INSTITUTO SUPERIOR TÉCNICO
DEPARTAMENTO DE MATEMÁTICA

SEMINÁRIO

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Persistência do atractor de Feigenbaum em famílias a um parâmetro

por

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DIA:	**2ª feira, 14 de Julho de 1997**
HORA:	**14h às 15h**
LOCAL:	**Departamento de Matemática Sala de Seminários (P3.10)**
APOIO:	**Centro de Análise Matemática e Sistemas Dinâmicos Projectos JNICT e PRAXIS XXI**
URL:	**http://www.math.ist.utl.pt/DSseminar/**

Persistência do atrator de Feigenbaum em famílias a um parâmetro

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RESUMO

Consideramos um mapa ψ de classe C^r (para r suficientemente grande) cujos iterados definem um sistema dinâmico discreto numa variedade M de dimensão $n \geq 2$. Assumimos que ψ_0 exibe um atractor de Feigenbaum.

Provamos que cualquer tal mapa ψ é um ponto numa subvariedade local $\mathcal{S} \subset C^r(M)$ de codimensão um, no spaço $C^r(M)$ das transformações C^r da variedade M, tal que todo mapa em $\mathcal{S}$ exibe também um atrator de Feigenbaum.

Como consequênça, o atrator de Feigenbaum persiste quando é perturbado ao longo duma família a um parâmetro, transversal à subvariedade local $\mathcal{S}$ em ψ_0.

Construímos uma tal família a um parâmetro para cualquer ψ dado que exiba um atrator de Feigenbaum, aínda que ele esté longe do endomorfismo analítico ψ_0 ponto fixo pela renormalicaçãode duplicação. Finalmente, aplicamos esa construção para provar a seguinte conjectura de J. Palis: um mapa que exibe um atrator de Feigenbaum pode ser perturbado para obter tangências homoclínicas.

Persistence of the Feigenbaum attractor in one-parameter families

Eleonora Catsigeras and Heber Enrich

THEOREM 0

In the space of C^r maps in n dimensions the existence of a Feigenbaum attractor is a codimension one phenomenon, locally, near the Feigenbaum map.

Dimension 1, analytic maps: FEIGENBAUM, COULLET-TRESSER, LANFORD (1978-1982)

Dimension n, analytic maps: COLLET-ECKMANN-KOCH (1981).

Dimension 1, C^r maps: DAVIE (1996).

Dimension n, C^r maps: C-E (1996).

QUESTION:

Is the Feigenbaum attractor also a codimension one phenomenon far away from the Feigenbaum map?

ANSWER: Yes. (Not obvious: the renormalization is not differentiable).

- **Definition:**

  Feigenbaum attractor: the Cantor set attractor exhibited by maps in the STABLE SET of the Feigenbaum map.

- **The Feigenbaum map is fixed by the doubling renormalization. Its LOCAL STABLE SET is a codimension-one manifold. What about its GLOBAL STABLE SET?**

THEOREM 1

If ψ_0 exhibits a Feigenbaum attractor, then it belongs to a codimension-one local submanifold in the C^r space, formed by maps also exhibiting Feigenbaum attractors.

CONSEQUENCES:

- **Generic one parameter families of maps near ψ_0 will have a parameter value for which the Feigenbaum attractor is exhibited.**
- **Example: perturbations of the quadratic family of n dimensional endomorphisms.** $\psi_a(x_1 \dots x_n) = (x_n, 0, \dots, 0, 1 - a x_n^2)$

For a dissipative saddle :

COROLLARY (Yorke-Alligood ++)

One parameter families generically unfolding a homoclinic tangency pass through a **pure** *sequence of period doubling bifurcations that accumulate in a map exhibiting a* **Feigenbaum attractor.**

COROLLARY ($\Leftarrow$ Colli)

Near a diffeomorphism having a homoclinic tangency there are dense sets of maps exhibiting ∞ many coexisting Feigenbaum attractors.

THEOREM 2

Any map ψ_0 exhibiting a Feigenbaum attractor belongs to a one-parameter family ψ_μ of maps, such that passes through:

** a pure sequence of period doubling bifurcations, for $\mu_n \to 0^-$,*

** a sequence of tangent homoclinic bifurcations, for $\hat{\mu}_n \to 0^+$.*

COROLLARY (Conjecture of Palis) *Maps exhibiting Feigenbaum attractor can be approximated with maps exhibiting homoclinic tangencies.*

- **THEOREM 2 ALREADY KNOWN IF ψ_0 IS NEAR THE FEIGENBAUM MAP:**

  ECKMANN-WITWER (1987) dim. 1 analytic maps.

  C (1995) dim. n analytic maps.

  ENRICH - C (1996) dim. n C^r maps.

- **THEOREM 2, FAR AWAY FROM THE FEIGENBAUM MAP, IS NOT OBVIOUS BECAUSE THE RENORMALIZATION IS NOT SURJECTIVE NOR INJECTIVE.**

THE FEIGENBAUM ATTRACTOR IN n DIMENSIONS.

Definition:

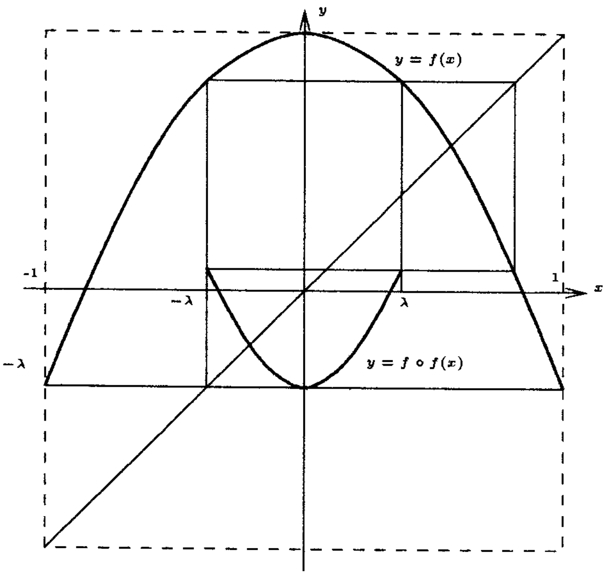
Infinitely doubling renormalizable maps whose renormalized maps eventually lie all in a C^r neighborhood of the Feigenbaum map in n dimensions. (And thus converge to the Feigenbaum map).

THE FEIGENBAUM MAP:

In dimension one: IN THE INTERVAL $[-1, 1]$.

$$\lambda^{-1} f_0 \circ f_0(\lambda x) = f_0(x)$$

$\lambda = -f_0(1)$



In n dimensions: ϕ_0 FROM A ~~SQUARE~~ CUBE, PROJECTS AND FOLLOWS THE GRAPH OF f_0.

$$\phi_0(x_1, x_2, \ldots, x_n) = (x_n, 0, \ldots, 0, f_0(x_n))$$

PROPOSITION (The Feigenbaum attractor).

If $\psi \in C^r$ is infinitely doubling renormalizable in a n dimensional ball, and $\mathcal{R}^m(\psi) \to \phi_0$, then, there exist

* *a* **minimal Cantor set** *K such that $\Psi(K) = K$,*

* *a neighborhood U of K, and, for each N large enough, a single periodic orbit of period 2^N in U, hyperbolic of saddle type, and* **no other periodic orbits** *in U.*

Besides:

* *K is* **the ω-limit of all the orbits in U, except** *those in the stable manifolds of the periodic orbits.*

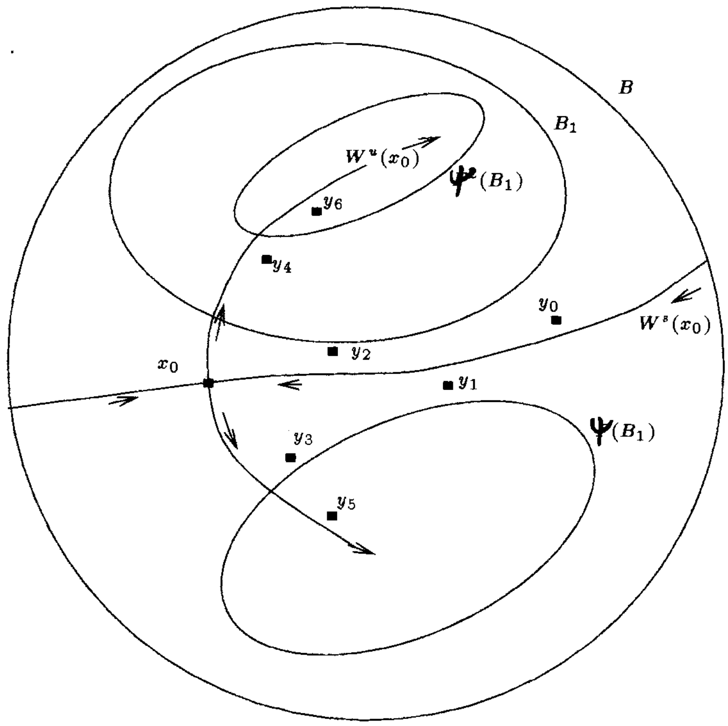
* *All the orbits in K are* **quasi-periodic and non-periodic.**

•

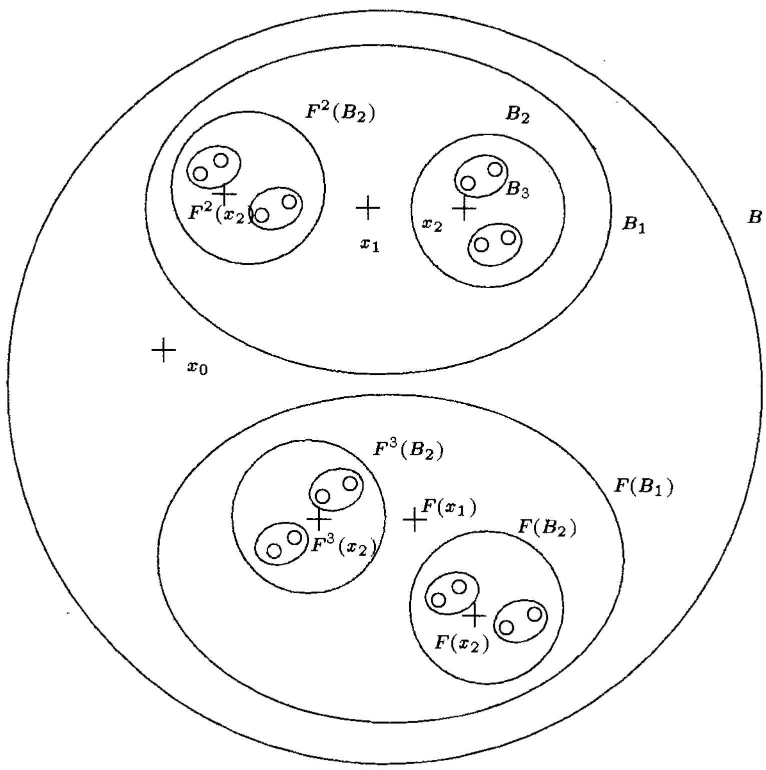
$$K = \bigcap_{m=1}^{\infty} \bigcup_{j=0}^{2^m-1} A_{j,m}$$

$A_{j,m}$ = **the j-th compact atom of generation m.**

- **Proposition: In the Feigenbaum attractor, the relation between the diameter of atoms of generation $m+1$ and of generation m converge to λ when $m \to \infty$.** ($\lambda = 0.3995\ldots$).



RENORMALIZACION: $\mathcal{R}\psi = \xi^{-1} \circ \psi \circ \psi \circ \xi$



Definition:

PERSISTENCE OF THE FEIGENBAUM ATTRACTOR IN ONE-PARAMETER FAMILIES NEAR $\Psi = \{\psi_t\}$

if there exists a real differentiable function a, such that:

*** for all family $X = \{\chi_t\}$ near Ψ, the map $\chi_{a(X)}$ exhibits a Feigenbaum attractor,**

***** $a(\Psi) = 0$, $a((+t_0) * \Psi) = -t_0$

PROPOSITION:

Last definition $\Leftrightarrow$ There exists $\mathcal{M}$, *a codimension-one local manifold in the space of C^r maps,*

intersecting transversally the given family Ψ at the map ψ_0, and such that

$\xi \in \mathcal{M} \Rightarrow \xi$ *exhibits a Feigenbaum attractor.*

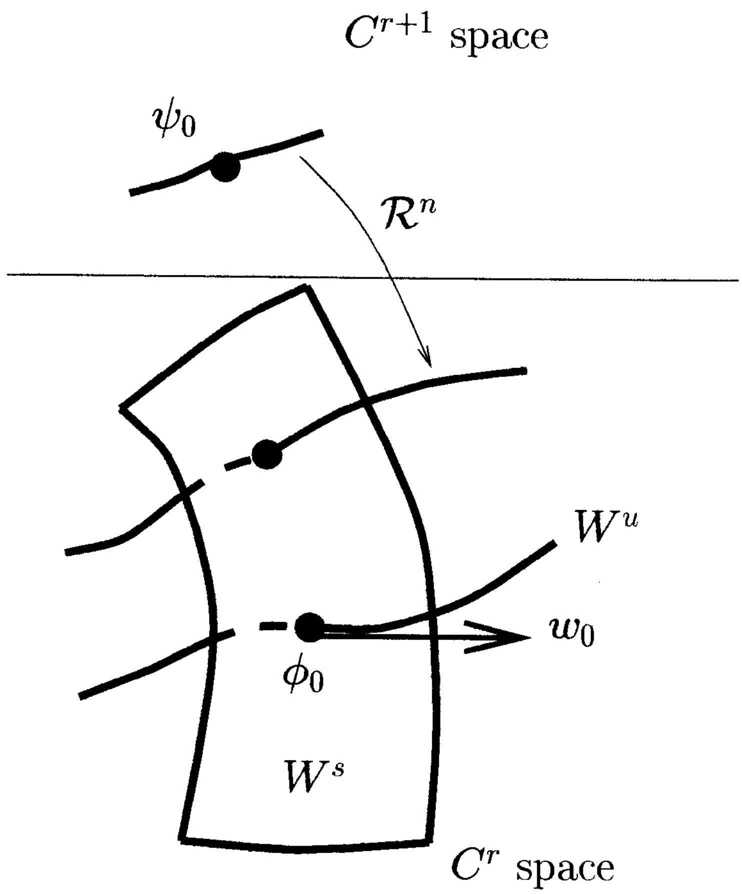
Proof of theorems 1 and 2:

- **FIRST STEP: If ψ_0 is contained in a good one- parameter family of C^r maps (that is, after renormalized, intersects transversally the local stable manifold of the Feigenbaum map), then the Feigenbaum attractor is persistent in one-parameter families.**
- **SECOND STEP: If ψ_0 has a Feigenbaum attractor, then it can be constructed a good one-parameter family containing ψ_0.**

FIRST STEP:

Difficulty: **Renormalization is not differentiable in the space of C^r maps.**

How to avoid it: **Sacrifice one degree of differentiability. The renormalization is differentiable from C^{r+1} to C^r.**



SECOND STEP:

Difficulty: $\mathcal{R}$ **is not surjective.**

Solution: **If w_0 is transversal to the local stable manifold of the Feigenbaum map, there exist a one parameter family passing through ψ_0 that after renormalized is tangent to a vector as near as wanted from w_0.**