h+1} \in \mathcal{K}$ spikes. Then, applying the action's rule (5), we obtain:

$$\begin{aligned} U > x_j(t_{h+1}) &= x_j(t_{h+1}^-) + \sum_{i \in I_{t_{h+1}}} A_{i,j} \geq \\ &\geq x_j(t_{h+1}^-) + A_{i_{h+1},j}. \end{aligned}$$

Joining with Inequality (19), we deduce:

$$\begin{aligned} U > x_j(t_{h+1}) &\geq L + \frac{(U-L)h}{N} + A_{i_{h+1},j} \geq \\ &\geq L + \frac{(U-L)h}{N} + \min\{A_{i,j}: i \in \mathcal{K}, j \in \mathcal{N}\}. \end{aligned}$$

Substituting the last term in the above inequality by the expression obtained from Inequality (9), we get

$$\begin{aligned} U > x_j(t_{h+1}) &\geq L + \frac{(U-L)h}{N} + \frac{U-L}{N} = \\ &= L + \frac{(U-L)(h+1)}{N}. \end{aligned}$$

This is Inequality (18) for $h+1$, ending the proof of Assertion (B).

Now, let us show that Assertion (B) implies Assertion (A). Arguing by contradiction, let us suppose that for all $1 \leq h \leq N$, at most $N-1$ cells spike at instant t_h . Then, we can apply Assertion (B) with $h = N$. From Inequality (17) we obtain:

$$\#(\mathcal{K} \setminus (\bigcup_{l=1}^N I_{t_l})) > N^2 - N(N-1) = N \geq 1.$$

So, there exists more than N neurons that have not spiked at instants t_1, t_2, \dots, t_N (including instant t_N). From Inequality (18) with $h = N$, we have:

$$\begin{aligned} U > x_j(t_N) &\geq L + \frac{(U-L)N}{N} = L + (U-L) = U \\ \forall j \in \mathcal{K} \setminus \left(\bigcup_{l=1}^h I_{t_l}\right). \end{aligned}$$

We conclude that more than N neurons, which we call j , do not spike at instant t_N , but nevertheless, the values $x_j(t_N)$ of their respective state variables at instant t_N are larger or equal than the threshold level U . This contradicts Definition 2 of the spiking instants. We have proved Assertion (A), which implies Lemma 11, as wanted. \square

Lemma 12 *In the hypothesis of Theorem 8, let $N \geq 1$ be the minimal positive natural number that satisfies Inequality (9). If $T_0 > 0$ is the first instant such that at least N neurons of the kernel \mathcal{K} spike simultaneously, then there exists at least one neuron $i_0 \in \mathcal{K}$ that does not spike during the time interval $[0, T_0)$ (excluding its right extreme T_0).*

Proof: Let $\{t_n\}_{n \geq 1}$ be the increasing sequence of instants of Lemma 10. Applying Lemma 11,

$$T_0 = t_{n_0} \text{ for some } 1 \leq n_0 \leq N.$$

If $n_0 = 1$, then by condition (ii) of Lemma 10, no spike occurs during the time interval $[0, t_1) = [0, t_{n_0}) = [0, T_0)$. So, in this case Lemma 12 holds trivially. If $n_0 \geq 2$, then we take into account that $T_0 = t_{n_0}$ is the first instant at which at least N neurons of \mathcal{K} spike. In other words:

At any instant

$$t_1, \dots, t_{n_0-1}$$

at most $N - 1$ neurons of the kernel \mathcal{K} spike, and besides, from condition (ii) of Lemma 10, no neuron spikes at the intermediate times $t \in (t_n, t_{n+1})$. We conclude that:

Assertion (C): There exist at most

$$(N - 1)(n_0 - 1) < N \cdot n_0 \leq N^2$$

neurons of the kernel \mathcal{K} that spike during the time interval $[0, t_{n_0}) = [0, T_0)$.

From the definition of N as the minimum positive natural number that satisfies Inequality (9), we deduce

$$\frac{U - L}{\min\{A_{i,j}, i \in \mathcal{K}, j \in \mathcal{N}\}} \geq N + 1. \quad (20)$$

By hypothesis, Inequality (7) holds. Combining Inequalities (7) and (20), we obtain:

$$\sqrt{k} \geq (N + 1) > N, \quad k > N^2$$

where k is the number of neurons of the kernel \mathcal{K} . Then, from Assertion (C) we deduce that there exist less than k neurons of \mathcal{K} that spike during the time interval $[0, T_0)$. But $k = \#\mathcal{K}$. We conclude that there exists at least one neuron of the kernel \mathcal{K} that does not spike during such an interval, ending the proof of Lemma 12. \square

3.3 End of the Proof of Theorem 8

The Proof of Theorem 8 Consider the minimal positive natural number $N \geq 1$ that satisfies Inequality (9). From Lemma 11, for any initial state of the network, there exists a first instant $T_0 \geq 0$ (which may depend on the initial state), such that at least N neurons of the kernel \mathcal{K} spike simultaneously at T_0 . Applying now Lemma 9, all the neurons of the network spike simultaneously at T_0 . Thus, at instant T_0 , from the reset rule (4) we have

$$x_i(T_0) = 0 \quad \forall i \in \mathcal{N}.$$

By a translation of the origin of time axis to T_0 , the new initial state of each neuron i is zero. By hypothesis, all the cells are identical: i.e. they satisfy (during their integration periods) the same differential equation (1). Since at instant T_0 all of them have the same state equal to zero, and from T_0 they are all governed by the same differential equation, we deduce that all of them arrive together to the threshold level U , at an instant $T_1 > T_0$. Thus, all the neurons of the network spike together (again) at time $T_1 > T_0$; and besides

$$x_i(t) = x_j(t) \quad \forall T_0 \leq t < T_1 \quad \forall i \neq j.$$

Applying again the reset rule (4) at instant T_1 , we obtain

$$x_i(T_1) = 0 \quad \forall i \in \mathcal{N}.$$

Arguing by induction, if all the cells spike simultaneously at an instant T_n , then from instant T_n and until they arrive again to the threshold level U at instant $T_{n+1} > T_n$, we have:

$$x_i(t) = x_j(t) \quad \forall T_n \leq t < T_{n+1} \quad \forall i \neq j. \quad (21)$$

We assert that

$$\lim_{n \rightarrow +\infty} T_n = +\infty. \quad (22)$$

In fact, $T_{n+1} - T_n$ is the constant time that any neuron i (whose state at instant T_n is $x_i(T_n) = 0$) takes to arrive to the threshold level $U > 0$, being governed uniquely by the differential equation (1). Since the instantaneous velocity at which $x_i(t)$ increases with t is dx_i/dt , we obtain:

$$\begin{aligned} T_{n+1} - T_n &\geq \frac{x_i(T_{n+1}^-) - x_i(T_n)}{\max\{dx_i(t)/dt : T_n \leq t \leq T_{n+1}\}} = \\ &= \frac{U - 0}{\max\{f(x) : 0 \leq x \leq U\}} = b > 0, \end{aligned}$$

where b is a constant. Therefore $T_{n+1} \geq T_n + b$ for all $n \geq 0$, which implies $T_n \geq nb \rightarrow +\infty$, when $n \rightarrow +\infty$. This proves Equality (22), as wanted.

Since Equality (21) holds for all $n \geq 0$, and $T_n \rightarrow +\infty$, we deduce that

$$x_i(t) = x_j(t) \quad \forall t \geq T_0, \quad \forall i \neq j.$$

So, by Definition 7, the network synchronizes for all $t \geq T_0$, proving the first part of Theorem 8.

Now, to end the proof of Theorem 8, it is only left to prove Inequality (8), which bounds from above the transitory time T_0 . As proved in the first part above, the network synchronizes (as a later time) at the first instant $T_0 \geq 0$ at which at least N neurons of the kernel \mathcal{K} spike simultaneously. Applying Lemma 12, there exists a neuron i_0 that does not spike during the time interval $[0, T_0)$. But, from Lemma 9, all the neurons of the network \mathcal{K} spike at instant T_0 . Then, T_0 equals the waiting time until the neuron i_0 spikes for the first time. Therefore, during the time interval $[0, T_0)$ the state $x_{i_0}(t)$ of the neuron i_0 is strictly increasing on time, governed by the differential equation (1) plus the positive discontinuity jumps that are produced on the state of i by the actions that come from other neurons $j \neq i$. Thus, the time T_0 is not larger than the time T_0^* that would take the variable x_{i_0} , to arrive from the lowest possible level $L < 0$ at instant 0, to the highest possible level $U > 0$ at instant T_0 , if it were only governed by the differential equation (1):

$$T_0 \leq T_0^*.$$

So, let us compute T_0^* as if x_{i_0} were only governed by (1). Applying the mean value theorem of the derivative, there exists $\tau \in [0, T_0^*]$ such that

$$\left. \frac{dx_{i_0}}{dt} \right|_{t=\tau} = \frac{U - L}{T_0^* - 0},$$

from which

$$\begin{aligned} T_0^* &= \frac{U - L}{(dx_{i_0}/dt)|_{t=\tau}} = \\ &= \frac{U - L}{f(x_{i_0}(\tau))} \leq \frac{U - L}{\min\{f(x): L \leq x \leq U\}}. \end{aligned}$$

We deduce that

$$T_0 \leq T_0^* \leq \frac{U - L}{\min\{f(x): L \leq x \leq U\}},$$

which is Inequality (8) as wanted, ending the proof of Theorem 8. \square

4 Applications and other methods of research

Diverse mathematical tools are used to analyze the emergent global dynamics of networks of interacting units. For instance, the Linear Matrix Inequality (LMI) approach, is used to study the stability and

Lyapunov exponents in applications to Control Engineering, Communications, Manufacturing and Management [20, 23], and in particular to prove synchronization of artificial neuronal networks [28].

Classical arguments in the theory of differential equations and abstract dynamical systems are used for networks with low dimensions, mostly those that are continuously coupled on time. For instance, phase locking (instead of full synchronization) is proved in the emergent dynamics of low-dimensional networks of interacting predator-prey communities [42, 13]. The dynamics of populations is also modeled by impulsive differential equations, as a pulsed coupled network, in some examples of vaccination and biological control (see for instance [24] and references therein). On small networks, equilibrium states, periodic orbits and synchronized periodic orbits are searched in the emergent dynamics of a network composed by two interacting populations in a social system [41].

Other mathematical approach to study the emergent behaviour of finite networks of interacting units is provided by the Game Theory. Finite games are also studied as networks (e.g. [31]). As a difference with the approach given by the Theory of Dynamical Systems, the Game Theory usually models the network taking into account the existence of different strategies according to which the cells interact dynamically (e.g. [16]). The excitatory interaction among the cells in a neuronal network work similarly as the so called *imitative strategy* in the theory of games. Combining arguments of Dynamics and Game Theory, the so called *evolutionary games*, under an imitative strategy hypothesis, are applied to study a social network of firms and workers [1]. The synchronization of coupled dynamical units appears also in the research of economic cycles [44]. Classical arguments of Dynamics under the hypothesis of Game Theory, is applied to study a nonlinear network composed by two interacting competitors with delay times in the insurance market [50].

Artificial Neuronal Networks (ANNs; see for instance [10] and the references therein) is a methodology that is being widely used in applications, mainly to simulate and predict the behaviour of physical or social systems with large complexity, and for engineering design of artificial intelligence and data processing. The methodology of research with ANNs, usually applies a combination of different mathematical tools, from Dynamics, Probability Theory and Statistics, to Numerical Analysis, Optimization and Computational algorithms. The purpose of an ANN is, in general, to process the information that is receives from the neurons in the so called *input layer*, provide results obtained from the neurons of the so called *output layer*, by means of a black box, which is composed by the neurons of one or more *hidden*

layers. The system is controlled by diverse feed-back or feed-forward couplings among the layers. A sequential learning process is produced by the ANN by the accumulation of the previous dynamical experiences that the information data provides. The main characteristic of an ANN that allows it to work as a learning machine and makes it useful in applications is its self-organizing capability. This ability is governed by the so called *self-organizing-mappings* (SOM) which are functional representations. Among the self-organizing features, some subsets of neurons in the network may self-synchronize, or at least periodically self-synchronize.

For instance, the self-synchronization of groups of an ANN is applied to model a network of the medieval society [5]. Also the SOM technique is applied in the research of the market of equity funds [9]. Different architectures of ANNs and SOM are applied in the prediction and surety bonding in Construction Engineering [2], in meteorologic predictions of the wind speed [14], in the optimization of the compression ratio and other criteria for image processing [21, 3], in the study of dataset clustering in Gastroenterology [11], to simulate and detect earning management on accounting and financial applications [17], etc. A particularly flexible application of ANNs is the Field-Programmable-Gate-Array (FPGA). It is the design of integrated-circuits based ANNs such that the architecture of the network and the self-organizing mappings that allow the network work as a learning machine, are programmable.

5 Final comments

In Neuroscience, the formation of groups of neurons into synchronized clusters is based on the mutual synaptical connections that are modeled as a bio-electrical circuit [15]. In general, in electrical circuits synchronization provides security and stability of the system, and in the case of data processing systems, it provides sufficient reliability [40].

Nevertheless, a synchronized orbit (and also any other periodic limit orbit) is usually not obtained for free. If attracting all or most orbits (as we have proved in Theorem 8), synchronization implies a sacrifice in the theoretic optimum amount of information that the system processes or memorizes with respect to the number of neurons that are used for the task. In fact, according to the abstract mathematical definition of entropy in the Ergodic Theory (e.g. [22]), if the global attractor is periodic, the system has zero entropy. In spite of this mathematical property, the null entropy only means that the limit *rate* of increasing information is zero, when time goes to infinite. But the system

may still be able to process or memorize a positive finite amount of information, while its transients last. In fact, by Formula (8) we find a bound from above of the transitory time T_0 . The bound works for any initial state, but the transitory time, and the patterns of clustered neurons that are formed during the transitory time intervals, depend on the initial state of the network (Lemmas 11 and 12).

As a consequence, if the network \mathcal{N} under observation, is a part of a larger network, and if this larger network sends signals to \mathcal{N} , then it modifies the instantaneous state of one or more of the cells in \mathcal{N} . Thus, \mathcal{N} will generically loose its synchronization as a consequence of the signals it receives from its exterior. From the lost of synchrony, and while the transitory time T_0 lasts, the network \mathcal{N} will show a specific pattern in the sequence $\{t_n\}_{n \geq 1}$ of spiking instants and in the clusters (or subsets) of spiking neurons (Lemmas 10 and Lemma 12). This pattern depends on the stimulus that \mathcal{N} received from the external macro-network. Since the system is assumed to be deterministic, each external stimulus provokes the reproduction (in \mathcal{N}) of each transitory pattern. The transitory time interval T_0 until the network recuperes its synchronization, and the pattern it shows, depend on the received stimulus, but may last much more than the stimulus itself. In fact, the (optimum) upper bound of the transitory time in Inequality (8) may be very large for certain choice of the parameters of the network \mathcal{N} .

Finally, we notice that the synchronization result would be false in general, if the model of each neuron (as an integrate and fire oscillator) were not one-dimensional. Many neurons in the nervous system are modeled by multi-dimensional differential equations (see for instance [39]). And not all the asymptotical dynamics of a model of a biological neuron in a multi-dimensional setting are globally stable. [12]. In fact, the one-dimensional phase-amplitude description of some neurons has been recently proved to be inadequate, in some cases, for the analysis of rhythms and oscillations [25]. If the limit set of a the dynamics of a neuron i has (for instance) more than one limit cycle, then the perturbation produced on the state of i , for example by the actions of other neurons in the network, may move the state from the basin of attraction of one limit cycle to the basin of a different limit cycle. The resultant dynamics may be chaotic, rather than periodic [26].

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