Simultaneous Continuation of Infinitely Many Sinks Near a Quadratic Homoclinic Tangency.

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Short Communication to the
III Congreso Latinoamericano de Matemáticos.
Chile Agosto 2009

Abstract

We consider $f \in \mathrm{Diff}^3(M)$ on a surface M, exhibiting infinitely many sinks near the generic unfolding of a quadratic homoclinic tangency Q_0 of a dissipative saddle. We prove that f belongs to an infinite dimensional submanifold $\mathcal{M} \subset \mathrm{Diff}^3(M)$ such that the infinitely many sinks of f have, along $\mathcal{M}$, simultaneous isotopic continuations. Complementary, if f is perturbed along a one-parameter family that unfold generically the tangency Q_0, then at most a finite number of those sinks have continuation.

1 Statement of the results

Let M be a two-dimensional C^∞ compact, connected riemannian manifold, and let $f_{t_{|t|<\varepsilon}} \subset \mathrm{Diff}^3(M)$ be a one-parameter family of C^3 diffeomorphisms on M . We assume that f_0 has a saddle periodic point P_0 and that it is dissipative in P_0. We also assume that the diffeomorphism f exhibits at Q_0 a quadratic tangency between the stable and unstable manifold of P_0 and that the family unfolds it generically for $t > 0$. The Theorems of Newhouse and Robinson [N 1970], [N 1974], [N 1979], [R 1983], assert that, given $\varepsilon > 0$, there exists an open interval $I \subset (0, \varepsilon)$ and a residual set $J \subset I$, of values of the parameter t, such that:

For all $t \in I$, f_t has a hyperbolic maximal subset $\Lambda(f_t)$ with persistence of homoclinic quadratic tangencies in an open set $V \subset M$ isolated from $\Lambda(f_t)$.

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For all $t \in J$, f_t exhibits infinitely many simultaneous sinks.

Let us suppose that $g_0 \in \mathrm{Diff}^3(M)$ exhibits a sink S_0. Consider $g_1 \in \mathrm{Diff}^3(M)$. We say that the sink S_1 of g_1 is the isotopic continuation of S_0 if there exists an isotopy $\{g_t\}_{t\in[0,1]} \subset \mathrm{Diff}^3(M)$ such that for all $t \in [0,1]$ there exists a sink $S_t = S(g_t)$ of g_t, and the transformation $t \in [0,1] \mapsto S_t \in M$ is of C^1 class.

Now, we state our first main result:

THEOREM 1 *Let $f_0 \in \mathrm{Diff}^3(M)$ exhibit a quadratic homoclinic tangency Q_0 of a dissipative saddle point P_0 on the surface M. Let $\{\widetilde{f}_t\}_{-\varepsilon\leq t\leq\varepsilon} \subset \mathrm{Diff}^3(M)$ be a one-parameter family, which generically unfolds the tangency at Q_0 exhibited by f_0.*

Then, there exist an open set $V \subset M$, an open interval $I \subset (-\varepsilon, \varepsilon)$, and a residual set $J \subset I$, such that, for all $f_\infty \in \{\widetilde{f}_t : \ t \in J\}$:

1. *f_∞ exhibits infinitely many coexisting sinks $S_i(f_\infty)$, $i \in \mathbb{N}$.*
2. *There exists a C^1 infinite-dimensional manifold $\mathcal{M} \subset \mathrm{Diff}^3(M)$, such that:*
   - *(a) $f_\infty \in \mathcal{M}$*
   - *(b) If $g \in \mathcal{M}$ then g exhibits the isotopic continuation $S_i(g) \in V$ of the infinitely many sinks $S_i(f_\infty)$.*

Theorem 1 states a condition for the simultaneous isotopic continuation of infinitely sinks: to move the diffeomorphism f_∞ along the infinite-dimensional manifold $\mathcal{M} \subset \mathrm{Diff}^3(M)$. Our next Theorem 2 provides an opposite result: if moving the diffeomorphism along the given family $\widetilde{f}_t$, such a simultaneous continuation is not possible.

THEOREM 2 *In the hypothesis of* Theorem 1, *the sets $V \subset M$, $J \subset I$ and the diffeomorphism $f_\infty \in \{\widetilde{f}_t : \ \ t \in J\}$, can be constructed such that f_∞ verifies the thesis 1. and 2. of* Theorem 1, *and besides:*

If $t \in I$ and $f_t \neq f_\infty$, then $\widetilde{f}_t$ exhibits, at most, a finite number of simultaneous isotopic continuations of the sinks $S_i(f_\infty)$.

From Theorem 2, we conclude that, if there exists some *maximal dimension-*manifold $\mathcal{M} \subset \mathrm{Diff}^3(M)$, verifying the thesis a. and b. of Theorem 1, then $\mathcal{M}$ has at least codimension one. On the other hand, in our proof of Theorem 1, we construct the manifold $\mathcal{M}$ of infinite dimension but also of infinite codimension.

An open question, which we can precise now, is the following:

REMARK 1.1 Let $f_\infty \in \mathrm{Diff}^3(M)$ and $\mathcal{M} \subset \mathrm{Diff}^3(M)$, verifying parts a. and b. of Theorem 1.

Has $\mathcal{M}$ necessarily infinite codimension?

2 Route of the proofs.

To prove Theorem 1 we follow six Steps:

Step 1: Applying Newhouse Theorem of persistence of homoclinic tangencies [N 1970], consider an open set $\mathcal{N} \subset \text{Diff}^3(M)$ such that for all $f \in \mathcal{N}$:

There exists a hyperbolic maximal set $\Lambda(f)$, an open neighborhood $U \subset M$ of $\Lambda(f)$, an open neighborhood $V \subset U$ isolated from $\Lambda(f)$, and a C^1 line of quadratic tangencies $L(f) \subset V$ between the stable and unstable manifolds of $\Lambda(f)$, such that f exhibits persistence of homoclinic tangencies along $L(f)$. This last means that, for a dense subset of diffeomorphisms f in $\mathcal{N}$, there exists in $L(f) \subset V$ a homoclinic tangency point of a saddle of $\Lambda(f)$.

Construct any sequence $\{f_i\}_{i \in \mathbb{N}}$ of diffeomorphisms, along the given one-parameter family, such that $f_i \in \mathcal{N}$ and f_i has a homoclinic tangency Q_i of a periodic saddle $P_i \in \Lambda(f_i)$.

Step 2: Consider, as in [PT 1993] and [M 1973], trivializing coordinates of the local stable and unstable foliations in the neighborhood U of $\Lambda(f)$, for all $f \in \mathcal{N}$. Prove a strong dissipative property of $\Lambda(f)$, from the dissipative hypothesis of the first saddle P_0. Conclude, using the $r-$ normality, that the stable foliation is of class C^3, while the unstable foliation is $C^{1+\theta}$.

Step 3: Compute the iteration $f^n|_V$ in the trivializing coordinates, as in [PT 1993], but adapting the computations, so the coordinates are chosen only once in the neighborhood U of $\Lambda(f)$, and are independent of the saddle point P_i. Generalizing to a wider context those computations in [PT 1993], prove the following key result:

Given $\epsilon > 0$, if the number n of iterates is large enough, then for all $f \in \mathcal{N}$ there exists a rectangle $V_{i,n} \subset V$ such that $f^n|_{V_{i,n}}$ is diffeomorphically conjugated to a C^1-perturbation F of the quadratic family:

$$F(x,y) = (y^2, y^2 + \mu) + G(x,y), \quad \forall\, (x,y) \in [-4,4]^2$$

for some constant $\mu \in [-4,4]$ and some map G of C^1- class such that $\|G\|_{C^3} < \epsilon$.

Step 4: From the key result above, if $\mu \in (-3/8, 1/8)$, the unique attracting fixed point of the quadratic map $(x,y) \mapsto (y^2, y^2 + \mu)$ persists as a sink for all its perturbations F, and so for the diffeomorphism $f^n|_{V_{i,n}}$ which is diffeomorphically conjugated with F. Conclude this result *unifomly for all $f \in \mathcal{N}$*. From the computations, get to a sufficient open condition for the diffeomorphisms $f \in \mathcal{N}$ to exhibit a sink $S_i \in V$ of period n, after unfolding the tangency Q_i.

Step 5: Construct infinite dimensional manifolds $\mathcal{M}$ in $\mathcal{N}$ such that, independently of i, and independently of how many sinks $\{S_i\}_{1 \leq i} \subset V$ a diffeomorphism $f \in \mathcal{M}$ has (if it has some sinks), all the difffeomorphisms $g \in \mathcal{M}$ exhibit isotopic continuations of all the sinks $S_i \in V$ of f. This is possible because the sufficient condition obtained in Steps 4 and 5, that allows the construction of the sinks, are uniform for all $f \in \mathcal{N}$. In the Lemmas that prove that uniformity, we use the C^3 smoothness of the stable foliation.

Step 6: Along the given one-parameter family $\widetilde{f}_t$, construct f_∞ exhibiting infinitely many sinks, reproducing the proof of Newhouse and Robinson [N 1974], [R 1983]. Join with the result of Step 5, and conclude the thesis of Theorem 1.

The proof of Theorem 2 follows applying the same six Steps as above, after observing two facts:

First: The Step 4 in the proof above, can be also stated as follows:

If $\mu \notin [-1, 1]$, then, neither exist an attracting fixed point of the quadratic map $(x, y) \mapsto (y^2, y^2 + \mu)$, nor exist for its perturbations F. So we prove the following claim:

For any $g \in \mathcal{N}$, if the diffeomorphism $g^n|_{V_{i,n}}$, after applying the computations in Step 3, approaches to $(x, y) \mapsto (y^2, y^2 + \mu)$ for some $\mu \notin [-1, 1]$, then g has not a fixed point being a sink in $V_{i,n}$. If some $f \in \mathcal{N}$ has such a sink $S_i(f) \subset V$ of period n, then $S_i(f)$ has not an isotopic continuation to a sink of g.

Second: Prove that, along the given family $\{\widetilde{f}\}_t$, if $\widetilde{f}_{t_1} = f_\infty$, then for $t \neq t_1$ the diffeomorphism $\widetilde{f}_t$ verifies the assumption for g in the claim above.

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Communicación en el
III Congreso Latinoamericano de Matemáticos
August 31th.-September 4th. 2009

SIMULTANEOUS CONTINUATION OF INFINITELY MANY SINKS NEAR A QUADRATIC HOMOCLINIC TANGENCY

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ASSUMPTIONS:

- M is a 2-dimensional compact manifold

- $f_0 \in \mathrm{Diff}^3(M)$ has a DISSIPATIVE SADDLE point P_0 with a HOMOCLINIC QUADRATIC TANGENCY POINT Q_0

- $\{f_t\}_{-\varepsilon \leq t \leq \varepsilon} \subset \mathrm{Diff}^3(M)$ is a one-parameter family GENERICALLY UNFOLDING THE HOMOCLINIC TANGENCY.

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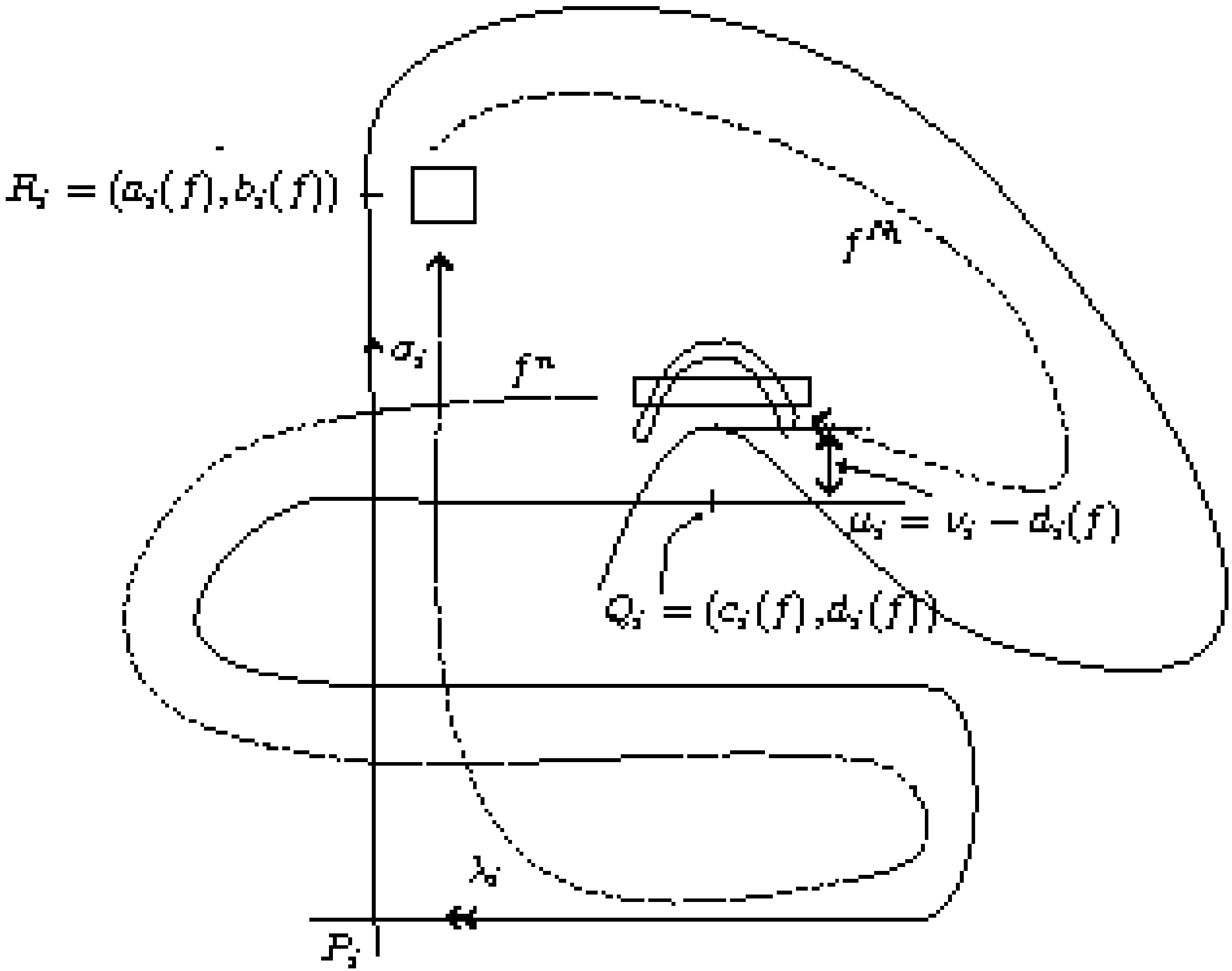
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f^M
σ_i
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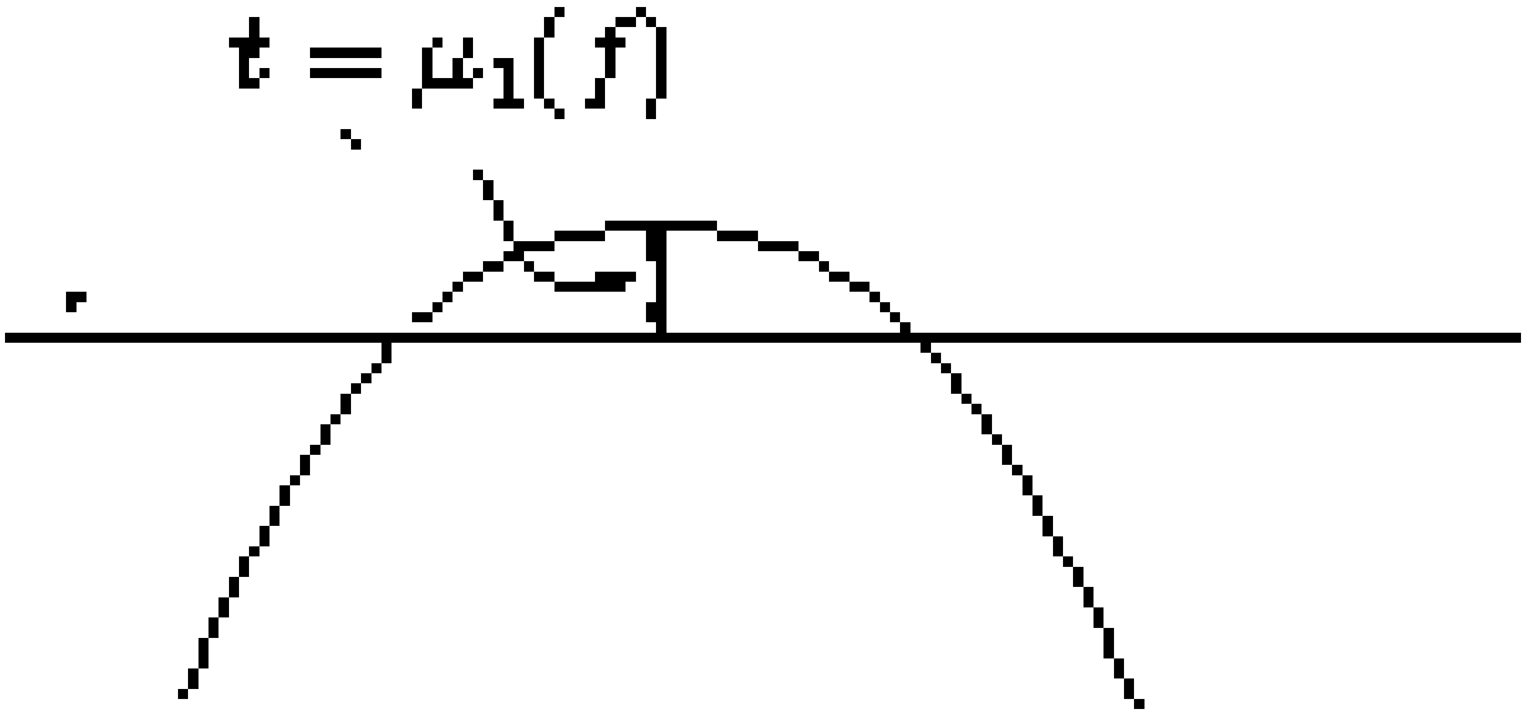
$t = \mu_1(f)$

ASSUMPTIONS:

- M is a 2-dimensional compact manifold
- $f_0 \in \mathrm{Diff}^3(M)$ has a dissipative saddle periodic point P_0 with a homoclinic quadratic tangency Q_0
- $\{\tilde{f}_t\}_{-\varepsilon \leq t \leq \varepsilon} \subset \mathrm{Diff}^3(M)$ is a one-parameter family generically unfolding the homoclinic tangency

Theorems of Newhouse 1970, 1974, 1979 and Robinson 1983:

there exists an open interval $I \subset (0, \varepsilon)$ and a residual set $J \subset I$, of values of the parameter t, such that:

For all $t \in I$, f_t has a hyperbolic maximal subset $\Lambda(f_t)$ with persistence of homoclinic quadratic tangencies in an open set $V \subset M$ isolated from $\Lambda(f_t)$.

For all $t \in J$, f_t exhibits infinitely many simultaneous sinks.

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OPEN QUESTION:

HAVE THE INFINITELY MANY COEXISTING SINKS SIMULTANEOUS CONTINUATIONS IN SOME OPEN NEIGHBORHOOD IN THE FUNCTIONAL SPACE?

THEOREM 1: There exists an open interval $I \subset (0,\varepsilon)$ of parameter values ,and a residual subset $J \subset I$ such that:

For all $t \in J$

1. f_t exhibits infinitely many coexisting sinks $S_i(f_t)$ (Newhouse-Robinson)

2. There exists a infinite-dimensional manifold $\mathcal{M} \subset \mathrm{Diff}^3(M)$ such that:

$f_t \in \mathcal{M}$

If $g \in \mathcal{M}$ *then* g *exhibits the isotopic continuation* $S_i(g)$ *of the infinitely many sinks* $S_i(f_t)$.

THEOREM 2:

the sets $J \subset I$ *and the diffeomorphism* $f_\infty \in \{f_t : \quad t \in J\}$, *can be constructed such that* f_∞ *verifies the thesis 1. and 2. of* Theorem 1, *and*

besides:

If $t \in I$ *and* $f_t \neq f_\infty$, *then* f_t *exhibits, at most, a finite number of simultaneous isotopic continuations of the sinks* $S_i(f_\infty)$.

Route of the proofs of Theorems 1 and 2 In SIX STEPS:

STEP 1:

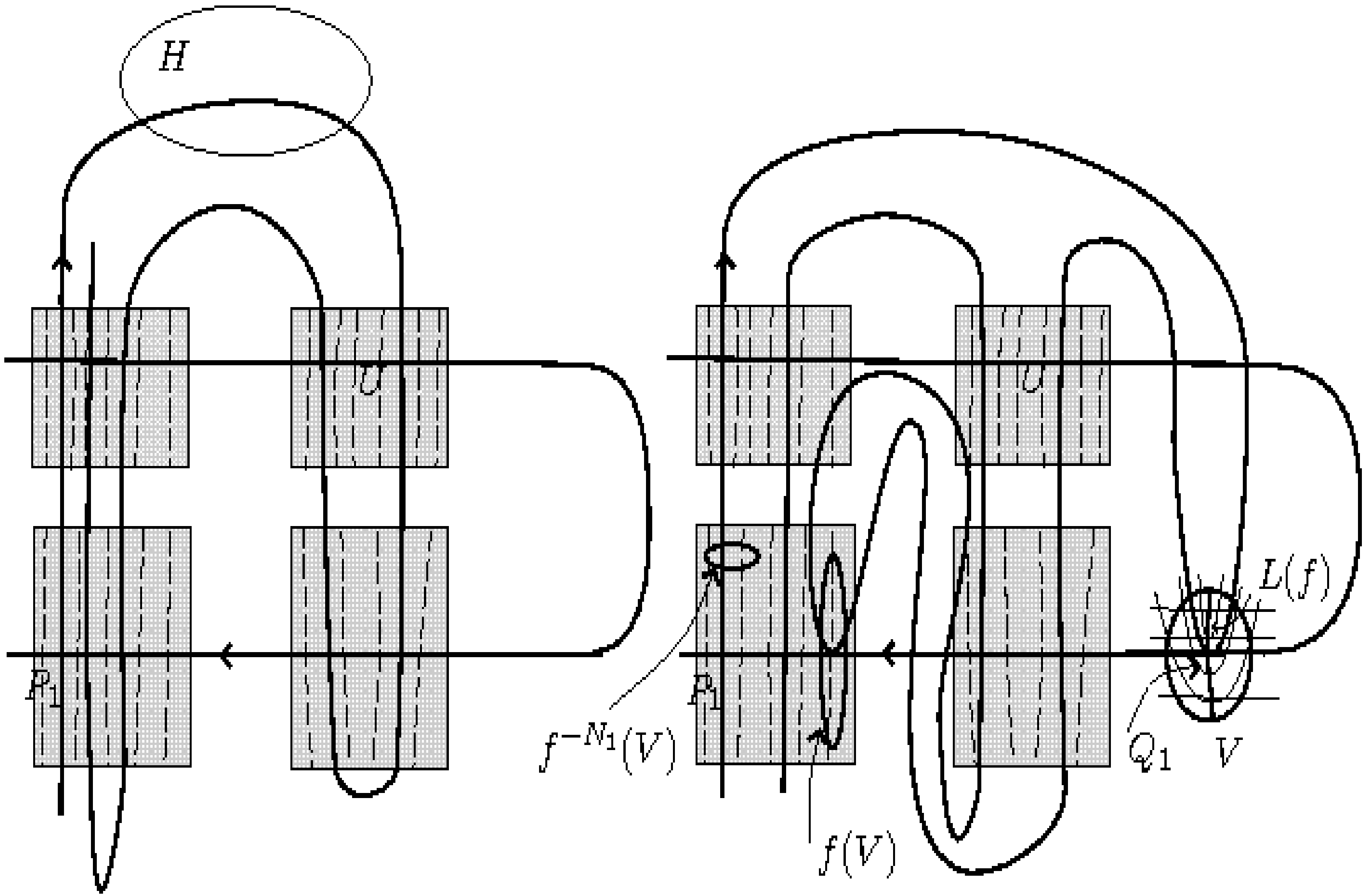
Apply Newhouse Theorem:

open set $\mathcal{N} \subset \mathrm{Diff}^3(M)$
for all $f \in \mathcal{N}$: There exist

hyperbolic maximal set $\Lambda(f)$

an open neighborhood $U \subset M$ of $\Lambda(f)$,
open set $V \subset M$ isolated from $\Lambda(f)$

C^1 line of quadratic tangencies
$L(f) \subset V$ between the stable and unstable manifolds of $\Lambda(f)$, such that f exhibits persistence of homoclinic tangencies along $L(f)$



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$f^{-N_1}(V)$
$f(V)$
$L(f)$
Q_1
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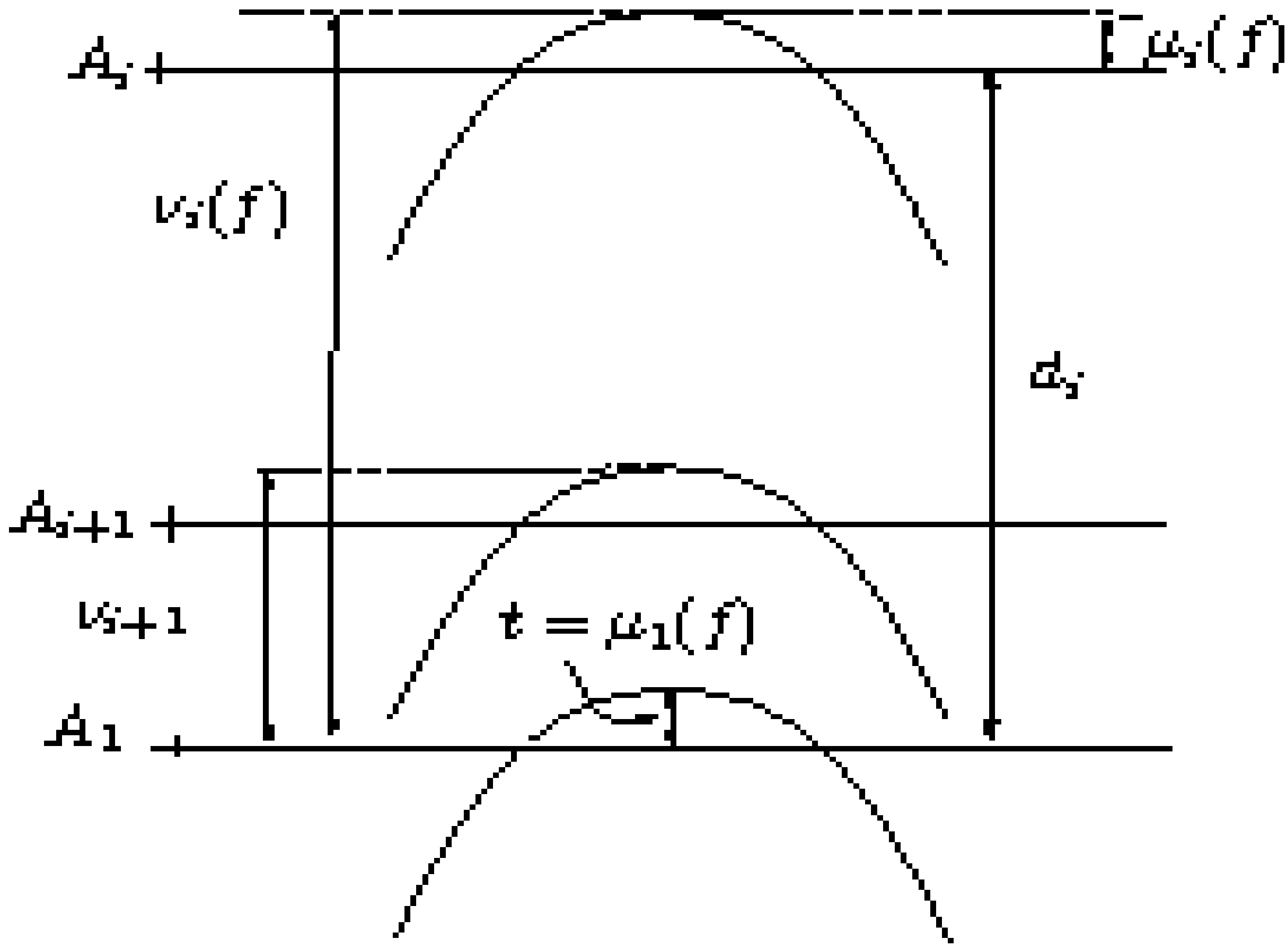
A_i
$\mu_i(f)$
$\nu_i(f)$
d_i
A_{i+1}
ν_{i+1}
$t = \mu_1(f)$
A_1

STEP 2:

Consider TRIVIALIZING COORDINATES OF THE INVARIANT FOLIATIONS in the neighborhood

$$U \subset M \text{ of } \Lambda(f)$$

Use dissipative hypothesis of the given hyperbolic periodic point exhibiting a quadratic homoclinic tangency in f sub0, to obtain STRONG DISSIPATIVE CONDITION of the hyperbolic set for all

$$f \in \mathcal{N} \subset \mathrm{Diff}^3(M)$$

Apply r-normality to conclude that

The STABLE LOCAL FOLIATION is C^3

while the unstable foliation is not necessarily more than $C^{1+\varepsilon}$

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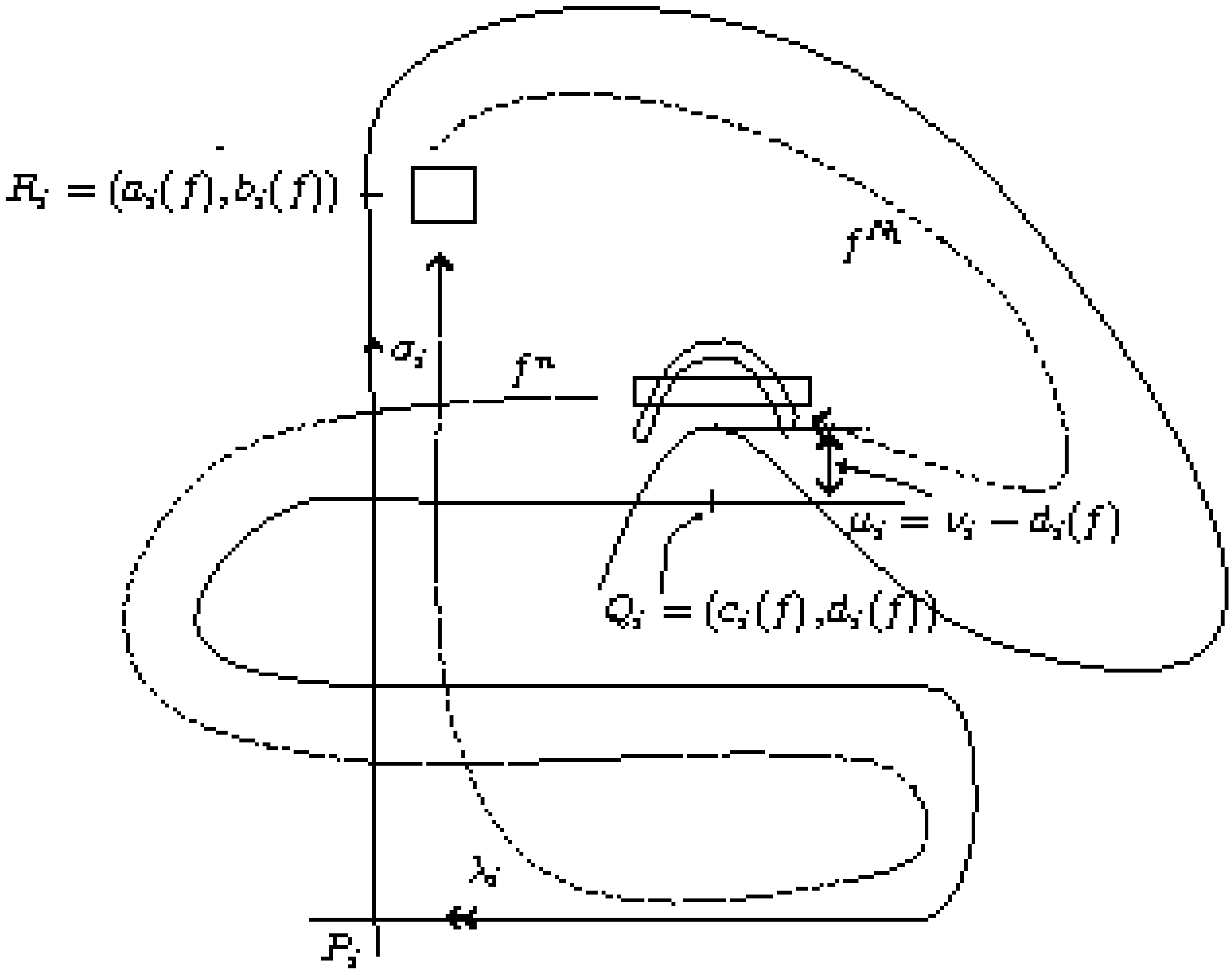
STEP 3:

Compute the iteration of f in the trivializing coordinates as in [PT 1993] (up to non trivial adaptations, using C3 differentiability), to conclude the following :



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λ_i
P_i

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Compute the iteration of f in the trivializing coordinates as in [PT 1993] (up to non trivial adaptations, using C3 differentiability), to conclude the following :

LEMMA

(Uniform approximation to the quadratic family):

The open sets $\mathcal{N} \subset Diff^3(M)$ *and* $V \subset M$ *can be chosen so:*

Given $\varepsilon > 0$ *and the periodic saddle* $P_i(f) \in \Lambda(f)$, *(such that some* $f_i \in \mathcal{N}$ *has a quadratic homoclinic tangency* $Q_i \in V$*), there exists* $N \geq 1$ *such that:*

- *For all $f \in \mathcal{N}$ and for all $n \geq N$ there exists a rectangle* $V^* = V^*(\varepsilon, i, n, f) \subset V$ *such that*

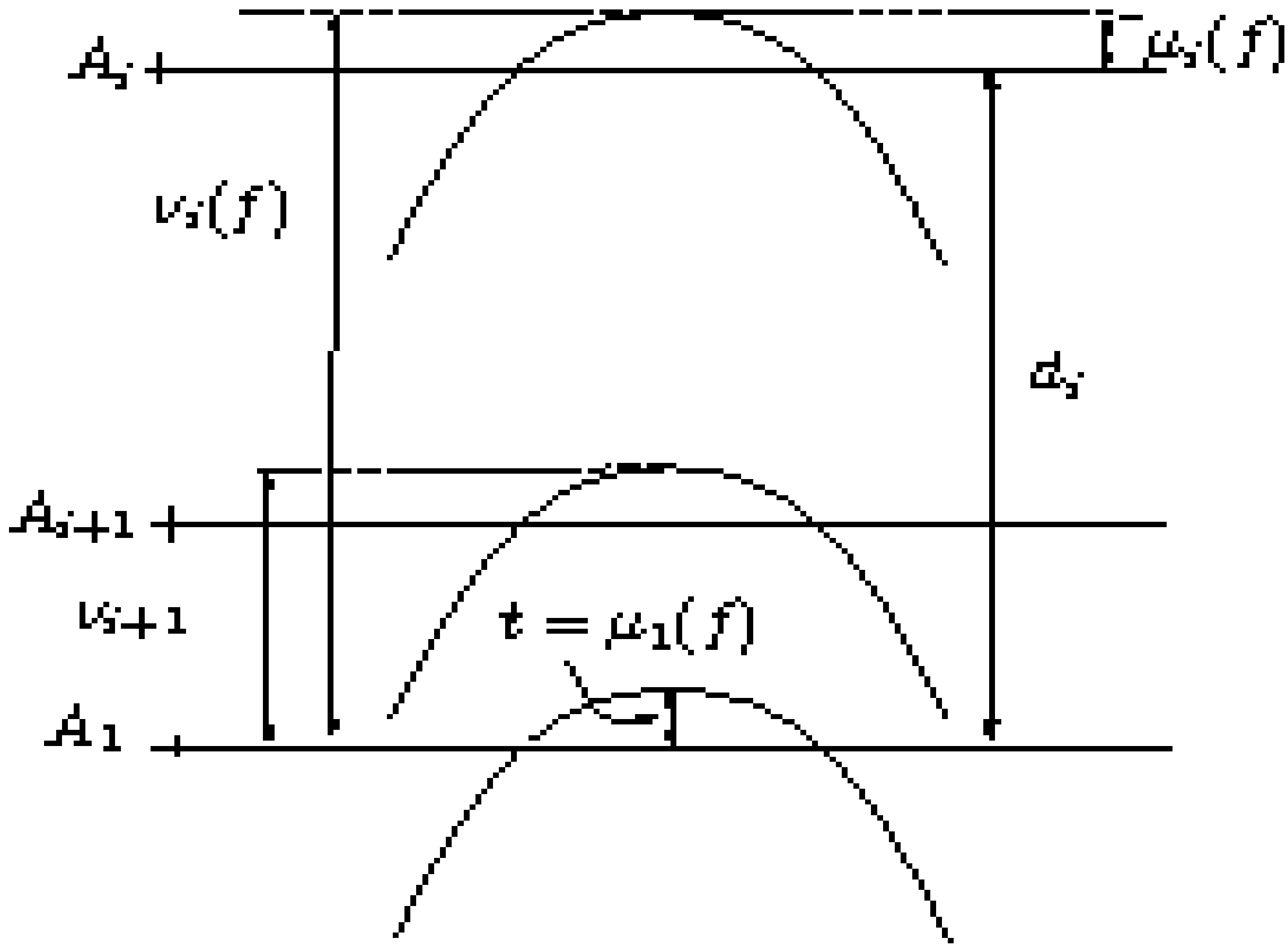
$$f^n|_{V^*}$$

is diffeomorphically conjugated to

$$F(x,y) = (y^2, y^2 + \mu) + G(x,y), \quad \forall (x,y) \in [-4,4]$$

where $\mu = \mu(P, n, f) \in [-4,4]$ *and* G *is a* C^3 *map*

$$\|G\|_{C^3} < \varepsilon.$$



A_i
$\mu_i(f)$
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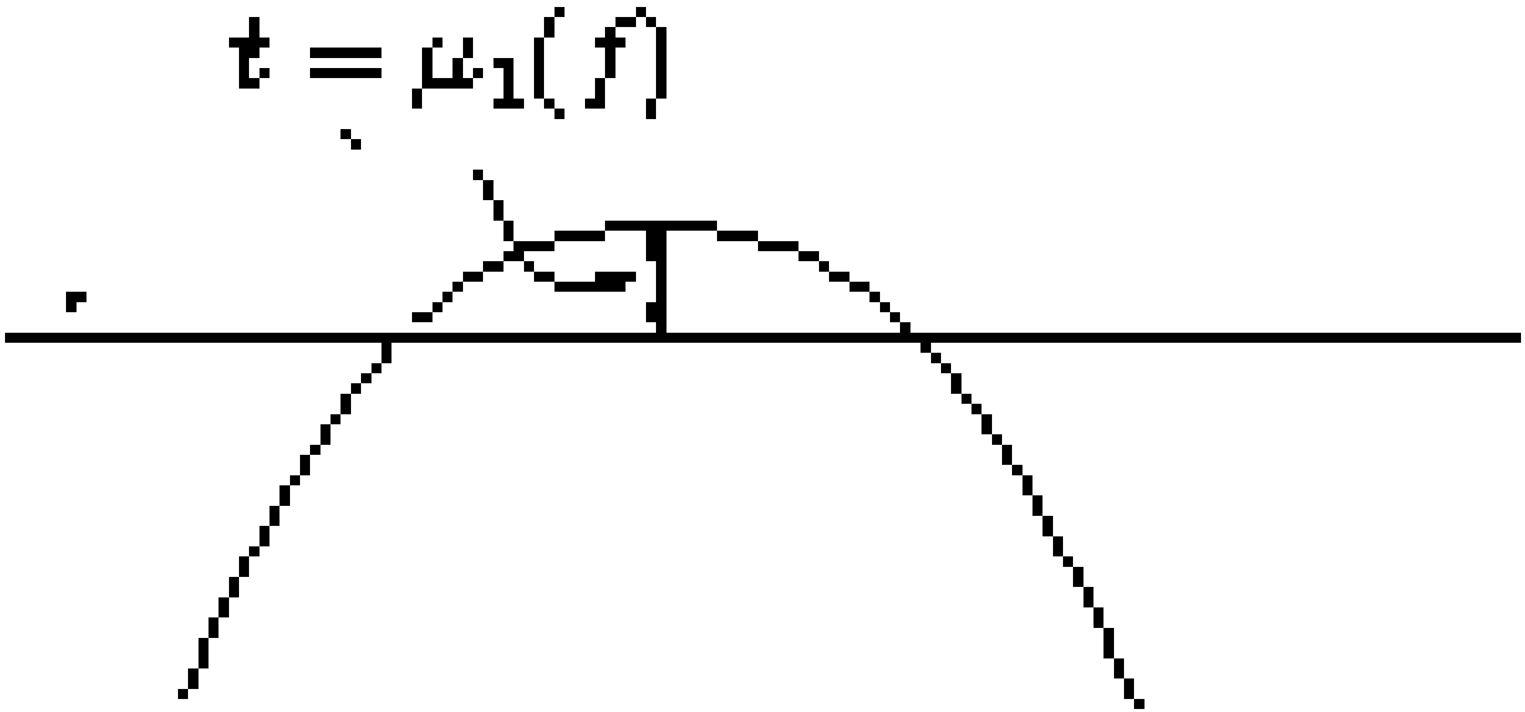
A_1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LEMMA
(Uniform approximation to the quadratic family):

- *For all* $f \in N$ *and for all* $n \geq N$ *there exists a rectangle* $V^* = V^*(\varepsilon, i, n, f) \subset V$ *such that*

$$f^n|_{V^*}$$

is diffeomorphically conjugated to

$$F(x,y) = (y^2, y^2+\mu)+G(x,y), \quad \forall (x,y) \in [-4,4]$$

where $\mu = \mu(P, n, f) \in [-4,4]$ *and* G *is a* C^3 *map*

$$\|G\|_{C^3} < \varepsilon.$$

STEP 4

Consequences of Lemma:

Fix adequate $\varepsilon_0 > 0$ $\qquad P_i(f)$

obtain a REAL FUNCTIONAL INEQUATION

$$\mu(i, n, f) \in \left(-\frac{3}{8}, \frac{1}{8}\right) \qquad (4a)$$

Due to Lemma in STEP 3 and the properties of the 1-dimensional quadratic family:

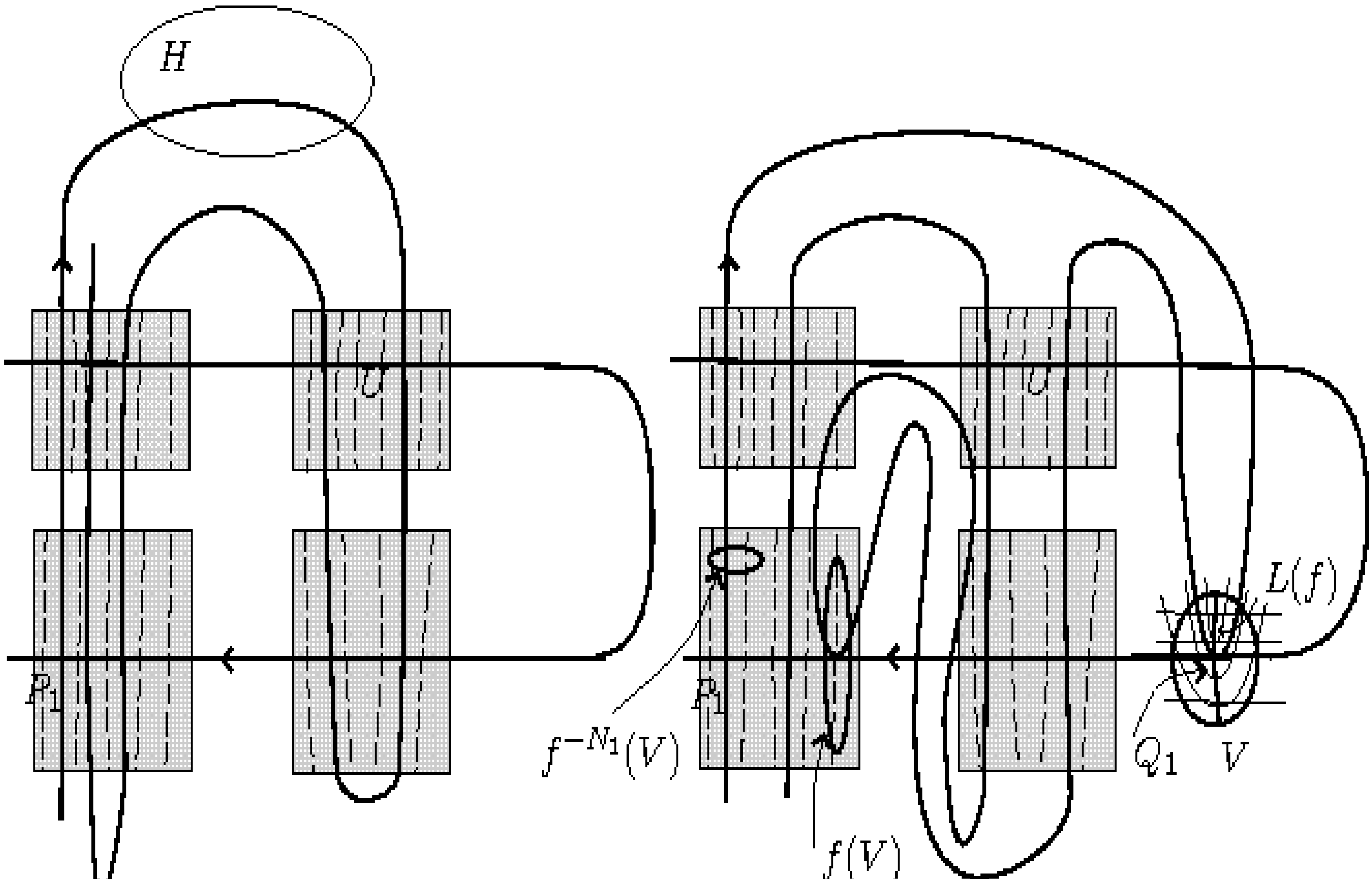
- *If $f \in \mathcal{N}$ verifies INEQUATION (4a) then f has a sink $S_i(f)$ of period n.*
- *If f and g verify INEQUATION (4a) for the same values of i, n, then the sink $S_{i'}(g)$ is the isotopic continuation of the sink $S_i(f)$*

STEP 4

Consequences of Lemma:

$$\mu(i, n, f) \notin [-1, 1] \qquad (4b)$$

- *If f verifies INEQUATION (4a) and g verifies (4b) (for the same values of i, n, then the sink $S_i(f)$ of f has no isotopic continuation to a sink of g.*



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U
$f^{-N_1}(V)$
P_1
$f(V)$
$L(f)$
Q_1
V

STEP 5:

For each $f_1 \in \mathcal{N}$ fixed (for instance in the given one-parameter family), construct a local infinite dimensional manifold $\mathcal{N}_1 \subset \mathcal{N}$ (we did it of infinite codimension)

and foliate $\mathcal{N}_1$ with submanifolds $\mathcal{M}_k$ (for real values of k near 0) of codimension one in $\mathcal{N}_1$, such that:

$f \in \mathcal{M}_k \Leftrightarrow f \in \mathcal{N}_1$, ***and***

$\mu(i, n, f) = \mu(i, n, f_1)$ ***+ k*** *for all* i, n.

STEP 5:

Conclude that

For all $f \in \mathcal{M}_0$ *there exists the isotopic continuations of all the sinks* $S_i(f_1)$ *constructed*

from the Inequality (4a) for f_1

STEP 6 (CONCLUSIONS):

End of the proof of Theorem 1:

construct a nested sequence of open real intervals $[t_i^-, t_i^+] : i \geq 1$

of parametervalues of the given family $\{f_t\}$

such that

$$t \in (t_i^-, t_i^+) \Rightarrow f_t \text{ exhibits a sink of period } n_i$$

$$\text{Take } F = f_T \qquad T \in \cap_{i \geq 1} [t_i^-, t_i^+] \text{ and}$$

the manifold $\mathcal{M} \subset Diff^2(M)$ *constructed in step 5.*

CONCLUSION Theorem 1

F and all $f \in \mathcal{M}$ have infinitely many coexisting sinks $S_i(f)$ such that are all the isotopic continuation of $S_i(F)$

STEP 6 (CONCLUSIONS):

End of the proof of Theorem 2:

For $f_t \neq F$ *in the given one-parameter family compute inequation (4b) to verify it for all* i, n *except at most a finite number of values of* i, n.

CONCLUSION along the given one parameter family $f_t \neq F$ *has at most a finite number of simulta-isotopic continuation of* $S_i(F)$