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# Generic $C^0$ maps of the interval: ergodic and pseudo-physical measures

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## Abstract

We study the ergodic properties of generic $C^0$ maps of the interval $I$. We prove that all ergodic measures are pseudo-physical, and that any pseudo-physical measure is in the weak$^*$-limit of the set of ergodic measures. Nevertheless, we also prove that the set of pseudo-physical measures is meager in the space of all invariant measures.

This is a joint work with Serge Troubetzkoy.

$I$ compact interval with nonempty interior, $f \in C(I)$.
$\mathcal{P}$: Borel probability measures on $I$ - weak$^*$-topology.
$\mathcal{P}_f \subset \mathcal{P}$: $f$-invariant prob. measures.

**Definitions:**

- *Empiric probabilities:* $$\sigma_{n,x} := \sum_{j=0}^{n-1} \delta_{f^j(x)}; \quad x \in I, \quad n \geq 1.$$
- *$P$-omega-limit of $x \in I$:*

$$p\omega(x) : = \{\mu \in \mathcal{P} : \lim \sigma_{n_j,x} = \mu \text{ for some } n_j \to +\infty\}$$

- *Physical measure $\mu$* if $$\text{Leb}\Big(\big\{x \in I : p\omega(x) = \{\mu\}\big\}\Big) > 0.$$
- *$\epsilon$-weak basin of statistical attraction of $\mu \in \mathcal{P}_f$:*

$$A_\epsilon(\mu) = \{x \in I : \text{dist}(p\omega(x), \mu) < \epsilon\}.$$

- *Pseudo-physical measure $\mu$* if $$\text{Leb}\Big(A_\epsilon(\mu)\Big) > 0 \ \forall \ \epsilon > 0.$$

**Theorem Abdenur-Anderson** (CMP, 2013)
*$C^0$-generically, $f$ has not physical measures, and for Lebesgue a.e. $x \in I$ there exists a (unique) measure $\mu_x \in \mathcal{P}_f$ such that $p\omega(x) = \{\mu_x\}$.*

Our first result:

**Theorem**

*$C^0$-generically:*
*• Any ergodic measure is pseudo-physical.*
*• Any pseudo-physical measure is in the closure of the ergodic measures, as well as in the closure of atomic measures.*
*• The subspace of pseudo-physical measures is a topologically meager subset of $\mathcal{P}_f$.*

SKETCH OF THE PROOF: on the board

Our results on the entropy of $C^0$-generic systems on the interval:

**Theorem**

*For $C^0$-generic $f\colon I \mapsto I$,*

- *The metric entropy function $\mu \in \mathcal{P}_f \to h_\mu(f)$ is everywhere neither upper semi-continuous nor lower semi-continuous.*
- *There exists non countably infinitely many pseudo-physical measures $\mu$ that are atomic, hence $h_\mu(f) = 0$*
- *For any natural number $m \geq 1$, there exists infinitely many pseudo-physical measures $\mu$ for which $h_\mu(f) = \log m$. Hence, the topological entropy is infinite.*
- *There exists infinitely many pseudo-physical measures $\mu$ for which $h_\mu(f) = \infty$.*

SKETCH OF THE PROOF (on the board)

## DEFINITION Periodic shrinking interval period $p \geq 1$



nonempty open interval $I \subset I_0$ such that

- $\{f^j(I)\}_{0 \leq j \leq p-1}$ pairwise disjoint.
- $\mathrm{Leb}(f^j(I)) < \mathrm{Leb}(I)$ $\forall\, 1 \leq j \leq p$
- $\boxed{f^p(I) \subset I}$

Eventually periodic shrink. interval

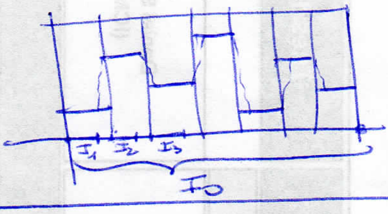
nonempty open interval $J \subset I_0$ such that

- $\exists\, n_0 \geq 0$ such that $\mathrm{Leb}(f^j(J)) < \mathrm{Leb}(J)$ $\forall\, 1 \leq j \leq n_0$
  $I \subset I_0$, $I$ periodic shrinking interval
  $\boxed{f^{n_0}(J) \subset I}$

**LEMMA 1** $C^0$-generically Leb-a.e. $x \in I_0$ belongs to a sequence $\{I_q\}_{q \geq 1}$ of eventually periodic or periodic shrink. intervals $I_q$ with length $(I_q) < \frac{1}{q}$

Route of the proof: $\mathcal{S}_{q,k} := \{ f \in C(I_0)$ such that $\exists\, \{I_1, I_2, \dots, I_{e_1}\}$

- $\mathrm{leb}(I_i) < \frac{1}{q}$
- $I_i$ shrinking interval
- $\mathrm{leb}(I \setminus \bigcup_{i=1}^{e_1} I_{e_1}) < 1/k$ $\}$




- $\mathcal{S}_{q,k}$ open in $C(I_0)$
- $\mathcal{S}_{q,k}$ dense in $C(I_0)$

$\bigcap_{k \geq 1} \bigcap_{q \geq 1} \mathcal{S}_{q,k}$ generic.

**COROLLARY** $C^0$ generically $f$ is Lebesgue a.e non expansive in the future

Def. leb a.e. $x \in I_0$, $\forall \alpha > 0$ $\mathrm{Leb}(\{y \in I_0 : \mathrm{dist}(f^n(y), f^n(x)) < \alpha \ \forall n \geq 0\}) > 0$

**COROLLARY** $\boxed{\mathcal{O}_f \subset \overline{\mathrm{Per}_f} = \overline{\mathcal{E}_f}}$

Koscielniak - Mazur - Oprocha - Pilarczyk (DCDS - 2014).

$\mu_0 \in \mathcal{O}_f$

$p\omega(x)$ for leb $> 0$ set of $x \in I_0$

$z_0$ periodic point

$I_1$ $f^j(x)$

$I_1$ periodic shrinking interval

$\bigcup_{j=0}^{p-1} f^j(I_1) \supset \mathrm{supp}\, \mu_x$ $\bigcup_{j=0}^{p-1} f^j(I_1) \supset \mathrm{orb}(z_0)$

## DEFINITION Shrinking ~~periodic~~ atomic MEASURES (supported on periodic orbits)



$\nu_0 \in Shr_q Per_f$ if $\cdot\ supp(\nu_0) \subset \bigcup_{j=1}^{p} f^j(I)$ where $I$ periodic shrinking interval (of period $p$)

$\cdot\ length(I) < \frac{1}{q}$.

## DEFINITION $\varepsilon$-Aprox shrinking atomic Measure

$\nu \in A_\varepsilon Shr_q Per_f$ if $\exists\, \nu_0 \in Shr_q Per_f$ such that $dist(\nu, \nu_0) < \varepsilon$

## COROLLARY of LEMMA 1

$C^0$-generically

$$\boxed{\bigcap_{\varepsilon>0} \bigcap_{q\geq 1} A_\varepsilon Shr_q Per_f \subset \mathcal{O}_f}$$

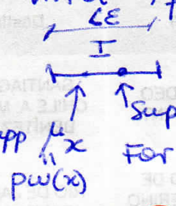
~~Plan of the proof~~ ~~$q$~~

Proof:

$\mu \in A_\varepsilon Shr_q Per_f$

$\nu \in Shr_q Per_f$

with $1/q < \varepsilon$



For Leb a.e. $x \in I$.

$\Rightarrow \mu \in \mathcal{O}_f$

## LEMMA 2

$C^0$-generically

$$\boxed{Per_f \subset \bigcap_{\varepsilon>0} \bigcap_{q\geq 1} A_\varepsilon Shr_q Per_f}$$

Route of the proof

$\mathcal{U}_{q,r} := \{ f \in C(I_0) : \exists \{U_1, U_2, \dots, U_h\}$

- $\mathcal{U}_{q,r}$ open in $C(I_0)$
- $\mathcal{U}_{q,r}$ dense in $C(I_0)$
- $\bigcap_{r\geq 1} \bigcap_{q\geq 1} \mathcal{U}_{q,r}$ generic

$\cdot\ Leb(U_i) < \frac{1}{q}$

$\cdot\ U_i \supset I_i$ periodic shrinking interval period that divides $r$.

$\cdot\ \bigcup_{i=1}^{h} U_i \supset \{x = f^r(x)\}$

## COROLLARY OF LEMMAS 1, 2 $\overline{Per_f} \subset \mathcal{O}_f$

~~END OF THE~~

③

# END OF THE PROOF OF THEOREM 1

we have proved: $\left.\begin{array}{l} \mathcal{O}_f \subset \overline{Per_f} = \overline{\mathcal{E}_f} \\ \overline{Per_f} \subset \mathcal{O}_f \end{array}\right\}$

$\boxed{\mathcal{O}_f = \overline{Per_f} = \overline{\mathcal{E}_f}}$ ; $\Rightarrow$ $\boxed{\mathcal{E}_f \subset \mathcal{O}_f}$

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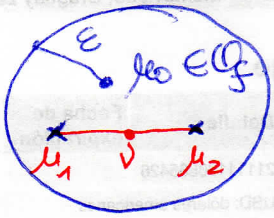
Now let us prove that $\mathcal{O}_f$ is megger in $\mathcal{G}_f$ (closed with empty interior)



$\mu_0$ not isolated in $\mathcal{O}_f$

$\exists\, \mu_1, \mu_2 \in \bigcap_q Shr_q\ Per_f$

$\mu_1 \neq \mu_2$

$I_1$ $\quad$ $I_2$

$\left.\begin{array}{l} I_1 \\ I_2 \end{array}\right\{$ periodic shrinking intervals, lengths $< 1/q$, orbit at distance $> 0$.

let $\nu = \lambda \mu_1 + (1-\lambda)\mu_2$ with $0 < \lambda < 1$

__we assert__ $A_\delta(\nu) = \emptyset$ for some $\delta > 0$, hence $\nu \notin \mathcal{O}_f$

Let $\varphi : I_0 \to [0,1]$ continuous $\quad \varphi(f^P(I_1)) = 1$

$\varphi(I_1^{\text{compl}}) = 0$

By contradiction assume $\exists y \in A_\delta(\nu)$ : $dist(p\omega(y), \nu) < \delta$

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(y)) - \int \varphi \, d\nu \right| < \delta$$

$0 < \lambda_P < 1$ $\quad$ (si elijo $\delta < \frac{\lambda}{2P}$)

$\Rightarrow \exists\, n_0 \geq 1$ such that $\varphi(f^{n_0}(y)) > 0$

$f^{n_0}(y) \in I_1 \Rightarrow f^{n_0 + P j}(y) \in f^P(I_1)$

$\varphi(f^{n_0 + jP}(y)) = 1 \quad \forall j$

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^{n_0 + j}(y)) = \frac{1}{P} \qquad \left(\frac{1}{P} - \frac{\lambda}{P}\right) < \delta$$

Absurdo si elijo $\delta < \frac{1-\lambda}{P}$.

THEOREM 2 c)

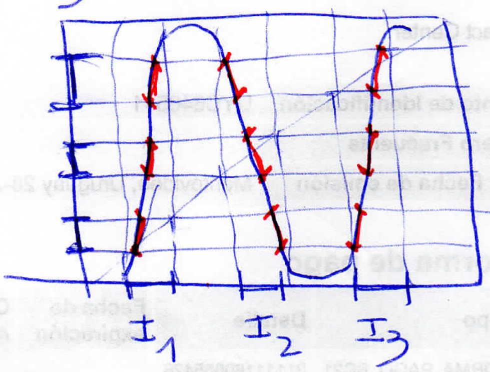
④

## $\boxed{\text{PSEUDO-PHYSICAL MEASURE } h_\mu(f) = \log m}$

• $\boxed{\text{DEFINITION}}$ "m-horseshoe"

pairwise disjoint nonempty open intervals $I_1 \quad I_2 \quad I_3 \quad \dots \quad I_m$ such that

$$\operatorname{int}(f(I_i)) \supset I_1 \cup I_2 \cup \dots \cup I_m$$




Atoms:

gen. 1 $\quad A_1 = \{ I_1, I_2, \dots, I_m \}$

gen. 2 $\quad A_2 = \{ I_{ij} \ \ 1 \le i,j \le m \}$

we choose ~~an~~ an interval $I_{ij}$

such that $I_{ij} \subset I_i$

$$\operatorname{int} f(I_{i,j}) \supset I_j$$

$$\Lambda_n := \bigcup_{A \in \mathcal{A}_n} A$$

$\Lambda$ set $\qquad \boxed{\Lambda := \bigcap_{n \ge 1} \Lambda_n}$

Def. $C^0$-hyperbolic m-horseshoe

$\boxed{\max_{A \in \mathcal{A}_n} \operatorname{length}(A) < \lambda^n}$ for ~~all~~ a constant $0 < \lambda < 1$

$\Rightarrow$ $\Lambda$ is a Cantor set

• Itinerary $\theta(x)$ of $x \in \Lambda$

$\theta: \Lambda \longrightarrow \{1, 2, \dots, m\}^{\mathbb{N}}$ homeomorfismo

conjugacion con shift.

Bernoulli measure $\mu$ ergodic ~~log~~, $h_\mu(f) = \log m$

$\boxed{\text{LEMMA 3}}$ $C^0$-generically for any point $x_0 = f(x_0)$ and for any $q \in \mathbb{N}^+$

$\exists$ an m-horseshoe ~~such that~~ contained in $\left[ x_0, x_0 + \frac{1}{q} \right]$.

Proof $\quad \mathcal{B}_{q,m} := \{ f \in C(I_0)$ such that $\exists \{ J_1, J_2, \dots, J_q \}$

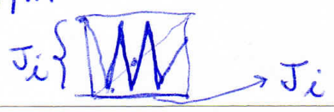
• $\operatorname{Leb}(J_i) < 1/q$

• $J_i \supset C^0$-hyperbolic m-horseshoe

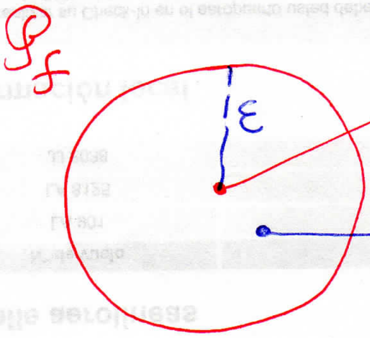
• $(\bigcup J_i)$ covers the fixed points of $f$

– $\mathcal{B}_{q,m}$ is $G_\delta$ in $C^0(I_0)$

– $\mathcal{B}_{q,m}$ is dense in $C(I_0)$



**Corollary** The entropy function is neither upper semi-continuous nor lower semi-continuous.




$\mu$ Bernoulli $\in \mathcal{E}_f \subset \mathcal{O}_f$

$\nu_k \in Per_f$

$h_{\nu_k}(f) = 0$

$\nu_k \to \mu$

$h_\mu(f) = \log m > 0$

$\boxed{h(f) \text{ is NOT LOWER Semi-cont.}}$

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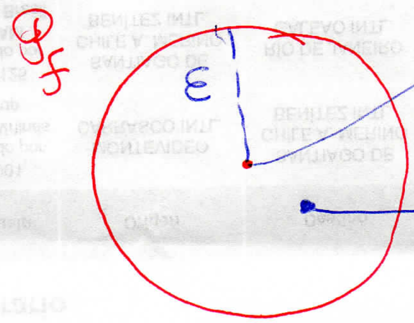
$\delta_{x_0}$ supported on a fixed point $x_0 = f(x_0)$

$\delta_{x_0} \in Per_f \subset \mathcal{O}_f$




$\mu_k$ Bernoulli $h_{\mu_k} = \log m$

$\mu_k \to \delta_{x_0}$

$h_{\delta_{x_0}}(f) = 0$

$\boxed{h(f) \text{ is NOT UPPER Semi-cont.}}$

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**THEOREM 2 d.**

$\boxed{\text{PSEUDO-PHYSICAL MEASURE: } h_\mu(f) = +\infty}$

- One constructs "An atom doubling cascade"
- Defines $\mu \in \mathcal{P}_f$ — $\mu$ is equidistributed in the atoms of each generation $n$

  $\#A_n = \frac{n(n+1)}{2}$
- Prove $\mu$ is ergodic
- Compute applying definition the metric entropy $h_\mu(f) = +\infty$.

*Thank you very much!*