PESIN'S ENTROPY FORMULA FOR C^1 EXPANDING MAPS OF THE CIRCLE

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Talk in the
XXV Jornadas Venezolanas de Matemáticas, Cumaná, Venezuela,
March 26th. to 29th., 2012

ABSTRACT. For any C^1 expanding map of the circle S^1 we prove that there exist equilibrium states for the potential $\psi = -\log|f'|$. Namely, the system necessarily has invariant measures that satisfy the Pesin's Entropy Formula. We state and prove several C^1 theorems relating SRB measures, SRB-like measures and measures satisfying the Pesin's Entropy Formula.

1. INTRODUCTION

In the SRB and Pesin's Theory for C^1 plus Hölder hyperbolic systems, in particular for C^1 plus Hölder expanding maps on the circle, it is proved that μ is SRB if and only if it satisfies the Pesin's Entropy Formula. And besides, for expanding maps on the circle, this occurs if and only if μ is equivalent to the Lebesgue measure, and there always exists one and only one of such measures.

There is large gap, which we try to reduce with this work, between the C^1-plus-Hölder theory of SRB measures and Pesin's theories. The behavior exhibited by several C^1 examples shows that the theorems for the C^1-plus-Hölder systems fail if weakening the hypothesis of regularity of the map to be only C^1. It is usually said that there is no chance to extend the theory for the C^1-case. We prove that, if a slight change in the definition of SRB measures is admitted, a substitutive theory can be developed. We state and prove the first theorems of this theory.

Why SRB and Pesin's Theory does not work in the C^1-scenario? Focusing on C^1-expanding maps of the circle, first, we recall that there exist transitive examples [Qu96] for which non countably many invariant measures satisfy the Pesin's Formula. So, transitivity does not imply uniqueness of the equilibrium state for the potential $-\log|f'|$.

Besides, the transitive examples of [Qu96] preserve the Lebesgue measure but it is non ergodic. So, transitivity does not imply ergodicity in the C^1 scenario, while for C^1 plus Hölder hyperbolic systems it does.

There also exist examples of C^1-expanding maps of the circle, for which a single measure satisfies the Pesin's Formula, but it is mutually singular with respect to Lebesgue. This is a large difference with the classical property of C^2 plus Hölder expanding maps. Indeed, this shows that Ledrappier-Young's characterization (with the absolute continuity property with respect to Lebesgue) of those measures that satisfy the Pesin's Entropy Formula, does not hold in the C^1-scenario.

Nevertheless we propose a substitutive theory, focused (by now) on the space $\mathcal{E}^1$ of all the C^1-expanding of the circle S^1, and prove the first theorems to compose this theory.

2. Definitions

We say that a C^1-map $f : S^1 \mapsto S^1$ is expanding if

$$|f'(x)| > 1 \ \ \forall \, x \in S^1.$$

We denote by $\mathcal{E}^1$ to the space of all C^1-expanding map of C^1.

An equilibrium state for the potential $\psi : -\log|f'|$, if it exists, is an invariant probability measure μ for which

$$h_\mu(f) + \int \psi \, d\mu = \sup_{\nu \in \mathcal{P}_f} h_\nu(f) + \int \psi \, d\nu,$$

where $h_\mu(f)$ denotes the theoretical measure entropy of f with respect to the probability measure μ, and $\mathcal{P}_f$ denotes the space of all the f-invariant probability measures.

We denote by $ES_f \subset \mathcal{P}_f$ to the (a priori maybe empty) set of all the equilibrium states of f for the potential $\psi = -\log|f'|$.

In [CQ01] it is proved that the sup in the above equation of the measures $\mu \in ES_f$, is necessarily equal to zero for all $f \in \mathcal{E}^1$. Therefore μ is an equilibrium state for the potential ψ if and only if it satisfies the following equality:

$$h_\mu(f) = \int \log|f'| \, d\mu.$$

On the other hand, for any f-invariant measure μ, and for any $f \in \mathcal{E}^1$, it is straightforward to check (applying Birkhoff Ergodic Theorem and Oseledec's Theorem), that

$$\int \log|f'| \, d\mu = \int \chi^+(x) \, d\mu,$$

where $\chi^+(x)$ is the (positive) Lyapunov of the orbit of x, which is defined for μ-a.e. $x \in S^1$.

Therefore, $\quad \mu \in ES_f \quad \Leftrightarrow \quad h_\mu(f) = \int \chi^+(x) \, d\mu.$

The latter Equality is called the Pesin's Formula of the Entropy.

For any point $x \in S^1$ the sequence of empirical probabilities along the future orbit of x is defined by:

$$\sigma_n(x) := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \in \mathcal{P} \quad \forall\, n \geq 1,$$

where δ_y denotes the Dirac's delta probability supported on the point $y \in S^1$, and $\mathcal{P}$ denotes the space of all the (non necessarily f-invariant) Borel-probabilities. In general, except if x is a fixed point, $\sigma_n(x) \notin \mathcal{P}_f \;\; \forall\, n \geq 2$.

The space $\mathcal{P}$ is endowed with the weak* topology, as a subset of the dual space of $C^0(S^1, \mathbb{R}^+)$. It is well known that $\mathcal{P}$, after endowed with the weak*-topology, is compact, sequentially compact and metrizable, and that is subspace $\mathcal{P}_f$ is compact. We denote by dist^* any chosen metric which induces the weak*-topology on $\mathcal{P}$. We denote by $\lim^*$ the limit of any convergent sequence in $\mathcal{P}$. We define the set

$$p\omega(x) = \{\mu \in \mathcal{P} : \; \exists n_j \to +\infty \text{ such that } \lim{}^*_{j\to+\infty} \sigma_{n_j}(x) = \mu\}.$$

It is easy to check that $p\omega(x) \subset \mathcal{P}_f$.

The SRB measures, also called physical measures, are defined as those probability measures $\mu \in \mathcal{P}_f$ such that the following set

$$B(\mu) := \{x \in S^1 : \; p\omega(x) = \{\mu\}\,\}$$

has positive Lebesgue measure. The set $B(\mu)$ is called the basin of statistical attraction of μ. In general, with this definition, an SRB measure is non necessarily ergodic: there are examples of such $f \in \operatorname{Diff}^\infty(M)$ with dimension of M larger than one. But for $f \in \mathcal{E}^1$ it can be proved that all SRB measures are ergodic (the proof of this fact is not easy).

In [CE11] we define: μ is SRB-like if for all $\epsilon > 0$ the following set

$$A_\epsilon(\mu) := \{x \in S^1 : \; \operatorname{dist}^*(p\omega(x), \mu) < \epsilon\}$$

has positive Lebesgue measure. We call the set $A_\epsilon(\mu)$ the basin of ϵ-weak statistical attraction of μ.

It is easy to check that SRB-like measures are f-invariant and that they do not depend of the chosen metric dist* in the space $\mathcal{P}$ of all the probability measures, provided that it induces the weak*-topology.

Immediately from the above definitions, all the SRB measures, if they exist, are SRB-like. In [CE11] we proved that there always exist at least one SRB-measure. From the definition, notice that SRB-like probabilities preserve a slightly weak physical property which generalizes the physical statistical attraction of SRB measures. It is standard to prove that the SRB-like measures coincide with the SRB measures if these latter exist and if the union of their basins of statistical attraction covers Lebesgue-almost all the orbits. In other case, after the results in [CE11], there still exist SRB-like invariant probabilities that are not SRB. In any case, we prove in [CE11] that the union of the basins of the statistical (weak)-attraction of all the SRB-like measures (including the SRB measures if they exist) cover Lebesgue-a.e.

3. Statement of the results.

Theorem 1. *For all $f \in \mathcal{E}^1$ there exist equilibrium states for the potential $\psi := \log |f'|$. Namely, the set of invariant measures that satisfy the Pesin's Entropy Formula is non empty.*

Theorem 2. (See also [CQ01] for the C^1-generic case in $\mathcal{E}^1$.)
For all $f \in \mathcal{E}^1$, if ES_f has a unique measure μ, then μ is necessarily SRB or physical, namely, its basin $B(\mu)$ of statistical attraction has positive Lebesgue measure. Moreover, $B(\mu)$ has full Lebesgue measure. Thus, no other SRB-like or SRB measure exists.

It is well known, from the classic Ruelle's Theorem (which requires more regularity than only C^1) that the space of expanding maps for which $\#ES_f = 1$ includes all the C^1-plus Hölder systems in $\mathcal{E}^1$, and besides that the unique $\mu \in ES_f$ for those systems is equivalent to the Lebesgue measure. It is also known, after Campbell and Quass Theorem [CQ01], that the above case for which $\#ES_f = 1$, is also the C^1-generic behavior, and that besides, C^1-generically such a unique equilibrium state μ is mutually singular with respect to the Lebesgue measure.

The main purpose is to study the non generic (namely, bifurcating) maps in $\mathcal{E}^1 \setminus \mathcal{E}^1 -$ plus Hölder for which $\#ES_f > 1$. We recall that the systems are necessarily transitive, since any $f \in \mathcal{E}^1$ is conjugated to a linear expanding map of the circle with degree larger or equal than 2. Nevertheless they are non necessarily ergodic [Qu96]:

Theorem 3. *If ES_f has finitely or countably infinitely many ergodic measures, then all of them are SRB, and the union of the basins of statistical attraction of all of them covers Lebesgue-almost all the orbits. Thus, the set of SRB measure coincides with the set of SRB-like measures*

This latter class of systems has non countably many ergodic measures satisfying the Pesin's Entropy Formula. No C^1 example still exist in this class, as far as we know, but C^0-examples of topologically expanding maps of the circle, with a similar property of non countably many ergodic measures which are SRB-like, were provided by [M05]).

Theorem 4. *If $f \in \mathcal{E}^1$ has non countably many ergodic measures satisfying the Pesin's Entropy Formula, then there are non countably many SRB-like probability measures that are non SRB.*

Theorem 5. *For any $f \in \mathcal{E}^1$, all the SRB-like measures (including the SRB measures if they exist) satisfy the Pesin's Entropy Formula.*

Acknowledgements: The first author was partially supported by CSIC of Universidad de la República and ANII of Uruguay.

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Pesin's Entropy Formula for C^1 Expanding Maps of the Circle

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XXV Jornadas Venezolanas de Matemática
Cumaná, Venezuela, from March 26th. to 29th., 2012

Definition

A C^1 map $f : S^1 \mapsto S^1$ is **expanding** if

$$|f'(x)| > 1 \;\; \forall\, x \in S^1.$$

Purpose: Extend or adapt known results for $C^{1+\alpha}$ expanding maps to the C^1 **maps**. Search for the relations between:

- SRB or physical measures or measures **statistically describing the asymptotic behavior** of Lebesgue-a.e.
- Equilibrium states for the potential $\psi := -\log|f'|$ or equivalently, invariant measures μ that satisfy **the Pesin's Formula of the Entropy**:

$$h_\mu(f) = \int \chi^+ \, d\mu,$$

where χ^+ is the **positive Lyapunov exponent**.

Notation: $x \in S^1$ initial state; $\{f^j(x)\}_{j\geq 0}$ future orbit of x, $\mathcal{P}$ space of Borel **probabilities in S^1 endowed with the weak* topology**; $\mathcal{P}_f \subset \mathcal{P}$ subspace of f-invariant probabilities.
Recall: $\mathcal{P}_f$ is nonempty and weak* compact.

- **Sequence** $\{\sigma_{n,x}\}_{n\geq 1}$ **of Empirical Probabilities** of the future orbit of x:
$$\sigma_{n,x} := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}$$
where $\delta_\cdot$ is the Dirac delta.
REMARK: $\sigma_{n,x} \notin \mathcal{P}_f$ **unless** $x \in Per(f)$.
- **P-omega-limit**, omega limit in the space of probabilities:
$$p\omega(x) := \{\mu \in \mathcal{P} : \ \exists\ n_j \to +\infty \text{ such that } \lim_j \sigma_{n_j,x} = \mu\}.$$
$$\Rightarrow \quad \emptyset \neq p\omega(x) \text{ is weak}^* \text{ compact and contained in } \mathcal{P}_f$$

For any given $\mu \in \mathcal{P}_f$

- **BASIN OF STATISTICAL ATTRACTION**

$$B(\mu) := \Big\{x \in S^1 : \ p\omega(x) = \{\mu\}\Big\}.$$

- **BASIN OF EPSILON-WEAK STATISTICAL ATTRACT.**

$$B_\epsilon(\mu) := \Big\{x \in S^1 : \ \mathsf{dist}^*(p\omega(x), \mu) < \epsilon\Big\}.$$

Notation: m is the (non necessarily invariant) Lebesgue measure normalized so $m(S^1) = 1$.

Definition

An invariant measure μ is **SRB OR PHYSICAL** if

$$m(B(\mu)) > 0.$$

An invariant measure μ is **SRB-LIKE OR PHYSICAL-LIKE** if

$$m(B_\epsilon(\mu)) > 0 \ \ \forall \ \epsilon > 0.$$

Definition

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SRB $=$ (maybe empty) set of SRB or physical measures
SRB-like $=$ **(never empty)** set of SRB-like or physical-like measures
EQ $=$ set of measures satisfying Pesin's Entropy Formula
$=$ set of equilibrium states μ for the potential $\psi = -\log|f'|$:

.
$$h_\mu(f) = \int \chi^+ \, d\mu = \int \log|f'| \, d\mu.$$

Theorem (Ruelle: Any $C^{1+\alpha}$ case.)

If $f : S^1 \mapsto S^1$ is $C^{1+\alpha}$ expanding, then #(EQ) $=$ 1 and

$$\text{SRB} = \text{SRB-like} = \text{EQ} = \{\mu \in \mathcal{P}_f : \ \mu \ll m\}.$$

Theorem (Ruelle: Any $C^{1+\alpha}$ case.)

If $f : S^1 \mapsto S^1$ is $C^{1+\alpha}$ expanding, then
#(EQ) = 1 and SRB = SRB-like = EQ $= \{\mu \in \mathcal{P}_f : \ \mu \ll m\}$.

Theorem (Some previously known C^1 cases)

- *(Cambpell-Quass 2001)* C^1*-generically*
  # EQ = 1, SRB= SRB-like = EQ, and $\mu \in$*EQ* $\Rightarrow \ \mu \perp m$.
- *(Restatement of Quass 1996)* $\exists f$ C^1*-expanding on* S^1 *such that #(SRB-like)* > 1 *and #EQ* $= \infty$

Theorem (C-E 2012: Any C^1 case)

For any C^1 expanding map $f : S^1 \mapsto S^1$:

- *#(SRB-like)* ≥ 1, $\mu \in$ *SRB-like* $\Rightarrow \mu \in$ *EQ.*
- *# EQ = 1* $\Rightarrow$ *SRB = SRB-like = EQ*

Theorem (C-E 2012: Any C^1 case)

For any C^1 expanding map $f : S^1 \mapsto S^1$:

- **(1)** $\#(\textit{SRB-like}) \geq 1$
- **(2)** $\mu \in$ *SRB-like* $\Rightarrow \mu \in$ *EQ.*
- **(3)** $\#$ *EQ = 1* $\Rightarrow$ *SRB = SRB-like = EQ*

The complete and detailed proof can be found in:
Equilibrium States and SRB-like measures of C1 Expanding Maps of the Circle, ArXiv:1202.6584v1 [math.DS] Preprint 2012

Route of the Theorem's proof, in 7 steps with 3 lemmas.

- **Step 1. LEMMA 1:**
  SRB-like $\neq \emptyset$ and it is the minimal weak* compact set $\mathcal{K} \subset \mathcal{P}_f$ such that

$$p\omega(x) \subset \mathcal{K} \text{ for } m\text{-a.e.} x \in S^1.$$

Ideas for the proof of Lemma 1: Use the definition of SRB-like measure and the sequential weak*-compactness of $\mathcal{P}$.

Note Lemma 1 implies Assertion (1) of the Theorem.

Theorem (C-E 2012: Any C^1 case)

For any C^1 expanding map $f : S^1 \mapsto S^1$:

- **(1)** *#(SRB-like)* ≥ 1
- **(2)** $\mu \in$ *SRB-like* $\Rightarrow \mu \in$ *EQ.*
- **(3)** *# EQ = 1* $\Rightarrow$ *SRB = SRB-like = EQ*

- **Step 2.** Check that (1) and (2) $\Rightarrow$ (3).

Proof: (1), (2) and # EQ = 1 $\Rightarrow$ SRB-like = EQ $=\{\mu\}$.
Lemma 1 and SRB-like $=\{\mu\} \Rightarrow \mu \in$ SRB = SRB-like. $\square$
Note Now it is only left to prove (2).

- **Step 3.** Construct a finite partition $\mathbf{P}$ of S^1 with arbitrarily small diameter and such that

$$\mu(\partial \mathbf{P}) = 0 \quad \forall\ \mu \in \mathcal{P}_f. \qquad (*)$$

Note: (*) the ONLY reason why this proof does not work if dim > 1.

- **Step 3.** Finite partition $\mathbf{P}$ such that diam$(\mathbf{P}) < \alpha$ (α= expans. constant of f), and $\mu(\partial\mathbf{P}) = 0 \; \forall \; \mu \in \mathcal{P}_f$. $\qquad (*)$.
- **Step 4. Lemma 2 (lower bound for the entropy)**
  $\star \; \forall$ sequence $\{\nu_n\}_{n\geq 1}$ of **(non invariant)** $\nu_n \in \mathcal{P}$,
  $\star \; \forall \; \mu \in \mathcal{P}_f$ such that $\mu = \lim_i^* \frac{1}{n_i} \sum_{j=0}^{n_i-1} (f^*)^j \nu_{n_i}$, ($n_i \to +\infty$),

$$\limsup_i \frac{1}{n_i} H(\vee_{j=0}^{n_i} f^{-j}(\mathbf{P}), \; \nu_{n_i}) \leq h_\mu.$$

Remark: $\nu_n \notin \mathcal{P}_f$ **since Lemma 2 will be applied to convex combinations of the empirical distributions.**

Ingredients for the proof of Lemma 2:

- Topological expansivity of f:
diam$(\mathbf{P}) < \alpha, \; \mu \in \mathcal{P}_f \quad \Rightarrow \quad h_\mu = \lim_{n\to+\infty} \frac{1}{n} H(\vee_{j=0}^{n-1} f^{-j}\mathbf{P}, \mu)$.
- Non decreasing property of H with respect to finer partitions:

$$H(\vee_{j=0}^{n} f^{-j}\mathbf{P}, \nu) \leq \sum_{j=0}^{n} H(\mathbf{P}, (f^*)^j \nu)$$

- Convexity of H with respect to the probability:

$$\frac{1}{n} \sum_{j=0}^{n-1} H(\mathbf{P}, (f^*)^j \nu) \leq H(\mathbf{P}, \frac{1}{n} \sum_{j=0}^{n-1} (f^*)^j \nu).$$

- **Step 5. Reformulate the problem via Keller's argument:** (ref. Keller's book on Equilibrium States, 1998):
  $\forall\ r \geq 0$ define
  $$\mathcal{K}_r = \Big\{\mu \in \mathcal{P}_f :\ h_\mu + \int (-\log|f'|)\, d\mu \geq -r\Big\}.$$
  Recall that $\int \log|f'|\, d\mu = \int \chi^+\, d\mu$.
  Due to Ruelle's Inequality: $EQ = \mathcal{K}_0 = \bigcap_{r>0} \mathcal{K}_r$.
  To prove (2) : SRB-like $\subset$ EQ, **it is enough to prove that**
  SRB-like $\subset \mathcal{K}_r \ \ \forall\ r > 0$ (to be proved).

- **Step 6. Lemma 3** *For all* $\epsilon > 0$

$$m(\{x \in S^1 :\ \text{dist}^*(\sigma_{n,x}, \mathcal{K}_r) \geq \epsilon\}) \leq e^{-n(r-\epsilon)}$$

Remark: To prove Lemma 3, the **Lemma 2 giving a lower bound of the entropy is essential**. Recall that Lemma 2 was obtained after the construction of a good finite partition $\mathbf{P}$ such that $\mu(\partial\mathbf{P}) = 0\ \forall\ \mu \in \mathcal{P}_f$.

- **Step 7. End of the proof of the Theorem:**
  It is only left to prove that: SRB-like $\subset \mathcal{K}_r \quad \forall\, r > 0$

• From Step 6 **Lemma 3**: *For all* $0 < \epsilon < r/2$

$$m(\{x \in S^1 : \ \text{dist}^*(\sigma_{n,x}, \mathcal{K}_r) \geq \epsilon\}) \leq e^{-n(r-\epsilon)} \leq e^{-nr/2}$$

$\Rightarrow \quad \sum_{n=1}^{\infty} m(C_n) < +\infty,$
where $\ C_n := \{x \in S^1 : \ \text{dist}^*(\sigma_{n,x}, \mathcal{K}_r) \geq \epsilon\}.$

• Borel-Cantelli Lemma implies $m\left(\bigcap_{n_0 \geq 1} \bigcup_{n \geq n_0} C_n\right) = 0.$

In other words, for m-a.e.$x \in S^1$ there exists $n_0 \geq 1$ such that $x \notin C_n \ \ \forall\, n \geq n_0.$

• Then $\text{dist}^*(\sigma_{n,x}, \mathcal{K}_r) < \epsilon \ \ \forall\, n \geq n_0.$

$\Rightarrow \ \forall\, \epsilon > 0 : \ \ p\omega(x) \subset \{\mu \in \mathcal{P} : \text{dist}^*(\mu, \mathcal{K}_r) \leq \epsilon\}\ m-$a.e.$x \in S^1.$

$\Rightarrow \quad p\omega(x) \subset \mathcal{K}_r \quad .$

• Step 1: SRB-like is the minimal weak*-compact set that contains $p\omega(x)$ for $m-$a.e.$x \in S^1$

$$\Rightarrow \quad \text{SRB-like} \subset \mathcal{K}_r \qquad \square$$

Thank you very much for your kind attention!

Announcement

Dynamical Systems in Montevideo 2012

August 13th. to 17th., 2012

An international congress in Dynamical Systems will be held in Montevideo, Uruguay.

It will be a satellite conference of the 4th Latin American Congress of Mathematicians that will take place in Córdoba, Argentina.

http://imerl.fing.edu.uy/sdm2012/