New Trends in Onedimensional Dynamics
Celebrating the 70^{th} anniversary of Welington de Melo

Rio de Janeiro, November 14 - 18, 2016

Title: ***Stochastic perturbations of piecewise continuous maps.***

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Abstract.

We consider a piecewise continuous map $f : X \mapsto X$ with a finite number of continuity pieces on a compact metric space X. We study the ergodic properties of the stochastic dynamical system (X, fP_ϵ), obtained by adding at each iterate of f a noise of level $\epsilon > 0$, namely, a stochastic perturbation of f with family $P_\epsilon = P_\epsilon(x, \cdot)$ of transition probabilities, supported on the ball $B_\epsilon(f(x))$ for each $x \in X$.

We construct a transfer operator $\mathcal{L}^*_\epsilon$ in the space of probability measures, whose fixed points are the stationary measures of (X, f, P_ϵ). Under mild hypothesis on the transition probabilities, we prove the existence and finitude of ergodic stationary probability measures μ. We also prove that for each ergodic μ there exists a unique maximal period $p \geq 1$ and a $\mathcal{L}^*_\epsilon$-periodic probability measure ν with period p, such that $\mu = (1/p)\sum_{j=0}^{p-1} \mathcal{L}^{*j}_\epsilon \nu$. Finally, we prove that the ergodic periodic measures ν of maximal period are weakly mixing and also ergodic for all the multiples of $\mathcal{L}^{*p}_\epsilon$.

This is a joint work with Pierre Guiraud, Arnaldo Nogueira and Sandro Vaienti.

Stochastic Perturbations of Piecewise Continuous Maps

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Notation:

X compact metric space, $\mathcal{A}$ is its Borel sigma-algebra.
$\mathcal{M}$ is the space of all the probability measures on $(X, \mathcal{A})$ endowed with the weak* topology;
$C_0(X, \mathbb{C})$ space of continuous functions $\varphi : X \mapsto \mathbb{C}$ with the sup norm;
L_∞ space of bounded measurable functions.

Definition

$f : X \mapsto X$ **is Piecewise Continuous** if there exists a "topological partition" $\{X_i\}_{1\le i\le N}$ (i.e. X_i open, $X_i \cap X_j = \emptyset$ if $i \neq j$, and $\bigcup \overline{X}_i = X$)
and continuous maps $f_i : \overline{X}_i \mapsto X$ such that

$$f|_{X_i} = f_i.$$

Notation: $\Delta =$ set of discontinuity points of f

Definition

A Stochastic System by perturbation of f with noise of level $\epsilon > 0$ is (X, f, P_ϵ) where

$$P_\epsilon = \{P_{\epsilon,i}(x,\cdot)\}_{x\in\overline{X}_i} 1 \leq i \leq N$$

is the *family of transition probabilities* $P_{\epsilon,i}(x,\cdot) \in \mathcal{M}$ *of the stochastic process* $\{x_n\}_{n\geq 0}$, i.e.
$P_{\epsilon,i}(x,A) = \text{prob.}\big(\{x_{n+1} \in A \,|\, x_n = x\}\big)$ for all $x \in \overline{X}_i$, and

$$\bullet \text{supp}(P_{\epsilon,i}(x,\cdot)) = \overline{B}_\epsilon(f_i(x)) \;\; \forall\, A \in \mathcal{A}, \;\; \forall\, x \in \overline{X}_i.$$

Hypothesis on the noise: • $x \in \overline{X}_i \;\mapsto\; P_{\epsilon,i}(x,\cdot) \in \mathcal{M}$ is continuous.

• $P_{\epsilon,i}(x,\Delta) = 0 \;\; \forall\, x.$

• If $A \subset \overline{B}_\epsilon(f_i(x)) \cap \overline{B}_\epsilon(f_j(x'))$, then $P_{\epsilon,i}(x,A) = 0$ if and only if $P_{\epsilon,j}(x',A) = 0.$

Definition

The transfer operator $\mathcal{L} : L_\infty \mapsto L_\infty$ is

$$(\mathcal{L}\varphi)(x) := \int \varphi(y)\, P_{\epsilon,i}(x, dy) \quad \forall\, x \in X_i$$

The dual transfer operator $\mathcal{L}^* : \mathcal{M} \mapsto \mathcal{M}$ is

$$\int \varphi\, d(\mathcal{L}^*\mu) := \int (\mathcal{L}\varphi\, d\mu \;\; \forall\, \mu \in \mathcal{M}.$$

A stationary measure $\mu \in \mathcal{M}$ is a fixed point of $\mathcal{L}^*$.
A periodic measure μ **with period** p is a fixed point of $\mathcal{L}^{*p}$ for a minimum natural value of $p \geq 1$.
A measure μ **of period** p **is ergodic** if for any $\mathcal{L}^p$-invariant set A (i.e. $\mathcal{L}^p\chi_A = \chi_A$ μ-a.e.) either $\mu(A) = 1$ or $\mu(A) = 0$.

Theorem

(Existence and finitude of ergodic stationary measures)
The set $\mathcal{E}$ of stationary ergodic measures for the stochastic system (X, f, P_ϵ) is nonempty and finite.

SKETCH OF THE PROOF (on the board)

Theorem

(Periodic ergodic measures of maximal period)
For each ergodic stationary measure μ there exists a unique maximal period $p \geq 1$ and a periodic and ergodic probability measure ν with period p, such that

$$\mu = \frac{1}{p}\nu + \mathcal{L}^*\nu + \ldots + \mathcal{L}^{*p-1}.$$

Besides, the ergodic periodic measures ν of maximal period are weakly mixing and also ergodic for all the multiples of $\mathcal{L}^{*p}$.

SKETCH OF THE PROOF (on the board)

Definition

A piecewise continuous map $f : X \mapsto X$ **is piecewise contracting** if there exists a constant $0 < \lambda < 1$ such that, for any continuity piece X_i of f:

$$\text{dist}(f_i(x), f_i(x')) \leq \lambda \cdot \text{dist}(x, x') \;\; \forall \; x, x' \in X_i.$$

A piecewise contracting map **is typically periodic** if its attractor Λ does not intersect the set Δ of discontinuity points. (In such a case, the attractor Λ is composed by a finite number of periodic orbits).

Theorem

If f is piecewise contracting and typically periodic, then, for all the stochastic perturbations of f with noise level $\epsilon > 0$ small enough, there exists an invertible correspondence ξ between the family of ergodic periodic measures with maximal period for the transfer operator $\mathcal{L}^*$, and the periodic orbits of the attractor of the deterministic system.
Besides,

$$\text{period}(\nu) = \text{period}(\xi(\nu)), \quad \text{and} \quad \xi(\nu) \in \text{supp}(\nu) \subset \overline{B}_{\epsilon/(1-\lambda)}(\xi(\nu)).$$

SKETCH OF THE PROOF (on the board)