Observable measures.

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Abstract

For continuous maps on a compact manifold M, particularly for those that do not preserve the Lebesgue measure m, we define the *observable invariant probability measures* as a generalization of the physical measures. We prove that any continuous map has observable measures and characterize those that are physical in terms of the observability. We prove that there exists physical measures whose basins cover Lebesgue a.e, if and only if the set of all observable measures is finite or infinite numerable. We define for any continuous map, its *generalized attractors* using the set of observable invariant measures where there is no physical measure and prove that any continuous maps defines a decomposition of the space in up to infinitely many generalized attractors whose basins cover Lebesgue a.e.

1 Introduction

It is an old problem to find out "good" probability measures for maps $f\colon M \mapsto M$, meaning for that, an invariant probability that resume in some sense, the asymptotic dynamics by future iterations of the map. Sometimes, the map is born with a good measure, as in the case of billiards. But this is not true in general, and it is not an easy question to determine, in general, a single or a few probability measures representing the dynamics of the map.

There have been proposed several ideas to define a "good" invariant probability measure μ:

1. Lebesgue a.e. point in a set is generic with respect μ, that is, $\mu = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta^j_f(x)$, where the convergence is in the weak* topology of the space $\mathcal{P}$ of probabilities on M.

2. The conditional measures of μ on unstable manifolds are absolutely continuous respect to the Lebesgue measure along those manifolds.

3. μ verifies the Pesin's formula: $h_\mu(f) = \int \sum_i \lambda_i^+(x) \dim E_i(x)\, d\mu(x)$, where $\dim E_i(x)$ is the multiplicity of $\lambda^+(x)$ in the Oseledec's decomposition.

4. The measure is the limit of measures which are invariant under stochastic perturbations.

A measure verifying 4 or 1 "concentrates" on points which are more visited.

We will call physical measure, a probability measure verifying 1, and denote stochastically stable, a probability measure verifying 4. We will call SRB (Sinai-Ruelle-Bowen) measure a probability verifying 2. And we denote as a Pesin measure, a probability verifying 3.

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In this work, we address ourselves to propose another concept of "good" probabilities, which we call observable measures, that are particularly meaningful for non differentiable maps.

The following purposes motivated this work: Characterize the continuous homeomorphism f conjugated to transitive Anosov (topologically chaotic), that have a physical measure attracting the time average of Lebesgue almost all orbit. If not possible, at least generalize as much as possible the result in [CE01]. In any case, the main question is:

Is it possible to describe probabilistically in the space, in some *minimal way,* the asymptotic behavior of the time averages of Lebesgue a.e. orbit? We could answer to this question in Theorem 2.5. Generalized ergodic attractors and observable measures always exist for any continuous map.(Theorems 2.3 and 2.8). On the other hand, physical measures and ergodic attractors do not always exist. It is largely known the difficulties to characterize, or just find, non hyperbolic or non $C^{1+\alpha}$ maps that do have physical measures. This is a hard problem even in some systems whose iterated topological behavior is known. ([C93], [E98], [H00], [HY95]). The difficulties appear when applying the known techniques for constructing the physical measures in a hyperbolic setting, to a weaker hyperbolic context([Pe77], [S72], [A67], [PS04], [V98], [BDV00]).

The following open question refers to the existence and finiteness of physical measures and to the convergence of the sequence of time averages of Lebesgue a.e. orbit. It is possed in [P99] and leads to a global understanding of the dynamics from an ergodic viewpoint:

1.1 Palis Conjecture *Most dynamical systems have up to finitely many physical measures (or ergodic attractors) such that their basins of attraction cover Lebesgue almost all points.*

This conjecture admits the following equivalent statement, that seems weaker. (In fact, the definition 2.1 of observability is certainly weaker than the definition 2.2 of physical measures.)

1.2 Equivalent formulation of Palis Conjecture: *The set of observable measures for most dynamical systems is finite.*

Note: To prove the equivalence of statements 1.1 and 1.2 it is enough to join our Theorems 2.3.b and 2.5.

2 Statement of the results.

Let $f\colon M \mapsto M$ be a continuous map in a compact, finite-dimensional manifold M. Let m be the Lebesgue measure normalized to verify $m(M) = 1$, and not necessarily f-invariant. We denote $\mathcal{P}$ the set of all Borel probability measures in M, provided with the weak* topology, and a metric structure inducing this topology. For any point $x \in M$ we denote $p\omega(x)$ to the set of the Borel probabilities in M that are the partial limits of the (not necessarily convergent) sequence

$$\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right\}_{n \in \mathbb{N}} \tag{1}$$

where δ_y is the Delta de Dirac probability measure supported in $y \in M$.

The set $p\omega(x) \subset \mathcal{P}$ is the collection of the spatial probability measures describing the asymptotic time average, provided the initial state is x. If the sequence (1) converges then the set $p\omega(x) = \{\mu_x\}$. But we will consider also the opposite case of the maps such that, for Lebesgue a.e. $x \in M$, the sequence (1) is not convergent.

The phenomena exhibited when (1) is not convergent are similar to the time-delayed specification properties studied by Bowen in ([BR75]), but here they are seen on the time average probabilities, instead on the points along the orbit.

We define:

Definition 2.1 (Observable probability measures.) A probability measure $\mu \in \mathcal{P}$ is *observable* if for all $\epsilon > 0$ the set $A_\epsilon = \{x \in M : \text{ dist }^*(p\omega(x), \mu) < \epsilon\}$ has positive Lebesgue measure $m(A_\epsilon) > 0$.

The set $A_\epsilon = A_\epsilon(\mu) \subset M$ is called *the ϵ- basin of partial attraction* of the probability μ.

We note that the definition above is independent of the choice of the distance in $\mathcal{P}$, provided that the metric structure induces its weak* topology. We also remark that observable measures are f-invariant, and that usually at most a few part of the space of invariant measures for f are observable measures.

Definition 2.2 (Physical probability measures.) A probability measure $\mu \in \mathcal{P}$ is *physical* (even if it is not ergodic), if the set $B = \{x \in M : p\omega(x) = \{\mu\}\}$ has positive Lebesgue measure $m(B) > 0$.

If μ is physical, the set $B = B(\mu) \subset M$ is called the *basin of attraction* of μ.

From the definitions above note that physical measures and observable measures are f-invariant. All physical measure is observable but not all observable measure is physical.

We prove the following starting results:

Theorem 2.3 (Existence of observable measures and physical measures.)

a) *For any continuous map f, the set $\mathcal{O}$ of all observable probability measures for f is non-empty and* weak*-*compact.*

b) *$\mathcal{O}$ is finite or countably infinite if and only if there exist* (resp. finitely or countable infinitely many) *physical measures of f attracting* (i.e. the union of their basins of attraction cover) *Lebesgue almost all orbits.*

Definition 2.4 (Basin of attraction.)

The *basin of attraction* $B(\mathcal{K})$ of a compact subset $\mathcal{K}$ of the space $\mathcal{P}$ of all the Borel probability measures in M, is the (maybe empty) subset of M defined as:

$$B(\mathcal{K}) = \{x \in M : p\omega(x) \subset \mathcal{K}\}$$

If the purpose is to study the asymptotic to the future time average behaviors of Lebesgue almost all points in M, then the set $\mathcal{O}$ of all observable measures for f is the exact necessary and sufficient solution. In fact we have the following:

Theorem 2.5 (Attracting minimality property of the set of observable measures.)

The set $\mathcal{O}$ of all observable measures for f is the minimal compact subset of the space $\mathcal{P}$ whose basin of attraction has total Lebesgue measure.

Due to the conjecture in 1.1 and Theorem 2.3, we are interested in partitioning the set $\mathcal{O}$ of observable measures, or to reduce it as much as possible, into different compact subsets whose basins of attractions have positive Lebesgue measure. Due to results in Theorem 2.5, no proper compact part of $\mathcal{O}$ has a total Lebesgue basin. We define:

Definition 2.6 (Generalized Attractors - Reductions of the space $\mathcal{O}$.) A *generalized attractor* $(A, \mathcal{A}) \subset \mathcal{M} \times \mathcal{O}$, (or a *reduction* $\mathcal{A}$ of the space $\mathcal{O}$ of all observable measures for f), is a compact subset $(A, \mathcal{A})$ such that the basin of attraction $B(\mathcal{A}) = \{x \in M : p\omega(x) \subset \mathcal{A}\}$ has positive Lebesgue measure in M, and A is the (minimal) compact support in M of all the probability measures in $\mathcal{A}$. We call $(A, \{\mu\})$ an *attractor* if it is a generalized attractor with a single invariant probability μ, i.e. μ is a physical measure.

To illustrate the difference between generalized attractor and attractor in the usual topological sense, take the C^2 almost Anosov, conjugated to a transitive Anosov in the torus, with a fixed non hyperbolic saddle p_0 with weak expansive direction and strong contraction. It has a unique generalized attractor supported on p_0. Even if topologically chaotic (conjugated to Anosov), statistically in the mean p_0 acts like a sink.

In spite a system could not exhibit a physical measure, still the reductions of the space of observable measures divide the manifold in the basins of *generalized attractors.*

Definition 2.7 Irreducibility

A generalized attractor $\mathcal{A} \subset \mathcal{P}$ is *irreducible* if it does not contain proper compact subsets that are also generalized attractors.

It is *trivial* or *trivially irreducible* if its diameter in $\mathcal{P}$ is zero , or in other words, if $\mathcal{A}$ has a unique observable measure μ.

Note that physical measures are trivially irreducible and conversely.

The following result is much weaker but related with the Palis' conjecture stated in paragraph 1.1:

Theorem 2.8 (Decomposition Theorem)

For any continuous map $f\colon M \mapsto M$ there exist a collection of (up to countable infinitely many) generalized attractors whose basins of attraction are pairwise Lebesgue-almost disjoint and cover Lebesgue-almost all M.

The continuous maps divide in two disjoint classes:

- *The generalized attractors of the decomposition are all irreducible and then the decomposition is unique.*
- *For all $\epsilon > 0$ there exist a decomposition for which the reducible generalized attractors have all diameter (in the weak* space of probabilities), smaller than ϵ* (and thus, for a rough observer, all the reducible generalized attractors of the decomposition act as physical measures).

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Observable Measures

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"Good" invariant probabilities

They resume in some minimal sense, the asymptotic dynamics.

- **Physical measure μ:** The point x in a **positive Lebesgue measure set**, is generic with respect μ, i.e:

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_f^j(x)$$

- **Sinai-Ruelle-Bowen (SRB) measure μ:** The conditional measures of μ on unstable manifolds are **absolutely continuous respect to the Lebesgue measure** along those manifolds.
- **The probability μ verifies the Pesin's formula:**

$$h_\mu(f) = \int \sum_i \lambda_i^+(x) \dim E_i(x)\, d\mu(x)$$

Palis' conjecture

Most dynamical systems have up to **finitely many physical measures** *(or ergodic attractors) such that their basins of attraction cover Lebesgue a.e.*

Equivalent formulation of Palis' Conjecture: The set of **observable measures** (to be adequately defined) for most dynamical systems is finite.

Notations.

- $f: M \mapsto M$ is a continuous map in a compact, finite-dimensional manifold M.
- m be the Lebesgue measure normalized to verify $m(M) = 1$, (not necessarily f-invariant).
- $\mathcal{P}$ the set of all Borel probability measures in M, provided with a metric structure inducing the weak* topology. The set of f invariant probabilities is $\mathcal{P}_f$
- For any point $x \in M$: $p\omega(x)$ is the set of the Borel probabilities in M that are the **partial limits** of the (not necessarily convergent) **sequence of empiric distribution:**

$$\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right\}_{n \in \mathbb{N}} \tag{1}$$

Definition of observable measure.

Definition

A probability measure $\mu \in \mathcal{P}$ is *observable* if for all $\epsilon > 0$ the set

$$A_\epsilon = \{x \in M : \text{ dist }^*(p\omega(x), \mu) < \epsilon\}$$

has **positive Lebesgue measure** $m(A_\epsilon) > 0$.
The set $A_\epsilon = A_\epsilon(\mu) \subset M$ is called *the ϵ- basin of partial attraction* of the probability μ.

Remarks: The definition above is independent of the choice of the distance in $\mathcal{P}$, (inducing the weak* topology). Observable measures are f-invariant. The Lebesgue measure of A_ϵ is **positive for all** $\epsilon > 0$, but may be is zero for $\epsilon = 0$.

Definition

(Physical probability measures.) *A probability measure* $\mu \in \mathcal{P}$ *is* physical *(even if it is not ergodic), if the set* $B = \{x \in M : p\omega(x) = \{\mu\}\}$ *has positive Lebesgue measure* $m(B) > 0$.

If μ *is physical, the set* $B = B(\mu) \subset M$ *is called the* basin of attraction *of* μ.

Remark: Any physical measure is observable but not all observable measure are physical.

Theorem

Existence of observable measures.

a) *For any continuous map f, the set $\mathcal{O}$ of observable measures of f is not empty and* weak**-compact.*

b) *$\mathcal{O}$ is finite or countably infinite if and only if there exist* (resp. finitely or countable infinitely many) *physical measures whose basins of attraction cover* Lebesgue a.e.

c) *There exists a unique physical measure whose basin is Lebesgue a.e. if and only if the set of observable measures has a unique probability.*

Minimality of the set of observable measures

Definition: The *basin of attraction* $B(\mathcal{K})$ of a compact subset $\mathcal{K}$ of the space $\mathcal{P}$ of all the Borel probability measures in M, is the (maybe empty) subset of M defined as:

$$B(\mathcal{K}) = \{x \in M : p\omega(x) \subset \mathcal{K}\}$$

If the purpose is to study the asymptotic to the future time average behaviors of Lebesgue almost all points in M, then **the set $\mathcal{O}$ of all observable measures for f is the exact necessary and sufficient set of probabilities.**

Theorem

The set $\mathcal{O}$ of all observable measures for f is the minimal compact subset of the space $\mathcal{P}$ whose basin of attraction has total Lebesgue measure.

Generalized Attractors.

Due to our motivation in the conjecture of Palis we are interested in partitioning the set $\mathcal{O}$ of observable measures, or to reduce it as much as possible, into different compact subsets whose basins of attractions have positive Lebesgue measure. But due to the Theorem of minimality, no proper compact part of $\mathcal{O}$ has a total Lebesgue basin.

Definition

(Generalized Attractors - Reductions of the space $\mathcal{O}$.) A *generalized attractor* $(A, \mathcal{A}) \subset \mathcal{M} \times \mathcal{O}$, *is a compact subset such that* **the basin of attraction**

$$B(\mathcal{A}) = \{x \in M : p\omega(x) \subset \mathcal{A}\} \subset M$$

has positive Lebesgue measure.

• A *is the (minimal) compact support in* M *of all the probability measures in* $\mathcal{A}$.

• $\mathcal{A} \subset \mathcal{O}$ *is also called a "reduction" of the set of observable measures.*

• *In particular we call* $(A, \{\mu\})$ *an* attractor *if it is a generalized attractor with a single invariant probability* μ, *i.e.* μ *is a physical measure.*

Examples: Conjugated to transitive Anosov 1.

• To illustrate the difference between generalized attractor and topological attractor: consider the C^2 almost Anosov, conjugated to a transitive Anosov in the two-torus, with a fixed non hyperbolic saddle p_0 with weak expansive direction (toplogically expanding but with eigenvalue 1) and strong (hyperbolic) contraction. It has a unique generalized attractor supported on p_0. (Hu and Young Ergod.Th.&Dyn.Sys. 15-1995) Even if topologically chaotic (conjugated to Anosov), statistically in the mean (i.e. attracting the sequences of empiric distributions) p_0 **acts like a sink.**

Examples: Conjugated to transitive Anosov 2.

• On the opposite situation, the example of Lewowicz (Journal of Diff.Eq. 38, 1980) in the two-torus, is a real analytic diffeomorphism, conjugated to a transitive Anosov, but non hyperbolic:

$$x = x + y + sen(2\pi x), \quad y = y + \mathrm{sen}(2\pi x)$$

It preserves the Lebesgue measure m which is besides ergodic (C.E.2001, Disc.Cont.Dyn.Sys.7): therefore, in this example, m is a physical measure and the unique observable measure.

• **Open question:** Characterize the non singular homeomorphisms (or at least the C^1 diffeomorphisms that are not $C^{1+\alpha}$) in the torus, that are conjugated to transitive Anosov and have a unique physical measure attracting Lebesgue a.e.

Examples: C^1 expanding maps 1.

- It is well known that $C^{1+\alpha}$ expanding maps have a single physical measure attracting Lebesgue a.e. that is absolute continuous respect to Lebesgue (Ruelle).
- If f is not $C^{1+\alpha}$ but is C^1 and expanding, then generically it also has a single physical measure attracting Lebesgue a.e., but it is **mutually singular respect to Lebesgue.** (Campbell and Quas, Commun.Math.Phys. 349, 2001).
- In this last case the physical measure verifies the Pesin's formula of the entropy (C.E.2010, using ideas of the book of Keller, 1998)

Examples: C^1 expanding maps 2.

• There exist C^1 expanding maps in the circle, that have no physical measure attracting Lebesgue a.e. : the Lebesgue measure is invariant but not ergodic (Quas, Ergod.Th. & Dyn.Sys.16 (1996). Thus, in this example the set of observable measures contains more than one probability, and is exactly the closure of the set of the ergodic components of the Lebesgue measure.

In spite a system could not exhibit a physical measure, still the reductions of the space of observable measures divide the manifold in the basins of **generalized attractors.**

Definition

Irreducibility

A generalized attractor $\mathcal{A} \subset \mathcal{P}$ is *irreducible* if it does not contain proper compact subsets that are also generalized attractors.
It is *trivial* or *trivially irreducible* if its diameter in $\mathcal{P}$ is zero , or in other words, if $\mathcal{A}$ has a unique observable measure μ. (Physical measures are trivially irreducible and conversely.)

The following result is much weaker but related with the Palis' conjecture:

Theorem

(Decomposition Theorem)
For any continuous map $f : M \mapsto M$ *there exist a collection of (up to countable infinitely many) generalized attractors whose basins of attraction are pairwise Lebesgue-almost disjoint and cover Lebesgue-almost all* M.
The continuous maps divide in two disjoint classes:
• *The generalized attractors of the decomposition are all irreducible and then the decomposition is unique.*
• *For all* $\epsilon > 0$ *there exist a decomposition for which the reducible generalized attractors have all diameter (in the weak* space of probabilities), smaller than* ϵ (and thus, for a rough observer, all the reducible generalized attractors of the decomposition act as physical measures).