TOPOLOGICAL ENTROPY ON POINTS WITHOUT PHYSICAL-LIKE BEHAVIOUR

ELEONORA CATSIGERAS, XUETING TIAN[1], AND EDSON VARGAS

Abstract. Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemannian manifold M. Let $\mathcal{O}_f$ denote the space of all SRB-like measures and for $x \in M$, $pw(x)$ denote the limit set of $\{\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}\}_{n\in\mathbb{N}}$ in weak* topology where δ_y is the Dirac probability measure supported at $y \in M$. We state a sufficient condition to prove that the set of points without physical-like behaviour

$$\Gamma_f = \{x : pw(x) \cap \mathcal{O}_f = \emptyset\}$$

has full topological entropy, even though in general it always has zero Lebesgue measure. In particular, this phenomena is valid for all C^1 transitive Anosov diffeomorphisms and time-1 maps of all C^1 transitive Anosov flows. We emphasize that the system is just required C^1. The proof ideas are mainly based on Pesin's entropy formula and variational principle of saturated sets.

1. Introduction

Let $f : M \to M$ be a continuous map on a compact manifold M. Let m be a Lebesgue measure normalized such that $m(M) = 1$, and not necessarily f- invariant. Let $\mathcal{P}$ denote the space of all probability measures, and $\mathcal{P}_f \subset \mathcal{P}$ denote the space of f-invariant probability measures, endowed with the weak* topology. For a point $x \in M$ we consider the following sequence

$$\Big\{\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}\Big\}_{n\in\mathbb{N}}$$

where δ_y is the Dirac probability measure supported at $y \in M$. Define the set $p\omega_f(x)$ of probability measures:

$$p\omega_f(x) = \Big\{\mu \in \mathcal{P} : \quad \exists\ n_i \to +\infty \text{ such that } \lim_{i\to+\infty}^{*} \frac{1}{n_i}\sum_{j=0}^{n_i-1}\delta_{f^j(x)} = \mu\Big\}.$$

We say that $p\omega_f(x)$ describes the asymptotic statistics of the orbit of x. It is standard to check that $p\omega_f(x) \subset \mathcal{P}_f$. From [11] we know that $p\omega_f(x)$ is always nonempty, compact and connected.

Recall that a measure $\mu \in \mathcal{P}$ is called *physical or SRB* (Sinai-Ruelle-Bowen), if the set

$$A(\mu) = \{x \in M : \ p\omega_f(x) = \{\mu\}\}$$

has positive Lebesgue measure. The set $A(\mu)$ is called basin of attraction of μ. Now let's recall the definition of SRB-like measure [9].

Definition 1.1. (**SRB-like measures**, c.f. [9]) A probability measure $\mu \in \mathcal{P}_f$ is *SRB-like* (or observable or physical-like) if for any $\varepsilon > 0$ the set

$$A_\varepsilon(\mu) = \{x \in M : \ \operatorname{dist}(p\omega_f(x), \mu) < \varepsilon\}$$

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[1]X. Tian is the corresponding author.

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has positive Lebesgue measure. The set $A_\varepsilon(\mu)$ is called basin of $\varepsilon-$attraction of μ.

We denote with $\mathcal{O}_f$ the set of all SRB-like measures for $f : M \mapsto M$. It is standard to check that every SRB-like measure for f is f-invariant. Let's recall some basic results related with SRB-like measures. We call basin of attraction $A(\mathcal{K})$ of any nonempty weak* compact subset $\mathcal{K}$ of probabilities, to

$$A(\mathcal{K}) := \{x \in M : p\omega_f(x) \subseteq \mathcal{K}\}.$$

The following theorem is a basic characterization of SRB-like measures, which is a reformulation of the main results of [9]:

Theorem 1.2. *Let $f : M \to M$ be a continuous map on a compact manifold M. Then the set $\mathcal{O}_f$ of all SRB-like measures for f is the minimal weak* compact subset of $\mathcal{P}$ whose basin of attraction has total Lebesgue measure.*

In other words: $\mathcal{O}_f$ is nonempty and weak* compact, and the minimal nonempty weak* compact set that contains, for Lebegue almost all the initial states $x \in M$, the limits of the convergent subsequences of $\{\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}\}_{n\in\mathbb{N}}$.

Define

$$\Delta_f = \{x : pw(x) \subseteq \mathcal{O}_f\}. \tag{1}$$

Note that Δ_f is the maximal set such that all limit points of empirical measure of points in this set are SRB-like. By Theorem 1.2, Δ_f has Lebesgue full measure. Let

$$\Gamma_f = \{x : pw(x) \cap \mathcal{O}_f = \emptyset\}. \tag{2}$$

The set Γ_f is called the set of points without physical-like behaviour. Obviously, $\Gamma_f \subseteq M \setminus \Delta_f$ and thus Γ_f has Lebesgue zero measure. However, in this paper we will show for lots of smooth dynamics, the topological entropy of Γ_f can be large and even equal to the full entropy.

Before stating main results we combine Γ_f with irregular set for consideration together. Let

$$I_f = \{x \in M : pw_f(x) \text{ is not a singleton}\},$$

called irregular set of f. By weak* topology, $x \in I_f$ if and only if there is some continuous function $\phi : X \to \mathbb{R}$ such that the ergodic average

$$\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i(x))$$

does not converge as $n \to +\infty$. By Birkhoff Ergodic theorem, irregular set I_f has zero measure for any ergodic measure and then by Ergodic Decomposition theorem so does for any invariant measure. Since Γ_f has Lebesgue zero measure, we have

Proposition 1.3. *Let $f : M \to M$ be a continuous map on a compact manifold M. Then the set $\Gamma_f \cap I_f$ has zero measure not only for Lebesgue measure but also for all invariant measures.*

In strong contrast, we observe that $\Gamma_f \cap I_f$ may be 'large' in the sense of topological entropy. Now let us start to state the main result, in which the concepts of g-almost product property and uniform separation will be introduced in section 2.2 and 2.3 respectively.

Theorem 1.4. *Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemannian manifold M. Suppose that:*
(H1) f has a dominated splitting $TM = E \oplus F$ for f such that E is not expanding

and F is not contracting in the sense of SRB-like measures, that is, for any $\mu \in \mathcal{O}_f$ and μ a.e. x,

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|D_x f^n|_E\| \leq 0, \quad \limsup_{n\to+\infty} \frac{1}{n} \log m(D_x f^n|_F) \geq 0;$$

(H2) there is an invariant measure ν_0 such that Pesin's entropy formula fails, that is, the metric entropy of ν_0 is not equal to the integral of the sum of all nonnegative Lyapunov exponents of ν_0 a.e. x;
(H3) f has g-almost product property and uniform separation.
Then the set $\Gamma_f \cap I_f$ has full topological entropy.

It seems that the assumptions of Theorem 1.4 are so many and very strong. However, it is suitable for all hyperbolic dynamics. That is,

Theorem 1.5. *Let M be a compact Riemannian manifold and let $f : M \to M$ be a C^1 transitive Anosov diffeomorphism on M or be time-1 map of a C^1 transitive Anosov flow on M. Then $\Gamma_f \cap I_f$ has full topological entropy.*

We emphasize that the system is just required C^1. From [1] we know for a C^1 transitive Anosov diffeomorphism, I_f carries full topological entropy so that Theorem 1.5 can be as a further refined observation of irregular points by combing the physical-like behaviour.

This paper is organized as follows. In section 2 we will introduce some concepts and some useful lemmas, and in section 3 we will prove Theorem 1.4 and Theorem 1.5.

2. Preliminaries

2.1. **Entropy.** Let $\mu \in \mathcal{P}_f$. Given $\xi = \{A_1, \cdots, A_k\}$ a finite measurable partition of M, i.e., a disjoint collection of elements of $\mathfrak{B}(M)$ whose union is M, we define the entropy of ξ by

$$H_\mu(\xi) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i).$$

The metric entropy of f with respect to ξ is given by

$$h_\mu(f, \xi) = \lim_{n\to\infty} \frac{1}{n} \log H_\mu(\bigvee_{i=0}^{n-1} f^{-i}\xi).$$

The metric entropy of f with respect to μ is given by

$$h_\mu(f) = \sup_{\xi} h_\mu(f, \xi),$$

where ξ ranges over all finite measurable partitions of M.

Let us recall the definition of entropy working for non-compact sets (see [6]). Let $\Lambda_n = \{0, 1, 2, \cdots, n-1\}$. Let $x \in M$. The dynamical ball $B_n(x, \varepsilon)$ is the set

$$B_n(x, \varepsilon) := \{y \in M | \max\{d(f^j(x), f^j(y)) | j \in \Lambda_n\} \leq \varepsilon\}.$$

Definition 2.1. For a general subset $E \subseteq M$, let $\mathcal{G}_n(E, \sigma)$ be the collection of all finite or countable covers of E by sets of the form $B_u(x, \sigma)$ with $u \geq n$. We set

$$C(E; t, n, \sigma, f) := \inf_{\mathcal{C} \in \mathcal{G}_n(E,\sigma)} \sum_{B_u(x,\sigma)\in\mathcal{C}} e^{-tu}$$

and

$$C(E; t, \sigma, f) := \lim_{n\to\infty} C(E; t, n, \sigma, f).$$

Then

$$h_{top}(E;\sigma,f):=\inf\{t:C(E;t,\sigma,f)=0\}=\sup\{t:C(E;t,\sigma,f)=\infty\}$$

and the Bowen's (Hausdorff) topological entropy of E is

$$h_{top}(f,E):=\lim_{\sigma\to 0}h_{top}(E;\sigma,f). \tag{3}$$

It was proved by Bowen that $h_{top}(f,M)$ equals to the classical $h_{top}(f)$.

Now let us first state a basic fact that

Theorem 2.2.

$$h_{top}(f,\Delta_f)\le\sup_{\mu\in\mathcal{O}_f}h_\mu(f);\ \ h_{top}(f,\Gamma_f)\le\sup_{\mu\in\mathcal{P}_f\setminus\mathcal{O}_f}h_\mu(f).$$

Proof. This can be deduced from the definition of Δ_f or Γ_f and the estimate of [6] for $t=\sup_{\mu\in\mathcal{O}_f}h_\mu(f)$ or $\sup_{\mu\in\mathcal{P}_f\setminus\mathcal{O}_f}h_\mu(f)$.

Theorem 2.3. [6, Theorem 2] *Let f be a continuous map of a compact metric space M. If we denote*

$$QR(t)=\{x:\exists\mu\in pw_f(x)\ s.t.\ h_\mu(f)\le t\},$$

then $h_{top}(f,QR(t))\le t$.

□

2.2. $g-$**almost product property.** Firstly we recall the definition of specification property which is stronger than $g-$almost product property, see [11, 17, 4, 5, 3, 19].

Definition 2.4. We say that the dynamical system f satisfies *specification property*, if the following holds: for any $\epsilon>0$ there exists an integer M_ϵ such that for any $k\ge 2$, any k points $x_1,\cdots,x_k$, any integers

$$a_1\le b_1<a_2\le b_2\cdots<a_k\le b_k$$

with $a_{i+1}-b_i\ge M_\epsilon\,(1\le i\le k-1)$, there exists a point $x\in M$ such that

$$d(f^j(x),f^j(x_i))<\epsilon,\quad for\ a_i\le j\le b_i,\,1\le i\le k. \tag{4}$$

The original definition of specification, due to Bowen [4], was stronger.

Definition 2.5. We say that the dynamical system f satisfies *Bowen's specification property*, if f satisfies specification and besides for any integer $p\ge M_\epsilon+b_k-a_1$, there exists a point $x\in M$ such that $f^p(x)=x$ satisfies (4).

Now we start to recall the concept $g-$almost product property in [16] (there is a slightly weaker variant, called almost specification, see [20]). It is weaker than specification property (see Proposition 2.1 in [16]). A striking and typical example of $g-$almost product property (and almost specification) is that it applies to every $\beta-$shift [16, 20]. In sharp contrast, the set of β for which the $\beta-$shift has specification property has zero Lebesgue measure [7].

Definition 2.6. Let $g:\mathbb{N}\to\mathbb{N}$ be a given nondecreasing unbounded map with the properties

$$g(n)<n\ \text{ and }\ \lim_{n\to\infty}\frac{g(n)}{n}=0.$$

The function g is called *blowup function.* Let $x\in M$ and $\varepsilon>0$. The $g-$blowup of $B_n(x,\varepsilon)$ is the closed set

$$\begin{aligned}B_n(g;x,\varepsilon):=\{\ y\in M\mid\quad&\exists\Lambda\subseteq\Lambda_n,\#(\Lambda_n\setminus\Lambda)\le g(n)\ \text{ and}\\&\max\{d(f^j(x),f^j(y))|\,j\in\Lambda\}\le\varepsilon\ \},\end{aligned}$$

where $\#\Gamma$ denotes the cardinality of a finite set Γ.

Definition 2.7. We say that the dynamical system f satisfies $g-$*almost product property* with blowup function g, if there is a nonincreasing function $m:\mathbb{R}^+\to\mathbb{N}$, such that for any $k\geq 2$, any k points $x_1,\cdots,x_k\in M$, any positive $\varepsilon_1,\cdots,\varepsilon_k$ and any integers $n_1\geq m(\varepsilon_1),\cdots,n_k\geq m(\varepsilon_k)$,

$$\bigcap_{j=1}^{k} f^{-M_{j-1}}B_{n_j}(g;x_j,\varepsilon_j)\neq\emptyset,$$

where $M_0:=0, M_i:=n_1+\cdots+n_i, i=1,2,\cdots,k-1$.

2.3. **Uniform separation.** Now we recall the definition of uniform separation property [16]. For $x\in M$, define

$$\Upsilon_n(x):=\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}$$

where δ_y is the Dirac probability measure supported at $y\in M$. For $\delta>0$ and $\varepsilon>0$, two points x and y are (δ,n,ε)-separated if $\#\{j: d(f^jx,f^jy)>\varepsilon,\ j\in\Lambda_n\}\geq\delta n$. A subset E is (δ,n,ε)-separated if any pair of different points of E are $(\delta,n,\varepsilon)-$separated. Let $F\subseteq\mathcal{P}$ be a neighborhood of $\nu\in\mathcal{P}_f$. Define

$$M_{n,F}:=\{x\in M|\,\Upsilon_n(x)\in F\},$$

and define

$$N(F;\delta,n,\varepsilon):=\text{maximal cardinality of a }(\delta,n,\varepsilon)-\text{separated subset of }M_{n,F}.$$

Definition 2.8. We say that the dynamical system f satisfies *uniform separation property*, if following holds. For any $\eta>0$, there exist $\delta^*>0,\epsilon^*>0$ such that for μ ergodic and any neighborhood $F\subseteq\mathcal{P}$ of μ, there exists $n^*_{F,\mu,\eta}$, such that for $n\geq n^*_{F,\mu,\eta}$,

$$N(F;\delta^*,n,\epsilon^*)\geq 2^{n(h_\mu(f)-\eta)}.$$

Now let us recall a basic relation between expansiveness and uniform separation in [16].

Theorem 2.9. [16, Theorem 3.1] *Let f be a continuous map of a compact metric space M. If f is expansive (or asymptotically h-expansive), f satisfies uniform separation.*

2.4. **Variational Principle for saturated sets.** Now we recall a result from [16]. The system f is said to be *saturated* (or f has saturation property), if for any compact connected nonempty set $K\subseteq\mathcal{P}_f$,

$$h_{top}(f,G_K)=\inf\{h_\mu(f)\,|\,\mu\in K\},$$

where $G_K=\{x\in M|\,pw_f(x)=K\}$.

Lemma 2.10. (Variational Principle, [16, Theorem 1.1])
Let f be a continuous map of a compact metric space M with $g-$almost product property and uniform separation property. Then f is saturated.

Remark that Lemma 2.10 is one key tool for all the proofs of our main results in present article. It allows the entropy estimates to be reduced to the problem of describing the various gap sets in terms of $pw_f(x)$.

On the other hand, from [16] if one does not have uniform separation property, then the saturated property just holds for any singleton K. For convenience to compare saturated property, we give a following notion called single-saturated property. We say f is *single-saturated*, if $h_{top}(f,G_\mu)=h_\mu(f)$ holds for any $\mu\in\mathcal{P}_f$, where $G_\mu=\{x\in M|\,pw_f(x)=\{\mu\}\}$.

Lemma 2.11. (Variational Principle, [16, Theorem 1.2])
Let f be a continuous map of a compact metric space M with $g-$almost product property. Then f is single-saturated.

Remark that for any continuous map f of a compact metric space M, there is a general fact (see Theorem 4.1 (3) in [16]): for any compact connected nonempty set $K \subseteq M(M,f)$,

$$(5) h_{top}(f,G_K) \leq \inf\{h_\mu(f)\,|\,\mu \in K\}, \quad \text{where} \quad G_K = \{x \in M|\, pw_f(x) = K\}.$$

In particular, for any $\mu \in M(M,f)$, we have

$$(6) \qquad h_{top}(f,G_\mu) \leq h_\mu(f), \quad \text{where} \quad G_\mu = \{x \in M|\, pw_f(x) = \{\mu\}\}.$$

Theorem 2.12. *If f satisfies g-almost product property, then*

$$h_{top}(f,\Delta_f) = \sup_{\mu \in \mathcal{O}_f} h_\mu(f); \;\; h_{top}(f,\Gamma_f) = \sup_{\mu \in \mathcal{P}_f \setminus \mathcal{O}_f} h_\mu(f).$$

Proof. By Lemma 2.11, every invariant measure μ satisfies that $h_{top}(f,G_\mu) = h_\mu(f)$. Since for any $\mu \in \mathcal{O}_f$, $G_\mu \subseteq \Delta_f$, then

$$h_{top}(f,\Delta_f) \geq \sup_{\mu \in \mathcal{O}_f} h_{top}(f,G_\mu) = \sup_{\mu \in \mathcal{O}_f} h_\mu(f).$$

Together with Theorem 2.2, one has $h_{top}(f,\Delta_f) = \sup_{\mu\in\mathcal{O}_f} h_\mu(f)$.
Similarly, one gets the other equality. □

2.5. **Dominated Splitting** & **Pesin Entropy Formula.**

Definition 2.13. **(Dominated Splitting)** *Let $f: M \to M$ be a C^1 diffeomorphism on a compact Riemannian manifold M. Let $TM = E \oplus F$ be a Df-invariant and continuous splitting such that $dim(E) \cdot dim(F) \neq 0$. We call $TM = E \oplus F$ to be a $(\sigma-)$dominated splitting if there exists $\sigma > 1$ such that*

$$\frac{\|Df|_{E(x)}\|}{m(Df|_{F(x)})} \leq \sigma^{-1}, \forall x \in M,$$

where $m(A) = \|A^{-1}\|^{-1}$ for linear map A.

Remark that the continuity of the splitting in the definition is not necessary because it can be naturally deduced from the required inequality in the dominated splitting(for example, see [2]). Remark that

$$\frac{\|Df^k|_{E(x)}\|}{m(Df^k|_{F(x)})} \leq \prod_{i=0}^{k-1} \frac{\|Df|_{E(f^i(x))}\|}{m(Df|_{F(f^i(x))})} \leq \sigma^{-N} \leq \sigma^{-1}, \forall x \in M.$$

This means if $T_M M = E \oplus F$ is a $\sigma-$dominated splitting of f for some $\sigma > 1$, then for any integer $k \geq 1$, $T_M M = E \oplus F$ is a $\sigma-$dominated splitting for system f^k.

There is an equivalent statement of dominated splitting. $TM = E \oplus F$ is a *dominated splitting* if there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})} \leq C\lambda^n, \forall x \in M, \; n \geq 1.$$

Remark that Gourmelon ([10]) proved that there always exists an adapted metric for which $C = 1$.

Now let us recall a result from [8].

Theorem 2.14. [8, Theorem 1] *Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemanian manifold M. If there is a dominated splitting $T_M M = E \oplus F$, then for any SRB-like measure μ, one has*

$$h_\mu(f) \geq \int \sum_{i=1}^{\dim(F)} \chi_i(x) d\mu = \int \log|\det Df|_F|\, d\mu, \tag{7}$$

where $\chi_1(x) \geq \chi_2(x) \cdots \geq \chi_{\dim(M)}(x)$ denote the Lyapunov exponents of $x \in M$.

Combining with Ruelle's inequality, Theorem 2.14 have a direct corollary as follows. Let PE_f denote the set of all invariant measures satisfying Pesin's entropy Formula, that is,

$$PE_f := \{\mu \in \mathcal{P}_f :\ h_\mu(f) = \int \sum_{\chi_i(x) \geq 0} \chi_i(x) d\mu\},$$

where $\chi_1(x) \geq \chi_2(x) \cdots \geq \chi_{\dim(M)}(x)$ denote the Lyapunov exponents of $x \in M$.

Corollary 2.15. *Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemanian manifold M with a dominated splitting $T_M M = E \oplus F$. Let $\mu \in \mathcal{O}_f$. If for μ a.e. x,*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|D_x f^n|_E\| \leq 0, \quad \limsup_{n\to+\infty} \frac{1}{n} \log m(D_x f^n|_F) \geq 0;$$

then $\mu \in PE_f$, that is,

$$h_\mu(f) = \int \sum_{\chi_i(x)\geq 0} \chi_i(x) d\mu = \int \sum_{i=1}^{\dim(F)} \chi_i(x) d\mu = \int \log|\det Df|_F|\, d\mu. \tag{8}$$

Moreover, if $f \in C^{1+\alpha}$, by classical Pesin theory ([13]*) $\mu \in PE_f$ implies (in fact, is equivalent) that μ has absolutely continuous conditional measures on unstable manifolds.*

In particular, we have a following consequence of Corollary 2.15, since by Theorem 1.2 we know that $\mathcal{O}_f \neq \emptyset$.

Corollary 2.16. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on a compact Riemanian manifold M with a dominated splitting $T_M M = E \oplus F$. If for any invariant measure μ and μ a.e. x,*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|D_x f^n|_E\| \leq 0, \quad \limsup_{n\to+\infty} \frac{1}{n} \log m(D_x f^n|_F) \geq 0;$$

then there is at least one invariant measure which is SRB-like, satisfies Pesin's entropy formula and has absolutely continuous conditional measures on unstable manifolds.

2.6. **Basic description of PE_f.**

Theorem 2.17. *Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemanian manifold M. Then PE_f is a convex subset of $\mathcal{P}_f$, and PE_f either is equal to $\mathcal{P}_f$ or it can not contain interior points.*

Proof. It is easy to check that PE_f is convex. Now we prove the other part. Suppose $PE_f \neq \mathcal{P}_f$. By contradiction, assume that there exists $\mu \in PE_f$ and a neighbourhood $\mathcal{U} \subseteq PE_f$ of μ. Take $\nu \in \mathcal{P}_f \setminus PE_f$. Then it is easy to check that

$$\{\theta\mu + (1-\theta)\nu :\ \theta \in [0, 1)\} \subseteq P_f \setminus PE_f.$$

However, if taking θ close to 1, $\theta\mu + (1-\theta)\nu$ should be in $\mathcal{U} \subseteq PE_f$, which is a contradiction. ☐

Theorem 2.18. *Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemanian manifold M with a dominated splitting $T_M M = E \oplus F$. If for any invariant measure μ and μ a.e. x,*

$$\liminf_{n\to+\infty} \frac{1}{n}\log \|D_x f^n|_E\| \le 0, \quad \limsup_{n\to+\infty} \frac{1}{n}\log m(D_x f^n|_F) \ge 0;$$

and the entropy function $h_{\cdot}(f) : \mathcal{P}_f \to [0,+\infty), \mu \to h_\mu(f)$ is upper-continuous, then PE_f is non-empty, convex and compact.

Proof. By assumption and Corollary 2.15 PE_f is non-empty. By Theorem 2.17 PE_f is convex. By assumption, for any invariant measure $\mu \in \mathcal{P}_f$,

$$\int \sum_{\chi_i(x)\ge 0} \chi_i(x) d\mu = \int \sum_{i=1}^{\dim(F)} \chi_i(x) d\mu = \int \log|\det Df|_F|\, d\mu.$$

Suppose $\mu_n \in PE_f$ and $\mu_n \to \mu$. Since dominated splitting is always continuous [2], then

$$\int \sum_{\chi_i(x)\ge 0} \chi_i(x) d\mu = \int \log|\det Df|_F|\, d\mu = \lim_{n\to\infty} \int \log|\det Df|_F|\, d\mu_n$$

$$= \lim_{n\to\infty} \int \sum_{\chi_i(x)\ge 0} \chi_i(x) d\mu_n = \lim_{n\to\infty} h_{\mu_n}(f) \le h_\mu(f).$$

On the other hand, by Ruelle's inequality,

$$h_\mu(f) \le \int \sum_{\chi_i(x)\ge 0} \chi_i(x) d\mu.$$

Thus $\mu \in PE_f$. □

Remark 2.19. (1) It is well-known that any Anosov diffeomorphism satisfies the assumptions of Theorem 2.18 and every periodic measure does not satisfy Pesin' entropy formula. Thus for Anosov case, PE_f is always non-empty, convex, compact and does not contain interior point.
(2) Let f be the time-t ($t \neq 0$) map of a Anosov flow of a compact Riemannian manifold X. In this case, f is partially hyperbolic with one-dimension central bundle and satisfies conditions restricted on the sum of stable and central bundle, and unstable bundle. Then f is far from tangency so that f is entropy-expansive which implies the upper-continuity of entropy function $h_{\cdot}(f)$ (see [14] or see [12, 15]). Note that every periodic measure of flow is still invariant for f and does not satisfy Pesin's entropy formula. Thus for time-t map f of a Anosov flow, PE_f is always non-empty, convex, compact and does not contain interior point.

3. Proof of Theorems 1.4 and 1.5

Firstly let us prove a general proposition. Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemannian manifold M. Define $\Gamma_f^* := \{x : pw(x) \cap PE_f = \emptyset\}$. Given a continuous function $\phi : X \to \mathbb{R}$, let

$$R_f^\phi := \{x \in X \mid \text{ergodic averages } \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x)) \text{ converge as } n \to +\infty\}.$$

For convenience, we call R_f^ϕ to be regular set w.r.t. ϕ (simply, ϕ-regular set). Define the ϕ-irregular set $I_f^\phi = X \setminus R_f^\phi$. If $C^0(X)$ denotes the space of all continuous functions on X, note that

$$I_f = \bigcup_{\phi \in C^0(X)} I_f^\phi. \tag{9}$$

Proposition 3.1. *Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemannian manifold M. Suppose that f is saturated, $PE_f \neq \emptyset$ and $\mathcal{P}_f \setminus PE_f \neq \emptyset$. Then*

(A). *For any $\phi \in C^0(X)$, if*

$$\inf_{\omega\in\mathcal{P}_f} \int \phi d\omega < \sup_{\omega\in\mathcal{P}_f} \int \phi d\omega, \tag{10}$$

then $\Gamma_f^* \cap I_f^\phi$ carries full topological entropy.

(B). *The set $\Gamma_f^* \cap I_f$ has full topological entropy.*

Proof of Proposition 3.1 (A). We divide the proof into two cases.

Case 1. $\sup_{\omega\in PE_f} h_\omega(f) < h_{top}(f)$.
Fix h satisfying $\sup_{\omega\in PE_f} h_\omega(f) < h < h_{top}(f)$. By Variational Principle [21], there is $\mu_1 \in \mathcal{P}_f$ such that $h_{\mu_1}(f) > h > \sup_{\omega\in PE_f} h_\omega(f)$. Thus $\mu_1 \in P_f \setminus PE_f$. Take $\tau \in (0,1)$ close to 1 enough such that $\tau h_{\mu_1}(f) > h$. By assumption of (10) we can take $\rho_0 \in \mathcal{P}_f$ such that $\int \phi d\rho_0 \neq \int \phi d\mu_1$. Let $\mu_2 = \tau\mu_1 + (1-\tau)\rho_0$. Then

$$\int \phi d\mu_1 \neq \int \phi d\mu_2 \tag{11}$$

and

$$h_{\mu_2}(f) = \tau h_{\mu_1}(f) + (1-\tau)h_{\rho_0}(f) \geq \tau h_{\mu_1}(f) > h.$$

It follows that $\mu_2 \in P_f \setminus PE_f$.

Let

$$K = \{\theta\mu_1 + (1-\theta)\mu_2 : \ \theta \in [0,1]\}.$$

Then each $\omega \in K$ satisfies that $h_\omega(f) \geq \min\{h_{\mu_1}(f), h_{\mu_2}(f)\} > h$ and thus $\omega \in P_f \setminus PE_f$. It follows that $G_K \subseteq \Gamma^* \cap I_f^\phi$. In other words, for $x \in G_K$, $pw_f(x) = K$ so that using (11) we have $x \in I_f^\phi$ by weak* topology and moreover, $pw_f(x) = K \subseteq P_f \setminus PE_f$ which implies $x \in \Gamma_f^*$.

On the other hand, since f is saturated, then

$$h_{top}(f, \Gamma_f^* \cap I_f^\phi) \geq h_{top}(f, G_K) = \inf_{\omega\in K} h_\omega(f) = \min\{h_{\mu_1}(f), h_{\mu_2}(f)\} > h.$$

By arbitrariness of h, we complete the proof of Case 1.

Case 2. $\sup_{\omega\in PE_f} h_\omega(f) = h_{top}(f)$.
By assumption we can take $\mu_0 \in PE_f$ and $\eta_0 \in \mathcal{P}_f \setminus PE_f$..

Fix $\epsilon > 0$. Take $\mu_1 \in PE_f$ such that $h_{\mu_1}(f) > h_{top}(f) - \epsilon$. If there is some $\omega_0 \in \mathcal{P}_f \setminus PE_f$ such that $\int \phi d\omega_0 \neq \int \phi d\mu_1$, take $\nu_0 = \omega_0$. Otherwise, $\int \phi d\eta_0 = \int \phi d\mu_1$ and by assumption of (10) we can take $\rho_0 \in \mathcal{P}E_f$ such that $\int \phi d\rho_0 \neq \int \phi d\mu_1$. Let $\nu_0 = \frac{1}{2}\eta_0 + \frac{1}{2}\rho_0$. Then for both cases of the chosen measure ν_0, we have $\nu_0 \in \mathcal{P}_f \setminus PE_f$ and $\int \phi d\mu_1 \neq \int \phi d\nu_0$.

Take $t_1 \neq t_2 \in (0,1)$ close 1 enough such that $\min\{t_1, t_2\} h_{\mu_1}(f) > h_{top}(f) - \epsilon$. Let $\mu = t_1\mu_1 + (1-t_1)\nu_0$ and $\nu = t_2\mu_1 + (1-t_2)\nu_0$. Then

$$\int \phi d\mu \neq \int \phi d\nu. \tag{12}$$

Let

$$K = \{\theta\mu + (1-\theta)\nu : \ \theta \in [0,1]\}.$$

Since $\mu_1 \in PE(f)$ but $\nu_0 \in P_f \setminus PE_f$, then for any $\tau \in [0,1)$, $\tau\mu_1 + (1-\tau)\nu_0 \in P_f \setminus PE_f$. In particular, $\mu, \nu \in P_f \setminus PE_f$. It follows that for any $\omega \in K$, $\omega \in P_f \setminus PE_f$. It follows that $G_K \subseteq \Gamma^* \cap I_f^\phi$. More precisely, for $x \in G_K$, $pw_f(x) = K$ so that using (12) we have $x \in I_f^\phi$ by weak* topology and moreover, $pw_f(x) = K \subseteq P_f \setminus PE_f$ which implies $x \in \Gamma_f^*$.

On the other hand, since f is saturated, then

$$h_{top}(f, \Gamma_f^* \cap I_f^\phi) \geq h_{top}(f, G_K) = \inf_{\omega \in K} h_\omega(f) = \min\{h_\mu(f), h_\nu(f)\} > h_{top}(f) - \epsilon.$$

By arbitrariness of ϵ, we complete the proof of Case 2.

(B). By assumption there are two different invariant measures $\mu \neq \nu$. By weak* topology, there is a continuous function $\phi \in C^0(X)$ such that $\int \phi d\mu \neq \int \phi d\nu$. In other words,

$$\inf_{\omega \in \mathcal{P}_f} \int \phi d\omega < \sup_{\omega \in \mathcal{P}_f} \int \phi d\omega.$$

By (A) $\Gamma_f^* \cap I_f^\phi$ carries full topological entropy and thus $\Gamma_f^* \cap I_f$ also has full topological entropy, since $I_f^\phi \subseteq I_f$ by (9). $\square$

Proof of Theorem 1.4 From assumption (H2), $\mathcal{P}_f \setminus PE_f \neq \emptyset$. By assumption (H1) and Corollary 2.15 PE_f contains $\mathcal{O}_f$ and thus is non-empty. Moreover, $\mathcal{O}_f \subseteq PE_f$ implies that

$$\Gamma_f^* \subseteq \Gamma_f. \tag{13}$$

By assumption (H3) and Lemma 2.10, f is saturated. Thus, one can use (13) and the case (B) of Proposition 3.1 to complete the proof. $\square$

Proof of Theorem 1.5. We firstly verify the assumptions of Theorem 1.4.

(1) *diffeomorphism case.* Condition (H1) is obvious. It is well-known that transitive Anosov diffeomorphism has specification property which is stronger than g-almost product property and it is expansive which implies uniform separation (by Theorem 2.9) so that condition (H3) holds. For the condition (H2), one just consider a periodic measure.

(2) *flow case.* As discussed in the second part of Remark 2.19, condition (H1) and (H2) holds, and moreover the time-1 map f is entropy-expansive which implies unform separation (by Theorem 2.9). From [19] we know f satisfies specification (even though the shadowing point may be not periodic) which is stronger than g-almost product property.

Then, we can complete the proof by using Theorem 1.4. $\square$

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(E. Catsigeras) Instituto de Matemática y Estadística "Rafael Laguardia"(IMERL), Facultad de Ingeniería, Universidad de la República. Uruguay
E-mail address: eleonora@fing.edu.uy

(X. Tian) School of Mathematical Science, Fudan University, Shanghai 200433, People's Republic of China
E-mail address: xuetingtian@fudan.edu.cn
URL: http://homepage.fudan.edu.cn/xuetingtian

(E. Vargas) Departamento de Matematica, IME-USP, Sao Paulo, Brasil
E-mail address: vargas@ime.usp.br