THE DYNAMICAL SYSTEMS, ERGODIC THEORY, AND PROBABILITY CONFERENCE DEDICATED
TO THE MEMORY OF NIKOLAI CHERNOV
University of Alabama at Birmingham, May 18-20, 2015

CONDITIONS FOR POSITIVE ENTROPY OF DIFFEOMORPHISMS WITH DOMINATED SPLITTING

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in a joint work with Xueting Tian (2)

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Parallel Session - Talk

Eleonora Catsigeras in a joint work with Xueting Tian

Title:

Conditions for positive entropy of
diffeomorphisms with dominated splitting.

Abstract:

We find several conditions that imply positive entropy for C1 diffeomorphisms with dominated splitting. These conditions are related with Pesin's Formula of the Entropy which is satisfied by all the (always existing) SRB-like measures.
This is a joint work with Xueting Tian.

Subject Area: 37D30; 37A35; 37B40

Lebesgue-essential exponents and positive entropy of C^1-diffeomorphisms with dominated splitting.

Eleonora Catsigeras* and Xueting Tian**

Dedicated to the Memory of Prof. Nikolai Chernov

Abstract

We study C^1-diffeomorphisms on compact manifolds that have global dominated splitting. We define the Lebesgue-essential exponents by considering the exponential rates according to which the differential of the Lebesgue measure asymptotically changes to the future and to the past. We find lower negative bounds of the Lebesgue-essential exponents that suffice for the topological entropy be positive. Finally, we prove some corollaries in particular cases: when a smooth measure is preserved, or, more generally, when the Lebesgue measure is recurrent.

Key words: Dominated Splitting; Entropy; Volume Change; SRB-like Measures
MSC 2010: 37D30; 37B40; 37D25; 37A35;

1 Introduction

Consider a compact and connected C^1-Riemannian manifold M without boundary and let $f \in \mathrm{Diff}^1(M)$ be a C^1 diffeomorphism on M.

Definition 1.1. (Dominated Splitting) The diffeomorphism f has a *dominated splitting* $TM = E \oplus F$ if this splitting is defined in all the points of the tangent bundle, is continuous and non trivial (i.e. $\dim(E), \dim(F) \neq 0$), and there exists a constant $\alpha < 1$ such that

$$\|Df|_{E(x)}\| \cdot \|Df^{-1}|_{F(f(x))}\| \leq \alpha, \forall x \in M.$$

We call E and F the *dominated and dominating subbundles* respectively. We call α the *domination constant.*
Note: In the above definition the continuity of the splitting is a redundant condition [1].

Definition 1.1 is a generalization of uniform hyperbolicity and also of partial hyperbolicity. Since uniform and partial hyperbolic systems have positive topological entropy, we first pose the following question:

¿Have all C^1 diffeomorphisms with dominated splitting positive entropy?

The answer is negative. In fact, Gourmelon and Potrie [4] have recently constructed a counterexample on the torus $\mathbb{T}^2$. Nevertheless, it is known that the answer is positive under some additional restrictive hypothesis of f. For instance, if the dominated splitting is partially hyperbolic, then the topological entropy of f is positive, as proved by Saghin, Sun

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and Vargas in [5]. Also, if f has a dominated splitting and preserves the Lebesgue measure (or a finite measure that is absolutely continuous with respect to Lebesgue), then it necessarily has positive entropy (see Corollary 3.4 at the end of this paper). This latter result is also a consequence of a Pesin-like formula for the metric entropy in the C^1-context, proved in [6].

To generalize the results of positive topological entropy for diffeomophisms with dominated splitting that do not necessarily preserve the Lebesgue measure and are not necessarily partially hyperbolic, we will focus on the assymptotic exponential derivative of the Lebesgue measure when applying f or f^{-1}. We will find a relation among those Lebesgue-essential exponents and the positiveness of the topological entropy. To do so, we introduce the following definition:

Definition 1.2. (Lebesgue-essential exponents)

For any point $x \in M$, we define the Lebesgue exponents $\lambda^+(x)$ and $\lambda^-(x)$ at x, to the future and the past respectively, by

$$\lambda^+(x) := \limsup_{n\to+\infty} \frac{1}{n} \log |\det Df^n_x|, \qquad \lambda^-(x) := \limsup_{n\to+\infty} \frac{1}{n} \log |\det Df^{-n}_x|.$$

Now, we define the *Lebesgue-essential exponents* λ^+_{ess} and λ^-_{ess}, to the future and the past respectively, by:

$$\lambda^+_{ess} := \text{Leb-ess sup } \lambda^+(x), \qquad \lambda^-_{ess} := \text{Leb-ess sup } \lambda^-(x),$$

where Leb-ess sup $u(x)$ denotes the essential supremum with respect to the Lebesgue measure of the measurable real function u.

We are ready to state the main result of this paper:

Theorem 1. *Let $f \in \mathrm{Diff}^1(M)$ have a dominated splitting $TM = E \oplus F$, where E and F are the dominated and dominating sub-bundles respectively. Let $0 < \alpha < 1$ be the domination constant. If*

$$\lambda^+_{ess} > -\dim(E) \log \alpha^{-1} \quad \text{or} \quad \lambda^-_{ess} > -\dim(F) \log \alpha^{-1}, \tag{1}$$

then the topological entropy of f is positive.

In Section 2 we will prove Theorem 1, and in Section 3 we will state and prove its corollaries.

2 Proof of Theorem 1.

To prove Theorem 1, we will construct an f-invariant probability measure with positive metric entropy. Thus, applying the variational principle, this construction implies that the topological entropy of f is positive, as wanted. The construction of such a probability measure will be based on the theory of pseudo-physical or SRB-like measures for C^1 maps, which was introduced in [3]. We will apply a result in [2] (generalizing a theorem in [6]): it provides a Pesin-like formula for the entropy to all the pseudo-physical or SRB-like measures of any $f \in \mathrm{Diff}^1(M)$ with dominated splitting.

In the sequel, we denote by $\mathcal{M}$ the space of all the Borel probability measures on M endowed with the weak*-topology. We denote by $\mathcal{M}_f$ the set of f-invariant measures contained in $\mathcal{M}$. Recall that that $\mathcal{M}$ and $\mathcal{M}_f \subset \mathcal{M}$ are nonempty, weak*-compact, sequentially compact, and convex metric spaces.

Definition 2.1. **(p-omega limit of x)** For any point $x \in M$ we construct the sequence $\{\sigma_{f,n}(x)\}_{n\geq 1} \subset \mathcal{M}$ of *empiric probabilities* along the finite pieces of the future orbit of x, by

$$\sigma_{f,n}(x) := \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}, \tag{2}$$

where, δ_y denotes the Dirac-delta probability measure supported on $y \in M$.

We define the *the p-omega limit* $p\omega_f(x) \subset \mathcal{M}_f$ by:

$$p\omega_f(x) := \{\rho \in \mathcal{M} \colon \exists\, n_j \to +\infty \text{ such that } \lim_{j\to+\infty}\sigma_{f,n_j}(x) = \rho\}.$$

Now, we fix a metric dist* in $\mathcal{P}$ that endows the weak*-topology, and recall the following definition taken from [3]:

Definition 2.2. **(Pseudo-physical or SRB-like measures)** Fix $\mu \in \mathcal{M}_f$ and $\epsilon > 0$. We define the *basin* $B_\epsilon(\mu)$ *of* $\epsilon-$*weak attraction of* μ by:

$$B_\epsilon(\mu) := \{x \in M \colon \ \mathrm{dist}^*(p\omega_f(x), \mu) < \epsilon\}.$$

We call the probability measure μ *pseudo-physical or SRB-like* for f if

$$\mathrm{Leb}(B_\epsilon(\mu)) > 0 \ \ \forall\, \epsilon > 0.$$

The following previous results are taken from [3], [2] and [6]:

Theorem 2.3. **(C.-Enrich [3])**

For any continuous map $f : M \mapsto M$ the set of pseudo-physical probability measures is nonempty, weak*-compact, does not depends of the chosen weak* metric, and contains $p\omega_f(x)$ for Lebesgue almost all $x \in M$.

Theorem 2.4. **(Pesin-like formula, C.-Cerminara-Enrich [2] and Sun-T. [6])**

If $f \in \mathrm{Diff}^1(M)$ has a dominated splitting $TM = E \oplus F$, where E and F are the dominated and dominating sub-bundles respectively, and if $\mu \in \mathcal{M}_f$ is pseudo-physical, then

$$h_\mu(f) \geq \int \sum_{i=1}^{\dim(F)} \chi_i(x)d\mu = \int \log|\det Df|_F|\, d\mu, \tag{3}$$

where $\chi_1 \geq \chi_2 \cdots \geq \chi_{\dim(M)}$ denote the Lyapunov exponents defined μ-a.e.

We are ready to start the proof of Theorem 1:

Let $f \in \mathrm{Diff}^1(M)$ have a dominated splitting $TM = E \oplus F$, where E and F are the dominated and dominating sub-bundles respectively. Define

$$\lambda^F_{ess} := \ \text{Leb-ess}\sup\ \lambda^F(x),$$

where

$$\lambda^F(x) := \limsup_{n\to+\infty}\frac{1}{n}\log|\det Df^n_x|_{F(x)}|.$$

Lemma 2.5. *If $\lambda^F_{ess} > 0$ then the topological entropy of f is positive.*

Proof. Consider the set $A := \{x \in M : \limsup_{n\to+\infty} \frac{1}{n} \log|\det Df^n_x|_{F(x)}| > 0\}$. From the condition $\lambda^F_{ess} > 0$, we deduce that Leb(A) > 0. Denote by $\mathcal{P}_f \subset \mathcal{M}_f$ the non-empty set of physical-like measures for f, and apply Theorem 2.3:

$$p\omega_f(x) \subset \mathcal{P}_f \ \ \text{Leb.- a.e.} x \in M.$$

Choose and fix a point $x \in A$ such that $p\omega_f(x) \subset \mathcal{P}_f$, and fix a sequence $n_j \to +\infty$ such that

$$\lim_{j\to+\infty} \frac{1}{n_j} \log|\det Df^{n_j}_x|_{F(x)}| = a > 0. \tag{4}$$

By choosing an adequate subsequence, there exists $\mu \in \mathcal{M}_f$ such that:

$$\lim_{j\to+\infty} \sigma_{f,n_j} = \mu \in \mathcal{M}_f. \tag{5}$$

After Definition 2.1, $\mu \in p\omega_f(x) \subset \mathcal{P}_f$. So, applying Theorem 2.4:

$$h_\mu(f) \geq \int \psi\, d\mu, \ \text{where } \psi := \log|\det Df|_F|. \tag{6}$$

By the definition of the weak* topology in $\mathcal{P}$ (since ψ is a continuous real function), and from equalities (5) and (4), we deduce:

$$\int \psi\, d\mu = \lim_{j\to+\infty} \int \psi\, d\sigma_{f,n_j}(x) = \lim_{j\to+\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \psi(f^i(x)) =$$

$$\lim_{j\to+\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} \log|\det Df_{f^i(x)}|_{F(f^i(x))}| = \lim_{j\to+\infty} \frac{1}{n_j} \log|\det Df^{n_j}_x|_{F(x)} = a > 0. \tag{7}$$

Joining inequalities (6) and (7) we obtain $h_\mu(f) > 0$ as wanted. □

End of the proof of Theorem 1

Proof. By hypothesis $\lambda^+_{ess} > -\dim(E)\log(\alpha^{-1})$ or $\lambda^-_{ess} > -\dim(F)\log(\alpha^{-1})$. It is not restrictive to assume that the first inequality holds. If not we would apply the same proof with the second inequality instead of the first one, and with f^{-1}, F, E in the place of f, E, F respectively.

Arguing as in the proof of Lemma 2.5 there exists a point $x \in M$, a sequence $n_j \to +\infty$, and a physical-like probability measure μ such that

$$\lim_{j\to+\infty} \frac{1}{n_j} \log|\det Df^{n_j}_x| = b > -\dim(E)\log\alpha^{-1}, \tag{8}$$

$$h_\mu(f) \geq \int \log|\det Df_F|\, d\mu, \tag{9}$$

$$\int |\det Df|\, d\mu = \lim_{j\to+\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} |\det Df_{f^i(x)}| = \lim_{j\to+\infty} \frac{1}{n_j} \log|\det Df^{n_j}_x| = b. \tag{10}$$

Since $E \oplus F = TM$ is a Df-invariant splitting and μ is an f-invariant measure, applying Oseledets Theorem we obtain:

$$\int \log|\det Df|\,d\mu = \int \sum_{k=1}^{\dim M} \chi_k\,d\mu = \int \sum_{k=1}^{\dim F} \chi_k\,d\mu + \int \sum_{k=\dim F+1}^{\dim M} \chi_k\,d\mu =$$

$$\int \log|\det Df|_F|\,d\mu + \int \log|\det Df|_E|\,d\mu.$$

Thus,

$$\int \log|\det Df|_F|\,d\mu = \int \log|\det Df|\,d\mu - \int \log|\det Df|_E|\,d\mu \tag{11}$$

Besides, applying the definition of dominated splitting, we obtain:

$$\log|\det Df_x|_{E(x)}| \le \dim(E)\log\|Df_x|_{E(x)}\| \le \dim(E)\cdot\log\Big(\frac{\alpha}{\|Df^{-1}_{f(x)}|_{F(f(x))}\|}\Big) \le$$

$$-\dim(E)\,\log\alpha^{-1} + \dim(E)\cdot\frac{\log|\det Df_x|_{F(x)}|}{\dim(F)}. \tag{12}$$

We recall that the dimensions of E and F are constant, because these sub-bundles are continuous and the manifold M is connected. Joining equality (11) with inequality (12):

$$\Big(1+\frac{\dim(E)}{\dim(F)}\Big)\int \log|\det Df|_F|\,d\mu \ge \int \log|\det Df|\,d\mu + \dim(E)\log\alpha^{-1}. \tag{13}$$

Finally, from inequalities (8), (9), (10) and (13) we conclude:

$$\Big(1+\frac{\dim(E)}{\dim(F)}\Big)h_\mu(f) \ge \Big(1+\frac{\dim(E)}{\dim(F)}\Big)\int \log|\det Df|_F| \ge \log|\det Df|\,d\mu + \dim(E)\log\alpha^{-1}$$

$$= b + \dim(E)\log\alpha^{-1} > -\dim(E)\log\alpha^{-1} + \dim(E)\log\alpha^{-1} = 0.$$

So, we conclude that $h_\mu(f) > 0$ as wanted. □

3 Corollaries.

In this section we apply Theorem 1 to some particular cases:

Corollary 3.1. *If $f \in \mathit{Diff}^1(M)$ has a dominated splitting and preserves the Lebesgue measure, then the topological entropy of f is positive.*

Proof. In fact, since $|\det Df| = 1$, the Lebesgue-essential exponents are zero. Thus, the condition 1 holds; hence Theorem 1 implies $h_{\text{top}}(f) > 0$. □

We will generalize Corollary 3.1 to cases for which the Lebesgue measure is not f-invariant, but the Lebesgue-essential exponents are still zero. To so do, we need the following definition:

Definition 3.2. (Recurrent measures)

Let $\rho \in \mathcal{M}$ (i.e. *rho* is a non necessarily invariant, Borel probability measure on M). We call ρ *a recurrent measure* if there exists a real number $0 < \delta < 1$ such that for any measurable set $B \subset M$, if $\rho(B) \ge 1-\delta$, then there exists $n_j \to +\infty$ such that

$$B \cap f^{n_j}(B) \ne \emptyset \quad \forall\, j \in \mathbb{N}.$$

Note that, due to Poincaré Lemma any f-invariant measure is recurrent (in such a case, δ can be arbitrarily chosen in the open interval $(0,1)$).

In the sequel, we denote by Leb the Lebesgue measure on M after a rescaling to make it a probability measure.

Corollary 3.3. *If $f \in \text{Diff}^1(M)$ has a dominated splitting and the Lebesgue probability measure is recurrent, then the topological entropy of f is positive.*

Proof. Applying Theorem 1, it is enough to prove that $\lambda^+_{ess} \geq 0$ or $\lambda^-_{ess} \geq 0$. Arguing by contradiction, and recalling Definition 1.2, assume that there exists a real number $-a < 0$ such that the $\text{Leb}(A) = 1$, where

$$A := \Big\{x \in M : \ \limsup_{n\to+\infty} \frac{\log|\det Df^n_x|}{n} < -a, \quad \limsup_{n\to+\infty} \frac{\log|\det Df^{-n}_x|}{n} < -a\Big\}.$$

For any natural number $N \geq 1$ and define

$$A_N := \Big\{x \in M : \ \frac{\log|\det Df^n_x|}{n} < -a, \quad \frac{\log|\det Df^{-n}_x|}{n} < -a \ \ \forall\, n \geq N\Big\}.$$

We have $A_N \subset A_{N+1}$ and $A = \bigcup_{N=1}^{+\infty} A_N$. So, $\lim_{N\to+\infty} \text{Leb}(A_N) = \text{Leb}(A) = 1$. So, for any given $0 < \delta < 1$ there exists $N \geq 1$ such that

$$\text{Leb}(A_N) > 1 - \delta.$$

Consider any measurable set $C \subset A_N$. We obtain:

$$\text{Leb}(C \cap f^n(A_N)) = \int_{x\in f^{-n}(C)\cap A_N} |\det Df^n_x|\, d\text{Leb}(x) \leq e^{-na} \cdot \text{Leb}(f^{-n}(C) \cap A_N) \quad \forall\, n \geq N.$$

$$\text{Leb}(C \cap f^{-n}(A_N)) = \int_{x\in f^{n}(C)\cap A_N} |\det Df^{-n}_x|\, d\text{Leb}(x) \leq e^{-na} \cdot \text{Leb}(f^{n}(C) \cap A_N) \quad \forall\, n \geq N.$$

In particular, applying the above inequalities to $C_1 := A_N \cap f^n(A_N)$ and $C_2 := A_N \cap f^{-n}(A_N)$, we deduce $\text{Leb}(C_1) \leq e^{-na} \cdot \text{Leb}(C_2)$, $\text{Leb}(C_2) \leq e^{-na} \cdot \text{Leb}(C_1)$ $\forall\, n \geq N$. Thus, $\text{Leb}(C_1) = \text{Leb}(C_2) = 0$; hence

$$\text{Leb}(A_N \cap f^n(A_N)) = \text{Leb}(A_N \cap f^{-n}(A_N)) = 0 \quad \forall\, n \geq N. \tag{14}$$

Now, construct the measurable set $B := A_N \setminus \Big(\bigcup_{n=N}^{+\infty} f^n(A_N)\Big)$. From equalities (14) we obtain $\text{Leb}(B) = \text{Leb}(A_N) > 1 - \delta$. And by construction of B we obtain $B \cap f^n(B) = \emptyset$ $\forall\, n \geq N$. Since the above assertions hold for any $0 < \delta < 1$, we conclude that Leb is not recurrent, contradicting the hypothesis. □

Finally, we state and prove the following consequence of Theorem 1. It is a generalization of Corollary 3.1.

Corollary 3.4. *If $f \in \text{Diff}^1(M)$ has a dominated splitting and preserves a smooth probability measure ρ (i.e. $\rho \ll$ Leb), then the topological entropy of f is positive.*

Proof. By hypothesis $\rho \ll \text{Leb}$. Let us prove that there exists $0 < \delta < 1$ such that $\rho(C) < 1$ for any measurable set $C \subset M$ such that $\text{Leb}(C) < \delta$. In fact, arguing by contradiction, if the latter assertion were false, then for any natural number $n \geq 2$ there would exist $C_n \subset M$ such that $\text{Leb}(C_n) < 1/n$ and $\rho(C_n) = 1$. So, taking $A = \bigcap_{n=1}^{+\infty} C_n$ we would obtain $\rho(A) = 1$ and $\text{Leb}(A) = 0$, which contradicts the hypothesis $\rho \ll \text{Leb}$. So, we have proved the existence of a real number $0 < \delta < 1$ satisfying the assertion at the beginning.

Take any measurable set $B \subset M$ such that $\text{Leb}(B) > 1 - \delta$. Thus $\text{Leb}(M \setminus B) < \delta$. By construction of δ we obtain $\rho(M \setminus B) < 1$; hence $\rho(B) > 0$. But besides $\rho \in \mathcal{M}_f$. Thus, from Poincaré Recurrence Lemma we deduce that there exists infinitely many future iterates of B that intersect B. In brief, we have proved that for any measurable set B such that $\text{Leb}(B) > 1 - \delta$ there exists $n_j \to +\infty$ such that $B \bigcap f^{n_j}(B) \neq \emptyset$. Applying Definition 3.2, the Lebesgue measure is recurrent. Finally, from Corollary 3.3, we conclude that the topological entropy of f is positive. □

Acknowledgements

X. Tian thanks the support of the National Natural Science Foundation of China (No. 11301088). E. Catsigeras thanks the Department of Mathematics of the University of Alabama at Birmingham, Prof. Alexander Blokh , and the scientific and organization committees of The Dynamical Systems, Ergodic Theory and Probability Conference Dedicated to the Memory of Nikolai Chernov. She also thanks the partial support of Agencia Nacional de Investigación e Innovación (ANII), Comisión Sectorial de Investigación Científica (CSIC) of Universidad de la República, and Project L'Oréal-Unesco-Dicyt of Uruguay.

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