# SUBCIRCUITS WITH PREDOMINANT EXCITATORY PULSED INTERACTIONS: MATHEMATICAL SUFFICIENT CONDITIONS TO SYNCHRONIZE SPIKES

ELEONORA CATSIGERAS[1]

Talk in the International Symposium on
Neurons, Circuits and Systems
Montevideo, Nov.30th – Dec.3rd, 2014

**Abstract**

We study a simplified mathematical model of neural subcircuits with synaptical interactions that are predominantly excitatory.

We prove that some inequalities relating parameters of the subcircuit in the model, are enough to produce the recurrent (but not necessarily periodic) synchronization of the spikes of all the neurons of the subcircuit.

The mathematical inequalities which are sufficienti conditions for recurrent syncrhonization take into account the following parameters of the subcircuit:

- Number of excitatory neurons of the subcircuit.
- The Graph mathematical Structure, which represents the oriented and weighted synaptical connections among neurons in the subcircuit.
- Minimum positive weight of the excitations in the subcircuit.
- Maximum absolute value of the negative weights of the inhibitions in the subcircuit.
- The maximum ISI (Interspike intervals) of the sub-subcircuit of excitatory neurons in the subcircuit.
- The minimum ISI of each inhibitory neuron in the subcircuit.

[1]Instituto de Matemática y Estadística “Rafael Laguardia”(IMERL), Fac. de Ingeniería, Universidad de la República, Av. Herrera y Reissig 565, C.P. 11300, Montevideo, URUGUAY
E-mail: eleonora@fing.edu.uy

# Subcircuits with predominant excitatory pulsed interactions: mathematical sufficient conditions to synchronize spikes

**Eleonora Catsigeras**

IMERL - Fac. Ingeniería
Universidad de la República - Uruguay

eleonora@fing.edu.uy

International Symposium on
Neurons, Circuits and Systems
Montevideo, Nov. 30th - Dec.3th., 2014

# ORGANIZATION OF THIS TALK

• Object of study: A mathematical system modeling a circuit of "neurons".
(Abstract – General – Simplified)

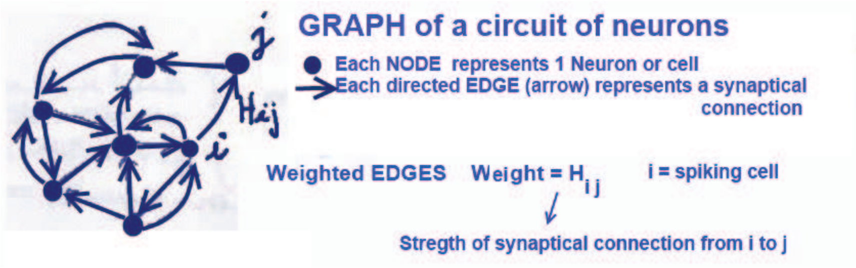
• Questions to research: Quantitative and qualitative dynamics of the mathematical system.

• Methodology of research: Logical deductive proofs (Rigorous proofs).

• Obtained results: Theorems 1 and 2 (their statements and their proofs).

• Mathematical proofs: Unfortunately not included in this talk, but they are the most enjoyable parts of the work.

• Conclusions: How to interpret the statements of Theorems 1 and 2 and their corollaries. They are necessarily true in the (simplified - general - abstract) mathematical model. But ¿do they necessarily hold for real biological neural networks?




**Definition**

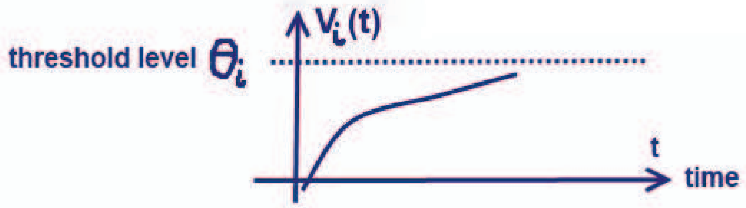
The cell $i$ (node $i$) is EXCITATORY if $H_{ij} \geq 0 \quad \forall\, j \neq i$.
The cell $i$ (node $i$) is INHIBITORY if $H_{ij} \leq 0 \quad \forall\, j \neq i$.

## Interspike Regime

(One-dimensional neuron)
Governed by a 1-dimensional differential equation:

$$\frac{dV_i}{dt} = F_i(V_i), \text{ where } F_i > 0, \text{ while } V_i(t) \leq \theta_i$$

The value $\theta_i$ is the *THRESHOLD LEVEL*.



## Interspike Regime

(One-dimensional neuron)

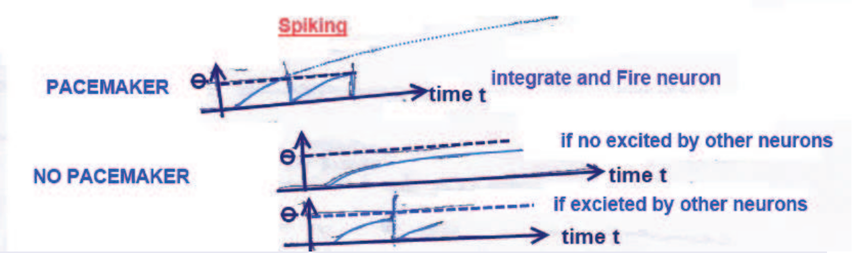
Governed by a 1-dimensional differential equation:

$$\frac{dV_i}{dt} = F_i(V_i), \text{ where } F_i > 0, \text{ while } V_i(t) \leq \theta_i$$

The value $\theta_i$ is the *THRESHOLD LEVEL*.

## Spiking Regime

(Pacemaker and No Pacemaker neuron)



## Interspike Regime

(d-dimensional neuron $i$)

Its state is described at each instant $t$ by the $d$ real variables, which are the component of a $d$-dimensional vector $\overline{x}_i(t)$.

Governed by a d-dimensional system of differential equations:

$$\frac{d\overline{x}_i}{dt} = F_i(\overline{x}_i), \text{ where } F_i : \mathbb{R}^d \mapsto \mathbb{R}^d, \text{ while } V_i(\overline{x}(t)) \leq \theta_i,$$

such that

$$\frac{dV_i(\hbar x_i(t))}{dt} = \nabla F_i \cdot \frac{d\overline{x}_i}{dt} > 0.$$

The value $\theta_i$ is the *THRESHOLD LEVEL*.

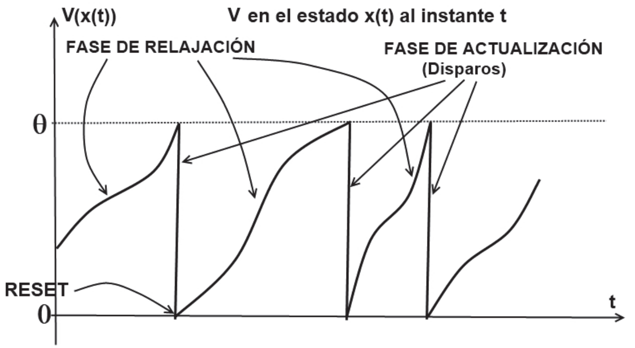
## Interspike Regime

(d-dimensional neuron)

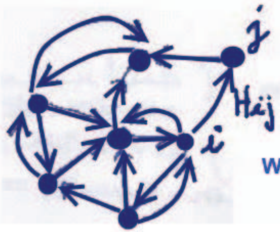
$$\frac{d\overline{x}_i}{dt} = F_i(\overline{x}_i), \text{ where } F_i : \mathbb{R}^d \mapsto \mathbb{R}^d, \text{ while } V_i(\overline{x}(t)) \leq \theta_i,$$

## Spiking Regime

(If neither excited nor inhibited by other neurons)




**REMARK: The spiking is not necessarily periodic**




## GRAPH of a circuit of neurons

- Each NODE represents 1 Neuron or cell
- Each directed EDGE (arrow) represents a synaptical connection

Weighted EDGES Weight = $H_{ij}$ i = spiking cell

↓

Stregth of synaptical connection from i to j

The cell $i$ (node $i$) is excitatory if $H_{ij} \geq 0 \quad \forall\, j \neq i$.
The cell $i$ (node $i$) is inhibitory if $H_{ij} \leq 0 \quad \forall\, j \neq i$.

### Synaptical Rule

*When neuron $i$ spikes (at instant $t_0$), the potential $V_j$ of neuron $j \neq i$ suffers and "instantaneous" jump:*

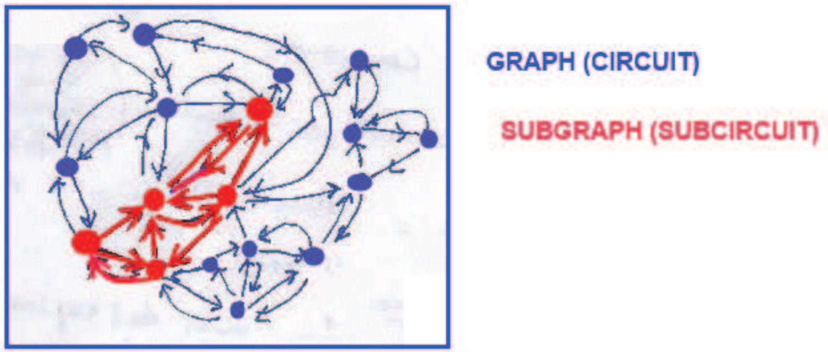
$$V_j(t_0) = V_j(t_0^-) + H_{i,j} \text{ if this number is } < \theta_j, \text{ or}$$

$$V_j(t_0) = 0 \text{ otherwise, and if so, also } j \text{ spikes at instant } t_0.$$

This math. model of the synaptical actions is SIMPLIFIED so:

- Instantaneous jump in the postsynaptical potential $V_j$.
- The REFRACTORY PHENOMENON holds for the spiking cells.

Graph of a circuit containing the graph of a subcircuit (in red) which is called a SUBGRAPH



## Parameters' Space

*Which are the "parameters" of the mathematical system modeling the circuit of neurons? Their values of some of them are NOT numbers but FUNCTIONS or other non numerical MATHEMATICAL STRUCTURES.*

For the relaxation and spiking regime of the neurons:

$$(m, F_1, F_2, F_3, \ldots, F_m, \theta_1, \theta_2, \theta_3, \ldots, \theta_m), \text{ where}$$

- $m$ is the number of neurons in the circuit
- $F_i$ is the (vectorial) function at the second member of the system of differential equations $d\overline{x}_i/dt = F_i(\overline{x}_i)$ governing the relaxation regime of the neuron $i$.
- $\theta_i$ is a real number: the threshold level of neuron $i$.

For the synaptical connections:

$$(G, \; H_{1,2}, H_{1,3}, \ldots, H_{m-1,m}), \text{ where}$$

- $G$ is a Graph Structure: the graph of the circuit, with $m$ nodes $i \in \{1, 2, \ldots, m\}$ and directed and weighted edges $(i, j) : \; i \neq j$
- The weights $H_{i,j}$ of the edges of the graph $G$: they are (positive or negative or zero) real numbers.

## Questions of Research: Dynamics of the Network (circuit or graph)

*Which are the qualitative and quantitative mathematical properties that one may obtain, BY LOGICAL DEDUCTION, from the general mathematical model defined above?*

EXAMPLES:

- Sequence of spiking instants
- ISI (interspike intervals) of a cell or of a subcircuit.
- Attractors and their basins - periodic orbits - limit cycles
- Synchronization of the spikes of several cells (periodic or non periodic synchronization)
- Waiting times until synchronization
- Recurrence

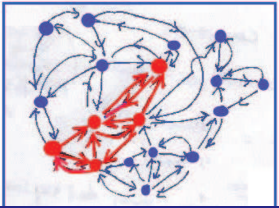
## Definition

Recurrent synchronization of spikes in a subcircuit $S$
if there exists instants $t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, \ldots$
such that at instant $t_n$ all the cells of the subcircuit $S$ spike simultaneously.

In Game Theory the phenomenon of synchronization is called "Grand Coalition".

**Remark:** Between the simultaneous synchronizations at instants $t_n$ and $t_{n+1}$, some neurons of the subcircuit may spike.

## Sufficient mathematical conditions for recurrent synchronization of the spikes in a subcircuit.




### Theorem 1

*If*

- *$S$ is complete and excitatory: $H_{i,j} > 0 \;\; \forall i \neq j$ in $S$,*
- *at least one cell in $S$ is pacemaker,*
- *the number $m$ of cells in $S$ is large enough in relation to the mininum excitatory weight:*

$$\sqrt{m} \;\geq\; \frac{\max_{j \in S} \theta_j}{\min_{i \neq j \text{ in } S} H_{i,j}},$$

*then*

*all the cells of the subcircuit $S$ recurrently synchronize spikes while $S$ does not receive inhibitions from the cells outside $S$.*

## Theorem 1

*If*

- *$S$ is complete and excitatory: $H_{i,j} > 0 \ \ \forall i \neq j$ in $S$,*
- *at least one cell in $S$ is pacemaker,*
- *the number $m$ of cells in $S$ is large enough in relation to the mininum excitatory weight:*

$$\sqrt{m} \ \geq \ \frac{\max_{j \in S} \theta_j}{\min_{i \neq j \text{ in } S} H_{i,j}},$$

*then*

*all the cells of the subcircuit $S$ recurrently synchronize spikes while $S$ does not receive inhibitions from the cells outside $S$.*

**Remark** This theorem holds:

- For any initial state of the cells
- Disregarding which are the functions $F_i$, and if they are similar or mutually very different, and which are the dimensions of vectorial states of the cells.
- No matter if the cells are mutually very different
- No matter if the interactions are mutually very different
- Disregarding how short or long are the refractory periods (but the refractory phenomenon must exist).

## Theorem 1

*If*

- $S$ *is complete and excitatory:* $H_{i,j} > 0 \ \ \forall i \neq j$ *in* $S$*,*
- *at least one cell in* $S$ *is pacemaker,*
- *the number* $m$ *of cells in* $S$ *is large enough in relation to the mininum excitatory weight:*

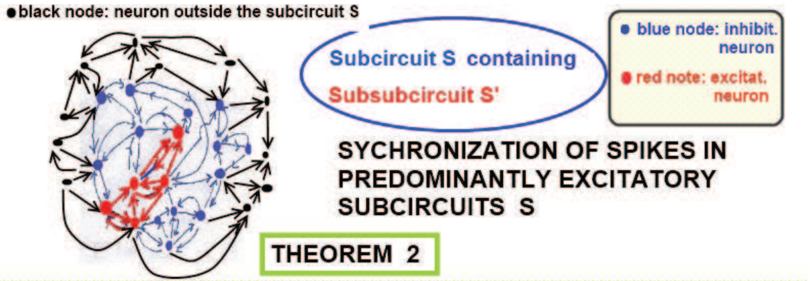
$$\sqrt{m} \geq \frac{\max_{j \in S} \theta_j}{\min_{i \neq j \text{ in } S} H_{i,j}},$$

*then*

*all the cells of the subcircuit* $S$ *recurrently synchronize spikes while* $S$ *does not receive inhibitions from the cells outside* $S$*.*

CHANGING THE CONNECTIONS OF THE SUBCIRCUIT $S$ to be non complete, but still excitatory, provided the number of non null connections is large enough, still produce recurrent synchronization, if certain other mathematical conditions and inequalities hold (work in progress).

Some HISTORY:
1992 Mirollo-Strogatz
1996 Bottani



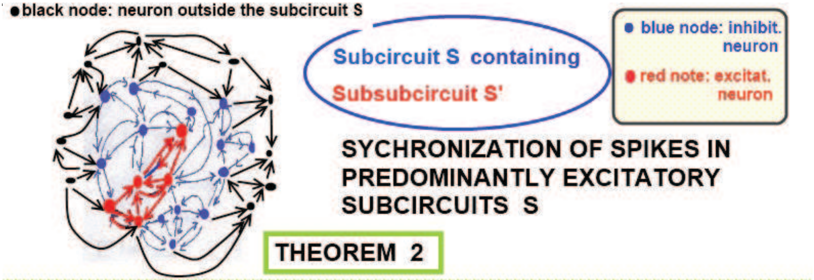

## Theorem 2

*If the subcircuit $S$ contains a sub-subcircuit $S'$ such that*

- *$S'$ is complete and excitatory*
- *all the cells in $S \setminus S'$ are inhibitory*
- *at least one neuron of S' es a pacemaker*
- *the number $m'$ of excitatory neurons in $S'$ satisfies the following inequality:*

$$\sqrt{m'} \geq \frac{\left(\max_{j \in S} \theta_j\right) + \left(\max ISI_{S'} / \min_{j \in S} ISI_j\right) \cdot \left(\max_{j \in S \setminus S'} |H_{j,i}|\right)}{\min_{i \neq j, i \in S', j \in S} H_{i,j}},$$

*then all the cells of the subcircuit $S$ recurrently synchronize spikes, while they do not receibe inhibitions from the cells outside $S$.*




## CONCLUSIONS

1. To avoid recurrent synchr. of spikes in the subcircuit $S'$ composed by excitatory cells connect the nodes of $S'$ with edges coming from inhibitory cells, BUT:
2. If the inhibitory cells connected to $S'$ are themselves excited by the cells of $S'$, then
   - they do not avoid the recurrent syncrhonization of $S'$.
   - Worst, the inhibitory cells also synchronize spikes with the excitatory cells of $S'$.
   - So, due to the refractory phenomenon, the inhibitory cells spiking simultaneously with those of $S'$ do not inhibit them.
3. Other cells outside $S$ (in black in the figure), that are inhibited by $S$ but not excited by $S'$, may turn off (do not spike).