Pesins Formula for C^1 diffeomorphisms with Dominated Splitting

E. Catsigeras, M. Cerminara and H. Enrich *

July 6th., 2012.

Talk in the 32o. EDAÍ, Rio de Janeiro, Julio 2012

Abstract

We consider C^1 diffeomorphisms with dominated splitting $E \oplus F$ that are topologically expansive and preserve no measure absolutely continuous with respect to Lebesgue. We characterize a never empty set of invariant probabilities (the SRB-like measures), by means of a quasi-physical property, for which the metric entropy is bounded from below by the integral of the sum of the Lyapounov exponents of $df|_F$. Joining this result with Ruelles inequality, we conclude that if all the positive Lyapunov exponents are included in those of $df|_F$, and these latter are all non negative, then any SRB-like measure satisfies the Pesins formula of the entropy.

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EDAÍ

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32º EDAÍ 6 de julho de 2012

Sala L856, Depto. de Matemática, Edifício Cardeal Leme, PUC–Rio.

Matinê: 14h30 – 15h30

Thue-Morse dynamical system

Christian Mauduit (Institut de Mathématiques de Luminy)

We give an introduction to symbolic dynamical systems by exploring the paradigmatic example of the Thue-Morse sequence, introduced by several mathematicians in different contexts since the 19th century, and defined as the the limit in $\{0,1\}^{\mathbb{N}}$ of the sequence of finite words $(T_k)_{k\in\mathbb{N}}$ defined by the recursion $T_0 = 0$, $T'_0 = 1$ and $T_{k+1} = T_k T'_k$, $T'_{k+1} = T'_k T_k$ for any non negative integer k.

References:

- M. Queffelec, Substitution Dynamical Systems - Spectral Analysis, Lecture Notes in Mathematics 1294, Springer.
- N. Pytheas Fogg, Substitutions in Dynamics, Arithmetics, and Combinatorics, Lecture Notes in Mathematics 1794, Springer.

Palestra 1: 15h45 – 16h45

Pesin's Formula for C^1 diffeomorphisms with Dominated Splitting

Eleonora Catsigeras (Universidad de la República)

We consider C^1 diffeomorphisms with dominated splitting $E \oplus F$ that are topologically expansive and preserve no measure absolutely continuous with respect to Lebesgue. We characterize a never empty set of invariant probabilities (the SRB-like measures), by means of a quasi-physical property, for which the metric entropy is bounded from below by the integral of the sum of the Lyapounov exponents of $df|_F$. Joining this result with Ruelle's inequality, we conclude that if all the positive Lyapunov exponents are included in those of $df|_F$, and these latter are all non negative, then any SRB-like measure satisfies the Pesin's formula of the entropy.

This is a joint work with M. Cerminara and H. Enrich.

Café: 16h45 – 17h15

Palestra 2: 17h15 – 18h15

Sobre expansividade (positiva) e medidas expansivas

Alexander Arbieto (UFRJ)

Apresentaremos conseqüências da expansividade (positiva). Também introduziremos o conceito de medidas expansivas e apresentaremos algumas conseqüências deste conceito.

Esta última parte é um trabalho em conjunto com Carlos Morales.

Confraternização: 19h00 – ∞

Garota da Gávea

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Pesin's Formula for C^1 Diffeomorphisms with dominated splitting

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Seminario EDAI, Río de Janeiro, 6 de Julio de 2012

$f \in \mathsf{Diff}^1(M) \quad \forall \mu \in \mathcal{P}_f \quad$ for μ-a.e. $x \in M$ the Lyapunov exponents are denoted by

$$\chi_1(x) \geq \chi_2(x) \geq \ldots \geq \chi_{\mathsf{dim}M}(x), \quad \chi_i^+(x) := \max\{\chi_i(x), 0\}$$

Ruelle's Inequality:

$$h_\mu \leq \int \sum_{i=1}^{\mathsf{dim}M} \chi_i^+ \, d\mu.$$

Definition

$\mu \in PF$ (μ satisfies Pesin's Formula of the Entropy) if

$$h_\mu = \int \sum_{i=1}^{\mathsf{dim}M} \chi_i^+ \, d\mu.$$

m Lebesgue measure on M

m^u Lebesgue measure along unstable manifolds.

- For Diff $^{1+\alpha}(M)$

  Pesin $\mu \in \mathcal{P}_f$ hyperbolic, $\mu \ll m \;\Rightarrow\; \mu^u \ll m^u \;\Rightarrow\; \mu \in PF$.

  Ledrappier-Young: $\mu \in PF \;\Rightarrow\; \mu^u \ll m^u$.

  Pugh-Shub: μ ergodic, hyperbolic and $\mu^u \ll m^u \;\Rightarrow\; \mu$ is SRB (namely, physical) $\;\Rightarrow\; \mu$ describes the asymptotic statistics of a Lebesgue-positive set of orbits.

- **General purpose:** Reformulate the results about measures $\mu \in PF$ for

$$f \in \text{Diff}^1(M) \; .$$

Search for the relations between:

- SRB or physical measures or "SRB-like" measures **statistically** describing the asymptotic behavior **of Lebesgue-a.e.**
- Invariant measures μ that satisfy **Pesin's Formula of the entropy ($\mu \in PS$)**.

- For Diff $^{1+\alpha}(M)$

  Pesin $\mu \in \mathcal{P}_f$ hyperbolic, $\mu \ll m \;\Rightarrow\; \mu^u \ll m^u \;\Rightarrow\; \mu \in PF$.

  Ledrappier-Young: $\mu \in PF \;\Rightarrow\; \mu^u \ll m^u$.

  Pugh-Shub: μ ergodic, hyperbolic and $\mu^u \ll m^u \;\Rightarrow\; \mu$ is SRB (namely, physical) $\;\Rightarrow\; \mu$ describes the asymptotic statistics of a Lebesgue-positive set of orbits.

- For Diff $^1(M)$

  Sun-Tian (2012): f with Dom. Split. $E \oplus F$ with $\lambda_F \geq 0, \lambda_E < 0$ $\mu \ll m \Rightarrow \mu \in PS$ (Mañé, 1981)

  Qiu (2011): f transitive Anosov: C^1-generically $\exists\,!\ \mu \in PF$. Besides: μ is SRB (namely physical). But $\mu \perp m$.

Theorem

Let $f \in \textit{Diff}^{\,1}(M)$ *with Dom. Split.* $E \oplus F$*:*

$$\|df^n|_{E(x)}\| \cdot \|df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n, \quad 0 < C,\ 0 < \lambda < 1.$$

Denote the Lyapunov exponents for any Oseledets' regular orbit by:

$$\chi_1 \geq \chi_{\dim F} > \chi_{\dim F+1} \geq \chi_{\dim F + \dim E}.$$

Then, there exist **"SRB-like"** *measures, such that*

1. *They have a pseudo physical property: the set of SRB-like measures minimally describe the asymptotic statistics Lebesgue a.e.*

2. *They satisfy the inequality:*

$$h_\mu(f) \geq \int \sum_{i=1}^{\dim F} \chi_i \, d\mu$$

Moreover: 1 $\Rightarrow$ 2.

3. *If besides* $\chi_{\dim F} \geq 0 \geq \chi_{\dim F+1}$, *then*

any SRB-like measure μ satisfies the Pesin's Formula of the Entropy ($\mu \in PF$)

- Fix $x \in M$ **Sequence of Empirical Probabilities** of x:

$$\sigma_{n,x} := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$$

- $p\omega$**-limit of** x:

$$p\omega(x) := \{\mu \in \mathcal{P} : \exists\, n_j \to +\infty \text{ such that } \lim_j \sigma_{n_j,x} = \mu\} \subset \mathcal{P}_f$$

For any given $\mu \in \mathcal{P}_f$

- **BASIN OF STATISTICAL ATTRACTION**

$$B(\mu) := \big\{x \in M : \; p\omega(x) = \{\mu\}\big\}.$$

- **BASIN OF ϵ-WEAK STATISTICAL ATTRACTION**

$$B_\epsilon(\mu) := \big\{x \in M : \; \text{dist}^*(p\omega(x), \mu) < \epsilon\big\}.$$

Definition

An invariant measure μ is **SRB OR PHYSICAL** if $m(B(\mu)) > 0$.
An invariant measure μ is **SRB-LIKE OR PSEUDO PHYSICAL** if $m(B_\epsilon(\mu)) > 0 \;\; \forall\, \epsilon > 0$.

Theorem

$f \in$ Diff$^1(M)$ with Dom. Split. $E \oplus F$, μ is SRB-like.

Then, $h_\mu(f) \geq \int \sum_{i=1}^{\dim F} \chi_i \, d\mu.$

Lemma 1. Upper bound of the Lebesgue measure m

$\forall \epsilon > 0 \; \exists \delta > 0$ s.t. $\forall$ finite partition α with $\text{diam}(\alpha) < \delta$

$\exists$ a sequence $\{\nu_n\}_{n \geq 0}$ of finite measures and $K > 0$ s.t.:

1) $\nu_n(X) < K \; \forall \; X \in \alpha^n = \bigvee_{j=0}^n f^{-j}(\alpha) \;\; \forall \; n \geq 0.$

2) $\forall \; C \in \mathcal{B}$: $m(C) \leq K e^{n\epsilon} I(\psi_n, C, \nu_n),$, where

$I(\psi_n, C, \nu_n) := \int_C e^{\psi_n} \, d\nu_n, \quad \psi_n(x) := -\log \left| \det df^n(x)|_{F_x} \right| =$

$\sum_{j=0}^{n-1} \psi \circ f^n(x), \quad \psi(x) := -\log \left| \det df(x)|_{F_x} \right|.$

Lemma 2 (Lower bound for the metric entropy)

$\forall \; \mu \in \mathcal{P}_f, \; \forall \; \epsilon, \; \delta > 0$ there exists a finite partition α satisfying diam$\alpha < \delta$ and there exists a sequence $\{C_n\}_{n \geq 0} \subset \mathcal{B}$ such that:

3) $\bigcap_{N \geq 0} \bigcup_{n \geq N} C_n \supset B_{\epsilon^*}(\mu)$ for some $\epsilon^* > 0$ for some dist* in $\mathcal{P}$.

4) $\forall$ sequence $\{\nu_n\}_{n \geq 0}$ of finite measures, if $\exists \; K > 0$ s.t.

$\nu_n(X) < K \; \forall \; X \in \alpha^n \; \forall \; n \geq 0$, then:

$$\limsup_{n \to +\infty} \frac{1}{n} \log I(\psi_n, C_n, \nu_n) \leq \epsilon + h_\mu(\alpha) + \int \psi \, d\mu$$

Theorem

$f \in Diff^1(M)$ with Dom. Split. $E \oplus F$, μ is SRB-like.

Then, $h_\mu(f) \geq \int \sum_{i=1}^{dimF} \chi_i \, d\mu.$

Proof of the Theorem using Lemmas 1 and 2: Let $\mu \in \mathcal{P}_f$ such that $h_\mu(f) - \int \sum_{i=1}^{\text{dim F}} \chi_i \, d\mu = -r < 0$. We will prove that $\mu \notin$ SRB-like. • Take $0 < \epsilon < r/4$. Consider $\delta > 0$ by Lemma 1. Construct the partition α and the sequence $\{C_n\}_{n \geq 0} \subset \mathcal{B}$ by Lemma 2. Construct the sequence $\{\nu_n\}_{n \geq 0}$ of finite measures and the constant $K > 0$, by Lemma 1.

• Apply Lemma 2:

$$\limsup_n \frac{1}{n} \log I(\psi_n, C_n, \nu_n) \leq \epsilon + h_\mu(\alpha) + \int \psi \, d\mu \leq (r/4) - r$$

• Apply Lemma 1: $m(C_n) \leq K e^{n\epsilon} I(\psi_n, C_n, \nu_n)$. Then:

$$\frac{1}{n} \log m(C_n) \leq \frac{\log K}{n} + \epsilon + \frac{1}{n} \log I(\psi_n, C_n, \nu_n)$$ with $0 < \epsilon < r/4$

• Join the two inequalities above:

$$\limsup_n \frac{1}{n} \log m(C_n) \leq (r/2) - r = -(r/2) < 0$$

• Apply Borel- Cantelli: $m\left(\bigcap_{N \geq 0} \bigcup_{n \geq N} C_n\right) = 0.$

• Use property (3) of Lemma 2:

$$m(B_{\epsilon^*}(\mu)) \leq m\left(\bigcap_{N \geq 0} \bigcup_{n \geq N} C_n\right) = 0$$

Then μ is not SRB-like. □

Lemma 1 (Upper Bound of the Lebesgue measure m)

$\forall\, \epsilon > 0\, \exists\, \delta > 0$ *s.t.* $\forall$ *finite partition* α *with diam*$(\alpha) < \delta$
$\exists$ *a sequence* $\{\nu_n\}_{n\geq 0}$ *of finite measures and* $K > 0$ *s.t.:*
1) $\nu_n(X) < K\ \forall\ X \in \alpha^n$ **2)** $\forall\ C \in \mathcal{B}:\ \ m(C) \leq K e^{n\epsilon} I(\psi_n, C, \nu_n)$,
where $I(\psi_n, C, \nu_n) := \int_C e^{\psi_n}\, d\nu_n, \quad \psi_n(x) := -\log\left|\det df^n(x)|_{F_x}\right|$.

Proof of Lemma 1:
1st. step) Prove Lemma 1 assuming Proposition 3
2nd. step) Prove Proposition 3 independently.

Proposition 3 (Pseudo-F local foliations.- Mañé 1981)

$\forall\, \epsilon > 0\, \exists\, \delta_0, K, n_0 > 0$, *and* δ_0*-local foliations* $\mathcal{L}$ *everywhere s.t.*
(A) $\mathcal{L}_i$ *is* C^1*- trivializable and its leaves are* $\dim F$*-dimensional*
(B) *dist*$\left(F_{f^n(x)}, T_{f^n(x)} f^n(\mathcal{L}_i(x))\right) < \epsilon\ \ \forall\ x,\ \forall\ n \geq n_0$
(C) and (D)$\forall\ n \geq 0$ *and* $\forall\ x, y$ *such that* $y \in B^n_{\delta_0}(x)$ *(dynamical ball):*
(C) $m^{\dim F}(f^n(\mathcal{L}_i(y) \bigcap B^n_{\delta_0}(x)) \leq K$
(D) $e^{-n\epsilon} K \leq \left|\det df^n_y | T_y(\mathcal{L}(y))\right| / \left|\det df^n_y | F_y\right| \leq K e^{n\epsilon}$

Lemma 1 (Upper Bound of the Lebesgue measure m)

$\forall\, \epsilon > 0\ \exists\, \delta > 0$ *s.t.* $\forall$ *finite partition* α *with diam*$(\alpha) < \delta$

$\exists$ *a sequence* $\{\nu_n\}_{n\geq 0}$ *of finite measures* *and* $\exists\, K > 0$ *s.t.:*

1) $\nu_n(X) < K\ \forall\ X \in \alpha^n$ **2)** $\forall\ C \in \mathcal{B}$: $m(C) \leq K e^{n\epsilon} \int_C e^{\psi_n}\, d\nu_n,$

where $\psi_n(x) := -\log\left|\det df^n(x)|_{F_x}\right|$.

1st. Step: Proof of Lemma 1 assuming Proposition 3.

$\forall\ A \in \alpha$ take a $\dim F$- local foliation $\mathcal{L}_A$ of Prop. 3

a $\dim E$- C^1-emb. submanifold W_A transversal to $\mathcal{L}_A$.

$\forall\ X_i \in \alpha^n\ X_i \subset A_i \in \alpha$. Denote $\mathcal{L}_i = \mathcal{L}_{A_i}$. Thus, Fubini's Theorem:

$$m(C) = \sum_{i=1}^{k_n} \int_{z\in W_{A_i}} d\mu^{W_{A_i}} \int_{y\in\mathcal{L}_i(z)} \mathbf{1}_{C\cap X_i}\, \phi_{A_i}\, dm^{\mathcal{L}_i(z)}$$

Denote $\widehat{y} = f^n(y) \in f^n(\mathcal{L}_i(z) \cap X_i) = \mathcal{L}_i^n(z)$:

$$\sum_{i=1}^{k_n} \int_{z\in W_{A_i}} d\mu^{W_{A_i}} \int_{\widehat{y}\in\mathcal{L}_i^n(z)} [\mathbf{1}_C\ \phi_{A_i}](f^{-n}(\widehat{y}))\left|\det df^{-n}|_{T_{\widehat{y}}\mathcal{L}_i^n}\right| dm^{\mathcal{L}_i^n(z)}$$

By (D) of Prop. 3: $\left|\det df^{-n}|_{T_{\widehat{y}}\mathcal{L}_i^n}\right| \leq K e^{n\epsilon} e^{\psi_n(f^{-n}(\widehat{y}))}$ Define ν_n:

$$\int h\, d\nu_n = \sum_{i=1}^{k_n} \int_{z\in W_{A_i}} d\mu^{W_{A_i}} \int_{\widehat{y}\in\mathcal{L}_i^n(z)} [h\ \phi_{A_i}](f^{-n}(\widehat{y})) dm^{\mathcal{L}_i^n(z)})$$

Then, $m(C) \leq K \int_C e^{\psi_n}\, d\nu_n$ Thus

$\nu_n(X_i) \leq K\ \max_{A\in\alpha}(\mu^{W_A}\, \|\rho_A\|_{C^0}) =$ constant. $\square$

Proposition 3 (Pseudo-F local foliations.- Mañé 1981)

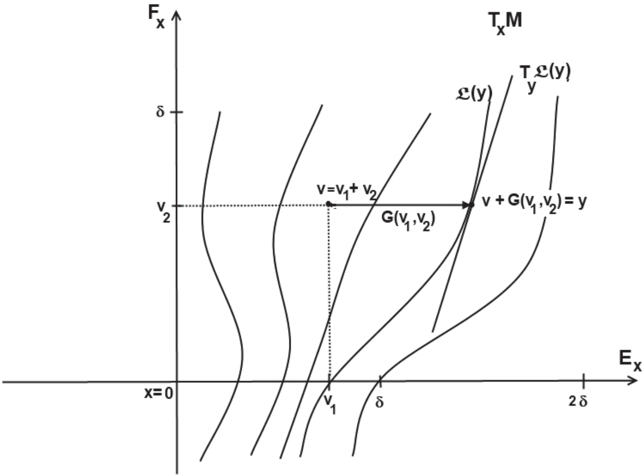
$\forall\, \epsilon > 0\, \exists\, \delta_0, K, n_0 > 0$, *and* δ_0*-local foliations* $\mathcal{L}$ *everywhere s.t.*
(A) $\mathcal{L}_i$ *is* C^1*- trivializable and its leaves are* $\dim F$*-dimensional*
(B) *dist*$\big(F_{f^n(x)}, T_{f^n(x)} f^n(\mathcal{L}_i(x))\big) < \epsilon \;\; \forall\, x, \; \forall\, n \geq n_0$
(C) and (D)$\forall\, n \geq 0$ *and* $\forall\, x, y$ *such that* $y \in B^n_{\delta_0}(x)$ *(dynamical ball):*
(C) $m^{\dim F}(f^n(\mathcal{L}_i(y) \bigcap B^n_{\delta_0}(x)) \leq K$
(D) $e^{-n\epsilon} K \leq \big|\det df^n_y | T_y(\mathcal{L}(y))\big| / \big|\det df^n_y | F_y\big| \leq K e^{n\epsilon}$

To prove Proposition 3, first take $\delta > 0$ small enough so any open set V of diameter smaller than 3δ is diffeomorphic by the exponential map $\exp_x$ (for any $x \in V$) to its image in $\exp_x^{-1} T_x M$.
Let us construct the (non invariant) foliation $\mathcal{L}$ from a Hadamard Graph.
So, before proving Proposition 3, let us recall what a Hadamard Graph is.

Notation: Fix $x \in M$. Denote $\mathbf{B}_\delta^{E_x, F_x}(\mathbf{0}) := \{v \in E_x, F_x : \|v\| \leq \delta\}$, $\mathbf{B}_\delta^x(\mathbf{0}) = \mathbf{B}_\delta^{E_x}(\mathbf{0}) \oplus \mathbf{B}_\delta^{F_x}(\mathbf{0}) \subset T_x M$.
$\forall\, v \in \mathbf{B}_\delta^x(\mathbf{0})$: $\pi_{E_x} v = v_1, \quad \pi_{F_x} v = v_2, \quad \gamma = \max_{x \in M}\{\|\pi_{E_x}\|, \|\pi_{F_x}\|\}$.

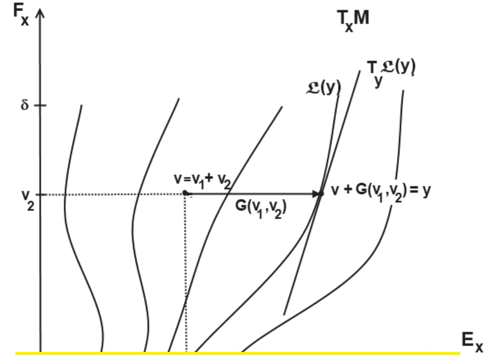
Definition: Hadamard Graph G

$G : \mathbf{B}_\delta^{E_x}(\mathbf{0}) \times \mathbf{B}_\delta^{F_x}(\mathbf{0}) \mapsto \mathbf{B}_\delta^{E_x}(\mathbf{0}), \qquad G(v_1, 0) = 0$
$\Phi : (v_1, v_2) \mapsto v_1 + v_2 + G(v_1, v_2) \in \mathbf{B}_{2\delta}^x(\mathbf{0})$ *is a C^1- diff. onto its image.*



Definition: Hadamard Graph G

$G : \mathbf{B}_\delta^{E_x}(\mathbf{0}) \times \mathbf{B}_\delta^{E_x}(\mathbf{0}) \mapsto \mathbf{B}_\delta^{E_x}(\mathbf{0})$, $\quad G(v_1, 0) = 0$ *and*
$\Phi : (v_1, v_2) \mapsto v_1 + v_2 + G(v_1, v_2) \in \mathbf{B}_{2\delta}^{x}(\mathbf{0})$ *is a* C^1*- diff. onto its image.*



Dispersion of G is $\text{disp}\ G = \max_{(v_1,v_2)} \|\partial G(v_1,v_2)/\partial v_2\|$, where $\partial G/\partial v_2$ is the Fréchet derivative of $G(v_1,\cdot) : \mathbf{B}_\delta^{F_x}(\mathbf{0}) \mapsto \mathbf{B}_\delta^{E_x}(\mathbf{0})$.

$T_y\mathcal{L}(y) = (Id|_{F_x} + \partial G/\partial v_2)\, F_x$

$\forall\ \epsilon > 0\ \exists c > 0$ *s.t.* $disp\, G \leq c \ \Rightarrow\ dist(T_y\mathcal{L}(y), F_x) < \epsilon/2\ \forall\ y \in Im(\Phi)$
$\exists\delta_0 > 0$ *s.t.* $dist\,(x,y) < \delta_0, \Rightarrow\ y \in Im(\Phi),\ \ dist(T_y\mathcal{L}(y), F_y) < \epsilon$

Once one has a Hadamard Graph G with disp $< c < 1/2$, and its associated foliation $\mathcal{L}$,
dist(E_x, E_y), dist(F_x, F_y) near zero.
Thus, the *same foliation* is associated to a Hadamard Graph $G^{\#}$ in the tangent space $T_y M$, satisfying

$$y = \exp_x(v_1 + v_2 + G(v_1, v_2) = \exp_y(u_1 + u_2 + G^{\#}(u_1, u_2)$$

$$u_1, u_2 \in \mathbf{B}_{\delta}^{E_y, F_y}(\mathbf{0}),$$

$$v_1, v_2 \in \mathbf{B}_{\delta}^{E_x, F_x}(\mathbf{0}),$$

Thus $\pi_{E(y)}(Id|_{F(x)} + \partial G^{\#}/\partial u_1)\pi_{F(x)}|_{T_y\mathcal{L}(y)}(\pi_{F(y)}|_{T_y\mathcal{L}(y)})^{-1} = \partial G^{\#}/\partial u_2$.
For any given $\epsilon' > 0$:
$2\|\pi_{E(y)}|_{F(x)}\| < \epsilon'$ if dist$(x, y) \leq \delta_0$ is small enough. Also
$\|\pi_{F(x)}|_{T_y\mathcal{L}(y)}(\pi_{F(y)}|_{T_y\mathcal{L}(y)})^{-1}\| < 1 + \epsilon' < 2$.

$\Rightarrow$ disp$G^{\#} \leq \epsilon' + 2\gamma$disp$G$

$\mathcal{L} : \Phi(v_1, v_2) = v_1 + v_2 + G(v_1, v_2) \ \forall\, (v_1, v_2) \in \mathbf{B}_\delta^{E_x}(\mathbf{0}) \times \mathbf{B}_\delta^{F_x}(\mathbf{0})$

Iterating by f^n the local foliation $\mathcal{L}$

$f^n(\mathcal{L} \cap B^n_{\delta_0}(x)) : \exp^{-1}_{f^n(x)} f^n \exp_x(v_1 + v_2 + G(v_1, v_2))$

where $\exp_x(v_1 + v_2 + G(v_1, v_2)) \in B^n_{\delta_0}(x)$, dynamical ball.

(1) • For small $\epsilon' > 0$ fix n_0 so $\|df^n|_{E(z)}\| \, \|df^{-n}|_{F(f^n(z))}\|\| < \epsilon' \ \forall n \geq n_0$

• Reduce $\delta_0 > 0$ so $\exists\ G^*_n$ defined as follows, and it is a Graph:

$f^n(\mathcal{L} \cap B^n_{\delta_0}(x)) : u_1 + u_2 + G^*_n(u_1, u_2), \ \forall\ 0 \leq n \leq n_0, \ \forall\ G$ s.t. $\text{disp}(G) < c$

Lemma

$\forall\ 0 < c < 1/2 \ \ \exists\ \delta_0, n_0 > 0$ s.t., if disp $G < c$ then:

G^*_n is a Graph for all $n \geq 0$ and disp $G^*_n < c$ for all $n \geq n_0$

Proof: **1.** $\forall\ \epsilon' > 0$ (to be fixed later) choose $n_0, \delta_0 > 0$ (depending on $\epsilon' > 0$) as in (1) above.

2. $\forall\ G$ s.t. disp $G < c$, $\forall\, n \geq 0$ s.t, G^*_n is a Graph and for all $y \in B^n_{\delta_0}$:
$\|\partial G^*_n(u_1, u_2)/\partial u_2\| \leq$
$\epsilon' + 2\gamma \|df^n_{E(y)}\| \, (\epsilon' + 2\gamma \|\partial G_x(v_1, v_2)/\partial v_2\| \, \|df^{-n}|_{F(f^n(y))}\|$.

3. Apply 2 and (1) with $n = n_0$ to fix $\epsilon' > 0$ (and thus to fix n_0, δ_0) s.t.
$\|\partial G_x(v_1, v_2)/\partial v_2\| \leq c \Rightarrow \left\|\partial G^*_{n_0}(u_1, u_2)/\partial u_2\right\| \leq c$

4. Apply **(1)** again to deduce that G^*_n is a Graph for all $n \geq 0$.

5. Apply step 2 to deduce that disp$G^*_n < c$ for all $n \geq n_0$. □

End of the proof of Proposition 3

Proposition 3 (Pseudo-F local foliations.- Mañé 1981)

$\forall\, \epsilon > 0 \,\exists\, \delta_0, K, n_0 > 0$, *and a finite number of* δ_0*-local foliations* $\mathcal{L}_i$ *s.t.*
(A) $\mathcal{L}_i$ *is* C^1*- trivializable and its leaves are* $\dim F$*-dimensional*
(B) *dist*$\big(F_{f^n(x)}, T_{f^n(x)} f^n(\mathcal{L}_i(x))\big) < \epsilon \ \ \forall\, x, \ \forall\, n \geq n_0$
$\forall\, n \geq 0$ *and* $\forall\, x, y$ *such that* $y \in B^n_{\delta_0}(x)$*:*
(C) $m^{\dim F}(f^n(\mathcal{L}_i(y) \bigcap B^n_{\delta_0}(x)) \leq K$
(D) $e^{-n\epsilon} K \leq \big|\det df^n_y | T_y(\mathcal{L}(y))\big| / \big|\det df^n_y | F_y\big| \leq K e^{n\epsilon}$

Proof: Choose $\delta > 0$ s.t. exp_x is a diffeo from $\mathbf{B}^x_\delta(\mathbf{0})$ onto its image $\forall\, x \in M$. Construct each local foliation $\mathcal{L}$ from a Graph $G \Rightarrow$ **(A)** holds.

$T_y(\mathcal{L}(y) = (Id + \partial G(v_1, v_2)/\partial v_2) F_x$

Thus, $\forall\, 0 < \epsilon' < \epsilon$, $\exists\, c > 0$ s. t. disp$(G) < c \Rightarrow$

$m^{\dim F}(\mathcal{L}(y)) \leq [(1 + c)\delta]^{\dim F} < \epsilon'$ **(C')** dist$(T_x(\mathcal{L}(x), F_x)) < \epsilon'$ **(B')**

Take δ_0, n_0 as in the above lemma: $f^n(\mathcal{L}(y) \cap B^n_{\delta_0}(x))$ is part of the foliation associated to a graph $f^*_n G$ for all $n \geq 0$; and

disp $f^*_n G < c \ \forall\, n \geq n_0$ **(E)**

(E), (C') $\Rightarrow$ **(C)** holds.
(E),(B') $\Rightarrow$ **(B)**, **(D)** hold for some constant $K > 0$. $\square$

Lemma 2 (Lower Bound of the Metric Entropy)

$\forall\ \mu \in \mathcal{P}_f,\ \forall\ \epsilon,\ \delta > 0$ there exists a finite partition α satisfying diam$\alpha < \delta$ and there exists a sequence $\{C_n\}_{n\geq 0} \subset \mathcal{B}$ such that:

3) $\bigcap_{N\geq 0}\bigcup_{n\geq N} C_n \supset B_{\epsilon^*}(\mu)$ for some $\epsilon^* > 0$ for some dist* in $\mathcal{P}$.

4) $\forall$ sequence $\{\nu_n\}_{n\geq 0}$ of finite measures, if $\exists\ K > 0$ s.t. $\nu_n(X) < K\ \forall\ X \in \alpha^n\ \forall\ n \geq 0$, then:

$\limsup_{n\to+\infty} \frac{1}{n} \log \int_{C_n} e^{\psi_n} d\nu_n \leq \epsilon + h_\mu(\alpha) + \int \psi\, d\mu$, *where*

$$\psi := -\log\left|\det df|_F\right|,\ \psi_n := \sum_{j=0}^{n-1} \psi \circ f^n$$

Proof of Lemma 2 Step 1. Construction (1a) Take $\delta_1 > 0$ s.t. dist$(x, y) < \delta_1 \Rightarrow |\psi(x) - \psi(y)| < \frac{\epsilon}{5}$.

(1b) Take α s.t. diam$(\alpha) \leq \min(\delta, \delta_1)$, $\mu(\partial X) = 0\ \forall\ X \in \alpha$

(1c) Fix $q \in \mathbb{N}^+$ s.t. $H(\alpha^q, \mu)/q \leq h_\mu(\alpha) + \epsilon/5$.

(1d) Choose $\{\varphi_i\}_{i\geq 1}$ dense in $C^0(M, [0, 1])$ and define dist* in $\mathcal{P}$:

$$\text{dist}^*(\mu_1, \mu_2) := |\mu_1\psi - \mu_2\psi| + \sum_{i=1}^{\infty} |\mu_1\varphi_i - \mu_2\varphi_i|/2^i.$$

(1e) Using (1b), fix $0 < \epsilon^* < \epsilon/5$ such that

$$\mu_n \in \mathcal{P},\ \text{dist}(\mu_n, \mu) \leq \epsilon^* \Rightarrow |H(\alpha^q, \mu_n) - H(\alpha^q, \mu)| \leq \epsilon/5.$$

(1f) Construct $C_n := \{x \in M : \text{dist}^*(\sigma_{n,x}, \mu) < \epsilon^*\}$.

Thus, Assertion (3) of Lemma 2 holds.

To end the proof of Lemma 2, one must prove that

$$\limsup_{n\to+\infty} \frac{1}{n} \log \int_{C_n} e^{\psi_n} d\nu_n \leq \epsilon + h_\mu(\alpha) + \int \psi \, d\mu$$

Second step. For each fixed $n \geq 0$, consider $\alpha^n \bigvee \{C_n\} = \{X_i \cap C_n\}_{1 \leq i \leq k_n}$ and choose one and only one point $x_i \in C_n \bigcap X_i$ for each $i \leq k_n$ (i.e. $C_n \bigcap X_i \neq \emptyset$).

$I_n := \int_{y \in C_n} e^{\psi_n(y)} d\nu_n(y) = \sum_{i=1}^{k_n} e^{n\epsilon/5} e^{\psi_n(x_i)} \nu(C_n \cap X_i)$

$$I_n \leq K e^{n\epsilon/5} \sum_{i=1}^{k_n} e^{\psi_n(x_i)}$$

Third step. Apply the Equality of Jensen:

$\log \sum_{i=1}^{k_n} e^{\psi_n(x_i)} = \sum_{i=1}^{k_n} \psi_n(x_i) p_i - \sum_{i=1}^{k_n} p_i \log p_i$

where $p_i = e^{\psi_n(x_i)}/L, \quad L = \sum_{i=1}^{k_n} e^{\psi_n(x_i)}$. Thus $\sum_{i=1}^{k_n} p_i = 1$

$$\log I_n \leq \log K + n\epsilon/5 + \sum_{i=1}^{k_n} \sum_{j=1}^{n-1} \int p_i \psi d\delta_{f^j(x_i)} - \sum_{i=1}^{k_n} p_i \log p_i$$

Fourth step. Define $\mu_n \in \mathcal{P}$ by

$\mu_n = \sum_{i=1}^{k_n} \sum_{j=1}^{n-1} \int p_i \ \delta_{f^j(x_i)} = \sum_{i=1}^{k_n} p_i \sigma_{n,x_i}$

$x_i \in C_n \Rightarrow \text{dist}^*(\sigma_{n,x_i}, \mu) \leq \epsilon^* \Rightarrow \text{dist}^*(\mu_n, \mu) \leq \epsilon^*$

$\Rightarrow \int \psi \, d\mu_n \leq \int \psi \, d\mu + \epsilon/5 \Rightarrow$

$$\log I_n \leq \log K + 2n\epsilon/5 + n \int \psi \, d\mu - \sum_{i=1}^{k_n} p_i \log p_i$$

To end the proof of Lemma 2, one must prove that

$\limsup_{n\to+\infty} \frac{1}{n}\log I_n \leq \epsilon + h_\mu(\alpha) + \int \psi\, d\mu$

At the end of the fourth step we have proved that

$\log I_n \leq \log K + 2n\epsilon/5 + n\int \psi\, d\mu - \sum_{i=1}^{k_n} p_i \log p_i$ where

$\sum_{i=1}^{k_n} p_i = 1, \quad \mu_n = \sum_{i=1}^{k_n} p_i \sigma_{n,x_i} = \frac{1}{n}\sum_{j=1}^{n-1}\sum_{i=1}^{k_n} p_i \delta_{f^j(x_i)}$

$x_i \in C_n \cap X_i \Rightarrow \mathrm{dist}^*(\sigma_{n,x_i},\mu) < \epsilon^* \Rightarrow \mathrm{dist}^*(\mu_n,\mu) < \epsilon^*$

Fifth step. Check the following

ASSERTION 5 $\exists n_0 \geq 1$ *such that*

$-\sum_{i=1}^{k_n} p_i \log p_i \leq n\epsilon/5 + nH(\alpha^q,\mu_n)/q \quad \forall\ n \geq n_0$

Sixth step. From Assertion 5 conclude the proof of Lemma 2:

Assertion 5 $\Rightarrow$ $\log I_n \leq \log K + 3n\epsilon/5 + n\int \psi\, d\mu + nH(\alpha^q,\mu_n)/q$

$\mathrm{dist}^*(\mu_n,\mu) < \epsilon^* \Rightarrow |H(\alpha^q,\mu_n)/q - H(\alpha^q,\mu)/q| \leq \epsilon/5 \Rightarrow$

$\log I_n \leq \log K + 4n\epsilon/5 + n\int \psi\, d\mu + nH(\alpha^q,\mu)/q$

$H(\alpha^q,\mu)/q \leq h_\mu(\alpha) + \epsilon/5 \Rightarrow$

$\log I_n \leq \log K + n\epsilon + n\int \psi\, d\mu + nh_\mu(\alpha)$ $\forall\ n \geq n_0$ $\square$

Assertion 5.- It is left to prove the following:

$\exists n_0 \geq 1$ *such that* $-\sum_{i=1}^{k_n} p_i \log p_i \leq n\epsilon/5 + nH(\alpha^q, \mu_n)/q \quad \forall\ n \geq n_0$

Proof: $k := \#\alpha$, $\alpha^n = \{X_i\}$, $x_i \in X_i$ Denote $\pi_n := \sum_{i=1}^{k_n} p_i \delta_{x_i} \Rightarrow$

$H(\alpha^n, \pi_n) = -\sum_{i=1}^{k_n} p_i \log p_i$; $\mu_n = \sum_{i=1}^{k_n} p_i \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f^j(x_i)}$;

$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f^{*j} \pi_n$ Fix $0 \leq l \leq q-1$;

$$\alpha^{n+l} = \left(\vee_{j=0}^{l-1} f^{-j}\alpha \right) \vee \left(f^{-l} \left(\vee_{j=0}^{n} f^{-j}\alpha \right) \right) \Rightarrow$$

$H(\alpha^n, \pi_n) \leq H(\alpha^{n+l}, \pi_n) \leq \sum_{j=0}^{l-1} H(\alpha, (f^*)^j \pi_n) + H(f^{-l}\alpha^n, \pi_n)$

$-\sum_{i=1}^{k_n} p_i \log p_i \leq q \log k + H(\alpha^n, f^{**l} \pi_n)$

$-\sum_{i=1}^{k_n} p_i \log p_i \leq n\epsilon/10 + H(\alpha^n, f^{*l} \pi_n)$ if $n \geq (10)q \log k(\epsilon)$

$n = Nq + s,\ 0 \leq s \leq q-1$:

$H(\alpha^n, f^{*l} \pi_n) \leq \sum_{h=0}^{N-1} H(\alpha^q, (f^*)^{hq+l} \pi_n) + \sum_{j=Nq}^{Nq+s} H(\alpha, f^{*j+l} \pi_n)$

$-\sum_{i=1}^{k_n} p_i \log p_i \leq n\epsilon/5 + \sum_{h=0}^{N-1} H(\alpha^q, (f^*)^{hq+l} \pi_n)$ $\Rightarrow$

$-q \sum_{i=1}^{k_n} p_i \log p_i \leq$ $nq\epsilon/5 + \sum_{h=0}^{N-1} \sum_{l=0}^{q-1} H(\alpha^q, (f^*)^{hq+l} \pi_n)$

$\leq nq\epsilon/5 + \sum_{j=0}^{n} H(\alpha^q, f^{*j} \pi_n)$ $\leq nq\epsilon/5 + nH(\alpha^q, \mu_n)$. $\square$