Seminario de Matemática
Instituto Venezolano de Investigaciones Científicas (IVIC)

CONFERENCIA

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Título:

Difeos C1 con "splitting dominado": condiciones suficientes para la positividad de su entropía topológica.

Resumen: Nos enfocamos en los difeomorfismos de clase C1 con "splitting dominado" en variedades compactas riemannianas de dimensión finita. Recientemente Gourmelon y Potrie contruyeron un ejemplo que muestra que la entropía topológica puede ser nula.

En este trabajo buscamos condiciones sufientes para que dichos difeos posean entropía positiva.
Encontramos varias de estas condiciones. Ellas relacionan el difeomorfismo con la medida de Lebesgue (no necesariamente invariante). Estas condiciones se cumplen como caso muy particular (no C1 genérico) cuando f es conservativo, pero también se cumplen en casos más generales en que f no preserva ninguna medida absolutamente continua respecto a la de Lebesgue.

Algunas de estas condiciones suficientes están relacionadas con una desigualdad que acota inferiormente la entropía métrica para ciertas medidas invariantes que llamamos "pseudofísicas" o "SRB-like".

En el caso particular de que el splitting sea parcialmente hiperbólico, esa desigualdad es una igualdad, la cual generaliza la conocida fórmula de Pesin para la entropía a un contexto de regularidad C1 y para las medidas pseudo-físicas o "SRB-like", en vez del contexto más restringido de los difeomorfismos C1 más Hölder, y en sustitución de las medidas físicas o SRB cuando estas no existen.

M compact manifold

$f \in \mathrm{Diff}^1(M)$.

DEFINITION DOMINATED SPLITTING (D.S.)

$TM = E \oplus F$ Df-invariant CONTINUOUS sub-bundles

Such that

$$\|Df_x|_{E_x}\| \cdot \|Df^{-1}_{f(x)}|_{F(f(x))}\| \leq \sigma^{-1} < 1 \quad \forall x \in M$$

$\sigma > 1$ CONSTANT.

Question

When a diffeo with D.S has positive entropy?

1st) NOT ALWAYS Gourmelon-Potrie Example in $\mathbb{T}^2$

2nd) If DS is partially hyperbolic : $\chi|_F \geq 0$ and $\chi|_E \leq 0$.

Saghin-Sun-Vargas

3rd) If Leb satisfies some conditions.

THEOREM 1 If f has D.S and Leb is recurrent then $h_f > 0$ (in particular if $\mathrm{Leb} \in \mathcal{P}_f$)

Def $\rho \in \mathcal{P}$ is "RECURRENT" if $\exists\ 0 < \delta < 1$ such that

$\rho(B) > 1 - \delta \Rightarrow f^{n_j}(B) \cap B \neq \emptyset$ for some $n_j \to +\infty$.

THEOREM 1 If f has D.S and Leb is recurrent (in particular if Leb $\in \mathcal{P}_f$) then $h_f > 0$

Def $\rho \in \mathcal{P}$ is "RECURRENT" if $\exists\ 0 < \delta < 1$ such that

$$\rho(B) > 1-\delta \Rightarrow f^{n_j}(B) \cap B \neq \emptyset \text{ for some } n_j \to +\infty.$$

THEOREM 2 If f has D.S and if at least ONE of the following holds then $h_f > 0$:

$$\lambda^{TM,f}_{ess} > -(\dim E)\cdot \log \sigma \quad (1) \qquad \lambda^{TM,f^{-1}}_{ess} > -(\dim F)\cdot \log \sigma$$

$$\lambda^{F,f}_{ess} > 0 \quad (2) \qquad \lambda^{E,f^{-1}}_{ess} > 0$$

Def

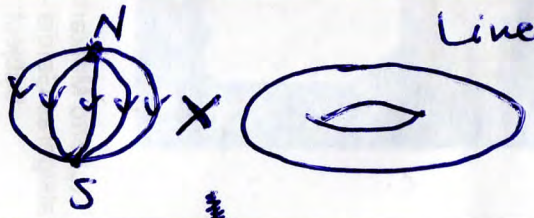
$$\lambda^{TM,f}(x) := \limsup_{n\to+\infty} \frac{1}{n} \log \left| \det df^n_x \right|$$

$$\lambda^{TM,f}_{ess} := \text{Leb ess. sup of } \lambda^{TM,f}(x)$$

$$\lambda^{F,f}(x) := \limsup_{n\to+\infty} \frac{1}{n} \log \left| \det df^n_x \right|_{F_x} \Big|$$

$$\lambda^{F,f}_{ess} = \text{Leb ess. sup of } \lambda^{F,f}(x)$$

Counter example converse of theorem 1 is false in S^2



THE PROOFS

Theorem 1 is indeed a Corollary of Theorem 2 because

Lemma : if Leb is recurrent then $\lambda_{ess}^{TM,f} \geq 0$ or $\lambda_{ess}^{TM,f^{-1}} \geq 0$

THEOREM 1 If f has D.S and Leb is recurrent then $h_f > 0$

THEOREM 2 If f has D.S and if at least ONE of the following holds then $h_f > 0$:

$$\lambda_{ess}^{TM,f} > -(\dim E)\cdot \log \sigma \quad (1)$$
$$\lambda_{ess}^{TM,f^{-1}} > -(\dim F)\cdot \log \sigma$$

$$\lambda_{ess}^{F,f} > 0 \quad (2)$$
$$\lambda_{ess}^{E,f^{-1}} > 0$$

THE PROOFS

Theorem 1 is indeed a Corollary of Theorem 2 because

Lemma: if Leb is recurrent then $\lambda_{ess}^{TM,f} \geq 0$ or $\lambda_{ess}^{TM,f^{-1}} \geq 0$

Proof of Lemma

By contradiction assume

$$\limsup_{n\to+\infty} \frac{1}{n} \log |\det df_x^n| < -\varepsilon$$

and

$$\limsup_{n\to+\infty} \frac{1}{n} \log |\det df_x^{-n}| < -\varepsilon \quad \text{Leb a.e. } x.$$

$\exists N = N(x)$ Leb a.e. x such that $|\det df_x^n| < e^{-n\varepsilon}$, $|\det df_x^{-n}| < e^{-n\varepsilon}$ $\forall n \geq N = N(x)$

$$A_N := \{x \in M : |\det df_x^n| < e^{-n\varepsilon},\ |\det df_x^{-n}| < e^{-n\varepsilon}\ \forall n \geq N\}$$

$A_N \subset A_{N+1}$ $\quad$ $\mathrm{Leb}(A_N) \xrightarrow[N\to+\infty]{} 1$ $\quad$ $\mathrm{Leb}(A_N) > 1-\delta$

$B \subset A_N$ $\quad$ $\mathrm{Leb}(A_N \cap f^n(B)) \leq e^{-n\varepsilon}\mathrm{Leb}(B)$; $\mathrm{Leb}(A_N \cap f^{-n}(B)) \leq e^{-n\varepsilon}\mathrm{Leb}(B)$

$B_1 = A_N \cap f^n(A_N)$

$B_2 = A_N \cap f^{-n}(A_N)$

$B \subset A_N$

$B_1 = A_N \cap f^{+n}(A_N)$

$B_2 = A_N \cap f^{-n}(A_N)$

$$\text{Leb}(A_N \cap f^{-n}(B_1)) = \text{Leb}(\underbrace{A_N \cap f^{-n}(A_N)}_{B_2}) \leq e^{-n\varepsilon}\,\text{Leb}(B_1)$$

$$\text{Leb}(\underbrace{A_N \cap f^{n}(B_2)}_{B_1}) \leq e^{-n\varepsilon}\,\text{Leb}(B_2) < e^{-4n\varepsilon}$$

$$\boxed{\text{Leb}(A_N \cap f^{n}(A_N)) = 0 \quad \forall n \geq N}$$

$\text{Leb}(A_N) > 1 - \delta$

$$C_N = A_N \setminus \bigcup_{n \geq N} f^n(A_N) \quad \Rightarrow \quad \text{Leb}(C_N) = \text{Leb}(A_N) > 1-\delta$$

$C_N \cap f^n(C_N) = \emptyset \quad \forall n \geq N \longrightarrow$ Leb no recurrent

Proof of THEOREM 2 (A)

PREVIOUS DEFINITIONS

$$p\omega(x) := \{\nu \in \mathcal{P} : \frac{1}{n_j}\sum_{i=0}^{n_j-1} \delta_{f^i(x)} \xrightarrow{\text{weak}^*} \nu \text{ for some } n_j \to +\infty\}$$

$p\omega(x)$ is weak* compact, nonempty $\subset \mathcal{P}_f$

Theorem 2 (A)

f with D.S.

If $\lambda^{F_1 f}_{ess} > 0$ then $\mu_f > 0$

where

$$\lambda^{F_1 f}(x) := \limsup_{n\to+\infty} \frac{1}{n} \log |\det df^n_x|_{F(x)}|$$

$$\lambda^{F_1 f}_{ess} = \text{Leb ess. sup of } \lambda^{F_1 f}(x)$$

Let $\mu \in \mathcal{P}$

BASIN OF STRONG STATISTICAL ATTRACTION OF μ is

$$B(\mu) = \{x \in M : p\omega(x) = \mu\}$$

BASIN OF ε-WEAK STATISTICAL ATTRACTION of μ is

$$A_\varepsilon(\mu) = \{x \in M : dist^*(p\omega(x), \mu) < \varepsilon\}$$

Def

μ is PHYSICAL OR SRB if $Leb(B(\mu)) > 0$

μ is PSEUDO-PHYSICAL OR SRB-like if $Leb(A_\varepsilon(\mu)) > 0 \ \forall \varepsilon > 0$

$(\mathcal{O}_f := \{\mu \in \mathcal{P} : \mu \text{ is SRB-like}\})$

PREVIOUS RESULTS:

Th. C-ENRICH Bull. Pol Acad. Sci 2011 If $f \in C^0(M)$

1) $\mathcal{O}_f$ is nonempty and weak* compact contained in $\mathcal{P}_f$

2) Leb a.e x $\quad p\omega(x) \subset \mathcal{O}_f$

Th. C-GERMINARA-ENRICH Erg Th. & DynSys 2015

If $f \in \mathrm{Diff}^1(M)$ has D.S. then $\forall \mu \in \mathcal{O}_f$

$$h_\mu(f) \geq \int \log |\det df|_F| \, d\mu = \int \left(\sum_{\chi_i \text{ of } df|_F} \chi_i \right) d\mu.$$

PROOF OF THEOREM 2 A

$\lambda^{F,f}_{ess} > 0 \quad C = \left\{ x \in M : \limsup_{n \to +\infty} \frac{1}{n} \log |\det df^n_x|_{F(x)}| > 0 \right\}$

$\hookrightarrow \mathrm{Leb}(C) > 0$

Leb a.e $x \in C \quad p\omega(x) \subset \mathcal{O}_f$

choose $x \in C \quad n_j \to +\infty$

$$\lim \frac{1}{n_j} \log |\det df^{n_j}_x| > 0$$

$$\lim{}^{*} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \delta_{f^i(x)} = \mu \in \mathcal{O}_f.$$

$$h_\mu(f) \geq \int \log |\det df|_F| \, d\mu = \lim \frac{1}{n_j} \sum_{i=0}^{n_j-1} \log |\det df_{f^i(x)}|_F| = \lim \frac{1}{n_j} \log |\det df^{n_j}_x| > 0.$$

□

Proof of Theorem 2

Hip) $\lambda^{TM,f}_{ess} > -(\dim E)\log\sigma$

$$C = \{ x \in M : \limsup_{n\to+\infty} \frac{1}{n} \log |\det df^n_x| > -(\dim E)\log\sigma \}$$

$\Rightarrow$ Leb $C > 0$ Leb a.e $x \in C$: $pca(x) \in \mathcal{O}_f$

$n_j \to +\infty$ such that

(1) $\lim \frac{1}{n_j} \log |\det df^{n_j}_x| > -(\dim E)\log\sigma$

(2) $\lim^* \frac{1}{n_j} \sum_{i=0}^{n_j-1} \delta_{f^i(x)} = \mu \in \mathcal{O}_f$

C-Germ. Ennichi:

$h_\mu(f) \geq \int \log|\det df|_F| \, d\mu$ if > 0 ✓

$$h_\mu(f) \geq \int \log|\det df| \, d\mu - \int \log|\det df|_E| \, d\mu$$

if $\int \log|\det df|_F| \, d\mu \leq 0$

$$h_\mu(f) \geq \int \log|\det df|\, d\mu - \int \log|\det df|_E|\, d\mu$$

if $\int \log|\det df|_F|\, d\mu \leq 0$

$$\blacksquare = \int \log|\det df| d\mu = \lim \int \log|\det df|\, d\, \frac{1}{n_j} \sum_{j=0}^{n_j-1} \delta_{f^j(x)} = \lim \frac{1}{n_j} \log|\det df^{n_j}_x|$$

$x \in \mathcal{O}_f$

$$\blacksquare > -(\dim E) \log \sigma$$

$$\blacktriangle = \int \log|\det df|_E|\, d\mu = \int \sum_{\substack{\text{Lyap exp} \\ \text{of } df|_E}} \chi_j \, d\mu < (\dim E) \left(\int \max_{\substack{\text{Lyap exp} \\ \text{of } df|_E}} \chi_j \, d\mu \right)$$

D.S. $\max_{\text{Lyap exp of } df|_E} \chi_j \leq \log \tilde{\sigma}^{-1} + \min_{\substack{\text{Lyap exp} \\ \text{of } df|_F}} \chi_i \pm$

$$\blacktriangle \leq -\dim E \log \sigma + \frac{1}{\dim F} \underbrace{\int \log|\det df|_F|\, d\mu}_{\leq 0} \leq -\dim E \log \sigma$$

□