Homoclinic tangencies near cascades of period doubling

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ABSTRACT

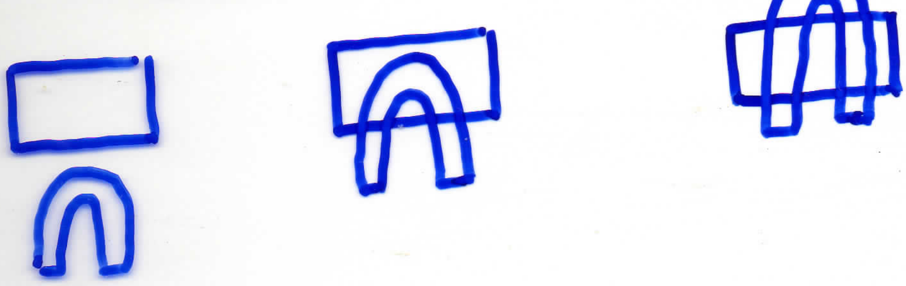
We study certain cascades of period doubling bifurcations in n dimensions whose periodic points have stable codimension one. We prove results of dimension reduction in two steps: first we reduce to dimension two, and second to dimension one, for maps that are uniformly dissipative with bounded geometry. We obtain theorem of approximation by homoclinic tangencies, for the Gambaudo and Tresser example and for cascades of period doublings that are analytic perturbations of the Feigenbaum map in dimension n.

$f: M^2 \circlearrowleft \quad C^r \quad r \geq 3$ homoclinic tangency

Generic one-parameter unfolding of a homoclinic tangency yields to:

- CASCADES of PERIOD DOUBLING BIFURCATIONS (YA 83)
- MAPS WITH INFINITELY MANY SINKS (N 79)
- HÉNON-LIKE ATTRACTORS (MV 92) (BC 91)

Creating a Horseshoe
$\rightarrow$ cascade of period doubling bifurcation



n dimensions
Unfolding of homoclinic tangency $\rightarrow$ cascades of period doubling bifurcations (M 91)

OPEN QUESTION:
ARE HOMOCLINIC BIFURCATIONS NECESSARY FOR GLOBAL INSTABILITY?

CONJECTURE (Palis)

The subset of diffeomorphisms that are either hyperbolic or homoclinic bifurcating is dense in the space of C^∞ surface diffeomorphisms.

PARTICULAR CASE :

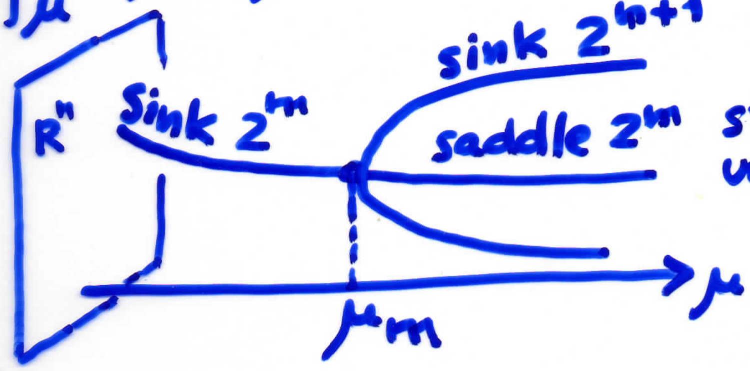
CAN CASCADES OF PERIOD DOUBLING BIFURCATIONS BE APPROXIMATED WITH MAPS EXHIBITING A HOMOCLINIC TANGENCY?

YES, in 2 cases of cascades :

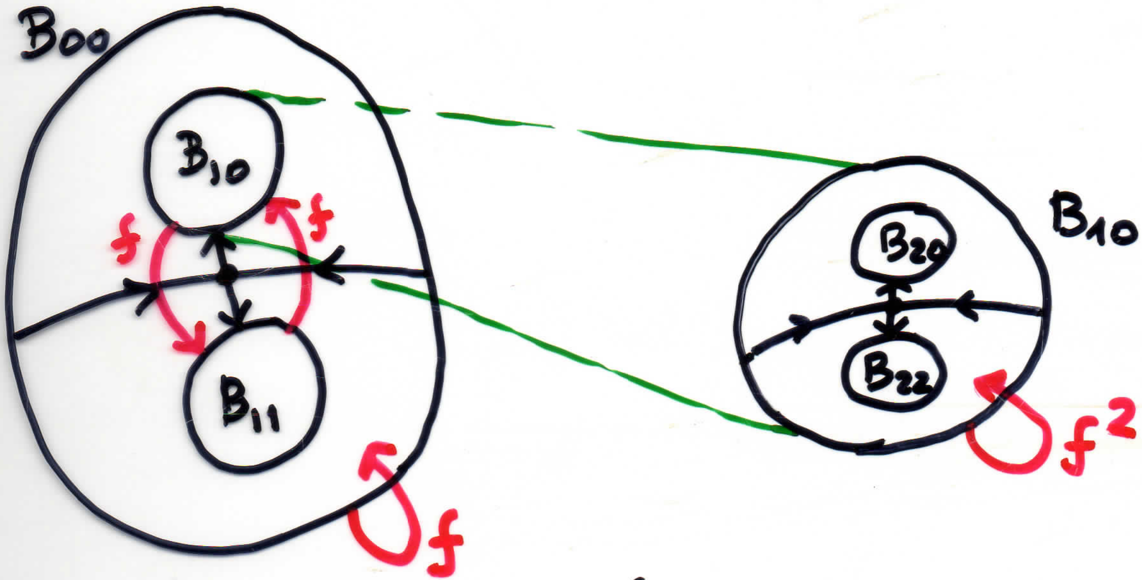
- ANALYTIC PERTURBATIONS OF THE FEIGENBAUM'S MAP IN n DIMENSIONS
- GAMBAUDO-TRESSER EXAMPLE (GT 92)

CASCADES OF PERIOD DOUBLING BIF. in n DIMENS.

$\{f_\mu\}_\mu$ $\quad \mu_m \to \mu_\infty \quad \mu_n$ p.d.b $\qquad f_\mu \in C^r \; r \geq 3$



"CASCADE" $\quad f = f_{\mu_\infty}$



FAMILY OF SUBDOMAINS $\{B_{m,j}\}$ $\begin{array}{l} 0 \leq j \leq 2^m - 1 \\ 0 \leq m \end{array}$

$f^{2^m} : B_{m,j} \circlearrowleft$

- fixed point $p_{m,j}$ of saddle type
- $f : B_{m,j} \to B_{m,j+1} \pmod{2^m}$ $\qquad$ disjoint
- ω-limit

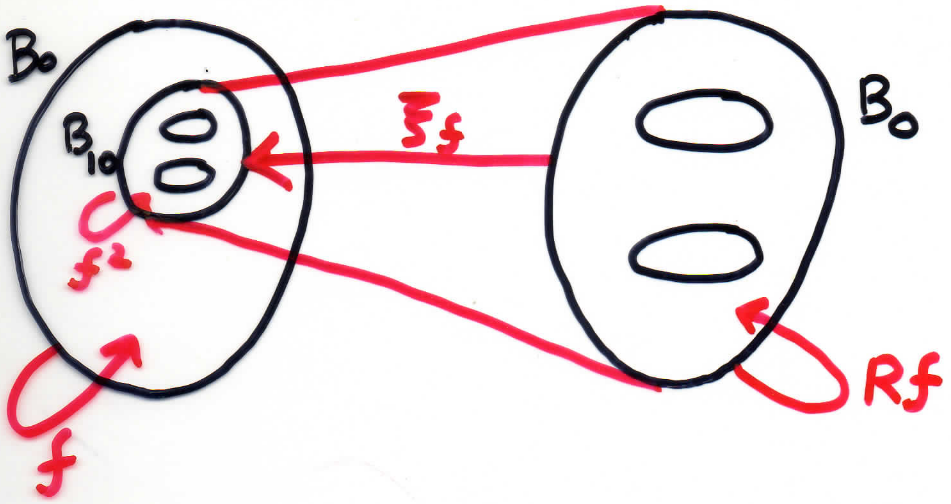
CONVERGENCE ASSUMPTION $\quad \operatorname{diam} B_{m,j} \xrightarrow[m\to\infty]{} 0$

(to avoid wandering sets)
(BGLT 93)

CANTOR SET ATTRACTOR

$$K = \bigcap_m \bigcup_j B_{mj}$$

RENORMALIZATION



ASSUMPTION : The subdomains are C^r diffeomorphic to B_0

$$\boxed{Rf = \xi_f^{-1} \circ f \circ f \circ \xi_f}$$

is a new cascade on B_0

2 CASES OF CASCADES

1. $R^m f$ approaches a ONE-DIM map

   Example : Perturbations of the Feigenbaum's map (CEK 81)

2. $R^m f$ does not lose its n-dimensional character

   Example: Gambaudo-Tresser case

   $$R^n f = f$$

f CASCADE IN DIM. 2 C^r $r \geq 3$

THEOREM (REDUCTION TO DIMENSION 1)

If, $R^m f$ $\xi_{R^m f}$ are C^r bounded

- f dissipative $(|\det Df(p)| \leq \alpha < 1)$
- Bounded geometry $\left(\begin{array}{l} \|D\xi_{R^m f}\|_{C^0} \leq \beta < 1 \\ |\det D\xi_{R^m f}| \geq \gamma > 0 \end{array}\right)$

Then:

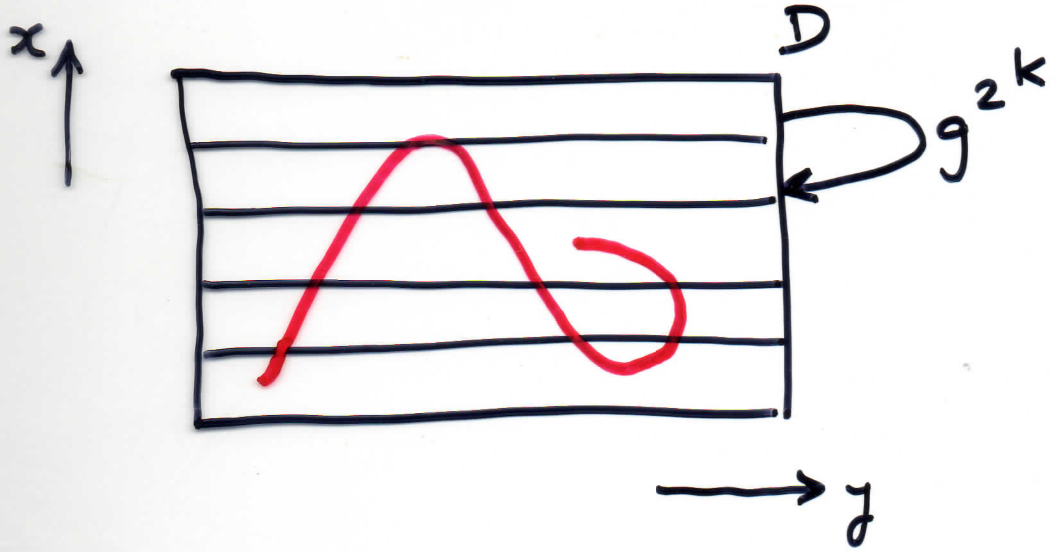
$$R^{m_j} f \xrightarrow{C^{r-1}} g$$

g endomorphism with one dim. image

More precisely $g^{2^k} : D \circlearrowleft$ $g^{2^k}(x,y) = (g_1(x), g_2(x))$

(x,y) C^{r-2} coordinates in D

g_1 map in the interval with critical point(s) and periodic orbits of period 2^n $n \geq 0$.



Idea of the proof :

- $R^{m_j} f \xrightarrow{C^{r-1}} g$ Arzelá - Ascoli

$$R^m f = \xi^{-1}_{R^{m-1}f} \circ \dots \circ \xi^{-1}_f \circ f^{2^m} \circ \xi_f \circ \dots \xi_{R^{m-1}f}$$

$$\det DR^m f = \prod_i \det D\xi^{-1}_{R^i f} \quad \det Df^{2^m} \quad \prod_i \det D\xi_{R^i f}$$

$$\leq K^m \alpha^{2^m} \longrightarrow 0$$

$$\boxed{\det Dg = 0 \quad \text{everywhere}}$$

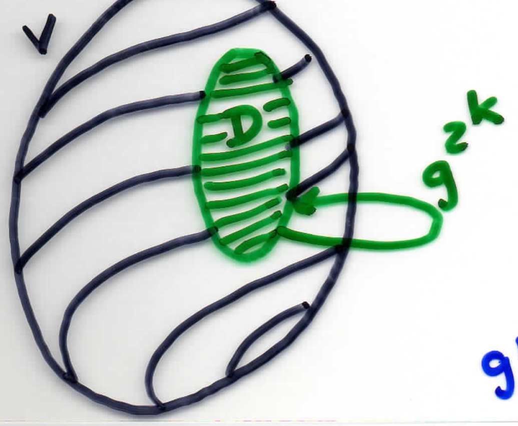
- p_i^m of period 2^i of $R^m f$
  subsequence m_j $\quad p_i^{m_j} \to p_i$

$$\boxed{g^{2^i}(p_i) = p_i}$$

$D(R^m f)^{2^i}\, p_i^m$ has expansive direction e.v. < -1
$Dg^{2^i}\, p_i$ has non contractive direction e.v. ≤ -1

- $\exists$ V open set such that Dg has a direction that is not contracted more than 1/2.

- In V : $\dim \operatorname{Ker} Dg = 1$; FOLIATION OF CLASS C^{r-2} ; each leaf $\xrightarrow{g}$ point



- $\exists D \quad g^{2^k} : D \circlearrowleft$
  trivialize the foliation in D

$$g^{2^k}(x,y) = (g_1(x), g_2(x))$$

- For some periodic point show
  $g_1' > 0$, for other $g_1' < 0$

GAMBAUDO-TRESSER EXAMPLE

THEOREM (GT 92)

$\exists f$ cascade in the unitary ball of R^n

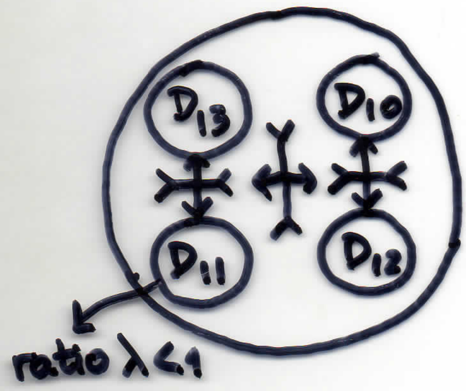
$f \in C^r \quad r = r(n) \xrightarrow[n]{} \infty$

$Rf = \Lambda^{-1} f^{2^n} \Lambda = f$

Λ homotecy of rate λ than can be chosen in an uiterval

CONSEQUENCE : No Feigenbaum's universality in dim. n.

Idea of the proof :



. isotopy Ψ_t

$$F_0^{2^n}\Big|_{D_{1,j}} = id$$

. Change F_0 inside $D_{1,j}$ to F_1 such that

$$\Lambda^{-1} F_1^{2^n}\Big|_{D_{10}} \Lambda = F_0$$

(use the isotopy Ψ_t and divide it in 2^n sections)

Step h Change F_{h-1} inside $D_{h,j}$ to F_h (dividing the isotopy in 2^{hn} sections)

. Prove that $\| F_h - F_{h-1} \|_{C^r} \leq k \left(\frac{1}{2^n \lambda^{r-1}} \right)^h$

Thus $F_h \xrightarrow{C^r} F$

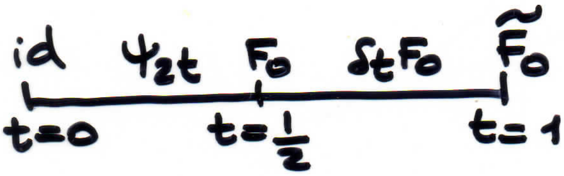
for r such that $2^n \lambda^{r-1} > 1$

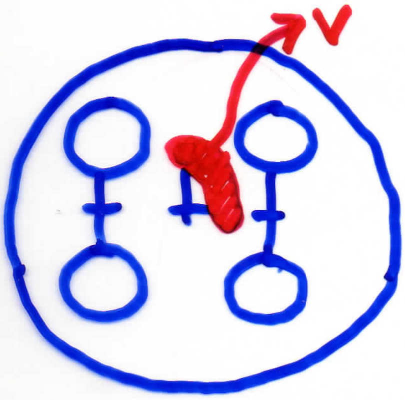
THEOREM

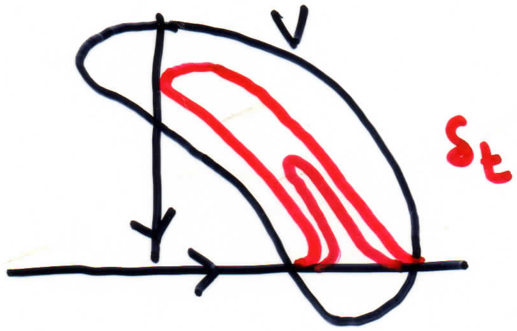
Let F be the cascade constructed above. Given $\varepsilon > 0$ there exists G exhibiting a homoclinic tangency, such that $\|G - F\|_{C^r} < \varepsilon$

Idea of the proof

- New isotopy $\tilde{\Psi}_t$







- Define $F_0, F_1 \dots\dots F_{h-1}$ as before
- Define G instead of F_h using $\tilde{\Psi}_t$ instead of Ψ_t

$$\Lambda_{h,j}^{-1} \circ G^{2^{hn}} \circ \Lambda_{h,j} = \tilde{F}_0$$

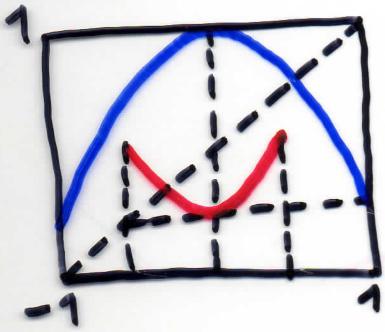
$\Rightarrow$ G exhibits a homoclinic tangency of a periodic point of period 2^{hn} (inside a subdisk $D_{h,j}$).

- $\|G - F_{h-1}\|_{C^r} \leq \tilde{k} \left(\frac{1}{2^n \lambda^{r-1}} \right)^h < \frac{\varepsilon}{2}$ if h is large

As $F_{h-1} \to F$: $\|G - F\|_{C^r} < \varepsilon$

ANALYTIC PERTURBATIONS OF THE FEIGENBAUM'S MAP IN M-DIMENSIONS

ONE-DIMENSIONAL THEORY (CT 78) (F 78) (L82)

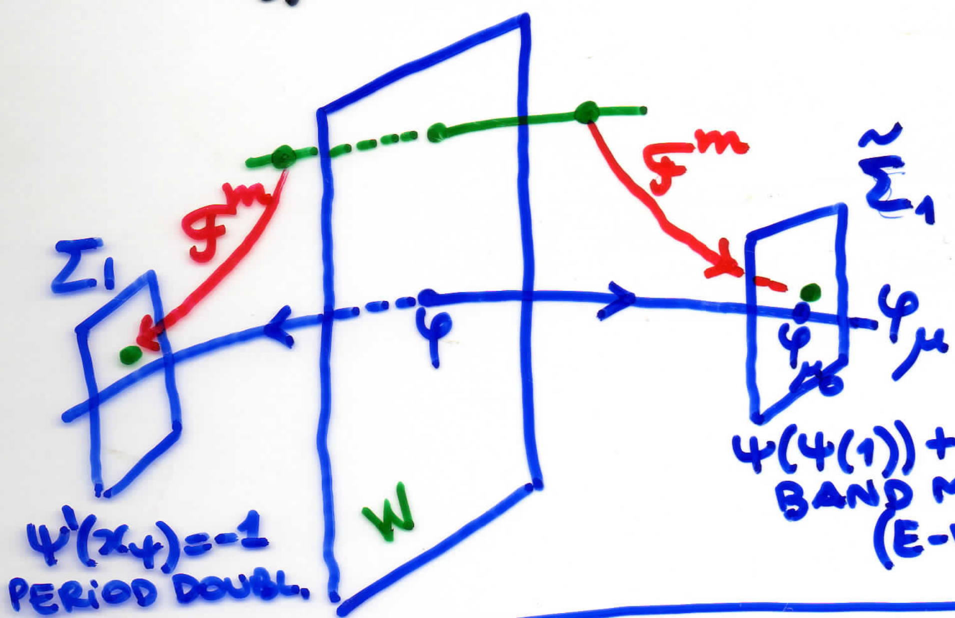


$\psi(x) = g(x^2)$ g analytic $g' < 0$ $g(1) < 0$

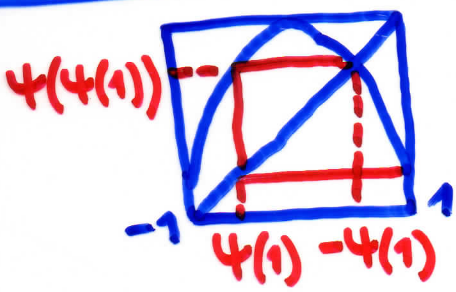
$$\boxed{(\mathcal{F}\psi)(x) = \psi(1)^{-1}\, \psi \circ \psi\,(\psi(1)x)}$$

$\exists\, \varphi$ FEIGENBAUM'S MAP
$\varphi(x) = f(x^2)$

- $\mathcal{F}\varphi = \varphi$
- $d\mathcal{F}_\varphi$ is hyperbolic compact operator of unstable dimension 1



BAND MERGING



C^r Feigenbaum's ONE DIM. THEORY LANFORD 88
DAVIE 92

FEIGENBAUM'S COULLET - TRESSER

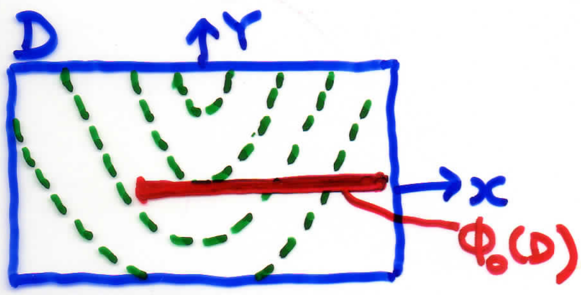
n-DIMENSIONAL THEORY (CEK 81)

D neighborhood of $[-1,1]\times\{0\}$ in $\mathbb{C}^n$

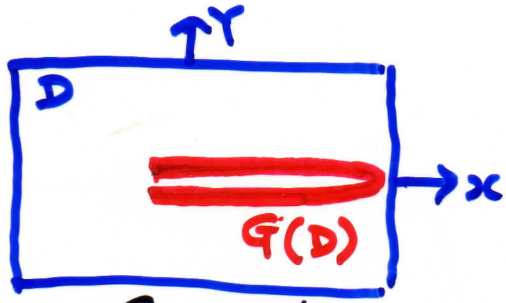
H_D analytic bounded maps

ϕ_0 Feigenbaum's map in n dimensions:

$\phi_0(x,Y) = (f(x^2-\alpha.Y), 0) \quad \alpha\in\mathbb{R}^{n-1}\ \alpha\neq 0$



ϕ_0



G near ϕ_0

THEOREM

In a neighborhood of ϕ_0 in H_D there exists a cod. 1 manifold W such that if $\{G_\mu\}_\mu \pitchfork W = G_{\mu_\infty}$ then

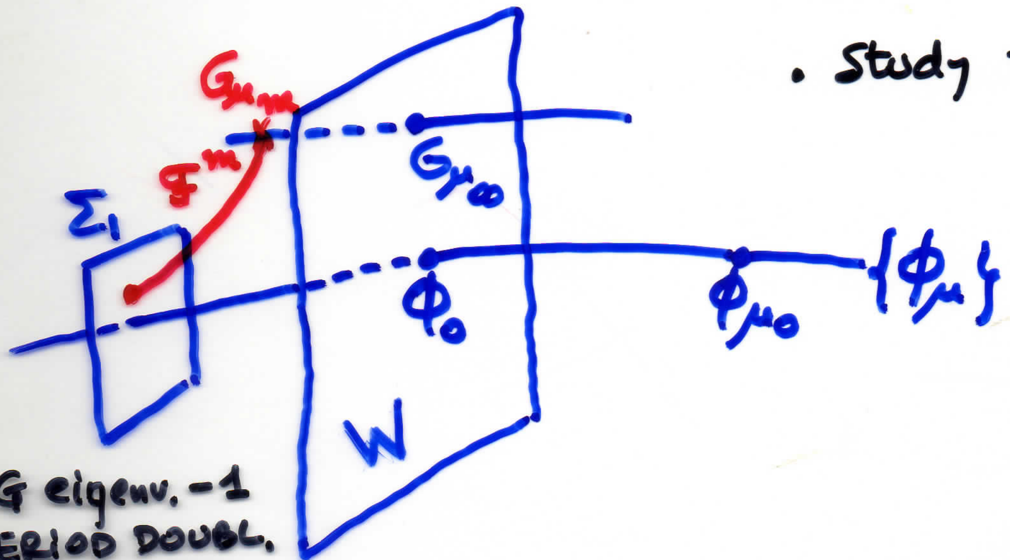
(1) $\exists\ \mu_m \to \mu_\infty$ monot. G_{μ_m} seq. of period doubling bifurcations (CEK 81)

(2) $\exists\ \bar{\mu}_m \to \mu_\infty$ monot. $G_{\bar{\mu}_m}$ homoclinic tangency

Idea of the proof of (1) (CEK 81)

Extend to dimension n the Coullet-Tresser-Feigenbaum's theory:

- Define $\mathcal{F}$
- Study the spectrum of $d\mathcal{F}_{\Phi_0}$



DG eigenv. -1
PERIOD DOUBL. BIFURC.

$$\mathcal{F}G = (I-\sigma_G)^{-1}\circ\Lambda_G^{-1}\circ G\circ G\circ\Lambda_G\circ(I-\sigma_G)$$

The unstable manifold $\{\Phi_\mu\}$ is the "same" of dimension one:

$$\Phi_\mu(x,Y) = (g_\mu(x^2-\alpha\cdot Y),\,0) \text{ where}$$

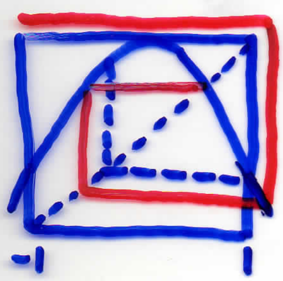
$$\varphi_\mu(x) = g_\mu(x^2) \quad \text{unst. manifold dim. 1.}$$

Idea of the proof of (2)

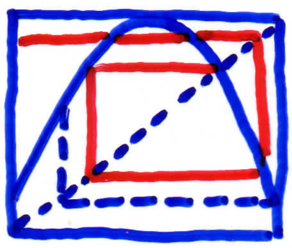
E.W 87

- Take $\Phi_{\mu_0}(x,Y)$ $\qquad \varphi_{\mu_0}$ is "band merging"
- For $\mu_1 < \mu_0$ near μ_0 $\qquad \varphi_{\mu_1}(\varphi_{\mu_1}(1)) + \varphi_{\mu_1}(1) > 0$

For $\mu_2 > \mu_0$ near μ_0 $\qquad \varphi_{\mu_2}(\varphi_{\mu_2}(1)) + \varphi_{\mu_2}(1) < 0$



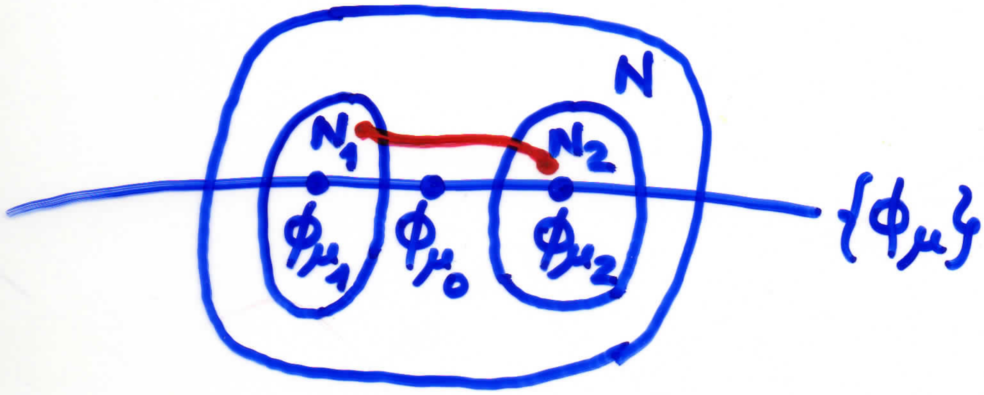
φ_{μ_1}



φ_{μ_2}

LEMMA

There exist neighborhoods N, N_1, N_2 in H_D



such that if $\{G_\mu\}_{\mu \in [a,b]} \subset N$ and

$G_a \in N_1 \quad G_b \in N_2$, then

$\{G_\mu\}_{\mu \in [a,b]}$ has a **HOMOCLINIC BIFURCATION WITH UNAVOIDABLE TANGENCY**.

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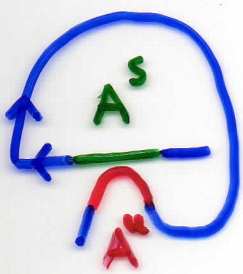
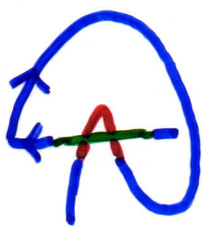
i.e: p_μ hyperbolic, saddle, stable cod 1

$A^s_\mu \subset W^s_\mu(p_\mu) \qquad A^u_\mu \subset W^u_\mu(p_\mu)$

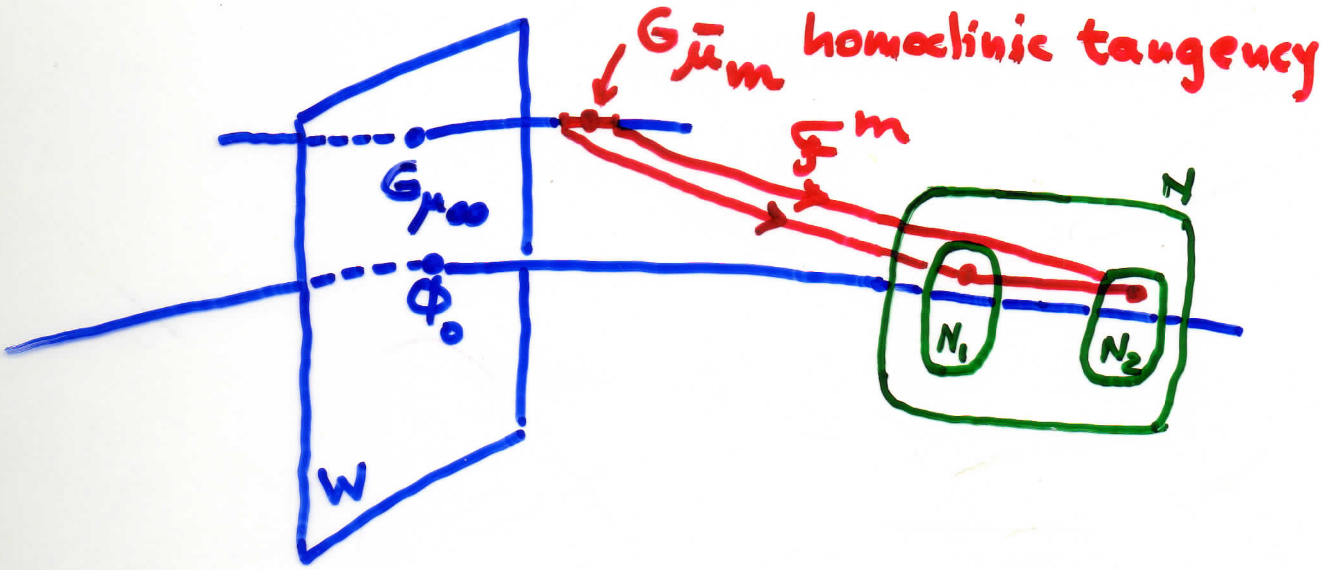
1. $\partial A^s_\mu \cap A^u_\mu = \partial A^u_\mu \cap A^s_\mu = \phi$
2. $\mu = a \qquad A^s_a \cap A^u_a = \phi$
3. $\mu = b \qquad A^s_b \pitchfork A^u_b \neq \phi$



$\mu = a$



$\mu = b$



$\exists \bar{\mu}_m \to \mu_\infty$ $\quad G_{\bar{\mu}_m}$ exhibits a homoclinic tangency

EXTENSION OF THE THEORY TO THE C^r TOPOLOGY

Use the ideas of (L88) (D92) to show that there is a topological hyperbolic behavior near Φ_0. Extend it to a n dimensional setting. Work in progress of EC

Main difficulty: $\mathcal{F}$ renormalization is not differentiable.