Abundant continuous dynamical systems with infinite entropy

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$C^0(X)$ = space of all the deterministic continuous dynamical systems $f : X \mapsto X$. Recurrent relation

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If a phenomenon P is generic or typical, then the family of systems that does not exhibit P is MEAGER, and the family of systems that does exhibit P is ABUNDANT (in the sense of Baire Cathegory Theory).

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Theorem 1. (Infinite metric entropy ergodic measures)

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INFINITE METRIC ENTROPY IS GENERIC OR TYPICAL
in the family of deterministic continuous dynamical systems. Thus, such systems ARE ABUNDANT.

Route of the proof of Theorem 1.

- In the box $[0,1]^m$: CONSTRUCT a nonempty G_δ-family $\mathcal{H} \subset C^0([0,1]^m)$ of continuous maps $h : [0,1]^m \mapsto [0,1]^m$, which we call **MODELS**.

Main Lemma

Any model h has an ergodic measure ν such that $h_\nu(h) = +\infty$.

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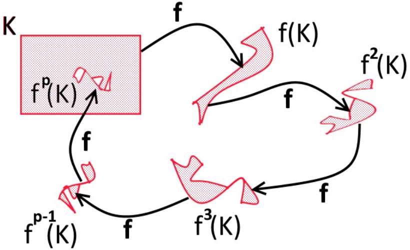
PERIODIC SHRINKING BOX $K \subset M$.

Let $f \in C^0(X)$.

Definition

Periodic shrinking box with period p is a compact set $K \subset M$ homeormorphic to $[0,1]^m$, such that

- $K, f(K), \ldots, f^j(K), \ldots, f^{p-1}(K)$ are pairwise disjoint,
- $f^p(K) \subset$ interior(K),
- diam$(f^j(K)) <$ diam(K) for all $j \geq 1$.



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Lemma 1

Generic $f \in C^0(X)$ has some periodic shrinking box K.

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Generic $f \in C^0(X)$ has some periodic shrinking box K such that the return map $f^p|_K : K \mapsto intK$ is conjugated to some model map h.

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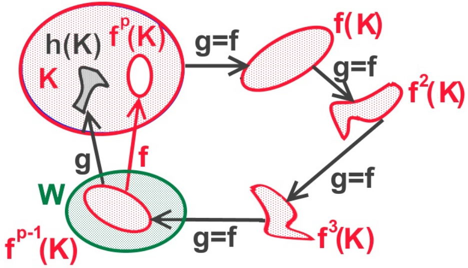
For a generic map $f \in C^0(M)$ there exists a periodic shrinking box K such that the return map $f^p|_K$ coincides, up to the conjugacy that transforms K onto the cube $D^m := [0,1]^m$, with a model map $h : D^m \mapsto D^m$.

Proof



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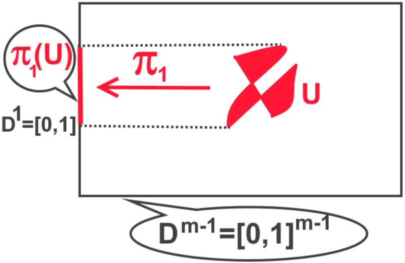
• END DE PROOF: Joining Lemma 2 and Main Lemma conclude that generic $f \in C^0(X)$ has an ergodic measure with infinite entropy.

Construction of the MODEL maps in the cube $D^m := [0,1]^m$.

Step 1: THE PROJECTION π_1 of D^m onto the interval $[0,1]$

$\pi_1 : D^m \mapsto D^1 := [0,1]$ is the following **projection**

$$\pi_1(x_1, x_2, \ldots, x_{m-1}, x_m) := x_m \in [0,1].$$



Construction of the MODEL maps in the cube $D^m := [0,1]^m$.

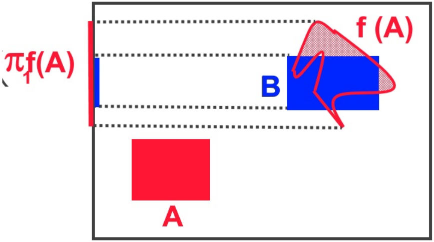
Step 2: The $\pi_1 f$-covering relation between boxes

Let A, B be two boxes in the interior of D^m. Let $f \in C^0(D^m)$.

Definition

A $\pi_1 f$-covers B; $\quad A \to_{\pi_1 f} B$ if

- interior$(f(A)) \cap B \neq \emptyset$,
- interior$(\pi_1 f(A)) \supset \pi_1 B$.



Construction of the MODEL maps in the cube $D^m := [0,1]^m$.

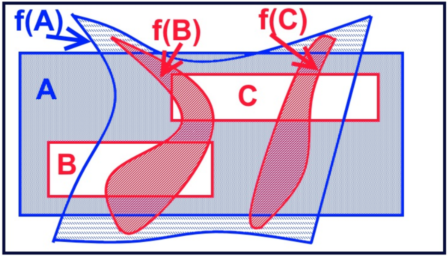
Step 3: ATOMS OF GENERATION 0 and 1.

Let A, B, C be three boxes in the interior of D^m such that

- $B, C \subset$ interior(A), $\quad B \cap C = \emptyset$,
- $B \to_{\pi_1 f} A$, $\quad C \to_{\pi_1 f} A$.

Definition

If so, we call **A the atom of generation 0**,
and **B, C the two atoms of generation 1**.



REMARK: The above condition is **OPEN** in $C^0(D^m)$. The **same** boxes A, B, C are also atoms of gen. 0 and 1 resp. $\forall\, g$ near enough f.

Construction of the MODEL maps in the cube $D^m := [0,1]^m$.

Step 5: ATOMS OF GENERATION n

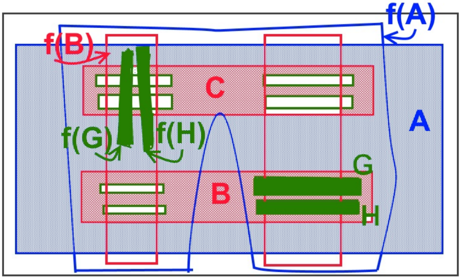
By induction on $n \geq 1$:

• $\mathcal{A}_n$ is a finite collection of exactly $2^{n(n+1)/2}$ pairwise disjoint compact boxes such that, for an adequate collection of pairs (B, C) of atoms of gen. $n-1$, there exists exactly two different boxes in $G, H \in \mathcal{A}_n$ such that

• $G, H \subset \text{int}(B)$ • $G \mapsto_{\pi_1 f} C, \quad H \mapsto_{\pi_1 f} C$.

Definition

If so, the boxes of $\mathcal{A}_n$ are called **the atoms of generation n**.



Construction of the MODEL maps in the cube $D^m := [0,1]^m$.

Final step: Definition of the MODEL

Definition

We call a map $f \in C^0(D^m)$ **a MODEL** if there exists a sequence

$$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n, \ldots$$

of finite collections $\mathcal{A}_n$ of pairwise disjoint boxes such that:

• For all $n \geq 0$, the boxes of $\mathcal{A}_n$ satisfy the definition of being atoms of generation n for f.

• $\lim_{n \to +\infty} \max_{A \in \mathcal{A}n} \text{diam}(A) = 0$.

Main Lemma

If $f \in C^0(D^m)$ is a model then it has an ergodic measure ν such that $h_\nu(f) = +\infty$.

Route of the proof.

• DEFINITION: The Λ-**set** is

$$\Lambda := \bigcap_{n \geq 0} \bigcup_{A \in \mathcal{A}_n} A.$$

• Λ is a Cantor set.
• Λ is f-invariant: $f(\Lambda) = \Lambda$.
• The Borel σ-algebra in Λ is generated by the atoms $A \cap \Lambda$.

• CONSTRUCT the pre-measure ν on Λ:

$$\nu(A \cap \Lambda) = \frac{1}{\# \mathcal{A}_n} = \frac{1}{2^{n(n+1)/2}} \qquad \forall\ A \in \mathcal{A}_n, \ \ \forall\ n \geq 0.$$

• The above pre-measure defines a unique Borel probability measure ν supported on Λ.
• ν is f-invariant and ergodic.

• Compute the **metric entropy of** ν and check that $h_\nu(f) = +\infty$.

Conclusions and further results: • Infinite metric entropy measures do not exist if $f \in C^{\text{Lips}}(M)$ because $h_{top}(f) < +\infty$.

• $h_{top}(f) = +\infty$ for generic $f \in C^0(M)$
(Yano, Inv. Math. 1980).

$\Rightarrow$ (1): *For all $K > 0$ there exists f-invariant μ_K such that* $h_{\mu_K}(f) \geq K$.

• (1) $\not\Rightarrow$ $\exists\ \mu$ such that $h_\mu(f) = +\infty$, because the metric entropy function is not upper semi-continuous.

• $h_{top}(f) = +\infty$ also for generic $f \in C^{\text{Hölder}}(M)$
(de Faria - Hazard - Tresser, ArXiv 2017).

• Does Theorem 1 also hold in $C^{\text{Hölder}}(M)$?
• If $\dim(M) \geq 2$, does Theorem 1 also hold in Homeo(M)?
• Theorem 1 is false in Homeo(M) if M is only a compact metric space but not a manifold (Akin-Glasner-Weiss, Trans. AMS, 2008).

THANK YOU!